HILBERT FUNCTIONS OF SQUARE-FREE MONOMIAL IDEALS

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Abstract. In commutative algebra, a common question is to ask how the number of generators of an ideal changes as you change the ideal. In this project, we concentrate on answering this question for square-free monomial (SFM) ideals. Specifically, we explore symbolic powers of an SFM ideal and ask how the number of generators changes as the power increases. The answer to this question is phrased as a Hilbert quasi-polynomial. We introduce several theorems to give insight on the quasi-polynomial of certain families of SFM ideals and how it changes as the ideal itself changes.

Contents

1. Introduction

In commutative algebra, a common question is to ask how the number of generators of an ideal changes as you change the ideal. In this paper, we will first introduce the class of ideals for which we want to study this question, which are known as *square-free monomial ideals*, and a particular way to raise a ideal to a power, called a *symbolic power*. Our main focus is to compute the minimal number of generators of symbolic powers of an ideal and ask how this number of generators changes as you increase the power.

It turns out this can be described by a *quasi-polynomial*, which we completely describe in this paper for every square-free monomial ideal of up to 4 variables, up to relabeling. Furthermore, we prove several theorems of two types: the first being characterizations of the quasi-polynomials of every ideal within a certain family, and the second being rules of how certain changes made to an ideal affect its quasi-polynomial. These theorems can be combined to describe more families in full.

Note: Any known results, including folklore results, have been labeled as “Exercises” when they have been included.

2. Preliminaries

There are many ideals worth studying in commutative algebra and algebraic geometry, and in this paper we will be focusing on a special family: square-free monomial ideals. In this section, we will introduce some basic definitions related to our research and some others to prepare for the tricks that we will later play. Throughout the entirety of this paper, we will be working in the ring $R = \mathbb{K}[x_1, \cdots, x_n]$ of polynomials over a field $\mathbb{K}$.

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2.1. **Squarefree Monomial Ideals and Simplicial Complexes.** We first define our objects of interest, the square-free monomial ideals.

Definition 2.1. An ideal $I \subset R$ is *generated by* a set $\{m_1, m_2, \dots, m_k\}$ if every $x \in I$ can be written as the sum of R-multiples of elements in $\{m_1, m_2, \dots, m_k\}$, i.e. $x = \sum_i r_i m_i$ for some $r_i \in R$. Each m_i is called a *generator* of I. And we write $I = (m_1, m_2, \dots, m_k)$.

Definition 2.2. We say an ideal $I \subset R$ is a *monomial ideal* if the following equivalent conditions hold:

(1) I can be generated by monomials
(2) If $f = f_1 + \cdots + f_k \in I$ where each f_i is a monomial, then $f_i \in I$ for all i

The following exercise illustrates how the elements in an ideal relate to the generators of the ideal.

Exercise 2.3. Let $I \subset R$ be a monomial ideal generated by monomials $m_1, \cdots, m_n$. Then a monomial $m \in I$ if and only if there exists an i such that $m_i | m$.

Proof. For the forward direction, consider $m \in I$. It can be written $m = \sum_{i=1}^{n} r_i m_i$. Since m is a monomial, $m = r_i m_i$. Thus, $m_i | m$. For the reverse direction, consider a monomial m such that $m_i | m$. Then $m = r m_i$ for some $r \in R$ so $m \in I$. $\square$

Now, we continue with the definition of the SFM ideal.

Definition 2.4. A *square-free monomial* has no variable with power greater than 1. A *square-free monomial (SFM) ideal* can be generated by only square-free monomials.

The following three definitions describe ideals which contain much information about the ideals they are related to, and are in general very useful in proving propositions related to ideals.

Definition 2.5. Let S be a commutative ring and $P \subset S$ be an ideal. We say P is a *prime ideal* if for all $a, b \in S$ such that $ab \in P$, we have either $a \in P$ or $b \in P$.

Definition 2.6. The *radical of an ideal* I in a ring S is the ideal $\sqrt{I} = \{f \in S | f^n \in I \text{ for some } n \in \mathbb{N}\}$; we say that an ideal is *radical* if $\sqrt{I} = I$

Definition 2.7. A ring S is *Noetherian* if all ascending chains of ideals $I_1 \subset I_2 \subset \dots$ in S stabilize, i.e. there exists some N such that $I_n = I_N$ for all $n \geq N$.

Note that $R = \mathbb{K}[x_1, \cdots, x_n]$ is Noetherian so we will be working with a Noetherian ring throughout the entirety of this paper.

Before we dive deeper into the research, we want to help readers better understand 1) the internal structure of the family of square-free monomial ideals and 2) some important facts that we may not use explicitly for the rest of the paper, but are essential to one's understanding of ideals and will be very helpful in future research.

Exercise 2.8. The two conditions in Definition 2.2 are equivalent.

Proof. ($1 \implies 2$) Suppose I is generated by monomials $m_1, \dots, m_n$, and let $f = f_1 + \cdots + f_k \in I$ where each f_i is a monomial. We know we can also write f as a sum of monomials of the form rm_i where $r \in R$, since I must be generated by $m_1, \dots, m_n$. Then, since we are in the polynomial ring, we must have f_i is a sum of some of the terms of this form, and since f_i we a monomial we have $f_i = r \cdot m_1^{a_1} \cdots m_n^{a_n}$, where $r \in R$ and the exponents are non-negative integers (and at least one is positive). Thus, $f_i \in I$ since each f_i can be generated by the generators of I.

($2 \implies 1$) Consider a polynomial generator m_i in I such that $m_i = m_{i1} + \cdots + m_{ij}$, where $m_{i1} \dots m_{ij}$ are all monomials. Then by assumption, $m_i = m_{i1} + \cdots + m_{ij} \in I$ and $m_{i1}, \dots, m_{ij} \in I$. Thus, we can replace m_i with the list $m_{i1}, \cdots$, and m_{ij}. Proceeding through each nonmonomial generator in this manner gives a set of monomial generators. $\square$

Exercise 2.9. If I is a monomial ideal, then $\sqrt{I}$ is a monomial ideal.

Proof. Let I be a monomial ideal, and consider some $r \in \sqrt{I}$ and monomials $r_1, r_2, \dots, r_n$ such that $r = r_1 + r_2 + \cdots + r_n$. Then there exists some power m such that $r^m \in I$. Note that r_i^m is a monomial term of r^m for every $i \in \{1, 2, \dots, m\}$. By Exercise 2.8, $r_i^m \in I$ for all i. This implies that $r_i \in \sqrt{I}$ for all i by the definition of the radical. It follows immediately by Exercise 2.8 that $\sqrt{I}$ is a monomial ideal, since $r \in \sqrt{I}$ implies $r_i \in \sqrt{I}$ for all i. □

Exercise 2.10. A monomial ideal is square-free if and only if it is radical.

Proof. Let us first consider the forward direction. Let I be a square-free monomial ideal in $\mathbb{K}[x_1, x_2, ..., x_n]$. We already know that $I \subseteq \sqrt{I}$, so we only need to show $I \supseteq \sqrt{I}$. Consider a generator $r \in \sqrt{I}$. Then by definition, there is some p such that $r^p \in I$. By Exercise 2.9, we may assume without loss of generality that r is a monomial. Then $r = x_{i_1} x_{i_2} \cdots x_{i_k}$ for some $i_1, \dots, i_k \in [1, n]$, so $r^p = x_{i_1}^p x_{i_2}^p \cdots x_{i_k}^p$. Since I is square-free and monomial, there exists some square-free monomial generator $m \in I$ such that $m | x_{i_1}^p x_{i_2}^p \cdots x_{i_k}^p$. Since m is square-free, this implies $m | x_{i_1} x_{i_2} \cdots x_{i_k}$, thus $m | r$ so $r \in I$.

For the reverse direction, we proceed by proving the contrapositive. Suppose I is a monomial ideal which is not square-free (we will show that $I \neq \sqrt{I}$). Then I has some minimal monomial generator which is not square-free, given by $m_0 = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$ for some powers $\alpha_1, \dots, \alpha_n$ not all less than 2. It follows that the square-free monomial $s = x_1^{\beta_1} x_2^{\beta_2} \cdots x_n^{\beta_n}$ is in $\sqrt{I}$ for $\beta_i = 1$ if $\alpha_i \geq 1$ and $\beta_i = 0$ if $\alpha_i = 0$. Since m_0 is minimal and $s | m_0$, we know s must not be in I. Thus, $I \neq \sqrt{I}$, so I is not radical. □

In order to prove more results like the previous one, we need to have a more concrete way to describe the structure of square-free monomial ideals. After some reading, we found that there is a one-to-one correspondence between square-free monomial ideals and simplicial complexes, which we now introduce.

Notation 2.11. For any set S, we denote by $P(S)$ the *power set* of S, or the set of all subsets of S.

Example 2.12. There are four subsets of the set with two elements. In particular:

$$P(\{x, y\}) = \{\varnothing, \{x\}, \{y\}, \{x, y\}\}.$$

Definition 2.13. A *simplicial complex* on n elements is a subset $\Delta \subset P(\{x_1, \dots, x_n\})$ such that if $S \in \Delta$ and $S' \subset S$ then $S' \in \Delta$.

