Periodic points on the regular n-gon

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Abstract

Using the transfer principle, we classify the periodic points on the regular n-gon and double n-gon translation surfaces and deduce consequences for the finite blocking problem on rational triangles that unfold to these surfaces.

1 Introduction

The group $\mathrm{GL}^+(2,\mathbb{R})$ acts on the moduli space of translation surfaces. This action is generated by complex scalar multiplication and Teichmüller geodesic flow. The stabilizer $\mathrm{SL}(X,\omega) < \mathrm{SL}(2,\mathbb{R})$ of a point (X,ω) in the stratum is called the *Veech group*. The Veech group is also the image of the affine diffeomorphism group $\mathrm{Aff}(X,\omega)$ under the map $D : \mathrm{Aff}(X,\omega) \to \mathrm{SL}(2,\mathbb{R})$ which sends an affine diffeomorphism to its constant derivative. If the Veech group is a lattice, (X,ω) is called a *Veech surface*.

Definition 1.1. Let (X,ω) be a Veech surface. A *periodic point* is a point $p \in (X,\omega)$ that is not a singularity of ω such that the orbit of p under the affine diffeomorphism group is finite.

Remark 1.2. A version of this definition which includes the singularities of ω first appeared in [**GHS03**]. Our definition is the one used in [**Api20**]. Under the original definition, an equivalent notion of a periodic point is a point marked by a holomorphic multisection of the universal curve over a Teichmüller curve. See [**Mö06**, Lemma 1.2] for details.

Consider the following translation surfaces. For n even, the *regular n-gon* is the regular n-gon with opposite sides identified, and for n odd, the *double n-gon* is two copies of a regular n-gon differing by a rotation by π with parallel sides glued together. The regular 8-gon and the double 5-gon are depicted in Figure 2. The rotation by π on both the regular n-gon and double n-gon is the hyperelliptic involution. The *Weierstrass points* are the fixed points of this involution. For the regular n-gon, the Weierstrass points are the center, midpoints of the sides, and the vertices. For the double n-gon, the Weierstrass points are just the midpoints of the sides and the vertices. In [**Vee89**], Veech proved that both of these families of translation surfaces are Veech surfaces.

This main theorem of our paper classifies the periodic points of the regular n-gon and double n-gon.

Theorem 1.3. *Let $n = 5$ or $n \geq 7$. The periodic points of the regular n-gon and double n-gon are exactly the Weierstrass points which are not singularities of the flat metric.*

Remark 1.4. This was already shown in the cases $n = 5, 8, 10$ by Möller in [**Mö06**, Theorems 5.1, 5.2].

The proof can be broken down into two main steps. First, we use the transfer principle to reduce the problem to classifying periodic points on an explicit set of line segments on (X, ω). Then we examine how these segments intersect different sets of cylinders and use flat surface techniques to deduce which points on these lines are periodic.

The regular n-gon and double n-gon are special cases of a larger infinite family of Veech surfaces called the Veech-Ward-Bouw-Möller surfaces [**BM10**]. An interesting next step would be to classify periodic points on these surfaces. We believe our methods would be applicable to these surfaces since they also have presentations as gluings of regular n-gons [**Hoo13, Wri13**].

Theorem 1.3 has immediate consequences for the finite blocking problem in translation surfaces and billiards.

Definition 1.5. Two points P, Q on a billiard table (resp. translation surface) M are *finitely blocked* if there is a finite set of points $S \subset M$ such that all billiard trajectories (resp. straight line segments which do not contain singularities in their interior) from P to Q pass through a point in S.

The following corollaries will be proven at the end of the paper.

