REU Report: Non-Axiomatizability of Categories in Finitary Logic

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Abstract

It was shown in [LRV23] that the category $\mathbf{Hilb}_m$ of Hilbert spaces with injective linear contractions as morphisms is not *finitely concrete*; namely, there is no faithful functor U from $\mathbf{Hilb}_m$ into $\mathbf{Set}$ that preserves directed colimits. In [Hen20],it was shown that if a category $\mathbf{K}$ is axiomatizable in a finitary logic, then there is a faithful functor from $\mathbf{K} \to \mathbf{Set}$ which preserves *directed colimits*. Therefore, $\mathbf{Hilb}_m$ is not axiomatizable in this sense.

Here, we extend this result and prove several related results. First, we show that finite concreteness in $\mathbf{Hilb}$ (the category of Hilbert spaces with all linear contractions as morphism) and $\mathbf{Hilb}_m$, and proceed using the finite concreteness of $\mathbf{Hilb}$. We then analyze the extent to which finite concreteness fails in $\mathbf{Hilb}$ (and $\mathbf{Hilb}_m$), and find that *all* faithful functors $U : \mathbf{Hilb} \to \mathbf{Set}$ are constant up to natural isomorphism. Further, we find that finite concreteness also fails in $\mathbf{Hilb}_r$, the category of Hilbert spaces with linear isometries as morphisms (Section 5).

Contents

1 Introduction

This project explores the concept of finite concreteness and its connection to axiomatizability of certain categories. It was proven in [LRV23] proved that the category $\mathbf{Hilb}_m$ of Hilbert spaces with injective linear contractions is not *finitely concrete*, which means there is no functor U from $\mathbf{Hilb}_m$ into $\mathbf{Set}$ that is faithful and preserves *directed colimits*. Faithful means that if $U : \mathcal{K} \to \mathbf{Set}$ is a functor, U is “injective” on the morphisms of $\mathcal{K}$. The functor U can be thought of as a functor mapping objects in $\mathcal{K}$ to an "underlying set". Directed colimit-preserving means that whenever there is a directed system $\langle X_i : i \in I \rangle$ in $\mathcal{K}$ with colimit X, then UX is the colimit of the directed system $\langle UX_i : i \in I \rangle$ in **Set**. Conceptually, this means that colimits in $\mathcal{K}$ are "set-like" in their behavior.

It was shown in [LRV23] that there is no such functor from $\mathbf{Hilb}_m \to \mathbf{Set}$. This means that, on the one hand, the normal underlying set functor (i.e. the functor mapping a Hilbert space to the set of its elements) does not have the desired properties of faithfulness and directed-colimit preserving. In the case of $\mathbf{Hilb}_m$, and in general, complete metric spaces, the underlying set of a directed colimit of complete metric spaces will be the completion of the union of the corresponding underlying sets, not the union itself. Furthermore, the result on finite concreteness of $\mathbf{Hilb}_m$ states that directed-colimit-preserving and faithfulness fails regardless of the choice of underlying set (i.e., for any functor from $\mathbf{Hilb}_m$ to **Set**). By results in [Hen20], this implies that $\mathbf{Hilb}_m$ is not axiomatizable in any $\mathbb{L}_{\infty,\omega}$ logic (and, in particular, any finitary first-order logic). In [LRV23], this result is extended to various categories in which $\mathbf{Hilb}_m$ can be embedded, including the category $\mathbf{CC^*Alg}$ of commutative unital C^*-algebras with unit-preserving $*$-homomorphisms.

In the following, we will prove several extensions of the main result on $\mathbf{Hilb}_m$ from from [LRV23] regarding the nonaxiomatizability of categories related to Hilbert spaces.

First, we prove a result establishing a link between $\mathbf{Hilb}_m$ (the category of Hilbert spaces with injective linear contractions) and **Hilb**, which has the same objects but with all linear contractions (Section 3). In particular, the proof of finite concreteness for $\mathbf{Hilb}_m$ in [LRV23] (outlined in Section 2.C) relies on the notion of *support* of an element of $x \in UA$ in a Hilbert space A in $\mathbf{Hilb}_m$. Briefly, x is supported on a subspace $A_0 \subseteq A$ iff any two morphisms $f, g : A \to B$ (for any other $B \in \mathbf{Hilb}_m$) that agree on A_0 also agree on x. We define an equivalent concept for **Hilb** (using morphisms in **Hilb**, i.e. linear contractions, instead of injective linear contractions). We then show that the two definitions are equivalent in the sense that any morphism in $\mathbf{Hilb}_m$ can be "transformed" into a morphism of **Hilb** (and vice versa) in a way that preserves the property in the definition of support. In other words, $x \in UA$ (considering A as an object of $\mathbf{Hilb}_m$) is supported on a subspace A_0 iff it is also supported on A_0 when considering A as an object of **Hilb**. This result means that all results on $\mathbf{Hilb}_m$ that rely on the definition of support (including the main result, Theorem 2.14), reduce to equivalent results on **Hilb**. We proceed to extend the result on **Hilb** with the understanding that they are also extensions of the results on $\mathbf{Hilb}_m$ (i.e., those in Section 2.C). This equivalence is useful because, unlike $\mathbf{Hilb}_m$, **Hilb** contains all of its directed colimits.