Simplicial complexes have nice, pictorial representations. For example, by definition of simplicial complexes, if Δ is a simplicial complex and any subset of Δ corresponds to a tetrahedron, then Δ will also contain subsets corresponding to each face, edge, and vertex of this tetrahedron. Geometrically, we can consider the following correspondence:

subset type	geometric shape
$P(\{x\})$	point
$P(\{x, y\})$	edge
$P(\{x, y, z\})$	filled triangle
$P(\{x, y, z, w\})$	solid tetrahedron

For a square-free monomial ideal I, we can associate a simplicial complex in the following way. Given I's unique prime decomposition (see the next subsection)

$$I = P_1 \cap \cdots \cap P_k$$

then I's simplicial complex is the simplicial complex with a simplex for each P_i. Namely, if $P_i = (x_{i_1}, \dots, x_{i_\ell})$ we include the corresponding simplex $P(\{x_{i_1}, \dots, x_{i_\ell}\})$. We will now further explain how we draw the pictures by an example.

Example 2.14. Let $R = \mathbb{K}[x_1, x_2, ..., x_n]$, and let $I = (x) \cap (y, z, w) = (xy, xz, xw)$ be a square-free monomial ideal in R. The picture of I's simplicial complex is in Figure 1.

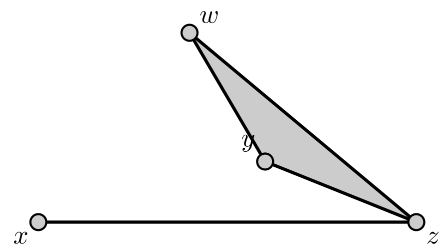

FIGURE 1. The simplicial complex for the (xy,xz,xw)

At the beginning of the research, the pictures for simplicial complexes help us collect all the cases in 4 variables and better understand the combinatorial structure of the ideals.

2.2. **Primary Decomposition and Powers.** In order to explore the Hilbert polynomials and Hilbert quasi-polynomials related to ideals, we must first understand powers of ideals. The *primary decomposition* of an ideal is useful in finding the *symbolic power* of square-free monomial ideals and is essential to classification of ideals through the previously mentioned simplicial complexes. So, we will proceed by discussing powers and primary decompositions.

There are two ways to take a power of an ideal: ordinary, which we will see gives a Hilbert polynomial, and symbolic, which gives a Hilbert quasi-polynomial. We first introduce the ordinary power:

Definition 2.15. Let I be an ideal of R. Then we define the ordinary power I^d to be the ideal generated by the product of any d elements of I, namely:

$$I^d := (\{a_1 \cdots a_d \mid a_i \in I \text{ for all } 1 \leq i \leq d\})$$

We can now say the following about the relationship between an ideal's minimal generators and its ordinary powers:

Exercise 2.16. If $I = (f_1, \dots, f_k)$ then I^d is generated by products of d generators. In other words, I^d is generated (possibly not minimally) by $\{f_{i_1} \cdots f_{i_d} \mid 1 \leq i_j \leq k\}$.

Proof. Let $I = (f_1, \dots, f_k)$ be an ideal of R. Then by definition, I^d is generated by $\{a_1 \cdots a_d \mid a_i \in I \text{ for all } 1 \leq i \leq d\}$. Note that each $a_i = r_1 f_1 + \cdots + r_k f_k$ for $r_i \in R$. Then it means that every possible (but not necessarily minimal) generators are in the form:

$$(r_{11} f_1 + \cdots + r_{1k} f_k) \cdots (r_{d1} f_1 + \cdots + r_{dk} f_k)$$

It is now clear that every possible generator can be generated by $\{f_{i_1} \cdots f_{i_d} \mid 1 \leq i_j \leq k\}$. Then it follows directly that I^d is generated by $\{f_{i_1} \cdots f_{i_d} \mid 1 \leq i_j \leq k\}$. □

The following exercise illustrates that powers of SFM ideals are still SFM, and therefore agree with everything we've discovered so far.

Exercise 2.17. If I is a monomial ideal, then I^d is also a monomial ideal for any $n \in \mathbb{N}$.

Proof. We will proceed by proving the contrapositive. Suppose I^d is not a monomial ideal. Then it must have at least one minimal generator that is not a monomial: call it m'. By Exercise 2.16, we know that I^d is generated by products of d generators of I, so m' must be a product of d generators of I. This implies that I must have a non-monomial generator, since the product of monomials is always a monomial but m' is a non-monomial product of generators of I. Thus, I must not be a monomial ideal. □

Recall that we are interested in studying the number of minimal generators as the power of an ideal changes. For convenience, we use the following notation:

Notation 2.18. We denote $\mu(I)$ to be the minimal number of generators of I.

We now introduce symbolic powers of an ideal.

Definition 2.19. The d-th symbolic power of a prime ideal $P \subset R = \mathbb{K}[x_1, \ldots, x_n]$ is the set $P^{(d)} = \{a \in R | sa \in P^d \text{ for some } s \notin P\}$.

A theorem of Zariski-Nagata gives that this is equivalent to a very geometric definition for those who know a bit of algebraic geometry.

Theorem (Zariski-Nagata, [Zar49],[Nag62]). *Let $I \subset R = \mathbb{K}[x_1, \ldots, x_n]$ be a prime ideal such that $\mathbb{V}(I) = X$ for an irreducible algebraic variety X, then*

$$I^{(d)} = \{f \in R, f \text{ vanishes to order at least } d \text{ on all of } \mathbb{V}(I)\}$$

However, don't be scared by the fancy word such as 'vanishes' since we don't need to know that much to solve our problem. Because we are working with SFM ideals, the computations of symbolic powers are relatively simple and rely on the *primary decomposition*.

For general ideals, we have the following definition, which we won't explain in detail, since we can simplify the problem for SFM ideals.

Definition 2.20. The d-th symbolic power of an ideal $I \subset R$ is

$$I^{(d)} = R \cap \bigcap_{P \in \mathrm{Ass}(P)} I^{(d)} R_P$$

For SFM ideals, this definition simplifies to a nice characterization in terms of their primary decompositions.

Definition 2.21. Let R be a Noetherian commutative ring, and I an ideal in R. Then I has an irredundant primary decomposition into primary ideals.

$$I = Q_1 \cap Q_2 \cdots \cap Q_n$$

A theorem by Lasker and Noether tells us that every ideal in a Noetherian ring can be decomposed as an intersection of finitely many primary ideals [Las05], [Noe21]. We will be interested in using this decomposition to study the symbolic powers of a SFM ideal and to classify different families of ideals. This definition is expanded upon in the following example:

Exercise 2.22. Let I be a square-free monomial ideal. Then I's primary decomposition is of the form $I = Q_1 \cap \cdots \cap Q_k$ where each Q_i is generated by a subset of the variables. Furthermore, if this decomposition is irredundant, i.e. if $Q_i \not\subset Q_j$ for all i and j, then this decomposition is unique up to reordering.

Proof. Observe that since I is a monomial ideal, if $I = (m_1 m_2, J)$ for some list of monomials J and $\gcd(m_1, m_2) = 1$, then $I = (m_1, J) \cap (m_2, J)$. By induction if $I = (x_1^{a_1} \cdots x_n^{a_n}, J)$, then $I = \bigcap_{i=0}^{n} (x_i^{a_i}, J)$. Thus, we may assume that the primary decomposition of I only has primary ideals of the form $(x_1^{a_1}, \cdots, x_n^{a_n})$ by breaking I down from its generators. In particular, since I is square-free, $a_i = 1$ as there were no higher exponents in the generators. The uniqueness follows from the uniqueness of intersections of subsets of the variables. □

The following exercise illustrates the simple computation of symbolic powers of SFM ideals.

Exercise 2.23. If I is a square-free monomial ideal and $I = Q_1 \cap \cdots \cap Q_k$ is a primary decomposition of I, then

$$I^{(d)} = Q_1^d \cap \cdots \cap Q_k^d.$$

This statement is Exercise 2.3 in [Gri19] so we defer the proof.

To further simplify symbolic power computations, we produced the following result:

Exercise 2.24. Let I, J, and K be three square-free monomial ideals such that $I = J \cap K$. Then $I^{(d)} = J^{(d)} \cap K^{(d)}$.

Proof. By Exercise 2.22, we know I, J, and K have primary decompositions, and these are unique up to including redundant ideals. Then let $P_1 \cap P_2 \cap \cdots \cap P_\ell$ be the unique irredundant primary decomposition of J and let $P_{\ell+1} \cap P_2 \cap \cdots \cap P_m$ be the unique irredundant primary decomposition of K. Then $I = J \cap K = P_1 \cap \cdots \cap P_m$. We note that some of the P_i here could be redundant since they could show up in both J and

K, but this will not affect our result since adding a term multiple times in an intersection does not change the intersection. By previous proposition,

$$I^{(d)} = P_1^d \cap \cdots \cap P_m^d = \left(P_1^d \cap \cdots \cap P_\ell^d\right) \cap \left(P_{\ell+1}^d \cap P_m^d\right) = J^{(d)} \cap K^{(d)}.$$

□

2.3. **Hilbert Polynomials and Quasi-Polynomials.** We will be interested in how the number of generators of the regular and symbolic powers of an ideal grows as we take higher and higher powers. It turns out that the number of generators grows like a (quasi-)polynomial. In order to see that, we now want to associate a polynomial to each graded module over a graded ring. We start by associating a function to each one.

Definition 2.25. The Hilbert function of a graded module M over the graded ring $R = \mathbb{K}[x_1, \ldots, x_n]$ is the function $h_M : \mathbb{Z}_{\geq 0} \to Z_{\geq 0}$ by $d \mapsto \dim_{\mathbb{K}} M_d$ where $\dim_{\mathbb{K}} M_d$ is the number of generators of M_d as a vector space over $\mathbb{K}$. We then have the following definition.

In the case that M is *standard graded*, i.e. every element in M is a sum of products of things in the degree 0 and 1 pieces. The following is a theorem of Hilbert, see for example, [BH93] Chapter 4 for details.