Corollary 1.6. *When $n = 5$ or $n \geq 7$, the pairs of finitely blocked points on the regular n-gon and double n-gon are any point that is not a singularity and its image under the hyperelliptic involution.*

Via the unfolding construction of Katok-Zemlyakov [**ZK75**], the billiard flow on the $(\frac{\pi}{2}, \frac{\pi}{n}, \frac{(n-2)\pi}{2n})$ triangle unfolds to the regular n-gon or double n-gon when n is even or odd respectively. The $(\frac{2\pi}{n}, \frac{(n-2)\pi}{2n}, \frac{(n-2)\pi}{2n})$ triangle unfolds to the regular n-gon, the double n-gon, or a double cover of one of these surfaces. Therefore, an immediate consequence of the previous corollary is the following.

Corollary 1.7. *When n is greater than or equal to 8 and even, the only pair of finitely blocked points on the $(\frac{\pi}{2}, \frac{\pi}{n}, \frac{(n-2)\pi}{2n})$ (resp. $(\frac{2\pi}{n}, \frac{(n-2)\pi}{2n}, \frac{(n-2)\pi}{2n})$) triangle is the vertex corresponding to the angle $\frac{\pi}{n}$ (resp. $\frac{2\pi}{n}$) and itself. When n is greater than or equal to 5 and odd, there are no finitely blocked points on the $(\frac{\pi}{2}, \frac{\pi}{n}, \frac{(n-2)\pi}{2n})$ (resp. $(\frac{2\pi}{n}, \frac{(n-2)\pi}{2n}, \frac{(n-2)\pi}{2n})$) triangles.*

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2 Proof

To classify the periodic points, it is useful to know the Veech group. In [**Vee89**, Definition 5.6, Theorem 5.8] (see also [**MT02**, Theorem 5.4]), Veech calculated the Veech groups of the regular n-gon and double n-gon. Let

$$r = \begin{pmatrix} \cos\frac{\pi}{n} & -\sin\frac{\pi}{n} \\ \sin\frac{\pi}{n} & \cos\frac{\pi}{n} \end{pmatrix}, \qquad s = \begin{pmatrix} 1 & 2\cot\frac{\pi}{n} \\ 0 & 1 \end{pmatrix}.$$

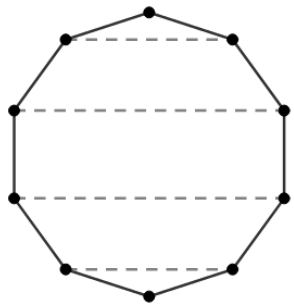

Figure 1: A cylinder decomposition for the regular n-gon when $n = 4k+2$.

Theorem 2.1. *For n even, the Veech group of the regular n-gon is generated by $\langle r^2, s, rsr^{-1}\rangle$ and isomorphic to the $(n/2, \infty, \infty)$ triangle group. For n odd, the Veech group of the double n-gon is generated by $\langle r, s\rangle$ and isomorphic to the $(2, n, \infty)$ triangle group. In particular these groups are lattices in $\mathrm{SL}(2,\mathbb{R})$. In the former case the quotient of $\mathrm{SL}(2,\mathbb{R})$ by the lattice has two cusps and in the latter case one cusp.*

Lemma 2.2. *Let (X,ω) be the regular n-gon or double n-gon. The ratios of circumferences (or heights) of any two cylinders in the same direction is irrational.*

Proof. When $n = 4k+2$, we may label the $k+1$ cylinders depicted in Figure 1 by $j = 1, \ldots, k+1$ from top to center. The circumferences are

$$c_j = 2\sin\frac{(2j-2)\pi}{n} + 2\sin\frac{2j\pi}{n}, \text{ for } j < k$$
$$c_{k+1} = 2\cos\frac{\pi}{n}.$$

We get that for $i, j < k+1$,

$$\frac{c_i}{c_j} = \frac{\sin(2i-1)\pi/n}{\sin(2j-1)\pi/n},$$

which is irrational by [**McM06**, Page 7]. When $j = k+1$, we have

$$\frac{c_i}{c_{k+1}} = 2\sin(2i-1)\pi/n,$$

which is irration by Niven's theorem. A similar trigonometric calculation shows the claim for all other values of n and other cylinder directions.