Having established that **Hilb** is not finitely concrete, i.e. there is no faithful and directed-colimit-preserving functor from $\mathbf{Hilb} \to \mathbf{Set}$, we prove a stronger version of this result in Section 4. In particular, we prove that any functor $U : \mathbf{Hilb} \to \mathbf{Set}$ is constant, up to natural isomorphism (in other words, any such functor is isomorphic to the constant functor which maps every Hilbert space A to $U(\{\vec{0}\})$).

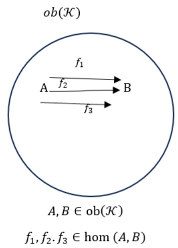
2 Background

2.A Categories and Functors

Definition 2.1. A *category* $\mathcal{K}$ consists of two parts:

(i) a collection of objects $\mathrm{ob}(\mathcal{K})$

(ii) for each pair of objects $A, B \in \mathrm{ob}(\mathcal{K})$, there is a collection of *morphisms* (maps) from $A \to B$.



Example 2.2. The following are examples of categories:

(a) **Set**: The category of sets (for set A, B, morphisms are functions $f : A \to B$)

(b) **Grp**: The category of groups (morphisms are group homomorphisms)

(c) **Ab**: The category of abelian groups

A category collapses all data for a certain type of structure (e.g. groups or sets) into one "item." When working with categories, we usually care less about the internal structure of each object in a category, and more about maps between objects (morphisms).

Definition 2.3 ([Ped89])**.** A *Hilbert space* is a vector space equipped with an inner product $\langle \cdot, \cdot \rangle$ which induces a norm $\|\cdot\|$ (where $\|\vec{v}\| = \sqrt{\langle \vec{v}, \vec{v} \rangle}$ for which the space is *complete*, i.e. the space contains all limits of Cauchy sequences.

A Hilbert space can be thought of as a generalization of normal Euclidean space which allows infinite dimensions.

Example 2.4. Examples of Hilbert spaces include

(a) $\mathbb{R}^n$, for any $n \in \mathbb{N}$.

(b) $\ell^2 := \left\{ (x_0, x_1, ...) \;\middle|\; x_i \in \mathbb{R}, \sum_{i \in \mathbb{N}} |x_i|^2 < \infty \right\}$

Note also that in [LRV23], Hilbert spaces are defined as complex Hilbert spaces. Here, we mainly consider real Hilbert spaces.

Definition 2.5. If A, B are Hilbert spaces, a *linear contraction* $f : A \to B$ is a linear map that does not "lengthen" any vectors, i.e., for all $\vec{v} \in A$, $\|f\vec{v}\| \leq \|\vec{v}\|$.

In this project, we look at various categories that all have Hilbert spaces as objects, but with different definitions of morphisms.

Example 2.6. The following are examples of categories:

(a) **Hilb**: the category of Hilbert spaces with linear contractions as morphisms

(b) $\mathbf{Hilb}_m$: the category of Hilbert spaces with injective linear contractions as morphisms

(c) $\mathbf{Hilb}_r$: the category of Hilbert spaces with *linear isometries* (norm-preserving maps) as morphisms

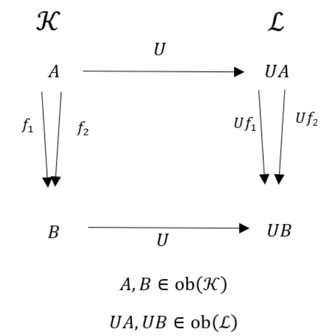
So far, we have looked at categories, consisting of objects and maps between them. We can also take another step out and look at a category as a single item, and consider maps between categories.

Definition 2.7. A *functor* is a map $U : \mathcal{K} \to \mathcal{L}$ between categories that

(i) maps objects in $\mathcal{K}$ to objects in $\mathcal{L}$

(ii) for objects A, B in $\mathcal{K}$, if $f : A \to B$ is a morphism, then U maps f to a morphism $Uf : U(A) \to U(B)$.

(iii) preserves identity and composition



Example 2.8. Examples of functors include:

(a) $U : \mathbf{Set} \to \mathbf{Set}$
$X \mapsto \mathcal{P}(X)$

(b) $U : \mathbf{Set} \to \mathbf{Vec}$
$\{e_1, ..., e_n\} \mapsto \langle e_1, ..., e_n \rangle$ (the vector space whose basis is $\{e_1, ..., e_n\}$)

Functors provide a way to "construct" one type of structure from another. In particular, for a functor $U : \mathcal{K} \to \mathbf{Set}$ that maps $A \mapsto S$, we can call S an "underlying set" of A. In some cases, there is a natural choice for such a functor. For example, $U : \mathbf{Grp} \to \mathbf{Set}$ maps a group G (equipped with homomorphism) to the set of its elements.

However, sometimes there are other choices that are more useful. We are interested in looking at ways to associate an underlying set to a Hilbert space.

Definition 2.9. A functor $U : \mathcal{K} \to \mathcal{L}$ is *faithful* if it is injective on the morphisms in $\mathcal{K}$, i.e. if A, B are objects of $\mathcal{K}$ and $f, g : A \to B$ are morphisms such that $Uf = Ug$, then $f = g$.