Definition 2.26. If M is standard graded, there is a polynomial H_M in the variable d such that $h_M(d) = H_M(d)$ for $d >> 0$. H_M is called the Hilbert polynomial of M.

In the case that M is finitely generated, but not by the degree 1 piece, then the Hilbert function is not a polynomial. It is, however, a quasi-polynomial.

Definition 2.27. A quasi-polynomial is a generalization of polynomials, whose coefficients come from a ring. The coefficients of quasi-polynomials are periodic functions with integral period.

$$q(k) = \alpha_d(k)k^d + \alpha_{d-1}(k)k^{d-1} + \cdots + \alpha_0(k)$$

where $\alpha_i(k)$ is a periodic function with integral period.

We will focus primarily on two cases for an ideal $I \subset \mathbb{K}[x_1, \ldots, x_n]$. In particular, let $\mathfrak{m} = (x_1, \ldots, x_n)$. Then we look at the Hilbert polynomials of

$$M = \bigoplus_{d \geq 0} I^d / \mathfrak{m} I^d$$

and the Hilbert quasi-polynomials of

$$M' = \bigoplus_{d \geq 0} I^{(d)} / \mathfrak{m} I^{(d)}$$

which are related to the Rees algebra $\bigoplus_{d \geq 0} I^d$ and symbolic Rees algebra $\bigoplus_{d \geq 0} I^{(d)}$ of I (they are each of these tensored with $R/\mathfrak{m} = \mathbb{K}$, and are typically called the special fiber ring and symbolic special fiber ring of I).

Since the the (symbolic) Rees algebra for a SFM ideal is finitely generated [HHT07], there is a Hilbert (quasi-) polynomial which computes the minimal number of generators of $(I^{(d)})$ I^d for $d >> 0$. In particular, we note that

$$\dim_{\mathbb{K}} I^d / \mathfrak{m} I^d = \mu(I^d)$$

and

$$\dim_{\mathbb{K}} I^{(d)} / \mathfrak{m} I^{(d)} = \mu(I^{(d)})$$

Throughout, we will call the polynomial $H_M(d) = \mu(I^d)$ for $d >> 0$ the "Hilbert polynomial of I" and denote it by $H_I(d)$ and we will call the quasi-polynomial $H_{M'}(d) = \mu(I^{(d)})$ for $d >> 0$ the "Hilbert quasi-polynomial of I" and denote it by $H_I^s(d)$ when we really mean the Hilbert (quasi-)polynomial of the (symbolic) special fiber ring of I.

3. First Results - Bounds of Regular Powers

Armed with those important facts, we come to the first stop in the journey - ordinary powers. To help readers build some intuition on what we are counting, we include a table of the ordinary powers of square-free monomial ideals in 4 variables.

Ordinary Powers

Name	I	$h_I(n) = (1,2,3,...)$	$H_I(n)$
4v1gA	$(xyzw)$	$1,1,1,...$	1
4v2gA	(x, yzw)	$2,3,4,5,...$	$n+1$
4v2gB	(xy, zw)	$2,3,4,5,...$	$n+1$
4v2gC	(xyz, xw)	$2,3,4,5,...$	$n+1$
4v2gD	(xyz, xyw)	$2,3,4,5,...$	$n+1$
4v3gA	(xy, z, w)	$3,6,10,15,...$	$\frac{(n+2)(n+1)}{2}$
4v3gB	(xy, xz, xw)	$3,6,10,15,...$	$\frac{(n+2)(n+1)}{2}$
4v3gC	(xy, xw, zw)	$3,6,10,15,...$	$\frac{(n+2)(n+1)}{2}$
4v3gD	(xy, xz, w)	$3,6,10,15,...$	$\frac{(n+2)(n+1)}{2}$
4v3gE	(xyz, xw, yw)	$3,6,10,15,...$	$\frac{(n+2)(n+1)}{2}$
4v3gF	(xyz, xyw, zw)	$3,6,10,15,...$	$\frac{(n+2)(n+1)}{2}$
4v3gG	(xyw, xzw, yzw)	$3,6,10,15,...$	$\frac{(n+1)(n+1)}{2}$
4v4gA	(x, y, z, w)	$4,10,20,35,...$	$\binom{n+3}{3}$
4v4gB	(xy, xz, yz, w)	$4,10,20,35,...$	$\binom{n+3}{3}$
4v4gC	(xy, xz, yw, zw)	$4,9,16,25,36,...$	$(n+1)^2$
4v4gD	(xw, yz, yw, zw)	$4,10,20,35,...$	$\binom{n+3}{3}$
4v4gE	(xyz, xw, yw, zw)	$4,10,20,35,...$	$\binom{n+3}{3}$
4v4gF	(xyz, xyw, xzw, yzw)	$4,10,20,35,...$	$\binom{n+3}{3}$
4v5gA	(xy, xz, xw, yz, yw)	$5,14,30,55,91,...$	$\frac{1}{6}(2n+3)(n+1)(n+2)$
4v6gA	(xy, xz, xw, yz, yw, zw)	$6,19,44,85,146,...$	$\frac{1}{3}(n+1)(2n^2+4n+3)$

The following three results arose from observations as we constructed the above table, using Python as an assistant in computing the Hilbert-polynomials.

Lemma 3.1. *Let $I \subset R$ be a square-free monomial ideal which is minimally generated by* 1 *element. Then:*

$$\mu(I^d) = 1$$

for all $d \in \mathbb{N}$.

Proof. Let m be the monomial which minimally generates I. Note that $m^d \in I^d$, by Def 2.15. Consider $h \in I^d$. Then there exist some $k_1, k_2, \dots, k_d \in I$ such that $\prod_{i=1}^d k_i = h$. Since m is the only generator of I, $m | k_i$ for all $i \in [1, d]$, thus $m^d | \prod_{i=1}^d k_i = h$. Thus, every element in I^d is divisible by m^d, hence I^d has only one minimal generator, m^d. □

Lemma 3.2. *Let $I \subset R$ be a square-free monomial ideal which is minimally generated by* 2 *elements. Then:*

$$\mu(I^d) = d + 1$$

for all $d \in \mathbb{N}$.

Proof. Let m_1, m_2 be the minimal generators of I. Note that $m_1 \nmid m_2$ and $m_2 \nmid m_1$.
By Exercise 2.16, I^d is generated by $m_1^a m_2^b$, with $a + b = d$ and $0 \le a, b \le d$. Before we simplify I^d, we see that these d+1 elements are in fact generators of the ideal. Now we want to show there are not any redundant ones. It's enough to show that:

$$m_1^a m_2^b \notin (\{m_1^d, m_1^{d-1} m_2, \dots, m_2^d\} \setminus \{m_1^a m_2^b\})$$

for any $a + b = d$. By the Exercise 2.3, this is equivalent to showing that for all $a' + b' = d$ with $a \neq a'$ and $b \neq b'$, we have:

$$m_1^{a'} m_2^{b'} \nmid m_1^a m_2^b.$$

Assume without loss of generality that $a' < a$. Note, this implies that $b' > b$. Let $k = \gcd(m_1, m_2)$, then $\gcd(\frac{m_1}{k}, \frac{m_2}{k}) = 1$. Then $k^{a'+b'}(\frac{m_1}{k})^{a'}(\frac{m_2}{k})^{b'} \nmid k^{a+b}(\frac{m_1}{k})^{a}(\frac{m_2}{k})^{b}$ if and only if $(\frac{m_1}{k})^{a'}(\frac{m_2}{k})^{b'} \nmid (\frac{m_1}{k})^{a}(\frac{m_2}{k})^{b}$. Cancelling from both sides, we see this is true if and only $(\frac{m_2}{k})^{b'-b} \nmid (\frac{m_1}{k})^{a-a'}$, which is true since $\gcd(\frac{m_1}{k}, \frac{m_2}{k}) = 1$. ☐

Lemma 3.3. *Let $I \subset R$ be a square-free monomial ideal which is minimally generated by 3 elements. Then:*

$$\mu(I^d) = \binom{d+2}{2}$$

for all $d \in \mathbb{N}$.

Proof. Let m_1, m_2, m_3 be the minimal generators of I. Note that they don't divide each other. We already know that the ideal is generated by $m_1^a m_2^b m_3^c$, with $a+b+c = d$, and $0 \leq a, b, c \leq d$, which gives us $\binom{d+3-1}{3-1}$ generators. We want to show that there is no redundant generators, i.e., for all $a'+b'+c' = d$ with $a \neq a'$, $b \neq b'$ and $c \neq c'$, we have:

$$m_1^{a'} m_2^{b'} m_3^{c'} \nmid m_1^a m_2^b m_3^c.$$

Assume for contradiction that $m_1^{a'} m_2^{b'} m_3^{c'} \mid m_1^a m_2^b m_3^c$. As I is a square-free monomial ideal, there is at least one variable that either divides only two generators or only one generator (otherwise the generators would be the same monomial). Assume without loss of generality those are m_1 and m_2 or just m_1.

If the variable w only divides m_1 and m_2 then $a + b = a' + b'$, then we arrive at the conclusion that $c = c'$. If the variable w only divides m_1, then it follows that $a = a'$.

This reduces us to the case when one of a, b, or c is equal to a', b', or c', respectively. Assume without loss of generality that $a = a'$, it follows from Lemma 1.10 that $m_2^{b'} m_3^{c'} \nmid m_2^b m_3^c$. ☐

We will see a similar combinatorial pattern in the Hilbert polynomials for some ideals. However, we note that it is not always this simple (i.e. we do not always have that $\mu(I^d) = \binom{d+k-1}{k-1}$ if I has k generators). Sometimes, we find that minimal generators may combine to create non-unique elements of I^d, as in the following example.