Now we prove the lemma for heights of cylinders. Define the *modulus* of a cylinder to be the ratio of the height and circumference. By [**Vee89**, Equations 5.2 and 5.4], the ratio of moduli of cylinders in the same direction is rational. Since the ratio of circumferences is irrational, the ratio of heights is irrational. □

Note that there are simpler proofs of Corollary 2.4 below than by using Lemma 2.3, but Lemma 2.2 will be used in the proof of Corollary 1.6.

Lemma 2.3. *Let (X,ω) be the regular n-gon or double n-gon. There is no translation surface (X',ω') such that there is a nontrivial translation cover $\pi : (X,\omega) \to (X',\omega')$.*

Proof. The cover π must send a cylinder with circumference c to one with circumference c/m, where m is some positive integer. By Lemma 2.2 any two cylinders in the same direction have an irrational ratio of circumferences, must be sent to distinct cylinders in (X',ω').

For $n = 4k+2$, the regular n-gon M belongs to $\mathcal{H}(k-1,k-1)$ which is a locus of genus k translation surfaces, and there is a cylinder direction on (X,ω) with $k+1$ cylinders. Thus, (X',ω') has $k+1$ cylinders in that direction and at most 2 singularities.

The shears of these $k+1$ cylinders span a $(k+1)$-dimensional subspace in $H^1(X,\Sigma;\mathbb{C})$. Since there are at most 2 singularities, the kernel of the projection $H^1(X,\Sigma;\mathbb{C}) \to H^1(X;\mathbb{C})$ is dimension at most 1, so the space of shears of the cylinders projects to an isotropic subspace of absolute cohomology of dimension at least k. Thus, (X',ω') has genus at least k, so π_ω must be degree 1. A similar argument holds for $n = 4k$ and $n = 2k+1$ since (X,ω) belongs to $\mathcal{H}(2k-2)$, and there exists a cylinder direction with k cylinders. $\square$

Corollary 2.4. *For the regular n-gon and double n-gon, the affine diffeomorphism group is isomorphic to the Veech group.*

Proof. Let $\mathrm{SL}(X,\omega)$ be the Veech group, $\mathrm{Aff}(X,\omega)$ the affine diffeomorphism group, and $\mathrm{Aut}(X,\omega)$ the group of translation automorphisms. They fall into an exact sequence

$$0 \longrightarrow \mathrm{Aut}(X,\omega) \longrightarrow \mathrm{Aff}(X,\omega) \longrightarrow \mathrm{SL}(X,\omega) \longrightarrow 0.$$

It suffices to show that $\mathrm{Aut}(X,\omega)$ is trivial. This follows from Lemma 2.3 since the translation cover $(X,\omega) \to (X,\omega)/\mathrm{Aut}(X,\omega)$ must be trivial. $\square$

By Corollary 2.4, the hyperelliptic involution, which corresponds to $-\mathrm{Id} \in \mathrm{SL}(X,\omega)$, is in the center of the affine diffeomorphism group. Thus, the Weierstrass points are periodic points. The difficulty in proving Theorem 1.3 lies in showing that all other points are not periodic. We first use the transfer principle to rule out all points except finitely many lines.

Proposition 2.5. *Let Γ be the Veech group of the regular n-gon or double n-gon (X,ω). If n is even, all periodic points must lie on the Γ orbit of one of the following two segments: the segment connecting the center of the polygon to a vertex of the n-gon and the segment connecting the center to the midpoint of an edge, as shown in Figure 2 left. If n is odd, all periodic points must lie on the Γ orbit of a horizontal saddle connection shown in Figure 2 right.*