We can think of a faithful functor as one that maps to an underlying set while remembering data from the original object. For example, we could map every group in **Grp** to the empty set. This is not faithful because every homomorphism maps to the empty function.

2.B Connection to Axiomatizability

Here we will explain the general connection between axiomatizability in various logics (denoted $\mathbb{L}_{\kappa,\lambda}$) and accessible categories.

Definition 2.10. A logical sentence is a finite string of symbols with *connectives* $\wedge, \vee, \to \leftrightarrow, \neg$, and quantifiers $\forall, \exists$

There are two parameters we can use to describe a logic:

(1) The maximum index size of a conjunction or disjunction $\bigwedge_{i \in I} \phi_i$ or $\bigvee_{i \in I} \phi_i$

(2) The maximum number of variables over which we quantify

We can formally denote a logic as $\mathbb{L}_{\kappa,\lambda}$, where the number of connectives is strictly less than κ, and the number of quantified variables is strictly less than λ.

There is a close connection between the existence of faithful, directed colimit-preserving functors from a category into **Set** and axiomatizability of that category in $\mathbb{L}_{\kappa,\lambda}$. The following are examples of results along these lines; see [AR94] for more details.

Theorem 2.11 ([AR94] Theorem 5.35)**.** *Accessible categories are precisely the categories equivalent to the categories of models of basic theories.*

Proposition 2.12 ([AR94] Proposition 5.39)**.** *Let λ be a regular cardinal and Σ be a λ-ary signature. Then each class of Σ-structures axiomatizable by a theory in L_λ is closed under*

(i) *λ-elementary substructures*

(ii) *λ-directed colimits of λ-elementary embeddings.*

Theorem 2.13 ([AR94] Theorem 5.44)**.** *A class $\mathcal{K}$ of σ-structures is axiomatizable (in L_∞ iff there exists regular cardinal λ such that $\mathcal{K}$ is accessible and accessibly embedded.*

The results here are specifically concerned with $\mathbb{L}_{\kappa,\omega}$ logics where κ is infinite, in which conjunctions and disjuntions of fewer than κ formulas are permitted. In [Hen20], it is shown that categories axiomatizable in such logics are all finitely concrete, and hence, failure of finite concreteness is enough to prove nonaxiomatizability in these logics.

2.C Main result on $\mathbf{Hilb}_m$

The following were shown in [LRV23]:

Theorem 2.14 ([LRV23] Theorem 18)**.** *No faithful functor from* $\mathbf{Hilb}_m$ *to* **Set** *preserves directed colimits.*

An outline of the argument follows:

Proof. Suppose a functor $U : \mathbf{Hilb}_m \to \mathbf{Set}$ is faithful and preserves directed colimits.

Definition 2.15 ([LRV23] Def. 8)**.** Let A be a Hilbert space, $x \in UA$, and A_0 a subspace on A. x is *supported* on A_0 if whenever $f, g : A \to B$ are morphisms (injective linear contractions) with $fi_{A_0,A} = gi_{A_0,A}$, then $f(x) = g(x)$, where $i_{A_0,A}$ is the inclusion map $A_0 \to A$.

It is then shown that finite-dimensional supports are closed under intersections, so that there is a unique minimal support of each $x \in UA$, using the assumption that U preserves directed colimits.

Then,

Lemma 2.16 ([LRV23] Lemma 14)**.** *If A is an infinite-dimensional Hilbert space, $x \in UA$ and A_0, A_1 finite-dimensional subspaces of A such that x is supported on A_0 and A_1, then x is supported on $A_0 \cap A_1$.*

Lemma 2.17 ([LRV23] Lemma 15)**.** *If A is a Hilbert space and $x \in UA$, then there is a unique minimal finite-dimensional subspace A_0 of A on which x is supported.*

Take a Hilbert space A with dimension λ, which is sufficiently large (in particular, $\lambda > \mu_0 + 2^{\aleph_0}$, with countable cofinality).

Lemma 2.17 shows that every $x \in UA$ has a unique minimal support in A, $\mathrm{supp}_A(x)$. Using this fact and a counting argument that shows $|UA|$ is strictly less than $\lambda^{\aleph_0}$, and that a Hilbert space can be partitioned into at least $\lambda^{\aleph_0}$ many lines, it is concluded that there are more lines than elements of UA. Therefore, there is at least one line $A_0 \subseteq A$ that is not equal to $\mathrm{supp}_A(x)$ for any $x \in UA$–and, in particular, for any $x \in UA_0$. However, every $x \in UA_0$ is supported on A_0, so $\mathrm{supp}_A(x)$ must be a proper subspace of A_0, i.e. the trivial space (since A_0 is a line).

However, this immediately contradicts the following fact that is due to the faithfulness of U:

Remark 2.18. For any nonzero subspace A_0 of an infinite-dimensional Hilbert space A, there is some $x_0 \in UA_0$ such that $i_{A_0,A}(x_0)$ has nontrivial support in A.