Example 3.4. Let $I = (xy, zw, xz, yw)$. Then note that:

$$(xy)(zw) = (xz)(yw),$$

i.e. we have a situation where $m_1 m_2 = m_3 m_4$.

The following result explains these observations.

Proposition 3.5. *Let $I \subset R$ be a square-free monomial ideal which is minimally generated by k elements. Then:*

$$\mu(I^d) \leq \binom{d+k-1}{k-1}$$

for all $d \in \mathbb{N}$. Furthermore, for $k = 1$, 2, or 3, we get equality.

Proof. The inequality is an immediate consequence of Exercise 2.16. When $k = 1$, 2, or 3, the equality follows from Lemmas 3.1, 3.2, and 3.3, respectively. ☐

We can also describe one clear instance when generators begin canceling each other out due to a pigeonhole-principle-like situation:

Proposition 3.6. *Let $R = \mathbb{K}[x_1, x_2, ..., x_n]$, and let I be a square-free monomial ideal in R which is minimally generated by k elements. If $k > n$, then $\mu(I^d) < \binom{d+k-1}{k-1}$.*

Proof. This is equivalent to the generators being algebraically dependent. As there are more generators than variables, they must be algebraically dependent and the result follows. ☐

Note, the Proposition 3.6 may seem like a contradiction to the previous lemmas, however when $n = 1$, 2, or 3 then any square-free monomial ideal is generated by at most 1, 2, or 3 elements respectively.

4. Symbolic Powers

4.1. **Rules: How generators change as we modify our ideal.** One way to think about SFM ideals is that in many cases ideals in n variables can be constructed from old ideals which have already studied in $n-1$ variables. In other words, what will happen if we add a new variable somewhere in one of the old ideals? This section studies the relation of these new ideals to the old ideals.

Theorem 4.1. *For a field $\mathbb{K}$, let L be a square-free monomial ideal in $R = \mathbb{K}[w, x_1, \dots, x_n]$ minimally generated by $m_1, \dots, m_k$, none of which are divisible by w. Consider the ideals $I = L \cap (w)$ and $J = L \cap (w, m_i)$ in R for a fixed $i \in \{1, 2, \dots, k\}$. Then $\mu(I^{(d)}) = \mu(J^{(d)}) = \mu(L^{(d)})$ for all d.*

Proof. Suppose $m = w^{\alpha_0} x_1^{\alpha_1} \cdots x_n^{\alpha_n} \in I^{(d)}$. By Exercise 2.23, we know that $(\alpha_0, \dots, \alpha_n)$ must satisfy some linear inequalities:

$$\begin{aligned} \alpha_0 &\geq d \\ L_1(\alpha_1, \dots, \alpha_n) &\geq d \\ &\vdots \\ L_k(\alpha_1, \dots, \alpha_n) &\geq d \end{aligned}$$

where $L_i(\alpha_1, \dots, \alpha_n)$ is a linear expression in $\alpha_1, \dots, \alpha_n$ corresponding to P_i in the primary decomposition of $L = P_1 \cap \cdots \cap P_k$ (and the first inequality corresponds to the (w) in the primary decomposition of $I = L \cap (w)$). If m is a minimal generator of J then $(\alpha_0, \dots, \alpha_n)$ must satisfy these inequalities minimally, meaning that there is no $(\alpha'_0, \dots, \alpha'_n)$ satisfying these inequalities with $\alpha'_i \leq \alpha_i$ for all i unless $\alpha'_i = \alpha_i$ for all i (otherwise, if $\alpha'_j < \alpha_j$ we could replace m with $\frac{m}{x_j}$).

Similarly, if $m = w^{\alpha_0} x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ is a minimal generator of $J^{(d)}$, then $(\alpha_0, \dots, \alpha_n)$ must still satisfy the linear inequalities above, except that we replace the first inequality with:

$$\begin{aligned} \alpha_0 + \alpha_{i_1} &\geq d \\ &\vdots \\ \alpha_0 + \alpha_{i_\ell} &\geq d \end{aligned}$$

where $m_i = x_{i_1} \cdots x_{i_\ell}$.

We claim that $m = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ is a minimal generator of $L^{(d)}$ if and only if $w^d m$ is a minimal generator of $I^{(d)}$ if and only if $mw^{d-\alpha} = w^{\alpha'_0} x_1^{\alpha'_1} \cdots x_n^{\alpha'_n}$ is a minimal generator of $J^{(d)}$, where $\alpha'_0 = d - \alpha$, $\alpha'_i = \alpha_i$ for $i \geq 1$, and $\alpha \in \{0, 1, \dots, d\}$ is the largest power so that there is some monomial f such that $m = f m_i^{\alpha}$ and $m_i \nmid f$.

The first if and only if in the claim is clear as the only inequality for $I^{(d)}$ that is not an inequality for $L^{(d)}$ is $\alpha_0 \geq d$ and it does not involve any of the other α_i. The second if and only if follows as the maximal power α determines the minimal α'_{i_j} in the new inequalities. Again, this generator is minimal because w does not divide any minimal generator of L. Informally, these say that once you have fixed the powers on the other variables in the minimal generator, the power on w is determined as 0, d, or $d - \min_j(\alpha_{i_j})$, respectively. □

This theorem will become useful later on in Corollary 4.14, when it allows us to expand our knowledge of one family of ideals using what we know about another family. The following two examples serve to better illustrate Theorem 4.1 in action.

Example 4.2. Consider the polynomial ring $\mathbb{K}[x, y, z]$ containing the ideals $I = (xz, yz) = (x, y) \cap (z)$ and $J = (x, yz) = (x, y) \cap (x, z)$. We have the case above, where $L = (x, y)$. We know

$$I^{(d)} = (\{x^\alpha y^\beta z^\gamma \mid \alpha + \beta \geq d,\ \gamma \geq d\})$$

$$J^{(d)} = (\{x^\alpha y^\beta z^\gamma \mid \alpha + \beta \geq d,\ \alpha + \gamma \geq d\})$$

Note that the minimal generators of $I^{(d)}$ are of the form $x^\alpha y^{d-\alpha} z^d$ where $0 \le \alpha \le d$, and the minimal generators of $J^{(d)}$ are of the form $x^\alpha y^{d-\alpha} z^{d-\alpha}$ where $0 \le \alpha \le d$. In this example, it is clear that there are $d+1$ generators in the d symbolic power of both I and J. For example, we get:

$$I^{(2)} = (x^2z^2, xyz^2, y^2z^2)$$

and

$$J^{(2)} = (x^2, xyz, y^2z^2).$$

Example 4.3. Now consider the polynomial ring $\mathbb{K}[w,x,y,z]$ containing the ideals $I = (xw, yw, zw) = (x,y,z) \cap (w)$ and $J = (x, yw, zw) = (x,y,z) \cap (x,w)$. Again, we have the case above, where $L = (x,y,z)$. We know

$$I^{(d)} = (\{x^\alpha y^\beta z^\gamma w^\delta \mid \alpha+\beta+\gamma \ge d,\ \delta \ge d\})$$
$$J^{(d)} = (\{x^\alpha y^\beta z^\gamma w^\delta \mid \alpha+\beta+\gamma \ge d,\ \alpha+\delta \ge d\})$$

Note that the minimal generators of $I^{(d)}$ are of the form $x^\alpha y^\beta z^{d-\alpha-\beta} w^d$ where $0 \le \alpha, \beta \le d$ and the minimal generators of $J^{(d)}$ are of the form $x^\alpha y^\beta z^{d-\alpha-\beta} w^{d-\alpha}$ where $0 \le \alpha, \beta \le d$. For example, we get:

$$I^{(2)} = (x^2w^2, xyw^2, y^2w^2, xzw^2, yzw^2, z^2w^2)$$

and

$$J^{(2)} = (x^2, xyw, y^2w^2, xzw, yzw^2, z^2w^2).$$

The following corollary expands on Theorem 4.1.

Corollary 4.4. *Define the ring R and the ideal $L = (m_1, \dots, m_k)$ in the same manner as above. Consider the ideal $J = L \cap (w, m_{i_1}, m_{i_2}, \dots, m_{i_\ell})$ for some fixed $i_1, i_2, \dots,$ and i_ℓ in $\{1, 2, \dots, k\}$. Then $\mu(J^{(d)}) = \mu(L^{(d)})$ for all d.*

Proof. Suppose $m = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \in L^d$. By Exercise 2.23, we know that $(\alpha_1, \dots, \alpha_n)$ must satisfy some linear inequalities:

$$L_1(\alpha_1, \dots, \alpha_n) \ge d$$
$$\vdots$$
$$L_k(\alpha_1, \dots, \alpha_n) \ge d$$

where $L_i(\alpha_1, \dots, \alpha_n)$ is a linear expression in $\alpha_1, \dots, \alpha_n$ corresponding to P_i in the primary decomposition of $L = P_1 \cap \cdots \cap P_k$. If m is a minimal generator of L then $(\alpha_1, \dots, \alpha_n)$ must satisfy these inequalities minimally.

Similarly, if $m = w^{\alpha_0} x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ is a minimal generator of $J^{(d)}$, then $(\alpha_1, \dots, \alpha_n)$ must still satisfy the linear inequalities above, and we also have some additional linear inequalities:

$$\alpha_0 + L_i'(\alpha_1, \dots, \alpha_n) \ge d$$
$$\vdots$$
$$\alpha_0 + L_\ell'(\alpha_1, \dots, \alpha_n) \ge d$$

where L_i' correspond to Q_i in the primary decomposition $(m_{i_1}, m_{i_2}, \dots, m_{i_l}) = Q_1 \cap \cdots \cap Q_\ell$.