Proof. First we relate the action of the affine diffeomorphism group $\mathrm{Aff}(X,\omega)$ on (X,ω) to the action of Γ on $\mathbb{R} - \{0\}$. Let $\phi \in \mathrm{Aff}(X,\omega)$ with derivative $A \in \Gamma$. Let α be a segment on (X,ω). Let P, Q be the endpoints of α, where P is a point fixed by $\mathrm{SL}(X,\omega)$. The holonomy $\mathrm{hol}(\alpha)$ lies in $\mathbb{R} - \{0\}$, and holonomy of the segment $\phi(\alpha)$ is $A\,\mathrm{hol}(\alpha)$. Since the endpoints of $\phi(\alpha)$ are P and $\phi(Q)$, if Q is a periodic point, then there is an $\epsilon > 0$ such that for any $\phi \in \mathrm{SL}(X,\omega)$ any segment connecting P and $\phi(Q)$ has holonomy greater than ϵ. Thus, we may determine that Q is not a periodic point by showing the orbit of $\mathrm{hol}(\alpha)$ under Γ is dense.

Let G, H be groups such that $G \times H$ acts on a space $\mathcal{X}$. Because these actions commute, we will denote the action by G with a left action and the action by H by a right action. The transfer principle states that the closed (resp. dense) orbits of $G \times H$ on $\mathcal{X}$ are in bijection with the closed

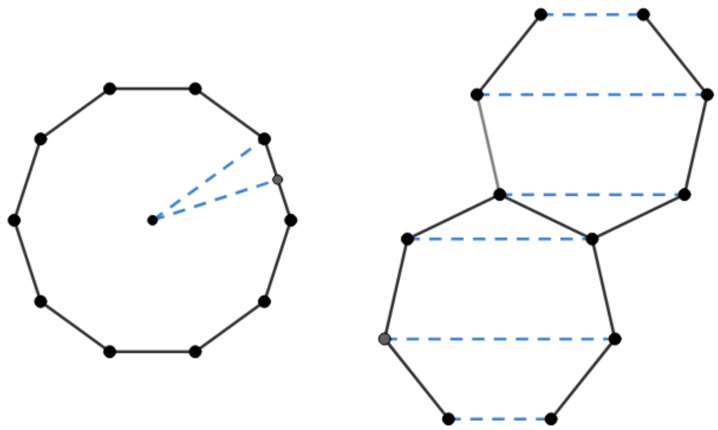

Figure 2: After using the transfer principle, the dashed lines are the only ones left to rule out.

(resp. dense) orbits of G on $\mathcal{X}/H$ and the closed (resp. dense) orbits of H on $G\backslash\mathcal{X}$. Under these correspondences, a $G \times H$ orbit of $x \in \mathcal{X}$ will be mapped to a G orbit of xH or an H orbit of Gx. In our context, G is Γ, $\mathcal{X}$ is $\mathrm{SL}(2,\mathbb{R})$, and H is the unipotent subgroup $U := \left\{ \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} : s \in \mathbb{R} \right\}$. It is a classical result that the only U orbits of $\Gamma\backslash\mathrm{SL}(2,\mathbb{R})$ are closed or dense, and the closed orbits are horocycles around the cusps.

Consider the case when n is even. Since Γ is the $(n/2, \infty, \infty)$ triangle group by Theorem 2.1, there are two cusps of $\Gamma\backslash\mathrm{SL}(2,\mathbb{R})$. The closed orbits in $\Gamma\backslash\mathrm{SL}(2,\mathbb{R})$ around the cusps are the U orbits of

$$\Gamma \begin{pmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{pmatrix} \text{ and } \Gamma \begin{pmatrix} \cos 2\pi/n & -\sin 2\pi/n \\ \sin 2\pi/n & \cos 2\pi/n \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{pmatrix},$$

where $\lambda > 0$. Under the bijection these correspond to the Γ orbits of

$$\begin{pmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{pmatrix} U \text{ and } \begin{pmatrix} \cos 2\pi/n & -\sin 2\pi/n \\ \sin 2\pi/n & \cos 2\pi/n \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{pmatrix} U,$$

We identify $\mathrm{SL}(2,\mathbb{R})/U$ with $\mathbb{R}^2 - \{0\}$ by sending a matrix $A \in \mathrm{SL}(2,\mathbb{R})$ to $A\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. These are the Γ orbits of the lines $y = 0$ and $y = \tan(\pi/n)x$ in $\mathbb{R}^2 - \{0\}$. We choose the point P to be the center of the n-gon, which is fixed by $\mathrm{Aff}(X,\omega) \cong \Gamma$ since it is fixed by the generators of Γ listed in Theorem 2.1. If Q is a periodic point, the holonomy of α must lie on the Γ orbit of either $y = 0$ and $y = \tan(\pi/n)x$, so Q must lie on the Γ orbit of one of the lines described in the proposition.