Therefore, no such functor $U : \mathbf{Hilb}_m \to \mathbf{Set}$ can be faithful and preserve directed colimits. In other words, $\mathbf{Hilb}_m$ is not finitely concrete. By Section 2.B, this means that the category $\mathbf{Hilb}_m$ is not axiomatizable in any finitary logic. □

3 Equivalence between $\mathbf{Hilb}_m$ and Hilb

3.A Supports in Hilb

The results in Section 2.C include the main result, Theorem 2.14, for the category $\mathbf{Hilb}_m$, whose morphisms are all injective linear contractions. In practice, it is often easier to work with the category **Hilb**, whose morphisms are all linear contractions. Here, we redefine some of the machinery in [LRV23] (in particular, the definition of *support*), and prove an equivalence between the results of section 3 for $\mathbf{Hilb}_m$ and **Hilb**.

For this section, as in the previous section, suppose $U : \mathbf{Hilb}_m \to \mathbf{Set}$ (or $U : \mathbf{Hilb} \to \mathbf{Set}$) is faithful and preserves directed colimits. The following is the definition of support for $\mathbf{Hilb}_m$, from [LRV23]:

Definition 3.1 ([LRV23] Definition 8)**.** Let A be a Hilbert space, $x \in UA$, A_0 a subspace on A. Then x is *supported* on A_0 if whenever $f, g : A \to B$ are morphisms (injective linear contractions) with $fi_{A_0,A} = gi_{A_0,A}$, then $f(x) = g(x)$, where $i_{A_0,A}$ is the inclusion map $A_0 \to A$.

In other words, x is supported on a subspace A_0 if whenever two morphisms f, g agree on A_0, they also agree on x.

The equivalent definition for **Hilb** is the following

Definition 3.2 (Definition 8b)**.** If A is a Hilbert space and $x \in UA$ and A_0 is a subspace of A, then x is *supported* on A_0 (in **Hilb**) if whenever $f, g : A \to B$ are morphisms of **Hilb** (linear contractions) with $fi_{A_0,A} = gi_{A_0,A}$, then $f(x) = g(x)$, where $i_{A_0,A}$ is the inclusion map $A_0 \to A$.

We now prove that these two definitions are equivalent: for a Hilbert space A, any $x \in UA$ is either supported on a subspace A_0 by both definitions or neither.

Lemma 3.3. *If A is a Hilbert space, $x \in UA$, A_0 a subspace on A, then the following are equivalent:*

(i) *Whenever $f, g : A \to B$ are injective linear contractions with $fi_{A_0,A} = gi_{A_0,A}$, then $f(x) = g(x)$.*

(ii) *Whenever $f, g : A \to B$ are linear contractions with $fi_{A_0,A} = gi_{A_0,A}$, then $f(x) = g(x)$.*

The proof relies on "transforming" morphisms in **Hilb** (linear contractions) to morphisms in $\mathbf{Hilb}_m$ (injective linear contractions), and vice versa.

Proof. ((i) ⇒ (ii)) Suppose (i) holds. Then it is sufficient to show that for any linear contractions $f, g : A \to B$ such that $fi_{A_0,A} = gi_{A_0,A}$, there are corresponding injective linear contractions $f', g' : A \to C$ (for some Hilbert space C) and $f'i_{A_0,A} = g'i_{A_0,A}$ (so that then by (i), $f'(x) = g'(x)$). From this, it follows that $f(x) = g(x)$.

Let $f, g : A \to B$ be linear contractions such that $fi_{A_0,A} = gi_{A_0,A}$ and define f', g' as follows:

$$\begin{aligned} f' &: A \to A \times B \\ &x \mapsto \frac{\sqrt{2}}{2}(x, f(x)) \\ g' &: A \to A \times B \\ &x \mapsto \frac{\sqrt{2}}{2}(x, g(x)) \end{aligned}$$

Since f and g are contractions (i.e. they scale vectors in A by at most 1), f' and g' are as well, since the norm $\|f(x)\|$ is at most $\frac{\sqrt{2}}{2}\sqrt{2\|x\|^2} = \|x\|$ for any $x \in A$, and likewise for g. We can also see that f' and g' are injective by construction, and that they agree on A_0 whenever f and g do.

Therefore, $f', g' : A \to B$ are injective linear contractions such that $f' i_{A_0,A} = g' i_{A_0,A}$, so by (i), $f'(x) = g'(x)$, i.e. $\frac{\sqrt{2}}{2}(x, f(x)) = \frac{\sqrt{2}}{2}(x, g(x))$, so that $f(x) = g(x)$.

((ii) ⇒ (i)) Immediate. □

Thus, the definition of support for $\mathbf{Hilb}_m$ is equivalent to that of **Hilb**, so that the following results (using Definition 3.1 for $\mathbf{Hilb}_m$) still hold for **Hilb**:

Remark 3.4. If A_0 is a subspace of a Hilbert space A and $x_0 \in UA_0$, then $i_{A_0,A}(x_0)$ is supported on A_0.

Proof. Let $C := i_{A_0,A}(x_0)$. Suppose $f, g : A \to B$ are morphisms such that $f i_{C,A} = g i_{C,A}$, so for all $c \in C$, $f i_{C,A}(c) = g i_{C,A}(c)$. In particular, if $x = U(x_0)$ for $x_0 \in A$, then $fi(x_0) = gi(x_0)$, and therefore $(Uf)(x) = U(f(x_0)) = U(g(x_0)) = (Ug)(x)$, so that x is supported on A_0. □

Continuing using **Hilb**, we derive the same results in section 3 as those for $\mathbf{Hilb}_m$.