Given a minimal generator of $L^{(d)}$, we have a minimal generator of $J^{(d)}$ with the same powers on x_1 through x_n and the power on w given by $d - \min_i(L_i'(\alpha_1, \cdots, \alpha_n))$, and every minimal generator of $J^{(d)}$ arises in this way. ∎

Theorem 4.1 and its following corollary describe a way to change ideals without changing their quasi-polynomials. We are now interested in describing one way to change an ideal so that its quasi-polynomial also changes in a predictable way. In particular, we focus on making changes to the primary decomposition. First, we explore what happens to the minimal generators of an ideal when we introduce a new variable into the primes of its primary decomposition.

Lemma 4.5. *Let I be a squarefree monomial ideal in $\mathbb{K}[x_0, x_1, \ldots, x_n]$ with unique irredundant primary decomposition $I = P_1 \cap P_2 \cap \cdots \cap P_k$, which exists by Exercise 2.22. Let $J = (P_1, x_0) \cap (P_2, x_0) \cap \cdots \cap (P_m, x_0) \cap P_{m+1} \cap \cdots \cap P_k$ for some $m \in \{0, \ldots, k\}$. If $x_0 \notin P_i$ for all i, then all the minimal generators of $I^{(d)}$ are also minimal generators of $J^{(d)}$.*

Proof. Suppose $x_0 \notin P_i$ for all i. For each $j \in \{1, \ldots, k\}$, consider the inequality

$$\sum_{i \in A_j} \alpha_i \geq d$$

where A_j is the set of all indices i such that x_i is a generator of P_j. Using Exercise 2.23, we can note that the generators of $I^{(d)}$ are of the form $x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$ where the exponents $\alpha_1, \ldots, \alpha_n$ satisfy the above inequality for every $j \in \{1, \ldots k\}$. Now consider $J^{(d)}$, whose generators are of the form $x_0^{\alpha_0} x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$. For each $j \in \{0, \ldots, m\}$, the exponents must satisfy the inequality

$$\alpha_0 + \sum_{i \in A_j} \alpha_i \geq d$$

where A_j is the set of all indices i such that x_i is a generator of P_j. Note that if $\alpha_0 = 0$, then the same values of $\alpha_1, \ldots, \alpha_n$ which satisfied the inequalities to be a generator of $I^{(d)}$ also satisfy the inequalities to be a generator of $J^{(d)}$. Thus, each generator of $I^{(d)}$ is a generator of $J^{(d)}$ where the exponent of x_0 is zero. They are minimal as x_0 does not divide them so they are in the intersection of $I^{(d)} \cap J^{(d)}$ and they are minimal in $I^{(d)}$. □

The following is an example which illustrates the lemma in action:

Example 4.6. Let $I = (y, z) \cap (y, w) \cap (z, w)$, $J_1 = (x, y, z) \cap (y, w) \cap (z, w)$, and $J_2 = (x, y, z) \cap (x, y, w) \cap (z, w)$. Then generators of $I^{(d)}$ are of the form $y^\alpha z^\beta w^\gamma$, where $\alpha + \beta \geq d$, $\alpha + \gamma \geq d$, and $\beta + \gamma \geq d$. We also have that generators of $J_1^{(d)}$ are of the form $x^\delta y^\alpha z^\beta w^\gamma$, where $\delta + \alpha + \beta \geq d$, $\alpha + \gamma \geq d$, and $\beta + \gamma \geq d$. Note that the same solutions for α, β, and γ that satisfy the inequalities for $I^{(d)}$ also satisfy the inequalities for $J_1^{(d)}$ when $\delta = 0$. Thus, all the generators for $I^{(d)}$ are also generators of $J_1^{(d)}$. The argument is the same for J_2, except for we have the inequality $\delta + \alpha + \gamma \geq d$ instead of $\alpha + \gamma \geq d$.

So, now we know that appending new variables into the primes of the primary decomposition increases the number of minimal generators, but by how much? The following result describes exactly how much the quasi-polynomial changes.

Proposition 4.7. *Let I be a square-free monomial ideal in $\mathbb{K}[x_0, x_1, \ldots, x_n]$ with unique irredundant primary decomposition $I = P_1 \cap P_2 \cap \ldots P_k$, which exists by Exercise 2.22. Let ℓ be the number of minimal generators of P_1, and let $J = (x_0, P_1) \cap P_2 \cap \cdots \cap P_k$. If I satisfies the following conditions:*

(1) $x_0 \notin P_i$ for all i
(2) P_1 contains all but one generator of each P_i

then $\mu\left(J^{(d)}\right) = \mu\left(I^{(d)}\right) + \binom{d+\ell-1}{\ell}$.

Proof. Suppose I satisfies those conditions, and let $J = (x_0, P_1) \cap P_2 \cap \cdots \cap P_k$. By Lemma 4.5, we know that every minimal generator of $I^{(d)}$ is also a minimal generator of $J^{(d)}$; in particular, they are all the generators in $J^{(d)}$ where the exponent of x_0 is zero. Thus, the rest of the minimal generators of $J^{(d)}$ must have $0 < \alpha_0 \leq d$. Let A_j is the set of all indices i such that x_i is a generator of P_j. Note that a minimal generator of $J^{(d)}$ must satisfy the inequalities

$$\alpha_0 + \sum_{i \in A_1} \alpha_i \geq d \quad \text{and} \quad \sum_{i \in A_j} \alpha_i \geq d$$

for all $j \in \{2, \ldots, k\}$. Note, we can see that the first inequality is in fact an equality since if $\alpha_0 > 0$ and $\alpha_0 + \sum_{i \in A_1} \alpha_i > d$, then the generator is not minimal (since we could divide by x_0 and still be in $J^{(d)}$). Without loss of generality, assume that $P_1 = (x_1, \cdots, x_l)$. It follows that we only need to count minimal solutions for $\alpha_0, \alpha_1, \ldots, \alpha_{\ell-1}$, since these will fix α_ℓ by the first equation and the rest of the powers are automatically determined once the first $\ell - 1$ of the α_i (for $i > 1$) are fixed by condition (2). The number of possibilities for $(\alpha_0, \ldots, \alpha_{\ell-1})$ are:

$$\sum_{\alpha_0=1}^{d} \sum_{\alpha_1=0}^{d-\alpha_0} \sum_{\alpha_2=0}^{d-\alpha_0-\alpha_1} \cdots \sum_{\alpha_{\ell-1}=0}^{d-\sum_{m=0}^{\ell-2} \alpha_m} 1 = \binom{d+\ell-1}{\ell}.$$

Alternatively, counting the set of α_0 through α_ℓ which satisfy these inequalities is equivalent to counting length ℓ partitions of d where the first integer is at least 1. That is equivalent to counting length ℓ partitions of $d-1$, and the number of those is given by $\binom{d-1+\ell}{\ell}$. Hence, $\mu\left(J^{(d)}\right) = \mu\left(I^{(d)}\right) + \binom{d-1+\ell}{\ell}$. $\square$

The following two examples illustrate exactly what is happening in these ideals.

Example 4.8. Consider again $I_1 = (y,z) \cap (y,w) \cap (z,w)$ and $J_1 = (x,y,z) \cap (y,w) \cap (z,w)$ (4v4gD in the second chart), where $\ell = 2$. Then we need to count generators of $J^{(d)}$ which are of the form $x^\alpha y^\beta z^\gamma w^\delta$ where $0 < \alpha \leq d$, $\alpha + \beta + \gamma \geq d$, $\beta + \delta \geq d$, and $\gamma + \delta \geq d$. Note that once we choose α and β for our minimal generators, we have that γ and δ are fixed at $\gamma = d - \alpha - \beta$ and $\delta = d - \beta$. So, we only need to count possible choices for α and β, which gives us

$$\sum_{\alpha=1}^{d} \sum_{\beta=0}^{d-\alpha} 1 = \sum_{\alpha=1}^{d} (d - \alpha + 1) = d^2 - \frac{d(d+1)}{2} + d = \binom{d+1}{2} = \binom{d+\ell-1}{\ell}.$$

Example 4.9. Now consider $I_2 = (y,z,w) \cap (y,z,v) \cap (y,w,v)$ and $J_2 = (x,y,z,w) \cap (y,z,v) \cap (y,w,v)$, where $\ell = 3$. Then we need to count generators of $J^{(d)}$ which are of the form $x^\alpha y^\beta z^\gamma w^\delta v^\sigma$ where $0 < \alpha \leq d$, $\alpha + \beta + \gamma + \delta = d$, $\beta + \gamma + \sigma \geq d$, and $\beta + \delta + \sigma \geq d$. We only need to count possible minimal solutions for α, β, and γ now, since it would follow that $\delta = d - \alpha - \beta - \gamma$ and $\sigma = d - \beta - \gamma$ in minimal generators. So, this count gives us

$$\sum_{\alpha=1}^{d} \sum_{\beta=0}^{d-\alpha} \sum_{\gamma=0}^{d-\alpha-\beta} 1 = \binom{d+2}{3} = \binom{d+\ell-1}{\ell}.$$

Our final result from this section describes the (symbolic) Hilbert function of an ideal with respect to the (symbolic) Hilbert functions of the ideals in a decomposition of our ideal into ideals in disjoint sets of variables. These ideals are the SFM ideals corresponding to the connected components of the ideals simplicial complex.