For n odd, we let P be the unique singularity of X. It is fixed by $\mathrm{Aff}(X,\omega)$. Since Γ is the $(2, n, \infty)$ triangle group, there is one cusp of $\Gamma\backslash\mathrm{SL}(2,\mathbb{R})$. By a similar argument as the case above, the closed orbits around cusps correspond to the Γ orbit of the line $y = 0$ in $\mathbb{R}^2 - \{0\}$. The only points that have this holonomy are the Γ orbits of the lines described in the statement of the proposition. $\square$

We will hold to the convention that all our cylinders are closed; that is, they contain their boundary.

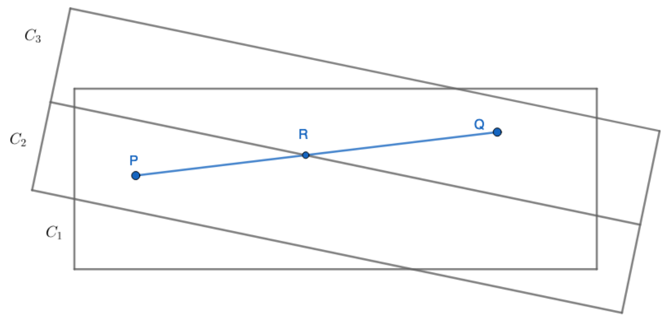

Figure 3: The three cylinders in Lemma 2.8.

Definition 2.6. Let C be a cylinder and B be one of the boundary circles of the cylinder. A point $p \in C$ has *rational height* in C if the distance between p and B is a rational multiple of the height of C.

The following lemma appeared in [**Api20**, Lemma 5.4].

Lemma 2.7. *Let (X, ω) be a translation surface and ϕ be a parabolic in the Veech group $SL(X, \omega)$. Let C be a cylinder in the corresponding parabolic direction. A point $p \in C$ has finite orbit under ϕ if and only if p has rational height in C.*

Proof. We can rotate the translation surface so that the cylinder is horizontal, so we may assume the parabolic is of the form $\begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$. Let h and c be the height and circumference of C respectively. Since ϕ must preserve C, it must be that $s = kc/h$ for some integer k. We may choose flat coordinates so that the bottom boundary of the surface is on the x-axis. Then ϕ maps a point $(x, y) \in C$ to $(x + ykc/h, y) \in C$, where the x coordinate is taken modulo c. Thus, (x, y) has finite orbit if and only if y is a rational multiple of h. □

Lemma 2.8. *Let C_2 and C_3 be two parallel cylinders sharing a boundary saddle connection B and let C_1 be another cylinder which intersects each of C_2 and C_3. Suppose that $\overline{PQ}$ is a line segment lying inside the intersection of C_1 and $C_2 \cup C_3$ and which intersects B at a single point R, so that $\overline{PR}$ lies inside $C_1 \cap C_2$ and $\overline{RQ}$ lies inside $C_1 \cap C_3$, as illustrated in Figure 3. Suppose that P has rational height in C_1 and C_2 and Q has rational height in C_1 and C_3. If the heights of the cylinders C_2 and C_3 do not have a rational ratio, then there are no other points on $\overline{PQ}$ that have rational height in C_1 and rational height in either C_2 or C_3.*