Lemma 3.5. *If A is a Hilbert space, $0 \le \delta \le 1$, and $x \in UA$, x is supported on A_0 (in* **Hilb***) iff whenever $f, g : A \to B$ are linear contractions with norm at most δ with $f i_{A_0,A} = g i_{A_0,A}$, then $f(x) = g(x)$.*

Proof. Similar to the result in $\mathbf{Hilb}_m$. □

Lemma 3.6 (Analogous to Lemma 2.16)**.** *If A is an infinite-dimensional Hilbert space, $x \in UA$ and A_0, A_1 finite-dimensional subspaces of A s.t. x is supported on A_0 and A_1, then x is supported on $A_0 \cap A_1$.*

Proof. Similar to proof of Lemma 2.16 □

Lemma 3.7 (Analogous to Lemma 2.17)**.** *If A is a Hilbert space and $x \in UA$, then there is a unique minimal finite-dimensional subspace A_0 of A on which x is supported.*

Proof. Similar to proof of Lemma 2.17 □

Lemma 3.8 (Analogous to Remark 2.18)**.** *For any nonzero subspace A_0 of an inifnite-dimensional Hilbert space A, there is some $x_0 \in UA_0$ such that $i_{A_0,A}(x_0)$ has nontrivial support in A.*

Proof. Similar to proof of Remark 2.18 □

The main result is the following:

Theorem 3.9 (Analogous to Theorem 2.14)**.** *No faithful functor from U : **Hilb** $\to$ **Set** preserves directed colimits.*

Proof. Similar to proof of Theorem 2.14 □

3.B $\mathbf{Hilb}_m$ does not contain all directed colimits

The result Theorem 3.9 is useful because the category **Hilb** contains all directed colimits. Note that [LRV23] mentions that $\mathbf{Hilb}_m$ is a category with directed colimits; however, consider the following counterexample:

Example 3.10. Let $H \in \mathbf{Hilb}_m$ be a nontrivial Hilbert space and set $H_n := H$ for all $n \in \mathbf{N}$. Let the connecting map $f_{n,n+1} = \frac{1}{2}\mathrm{Id}_H$. Suppose K is a colimit of this system, with morphisms $f_n : H = H_n \to K$. Then for any $h \in H$ and $n \in \mathbb{N}$,

$$f_1(h) = f_n \circ f_{1,n}(h) = f_n\Big(\frac{1}{2^{n-1}}h\Big)$$

where each f_n is an injective contraction, so then

$$||f_1(h)|| = ||f_n(2^{-(n-1)}h)|| = \frac{1}{2^{n-1}}h||f_n||$$

Because this holds for all n, the limit goes to 0 as $n \to \infty$, so f_1 is the zero map. However, $f_1 \equiv 0$ is not injective, so K is not a colimit in $\mathbf{Hilb}_m$.

On the other hand, **Hilb** does contain all of its directed colimits. In the proceeding sections, we consider **Hilb**, having established, by Lemma 3.3 that the results are equivalently true for $\mathbf{Hilb}_m$.

4 All functors $U : \mathbf{Hilb} \to \mathbf{Set}$ are constant

As outlined in Section 2.C, there is no functor $U : \mathbf{Hilb} \to \mathbf{Set}$ that is faithful and preserves directed colimits. In this section, we extend this to the stronger result that *any* functor from **Hilb** to **Set** that preserves directed colimits is constant, up to natural isomorphism.

4.A Main result

The main result in this section is the following:

Theorem 4.1 (Main result)**.** *Any functor $U : \mathbf{Hilb} \to \mathbf{Set}$ that preserves directed colimits is naturally isomorphic to the constant functor which takes $A \mapsto U(\{\vec{0}\})$.*

The proof relies on the following intermediate results:

Lemma 4.2. *If $A_0 \subseteq A$ is a closed subspace and $B \subseteq A_0$, then for all $x \in UA_0$, x is supported on B in A_0 iff x is supported on B in A.*

Proof sketch. For the forward direction, let $f, g : A \to C$ agree on B. Since x is supported on B in A_0, for any $f_0, g_0 : A_0 \to C_0$ that agree on B, f_0 and g_0 agree on x. In particular, we can take f_0 and g_0 to be f and g composed with the inclusion map from B into A.

For the reverse direction,the idea is similar: Let $f_0, g_0 : A_0 \to C_0$ be arbitrary and suppose $f_0 i_{B,A_0} = g_0 i_{B,A_0}$. Let $p : A \to A_0$ be the projection map from A to A_0, and take f, g to be f_0 and g_0 composed with p. □

A corollary of Lemma 4.2 is that for any closed subspace $A_0 \subseteq A$, $\text{supp}_{A_0}(x) = \text{supp}_A(x)$.