Proposition 4.10. *Let I be a square-free monomial ideal in $\mathbb{K}[x_1, \ldots x_n]$ that can be decomposed into $I = I_1 \cap \cdots \cap I_k$ so that each I_j can be written in distinct generators (i.e. if x_i divides a minimal generator in I_j, then x_i does not divide any of the minimal generator in I_ℓ if $\ell \neq j$). Let the corresponding Hilbert functions be $T_1, \ldots, T_k$. Then the Hilbert function for I is $T(d) = T_1(d)T_2(d)\ldots T_k(d)$ (and the same type of result also holds for the symbolic Hilbert function).*

Proof. By induction, this reduces to the case of $I = I_1 \cap I_2$ with the appropriate property on the generators. Then if I_1^d is minimally generated by $(a_1, \ldots, a_r)$ and I_2^d is minimally generated by $(b_1, \ldots, b_s)$, then $\text{lcm}(a_i, b_j) \in I_1^d \cap I_2^d$ and is equal to $a_i b_j$ since $\gcd(a_i, b_j) = 1$. We claim that $a_i b_j$ is a minimal generator of $I_1^d \cap I_2^d$ since anything that strictly divides a_i can't be the minimal generator; similarly, anything that strictly divides b_j can't be a minimal generator. Thus $I_1^d \cap I_2^d = I_1^d \cdot I_2^d = I^d$ and is minimally generated by all products of the generators. Thus, $T(d) = T_1(d)T_2(d)$ and the result follows. Similarly, $I^{(d)} = I_1^{(d)} \cap I_2^{(d)}$ by Exercise 2.24, and since all of the generators of $I_1^{(d)}$ and $I_2^{(d)}$ are in distinct variables, we have that $I_1^{(d)} \cap I_2^{(d)} = I_1^{(d)} \cdot I_2^{(d)}$ and $\mu(I^{(d)}) = \mu(I_1^{(d)}) \cdot \mu(I_2^{(d)})$ as m is a minimal generator of $I^{(d)}$ if and only if it is a product of minimal generators of $I_1^{(d)}$ and $I_2^{(d)}$, by the same argument as above. $\square$

4.2. **Families of SQM Ideals.** In this section, we compute the Hilbert quasi-polynomials giving the number of generators of the symbolic powers for several families of SFM ideals. As stated previously, we use the simplicial complex drawings to categorize ideals into families. Two of which are stated here:

Definition 4.11. *Edge ideals* are the ideals that correspond to simiplicial complexes consisting of only edges, or in other words whose primary decomposition consists of ideals of the form (x_i, x_j).

$$I = \bigcap_{i,j} (x_i, x_j)$$

Definition 4.12. *Triangle ideals* are the ones that correspond to simiplicial complexes consisting of only 'triangles', or in other words whose primary decomposition consists of ideals of the form (x_i, x_j, x_k).

$$I = \bigcap_{i,j,k} (x_i, x_j, x_k)$$

The following chart lists the quasi-polynomials of SFM ideals in 4 variables. In it, we observe similarities and differences which assist us in developing conjectures to explore further.

Symbolic Powers

Name	I	Decomposition	$H_I^s(d)$
4v1g	$(xyzw)$	$(x) \cap (y) \cap (z) \cap (w)$	1
4v2gA	(x, yzw)	$(x, y) \cap (x, z) \cap (x, w)$	$d + 1$
4v2gB	(xy, zw)	$(x, z) \cap (x, w) \cap (y, z) \cap (y, w)$	$d + 1$
4v2gC	(xyz, xw)	$(x) \cap (y, w) \cap (z, w)$	$d + 1$
4v2gD	(xyz, xyw)	$(x) \cap (y) \cap (z, w)$	$d + 1$
4v3gA	(xy, z, w)	$(x, z, w) \cap (y, z, w)$	$\frac{(d+2)(d+1)}{2}$
4v3gB	(xy, xz, xw)	$(x) \cap (y, z, w)$	$\frac{(d+2)(d+1)}{2}$
4v3gC	(xw, yz, yw)	$(x, y) \cap (y, w) \cap (z, w)$	$\frac{(d+2)(d+1)}{2}$
4v3gD	(xy, xz, w)	$(x, w) \cap (y, z, w)$	$\frac{(d+2)(d+1)}{2}$
4v3gE	(xyz, xw, yw)	$(x, y) \cap (x, w) \cap (y, w) \cap (z, w)$	$\begin{cases} \frac{1}{4}(d^2 + 6d + 4) & d \text{ even} \\ \frac{1}{4}(d^2 + 6d + 5) & d \text{ odd} \end{cases}$
4v3gF	(xyz, xyw, zw)	$(z, w) \cap (x, w) \cap (x, z) \cap (y, w) \cap (y, z)$	$\begin{cases} \frac{3}{2}d + 1 & d \text{ even} \\ \frac{3}{2}(d + 1) & d \text{ odd} \end{cases}$
4v3gG	(xyw, xzw, yzw)	$(x, z) \cap (y, z) \cap (y, x) \cap (w)$	$\begin{cases} \frac{3}{2}d + 1 & d \text{ even} \\ \frac{3}{2}(d + 1) & d \text{ odd} \end{cases}$
4v4gA	(x, y, z, w)	(x, y, z, w)	$\binom{d+3}{3}$
4v4gB	(x, yz, yw, zw)	$(x, y, z) \cap (x, y, w) \cap (x, z, w)$	$\begin{cases} \frac{3}{4}d^2 + 2d + 1 & d \text{ even} \\ \frac{1}{4}(3d^2 + 8d + 5) & d \text{ odd} \end{cases}$
4v4gC	(xy, xz, yw, zw)	$(x, w) \cap (y, z)$	$(d + 1)^2$
4v4gD	(xw, yz, yw, zw)	$(x, y, z) \cap (y, w) \cap (z, w)$	$\begin{cases} \frac{1}{2}d^2 + 2d + 1 & d \text{ even} \\ \frac{1}{2}d^2 + 2d + \frac{3}{2} & d \text{ odd} \end{cases}$
4v4gE	(xyz, xw, yw, zw)	$(z, y, x) \cap (w, x) \cap (w, y) \cap (w, z)$	$\begin{cases} \frac{1}{6}(3d^2 + 13d + 6) & [d]_3 = 0 \\ \frac{1}{6}(3d^2 + 13d + 8) & [d]_3 = 1 \\ \frac{1}{6}(3d^2 + 13d + 10) & [d]_3 = 2 \end{cases}$
4v4gF	(xyz, xyw, xzw, yzw)	$(x, w) \cap (y, w) \cap (y, x) \cap (z, y) \cap (z, w) \cap (z, x)$	$\begin{cases} 2d + 1 & d \text{ even} \\ 2d + 2 & d \text{ odd} \end{cases}$
4v5gA	(xz, xw, yz, yw, zw)	$(x, y, z) \cap (x, y, w) \cap (z, w)$	$\begin{cases} \frac{1}{8}(7d^2 + 18d + 8) & d \text{ even} \\ \frac{1}{8}(7d^2 + 20d + 13) & d \text{ odd} \end{cases}$
4v6gA	(xy, xz, xw, yz, yw, zw)	$(x, z, w) \cap (y, z, w) \cap (y, x, w) \cap (y, x, z)$	$\begin{cases} d^2 + \frac{8}{3}d + 1 & [d]_6 = 0 \\ d^2 + \frac{8}{3}d + \frac{7}{3} & [d]_6 = 1 \\ d^2 + \frac{8}{3}d^2 + \frac{2}{3} & [d]_6 = 2 \\ d^2 + \frac{8}{3}d^2 + 2 & [d]_6 = 3 \\ d^2 + \frac{8}{3}d^2 + \frac{4}{3} & [d]_6 = 4 \\ d^2 + \frac{8}{3}d^2 + \frac{5}{3} & [d]_6 = 5 \end{cases}$

We first work out the case where the simplicial complex of the ideal is the complete graph on n vertices.

Proposition 4.13. *If I is a square-free monomial ideal in $\mathbb{K}[x_1, x_2, \ldots, x_n]$ whose simplicial complex is a complete graph - that is, it can be decomposed into*

$$I^{(d)} = \bigcap_{1 \leq i < j \leq n} (x_i, x_j)^d = (x_1 \cdots x_{n-1}, \cdots, x_1 \cdots \hat{x_i} \cdots x_n, \cdots, x_2 \cdots x_n)^{(d)},$$

then

$$\mu(I^{(d)}) = \begin{cases} \frac{nd}{2} + 1 & d \text{ even} \\ \frac{n(d+1)}{2} & d \text{ odd} \end{cases}$$

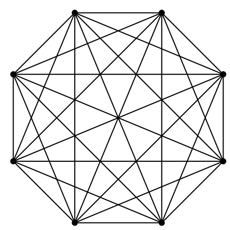

FIGURE 2. Complete Graph on Eight Vertices

Proof. We know $I^{(d)} = (\{x_1^{\alpha_1} \ldots x_n^{\alpha_n} | \alpha_i + \alpha_j \geq d, \text{ for each } 1 \leq i < j \leq n\})$ where $\alpha_1, \ldots, \alpha_n$ are nonnegative integers. Furthermore, $x_1^{\alpha_1} \ldots x_n^{\alpha_n}$ is a minimal generator of $I^{(d)}$ if and only if $(\alpha_1, \ldots, \alpha_n)$ satisfies these inequalities minimally, meaning that if $(\alpha_1', \ldots, \alpha_n')$ also satisfies these inequalities and $\alpha_i' \leq \alpha_i$ then we have $\alpha_i = \alpha_i'$ for each i. For any minimal generator, consider the power on x_1.