Proof. Assume for the sake of contradiction that there is a point S on $\overline{PR}$ other than P with rational height inside both C_1 and C_2. Let us construct a right triangle PQU inside C_1 as shown in Figure 4 with hypotenuse $\overline{PQ}$ and sides perpendicular and parallel to the boundary circles of C_1. Because the triangles SQT and PQU shown in Figure 4 are similar and the points P and Q are assumed to lie at a rational height in C_1, it follows that the length of $\overline{PS}$ is a positive rational multiple of the length of $\overline{PQ}$. Since $\overline{PS}$ also lies inside C_2, the same kind of argument implies that the length of $\overline{PS}$ is a rational multiple of the length of $\overline{PR}$. It follow that the lengths $\overline{PR}$ and $\overline{PQ}$

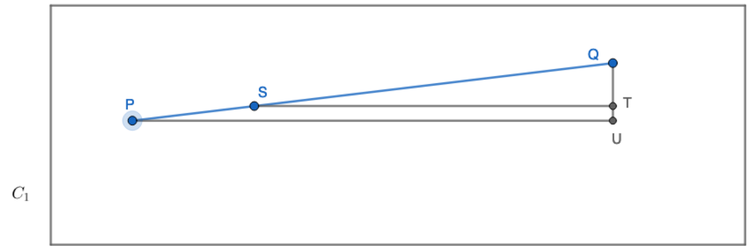

Figure 4: The triangles SQT and PQU are similar.

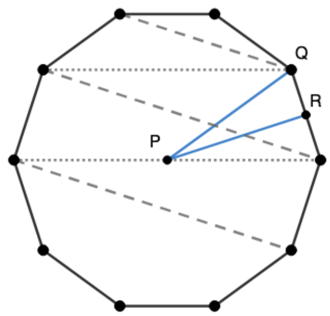

Figure 5: The cylinders dividing candidate lines in the even case.

have a rational ratio. Reversing the argument, this shows that the heights of C_2 and C_3 have a rational ratio, contradicting the assumption. The argument would be similar had S been inside C_1 and C_3. □

Proof of Theorem 1.3. Let Γ be the Veech group of the regular or double n-gon. By Proposition 2.5, any periodic points must lie on the Γ orbits of the line segments described in the proposition and depicted in Figure 2. Since Γ takes (non) periodic points to (non) periodic points, it suffices to consider only the segments pictured, which we call the "candidate lines". We will show that all points other than the Weierstrass points on the candidate lines cannot be periodic.

The case of the regular n-gon is depicted in Figure 5. The two candidate lines $\overline{PQ}$ and $\overline{PR}$ lie entirely inside a horizontal cylinder bounded by dotted lines in Figure 5 and each of them is divided in two by the cylinders delimited by dashed lines. By the Veech dichotomy [**Vee89**, Theorem 1.4] (see also [**MT02**, Theorem 5.10]), the Veech group contains parabolic elements which act by Dehn twists along each of these two cylinder directions. Since the endpoints of the candidate lines are periodic, it follows from Lemmas 2.7 and 2.8 that there can only be periodic points on the interiors of the candidate lines if the heights of the two parallel cylinders in Figure 5 have a rational ratio. However, by Lemma 2.2, the ratio of any two heights is irrational.

For the double n-gon, the candidate lines are permuted by the hyperelliptic involution. As a result, it suffices to prove that the candidate lines in a single n-gon contain no other periodic points. Consider a candidate line $\overline{PQ}$ which is not the topmost edge of the polygon, as depicted on the