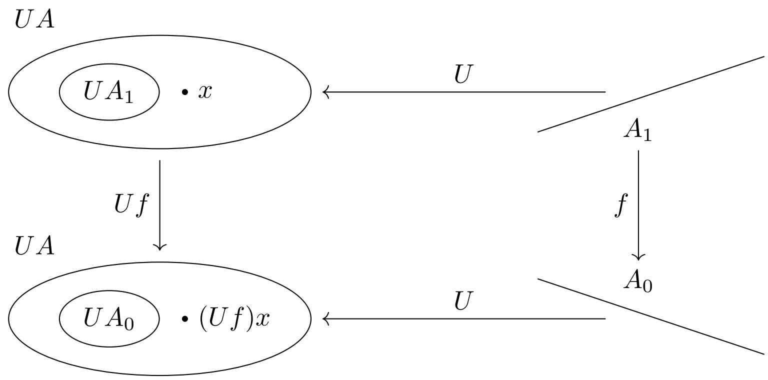
In the following, let $U : \mathbf{Hilb} \to \mathbf{Set}$ be a functor and let A be a Hilbert space as in Theorem 2.14, i.e. a λ-dimensional Hilbert space where λ is chosen to be greater than $\mu_0 + 2^{\aleph_0}$ with countable cofinality. Here, as in Theorem 2.14, μ_0 is chosen such that for any $\mu > \mu_0$, and U preserves μ-presentable objects.

The proof of the following lemma relies on a similar counting argument as the proof of Theorem 2.14.

Lemma 4.3. *No $x \in UA$ has $\text{supp}_A(x) = A_0$ for any one-dimensional $A_0 \subseteq A$.*

Proof sketch. Let A be a Hilbert space of dimension λ as in Theorem 18. Following the proof of Theorem 18: there are more distinct lines in A than there are elements of UA, so there is a line A_0 that is not the support of any $x \in UA$. Therefore, either there are some lines that are the support of elements x of UA and others that are not, or no line in A is the support of any $x \in UA$. We will prove that it must be the latter.

Suppose there is some line A_0 that is not the support of any element of UA, and another line $A_1 \subseteq A$ that is the support of some $x \in UA$. Let $f : A \to A$ be the natural rotation between A_1 and A_0 (this is a morphism in **Hilb**).



Since x is supported on A_1 in A, by definition for any $g, h : A \to C$, if $gi_{A_1,A} = hi_{A_1,A}$ then $g(x) = h(x)$. But then if $g_0, h_0 : A \to C_0$ and $g_0 i_{A_0,A} = h_0 i_{A_0,A}$, then fg_0 and fh_0 are maps from $A_1 \to C_0$ with $g_0 f i_{A_1,A} = h_0 f i_{A_1,A}$, so $g_0 f(x) = h_0 f(x)$. Therefore, $Uf(x)$ is also supported on A_0 in A, a contradiction. □

It follows from Lemma 4.3 that for any line A_0 in A and x in UA_0, $\text{supp}_{A_0}(x) = \{\vec{0}\}$ (the trivial space).

Also, for A as in Theorem 2.14 and for any $k \in \mathbb{N}$, there are at least $\lambda^{\aleph_0}$ many subspaces A_k of A with dimension k. This is because each $A_k \in \mathcal{A}_k$ has a basis $\{e_1, ..., e_k\}$ and each element $a \in A_k$ is of the form

$$a = \sum_{i=1}^{k} c_i e_i$$

where each $c_i \in \mathbb{R}$. There are $|\mathbb{R}| = 2^{\aleph_0}$ choices of c_i for each i, so the number of elements in A_k is

$$|A_k| = (2^{\aleph_0})^k = 2^{k\aleph_0} = 2^{\aleph_0}$$

Note that $A = \bigcup \mathcal{A}_k$. Then we have

$$\begin{aligned} \lambda^{\aleph_0} &= |A| \\ &= |\bigcup \mathcal{A}_k| \\ &\leq \sum_{A_k \in \mathcal{A}_k} |A_k| \\ &= |\mathcal{A}_k| 2^{\aleph_0} \end{aligned}$$

Since (as in Theorem 2.14) $\lambda^{\aleph_0} > 2^{\aleph_0}$, then $|\mathcal{A}_k|$ is at least $\lambda^{\aleph_0}$. So there are at least $\lambda^{\aleph_0}$ many k-dimensional subspaces of A.

Lemma 4.4. *For A with dimension λ and $x \in UA$, the support of x in A is trivial.*

Proof. This argument uses a counting argument similar to that of Theorem 2.14. For any $k > 0$, there are at least $\lambda^{\aleph_0}$ many k-dimensional subspaces of A. However, $|UA| \leq 2^{\aleph_0} < \lambda^{\aleph_0}$, so there are more k-dimensional subspaces of A than elements of UA, so not all k-dimensional subspaces can be the support of an element of UA, and therefore (following a similar argument as in Lemma 4.3 for lines), no k-dimensional subspaces can be the support of an element of UA. Since this is true simultaneously for all $k > 0$, the support of any $x \in UA$ in A must be trivial. □

From Lemma 4.4, it follows that for general A and $x \in UA$, the support of x in A is trivial.

Lemma 4.4 then implies that U is constant on pairs of morphisms $f, g : A \to B$ with the same domain and codomain:

Proof. By Lemma 4.4, if $A_0 \subseteq A$ is a line and $x \in UA_0$, then $\text{supp}_{A_0}\{x\} = \{\vec{0}\}$. This means that for all subspaces A' of A_0 (i.e. A_0 and the zero subspace $\{\vec{0}\}$), whenever f and g agree on A' (i.e. $fi_{A',A} = gi_{A',A}$), $Uf(x) = Ug(x)$. But when $x \in UA_0$, x is supported on A_0, so we always have $Uf(x) = Ug(x)$ for $x \in UA_0$.