If $\alpha_1 \leq \frac{d}{2}$, then we must have that $\alpha_2, \ldots, \alpha_n \geq d - \alpha_1$ (with equality if our generator is minimal) so that the sum of any two powers are greater than or equal to d. There are $\frac{d}{2} + 1$ possible choices for α_1 (and then $\alpha_2, \ldots, \alpha_n$ are fixed) if d is even and $\frac{d+1}{2}$ possible choices for α_1 (and again $\alpha_2, \ldots, \alpha_n$ are fixed) if d is odd.

If $\alpha_1 > \frac{d}{2}$, then $\alpha_2, \ldots, \alpha_n \geq d - \alpha_1$. However, we can have at most one of α_i such that $d - \alpha_1 \leq \alpha_i < \frac{d}{2}$ (otherwise, $2(d - \alpha_1) < \alpha_i + \alpha_j < d$). In fact, this is again back to the first case but with $0 \leq \alpha_i < \frac{d}{2}$ that gives us $\frac{d}{2}(n-1)$ possibilities if d is even and $\frac{d+1}{2}(n-1)$ possibilities if d is odd. Therefore, $\mu(I^{(d)}) = \frac{nd}{2} + 1$ if d is even and $\mu(I^{(d)}) = \frac{n(d+1)}{2}$ if d is odd. □

Then what if we delete only one edge?

Corollary 4.14. *If I is a square-free monomial ideal in $\mathbb{K}[x_1, \ldots x_n]$ whose simplicial complex is a complete graph missing 1 edge - (x_1, x_2), then $\mu\left(I^{(d)}\right) = \mu\left(J^{(d)}\right)$ where J is a square-free monomial ideal with the complete graph of $n-1$ variables.*

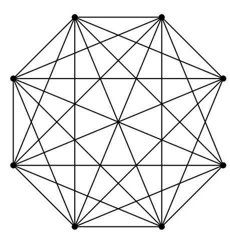

FIGURE 3. Complete Graph Missing 1 Edge

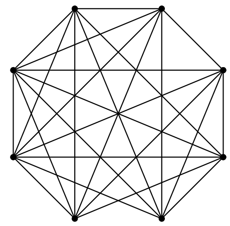

FIGURE 4. Complete Graph Missing 3 Non-intersecting Edges

Proof. First we have that

$$\begin{aligned} I &= \bigcap_{2\le i<j\le n} (x_i, x_j) \cap \bigcap_{i=1, j\neq 1,2,i}^{n} (x_i, x_j) \\ &= \bigcap_{i,j=2}^{n} (x_i, x_j) \cap (x_1, x_3x_4 \ldots x_n). \end{aligned}$$

Note that here

$$\begin{cases} \bigcap_{2\le i<j\le n} (x_i, x_j) \text{ is a complete graph for n-1 variables} \\ x_1 \text{ is a new variable} \\ x_3x_4\ldots x_n \text{ is a generator of } \bigcap_{2\le i<j\le n} (x_i, x_j). \end{cases}$$

So the proof follows directly from Theorem 4.1. □

We can now say something about the quasi-polynomial of the complete graph minus any set of disconnected edges, i.e. none of the edges share any vertices. (Check table ideals 4v2gB, 4v3gF for reference)

Corollary 4.15. *If I is a square-free monomial ideal in $\mathbb{K}[x_1, \ldots x_n]$ whose primary decompositions consists of all prime ideals generated by pairs of the variables except for the k ideals - $(c_{11}, c_{12}), \cdots, (c_{i1}, c_{i2}), \cdots, (c_{k1}, c_{k2})$ such that $c_{ij} \neq c_{wl}$ unless $i = w$ and $j = l$, then $\mu\left(I^{(d)}\right) = \mu\left(J^{(d)}\right)$ where J is a square-free monomial ideal with the complete graph of $n-k$ variables.*

Proof. Without loss of generality, we may assume that $c_{i1} = x_{2i-1}$ and $c_{i2} = x_{2i}$. We work by induction on the number of missing edges. The base cases of $k = 0$ and $k = 1$ are the previous two results. Assume the result is true if we move $k-1$ edges. Let I be the ideal whose simplicial complex is the complete graph on the variables x_1 to x_n minus the k edges $(x_1, x_2), (x_3, x_4), \ldots, (x_{2k-1}, x_{2k})$, and let J be the ideal whose simplicial complex is the complete graph on the variables x_2 to x_n minus the $k-1$ edges $(x_3, x_4), (x_5, x_6), \ldots, (x_{2k-1}, x_{2k})$. Then, we have that

$$\begin{aligned} I &= J \cap \bigcap_{i=1, j\neq 1,2,i}^{n} (x_i, x_j) \\ &= J \cap (x_1, x_3x_4 \ldots x_n). \end{aligned}$$

Note that here

$$\begin{cases} J \text{ is a complete graph for n-1 variables missing } k-1 \text{ edges} \\ x_1 \text{ is a new variable} \\ x_3x_4\ldots x_n \text{ is a generator of } \bigcap_{2\le i<j\le n} (x_i, x_j). \end{cases}$$

So the induction step follows directly from Theorem 4.1. By induction, the result holds. □

Then, to generalize more into the whole edge family, we notice that each ideal in the family consists of loop and chains. We are still studying them. See the future work section.

Now we consider another family of ideals: those whose simplicial complexes are the complete "2-graphs" on n vertices, by which we mean you include all possible simplices given by three of the vertices, as in the table ideal 4v4gB.

Proposition 4.16. *If $I \subset R$ is a square-free monomial ideal with n variables that can be decomposed into $I^{(d)} = \bigcap_{i,j,k=1}^{n}(x_i, x_j, x_k)^d = (x_1 \cdots x_{n-2}, \ldots, x_3 \cdots x_n)^{(d)}$, then*

$$\mu(I^{(d)}) = \begin{cases} \frac{n^2-n}{12}d^2 + \frac{n^2}{6}d + 1 & d \equiv 0 \mod 6 \\ \frac{n^2-n}{12}d^2 + \frac{n^2}{6}d^2 + \frac{n^2}{4} - \frac{5}{12}n & d \equiv 1 \mod 6 \\ \frac{n^2-n}{12}d^2 + \frac{n^2}{6}d + \frac{n(5-n)}{6} & d \equiv 2 \mod 6 \\ \frac{n^2-n}{12}d^2 + \frac{n^2}{6}d + \frac{n^2-3n+4}{4} & d \equiv 3 \mod 6 \\ \frac{n^2-n}{12}d^2 + \frac{n^2}{6}d + \frac{n}{3} & d \equiv 4 \mod 6 \\ \frac{n^2-n}{12}d^2 + \frac{n^2}{6}d + \frac{n(n+1)}{12} & d \equiv 5 \mod 6 \end{cases}$$

Proof. Consider a minimal generator $m = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ of $I^{(d)}$ where I is as in the statement of the proposition. Note that if all of the exponents are greater than $\frac{d}{3}$, than this generator would not be minimal so we can assume that there is one exponent less than or equal to $\frac{d}{3}$. Conversely, if at least three of the exponents are less than or equal to $\frac{d}{3}$, then in order for the inequalities to be satisfied, all of the exponents must be exactly $\frac{d}{3}$. This leaves us to count the case where exactly one or two of the exponents is less than or equal to $\frac{d}{3}$. We will do this by counting all of the ways for one exponent to be less than or equal to it, then subtracting off the doubly counted ways of having two of them less than or equal to it, and finally adding back in the case where all of the powers are $\frac{d}{3}$.

First, we count the ways for at least one of the exponents to be less than or equal to $\frac{d}{3}$. For simplicity, we treat the case where $\alpha_1 \leq \frac{d}{3}$. Then for all $2 \leq i < j < k \leq n$, we have

$$\begin{cases} \alpha_i + \alpha_j \geq d - \alpha_1 \\ \alpha_i + \alpha_k \geq d - \alpha_1 \\ \alpha_j + \alpha_k \geq d - \alpha_1 \end{cases}$$

If we add these together, we get: $2(\alpha_i+\alpha_j+\alpha_k) \geq 3(d-\alpha_1)$ which implies that $(\alpha_i+\alpha_j+\alpha_k) \geq \frac{3}{2}(d-\alpha_1) \geq d$. Thus, the only inequalities that we need to consider are the inequalities of the form $\alpha_1 + \alpha_i + \alpha_j \geq d$ or equivalently $\alpha_i + \alpha_j \geq d - \alpha_1$, since we will automatically have $\alpha_i + \alpha_j + \alpha_k \geq d$ for all $i, j, k \neq 1$ as long as we satisfy all the inequalities with α_1. After fixing α_1, we see that these equations are precisely the equations giving the minimal number of generators of $J^{(d-\alpha_1)}$ where

$$J = \bigcap_{2 \leq i < j \leq n} (x_i, x_j).$$

This gives an initial count:

$$\mu(I^{(d)}) \leq n \sum_{\alpha_1=0}^{\lfloor \frac{d}{3} \rfloor} \mu(J^{(d-\alpha_1)}),$$

but this has double counted the case when two of the exponents are less than or equal to $\frac{d}{3}$. Therefore, we want to subtract off this double count. For simplicity, we treat the case of $\alpha_1, \alpha_2 \leq \frac{d}{3}$. This forces $\alpha_i = d - \alpha_1 - \alpha_2$ for all other i. Thus, we need to subtract off the term $\binom{n}{2}\left(\lfloor \frac{d}{3} \rfloor + 1\right)^2$, corresponding to fixing two of our α_i between 0 and $\frac{d}{3}$.

But this has overcorrected for the term when all of the powers are exactly $\frac{d}{3}$. In particular, we have counted that monomial $n - \binom{n}{2}$ times so we need to add it back $\binom{n}{2} - n + 1$ times. Note, this term only occurs if $d \equiv 0 \pmod 3$.