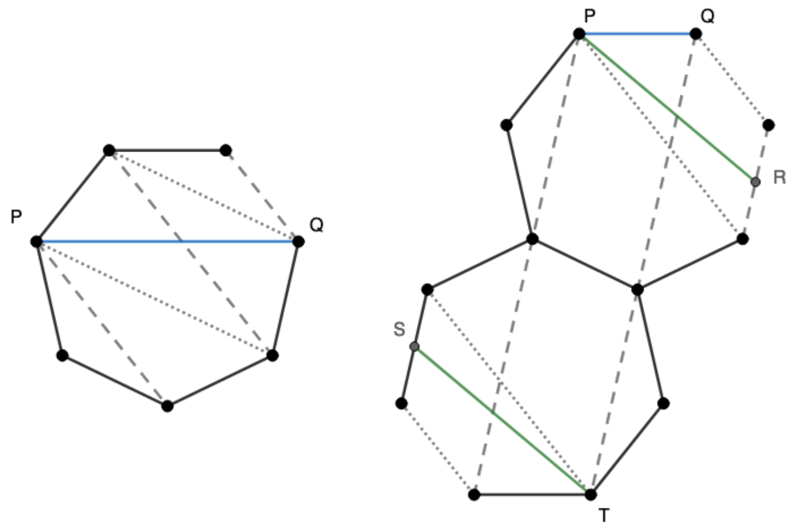

Figure 6: The cylinders dividing candidate lines in the odd case.

left in Figure 6. The argument of the previous paragraph applies to such a candidate line with the cylinders shown in the same diagram. For the edge of the polygon, we turn to the diagram on the right in Figure 6. By applying a Dehn twist along the cylinder marked with a dotted line, the candidate line $\overline{PQ}$ is sent to the two segments $\overline{PR}$ and $\overline{ST}$. The cylinder decomposition shown in the diagram now applies to these two segments. □

Proof of Corollary 1.6. Let (X, ω) be a translation surface that is not a translation cover of a torus. Let (X_{min}, ω_{min}) be the lowest genus translation surface such that there exists a translation cover $\pi_\omega : (X, \omega) \to (X_{min}, \omega_{min})$ and (Y_{min}, q_{min}) the minimal quadratic differential such that there exists a half translation cover $\pi_q : (X, \omega) \to (Y_{min}, q_{min})$. The surfaces (X_{min}, ω_{min}) and (Y_{min}, q_{min}) exist by [**Mö06**, Theorem 2.6] and [**AW17**, Lemma 3.3]. Theorem 3.5 in [**AW17**] states that the only pairs of finitely blocked points on (X, ω) are two points that have the same image under π_q, or pairs of points that are periodic points or singularities.

By Lemma 2.3, (X, ω) is not a torus cover, so (X_{min}, ω_{min}) exists, and in fact $(X, \omega) = (X_{min}, \omega_{min})$ exists by the same lemma. By [**AW17**, Lemma 3.3 and its proof], since $(X, \omega) = (X_{min}, \omega_{min})$ has an involution, (Y_{min}, q_{min}) is the quotient of (X, ω) be this involution.

By [**AW17**, Lemma 3.1], all pairs (p, q) where p is not a singularity and q is its image under the hyperelliptic involution are finitely blocked. (Note that the statement Lemma 3.1 does not cover the case when $p = q$ is a Weierstrass point that is not a singularity. However, the proof is identical.) Using [**AW17**, Theorem 3.5] and the computation of (Y_{min}, q_{min}), we get the only possible pairs of finitely blocked points not listed above are given by two points that are either periodic points or singularities, and by [**AW17**, Theorem 3.15] the blocking set must be the set of periodic points and singularities. By Theorem 1.3, these points are exactly the Weierstrass points and singularities of the flat metric. Because the regular n-gon is convex, a singularity can be connected to any singularity or Weierstrass point by a segment that does not have a Weierstrass point on the interior. Similarly, this is true for the double n-gon because all the singularities and Weierstrass points lie on a single convex n-gon. This shows that a singularity is not finitely blocked from any other point. Similarly, two distinct Weierstrass points are also not finitely blocked. Thus, the only finitely blocked points are the ones listed in the statement of the corollary. □

Proof of Corollary 1.7. Let T be a billiard table that unfolds to a translation surface (X, ω). Two points p and q on T are finitely blocked if and only if every preimage of p is finitely blocked from every preimage of q on (X, ω). The points listed are the only ones with this property. □

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