If A is an arbitrary Hilbert space, then we can write it as a directed colimit of finite-dimensional Hilbert spaces $A_0 \subseteq A_1 \subseteq ....$, i.e. A is the colimit of $\langle A_i : i \in I\rangle$. Since U preserves directed colimits, UA is the colimit of $\langle UA_i : i \in I\rangle$.

□

Lemma 4.5. *For all $A, B \in$ **Hilb** and morphisms $f : A \to B$, Uf is a bijection from $UA \to UB$.*

Proof. Let $A, B \in$ **Hilb**. By the previous remark, for all $f, g : A \to B$, $Uf = Ug$. In particular, since $U\text{id}_A = \text{id}_{UA}$, then for any $f : A \to A$, $Uf = \text{id}_{UA}$.

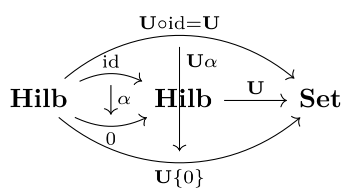
Note that in **Hilb**, there is at least one morphism between any two Hilbert spaces (e.g. the zero morphism), so let $g : B \to A$ be any morphism. Then $g \circ f$ is a morphism from $A \to A$, so that

$$\begin{aligned} U(g \circ f) &= \text{id}_A \\ &= Ug \circ Uf \end{aligned}$$

so Ug and Uf are both bijections (from $UB \to UA$ and $UA \to UB$, respectively). □

Now, the proof of Theorem 4.1 follows:

Proof. From Lemma 4.5, for each $f : A \to B$, Uf is a bijection, and each UA in **Set** has the same cardinality (in particular, for all $A \in$ **Hilb**, $|UA| = |U\{\vec{0}\}|$. There is a natural transformation α which maps $\text{id}_{\mathbf{Hilb}}$ to $0_{\mathbf{Hilb}}$. Then, since U is a functor from **Hilb** $\to$ **Set**, $U\alpha$ is a natural transformation that maps $U\text{id}_{\mathbf{Hilb}} = U$ to $U\{0\}$. Also, since U maps every morphism in **Hilb** to a bijection in **Set** (Lemma 4.5), $U\alpha$ is a natural isomorphism. Therefore, U is naturally isomorphic to $U\{\vec{0}\}$. □



5 Hilb$_r$

Here, we return to the main result (Theorem 2.14) from Section 2.C, and prove the same result for the category **Hilb**$_r$:

Definition 5.1. **Hilb**$_r$ is the category whose objects are Hilbert spaces and morphisms are isometries (linear maps between spaces that preserve norm).

5.A Main result on Hilb$_r$

We begin with the following lemmas regarding the behavior of rotations of Hilbert spaces in **Hilb**$_r$:

Lemma 5.2. *Let $A, B, C \subseteq \mathbb{R}^3$ be lines with $A \neq B$. Then there is a finite composition of rotations about A or B, which takes A to C.*

Proof sketch. By induction on the least $n \in \mathbb{N}$ such that the acute angle between A, B is $\geq 90°/2^n$. If $n = 0$, then $A \perp B$, so it suffices to take two rotations (since spherical coordinates can represent any point on the sphere). Now suppose the claim holds for n; we show it for $n+1$. There is a rotation f about B such that the angle between $A, f(A)$ is precisely $90°/2^n$ (by the intermediate value theorem, since a rotation by 180° makes the angle between $A, f(A)$ twice that between A, B). By the induction hypothesis, there is a finite composition of rotations about A or $f(A)$ taking A to C. Since a rotation about $f(A)$ is an f-conjugate of a rotation about A, we are done. ☐

Lemma 5.3. *Let H be a Hilbert space of dimension ≥ 3, and let $A, B \subseteq H$ be two distinct lines. Then every linear isometry $f : H \to H$ is a finite composition of isometries that fix either A or B.*

Proof. Let $g : H \to H$ be an invertible such finite composition taking A to $f(A)$, by the preceding lemma applied to a 3-dimensional subspace containing $A, B, f(A)$. Then $g^{-1} \circ f$ fixes A, and $f = g \circ g^{-1} \circ f$. ☐

We redefine the notion of support in terms of $\mathbf{Hilb}_r$:

Definition 5.4 (Analogous to Definition 3.1)**.** If A is a Hilbert space, $x \in UA$, and A_0 a subspace on A. x is *supported* on A_0 in $\mathbf{Hilb}_r$ if whenever $f, g : A \to B$ are isometries with $f i_{A_0,A} = g i_{A_0,A}$, then $f = g$, where $i_{A_0,A}$ is the inclusion map $A_0 \to A$.