Putting these three pieces together yields the expression

$$\mu\left(I^{(d)}\right) = \begin{cases} n \sum_{i=0}^{\lfloor \frac{d}{3} \rfloor} \mu(J^{(d-i)}) - \binom{n}{2}\left(\lfloor \frac{d}{3} \rfloor + 1\right)^2 + \left(\binom{n}{2} - n + 1\right) & \text{if } d \equiv 0 \pmod 3 \\ n \sum_{i=0}^{\lfloor \frac{d}{3} \rfloor} \mu(J^{(d-i)}) - \binom{n}{2}\left(\lfloor \frac{d}{3} \rfloor + 1\right)^2 & \text{if } d \equiv 1, 2 \pmod 3 \end{cases}$$

where we can use Proposition 4.13 to compute the summands. Letting $d = 6e + 3f + g$ where $0 \leq f \leq 1$ and $0 \leq g \leq 2$, we can write this expression as

$$\mu\left(I^{(d)}\right) = n \sum_{i=0}^{\frac{d-3f-g}{6}} \left(\frac{(n-1)(d-2i)}{2} + 1\right) + n \sum_{i=0}^{\frac{d-3f-g}{6}-1+f} \left(\frac{(n-1)(d-(2i+1)+1)}{2}\right)$$
$$-\binom{n}{2}\left(\frac{d-g}{3}+1\right)^2 + \left(\binom{n}{2} - n + 1\right),$$

if $d \equiv 0 \pmod 3$, and

$$\mu\left(I^{(d)}\right) = n \sum_{i=0}^{\frac{d-3f-g}{6}} \left(\frac{(n-1)(d-2i)}{2} + 1\right) + n \sum_{i=0}^{\frac{d-3f-g}{6}-1+f} \left(\frac{(n-1)(d-(2i+1)+1)}{2}\right) - \binom{n}{2}\left(\frac{d-g}{3}+1\right)^2,$$

if $d \equiv 1, 2 \pmod 3$ Substituting in for f and g gives the result $\square$

5. Comparison - Symbolic Powers vs Regular Powers

We observed that there are some interesting relations between the two powers. Some of them are turned into beautiful facts in this section, and some others still remain as conjectures. We will start with when the two powers always yield the same ideal.

Proposition 5.1. *If I is a square-free monomial ideal with 1 generator, then*
$$I^{(d)} = I^d$$
for all $d \in \mathbb{N}$

Proof. Let $I = (x_{i_1} \cdots x_{i_k})$. Then we can see that $I^d = (x_{i_1}^d \cdots x_{i_k}^d)$. Since $I = (x_{i_1} \cdots x_{i_k})$, we have that $I = (x_{i_1}) \cap \cdots \cap (x_{i_k})$. By Exercise 2.23, $I^{(d)} = (x_{i_1})^d \cap \cdots \cap (x_{i_k})^d = (x_{i_1}^d \cdots x_{i_k}^d) = I^d$. $\square$

Proposition 5.2. *If $I \subset R$ is a square-free monomial ideal with 2 generators, then*
$$I^{(d)} = I^d$$
for all $d \in \mathbb{N}$

Proof. Since I is square-free and generated by 2 elements (and R is commutative), we may assume (by grouping similar terms first and re-ordering the variables) that $I = (x_1 \cdots x_k x_{k+1} \cdots x_m, x_1 \cdots x_k x_{m+1} \cdots x_\ell)$. We know:

$$\begin{aligned}
I^{(d)} &= (x_1 \ldots x_k x_{k+1} \ldots x_m, x_1 \ldots x_k x_{m+1} \ldots x_\ell)^{(d)} \\
&= \bigcap_{s=1}^{k} (x_s)^d \cap (x_{k+1} \ldots x_m, x_{m+1} \ldots x_\ell)^{(d)} \\
&= \bigcap_{s=1}^{k} (x_s)^d \cap \bigcap_{\substack{k+1 \leq i \leq m \\ m+1 \leq j \leq \ell}} (x_i, x_j)^d \\
&= \bigcap_{s=1}^{k} (x_s)^d \cap (\{x_{k+1}^{\alpha_{k+1}} \ldots x_\ell^{\alpha_\ell} | \alpha_i + \alpha_j \geq d \text{ for all } k+1 \leq i \leq m < j \leq \ell\})
\end{aligned}$$

By observation, the defining equations are symmetric with respect to α_{k+1} through α_m and with respect to α_{m+1} through α_ℓ. Fix an element $x_1^{\alpha_1} \cdots x_n^{\alpha_n} \in I^{(d)}$. Then if i and j are both in the same subset (namely, if $i, j \leq m$ or $i, j \geq m+1$) and we also have $\alpha_i > \alpha_j$, then α_i can be replaced by $\alpha_i - 1$ and the defining equations would still be satisfied (since they are satisfied for α_j) so that monomial is not a minimal generator of the symbolic power. This implies that for the minimal generators, we have that $\alpha_{k+1} = \cdots = \alpha_m = a$ and $\alpha_{m+1} = \cdots = \alpha_\ell = b$, and $a + b = d$. This means that $I^{(d)}$ can also be written as

$$I^{(d)} = \left(\{(x_1 \ldots x_k)^d (x_{k+1} \ldots x_m)^{\alpha_m} (x_{m+1} \ldots x_\ell)^{d-\alpha_m} : 0 \leq \alpha_m \leq d\}\right) = I^d.$$

$\square$

But we cannot go on like this forever - the following quickly shows us that in three generators, the two powers do not yield the same ideal.

Example 5.3. For the ideal $I = (xy, yz, xz)$, the analogous statement to the previous two propositions fails in contrast to the ordinary power case.

Proof. We will show that $I^2 \neq I^{(2)}$. It is easy to see that $I^2 = (x^2y^2, x^2z^2, y^2z^2, x^2yz, xy^2z, xyz^2)$. To compute the symbolic power, we note that $I = (x, y) \cap (x, z) \cap (y, z)$. Then $I^{(2)} = (x, y)^2 \cap (x, z)^2 \cap (y, z)^2 = (x^2y^2, x^2z^2, y^2z^2, xyz)$. Thus, $I^2 \neq I^{(2)}$. □

6. Future work

Throughout our project, every answer opened several more questions. Because of the time constraints of the program, however, some of these questions currently remain unanswered. We list our conjectures here in hopes of resolving them in the future.

So far, we have characterized the Hilbert quasi-polynomials for some "edge ideals", by which we mean ideals whose simplicial complex corresponding to its prime decomposition is a graph which only contains edges (and vertices). In particular, we have characterized the case of edges ideals corresponding to complete graphs and edge ideals corresponding to complete graphs minus k disconnected edges. However, we have not fully characterized the Hilbert quasi-polynomials of all edge ideals. In order to do this, we propose looking at edge ideals whose simplicial complexes correspond to "chain" or "loops" like Figure 5.

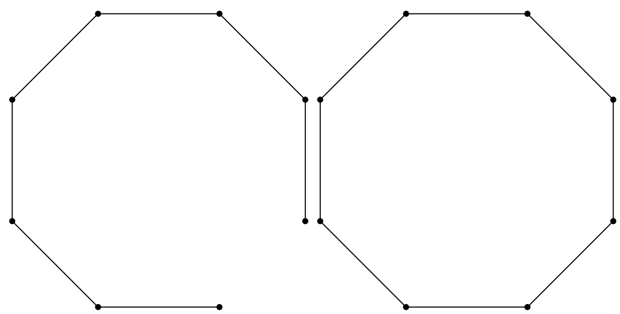

Figure 5. Chain and Loop Graph for Eight Variables

Conjecture 6.1. *If I is an edge ideal whose simplicial complex is a chain of k vertices and $k-1$ edges, then $h_I^s(n)$ is a quasi-polynomial of degree $\lceil \frac{k-1}{2} \rceil$.*

Here is some evidence for the conjecture.

#Vertices	Primary Decomposition	Quasi-polynomial	Potential Bound
2	(x, y)	d+1	$\lceil \frac{2-1}{2} \rceil$
3	$(x, y) \cap (y, z)$	d+1	$\lceil \frac{3-1}{2} \rceil$
4	$(x, y \cap (y, z) \cap (z, w)$	$\binom{d+2}{2}$	$\lceil \frac{4-1}{2} \rceil$

Conjecture 6.2. *If I is an edge ideal whose simplicial complex is a loop with k vertices and edges, then $h_I^s(n)$ is a quasi-polynomial of degree $\lceil \frac{k-2}{2} \rceil$.*

We are also very curious about the periods of SFM ideals whose simplicial complex consists of only edges and triangles.

Conjecture 6.3. *If I is an ideal whose simplicial complex consists only of edges or triangles, then $h_I^s(n)$ can be a quasi-polynomial of period less than or equal to 6.*

Although we mostly focus on Hilbert functions in this research, we are still interested in what the generators of the powers looks like - there are some approaches to the comparison of the generators of the two powers.

Conjecture 6.4. *Let $I \subset R$ be a square-free monomial ideal. Then for all minimal generator $\mathbf{x}^{\mathbf{a}}$ in $I^{(d)}$, there exists a minimal generator $\mathbf{x}^{\mathbf{b}}$ of I^d such that $\mathbf{x}^{\mathbf{a}} | \mathbf{x}^{\mathbf{b}}$.*

And hopefully, we can figure out the relation between the corresponding Hilbert polynomial and quasi-polynomial. And finally, we will continuing looking for the bounds of the degree for the Hilbert (quasi-) polynomials.

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