Using this definition and the preliminary results, we can derive the following results, following Section 2.C, for $\mathbf{Hilb}_r$:

Lemma 5.5 (Analogous to Lemma 2.16)**.** *Let A be infinite-dimensional and let $A_0, A_1 \subseteq A$ be lines. Then if $f, g : A \to B$ are isometries, there exists a sequence $f = k_1, \ldots, k_n = g : A \to B$ such that for each i, $k_i \sim k_{i+1}$ in the sense of Lemma 2.16 (i.e. k_i and k_{i+1} agree on either A_0 or A_1).*

Proof sketch. First, consider the case where at least one of f and g is invertible (WLOG, suppose g is invertible). By Lemma 5.2, we can write $g^{-1} \circ f = h_n \circ h_{n-1} \circ \ldots \circ h_1$. If $n = 1$, we have $g^{-1} \circ f = h_1$, where h_1 fixes A_0 or A_1 (i.e. if h_1 fixes A_i, $h_i|_{A_i} = \mathrm{id}|_{A_i}$). Then we have

$$\begin{aligned}
&(g^{-1} \circ f)|_{A_i} = \mathrm{id}_{A_i} \\
&\Rightarrow (g \circ g^{-1} \circ f)|_{A_i} = g \circ h_1|_{A_i} = g \circ \mathrm{id}|_{A_i} \\
&\Rightarrow g \circ h_1|_{A_i} = g|_{A_i}
\end{aligned}$$

A similar result follows for $n = 2$, and can be extended to work for any n. Therefore, if g is invertible, there exists a sequence $f = k_1, \ldots k_n = g : A \to B$ such that for each i, $k_i \sim k_{i+1}$.

Now, consider the case where neither f nor g is invertible. Let $C := \langle f(A) \cup g(A) \rangle \subseteq B$, the subspace generated by $f(A)$ and $g(A)$. Since A is infinite dimensional, $\dim(A) = \dim(C)$. Without loss of generality, we can restrict B to C (since C embeds into B via the inclusion isometry), so that there exists an invertible $h : A \to B$. Then, this case reduces to the previous case. ☐

Now, Lemma 5.5 proves the analogue of Lemma 2.16 for $\mathbf{Hilb}_r$ when A is infinite-dimensional and A_0, A_1 are lines. In particular, if x is supported on A_0 and A_1 in $\mathbf{Hilb}_r$, it is also supported on their intersection, and therefore $\mathrm{supp}_A(x) = \{\vec{0}\}$.

The main results for $\mathbf{Hilb}_r$ follow (along the lines of Remark 2.18 and Theorem 2.14):

Lemma 5.6 (Analogous to Remark 2.18)**.** *For any one-dimensional subspace A_0 of an infinite-dimensional Hilbert space A, there is some $x_0 \in UA_0$ such that $i_{A_0,A}(x_0)$ has nontrivial support in A.*

Proof. Follows the proof of Remark 2.18, as the maps $f, g : A \to B$,which send A to its copies in the left and right component, respectively, are already isometries. The remaining proof is identical. ☐

Theorem 5.7 (Analogous to Theorem 2.14)**.** *No faithful functor from U : $\boldsymbol{Hilb}_r \to \boldsymbol{Set}$ preserves directed colimits.*

Proof. Follows the proof of Theorem 2.14, replacing $\mathbf{Hilb}_m$ with $\mathbf{Hilb}_r$, and using the preceding lemmas and definition of support for $\mathbf{Hilb}_r$. ☐

6 Current and Future Work

We now have the results from Section 2.C for both **Hilb** (and $\mathbf{Hilb}_m$) and $\mathbf{Hilb}_r$ (Section 4). We have also strengthened this result in the case of **Hilb** and $\mathbf{Hilb}_r$ (Theorem 4.1). A natural next step would be to prove a similar result for $\mathbf{Hilb}_r$.

Note that we cannot prove the exact same result; it is not true that all functors from $\mathbf{Hilb}_r$ to **Set** are constant up to natural isomorphism. Consider the following example:

Example 6.1. Let $U : \mathbf{Hilb}_r \to \mathbf{Set}$ be defined by

$$A \mapsto \begin{cases} 2 & \text{if} \dim(A) \geq 2 \\ 1 & \text{otherwise} \end{cases}$$

For $A, B \in \text{ob}(\mathbf{Hilb}_r)$ with dimension at least 2 or A, B with dimension less than 2, if $f : A \to B$, then $Uf = \text{id}_{UA}$. If A has dimension less than 2 and B has dimension at least 2, then (WLOG) let $F : A \to \mathbf{2}$ map $0 \mapsto 0$.

This functor is not constant, but it is *eventually* constant, i.e. constant for large enough dimensional Hilbert spaces A. Here, we consider whether every functor from $\mathbf{Hilb}_r$ into **Set** exhibits this tendency:

Conjecture 6.2. *If $U : \mathbf{Hilb}_r \to \mathbf{Set}$ is a functor, then there is some κ for which U is constant on Hilbert spaces of dimension greater than κ.*

Evidence for this conjecture comes from the proof strategy in the previous sections, in which key lemmas are proven for Hilbert spaces of dimension λ, chosen as in theorem 18. However, we have no control over or indication of the size of λ. For **Hilb**, the desired result is generalized to all Hilbert spaces by Lemma 4.5; however, this proof relies on the fact that there exists at least one morphism between any two Hilbert spaces in **Hilb**. This is true in $\mathbf{Hilb}_r$, because morphisms must preserve norm, there are Hilbert spaces with no isometries between them.

Therefore, as of now, we can only prove an analogue of Theorem 2.14 for high enough dimensional Hilbert spaces in $\mathbf{Hilb}_r$, i.e. we can prove a version of Conjecture 6.2 with no control over the lower bound κ. A next step would be to find some restriction on the size of κ.

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