GRAVITATIONAL STABILITY OF A RIGID RING AND A SPHERE

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1. Introduction

In celestial mechanics, it is interesting to make the stability analysis of the orbital equilibria of a solid ring-planet system at certain spin rate. This is a classical problem which can be traced back to the era of Maxwell. In 1855, Maxwell published his paper: *On the stability of the motion of Saturn's Rings*, see [6]. In general, Saturn's ring is made of gas, dusts and asteroids in the universe. In that paper, he assumed Saturn's Ring was made of solids only and claimed: a solid ring around a planet must be unstable. He proved that claim using linear stability analysis, which was relatively prolix. In 2020, Dr.D.J.Scheeres provided a modern treatment to prove this classical result so called *the Angular Momentum Relative amended potential method*. He also applied the method to find new orbital equilibria. In this report, we will basically follow his approach and formulate the problem mathematically by constructing the ideal model of rigid ring and sphere shown in the following figure:

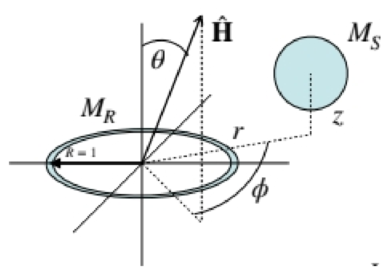

Figure 1. Ring-Sphere Formulation

Then the classical problem is corresponding to the following scenario $r = z = 0$. See the figure below: We will show:

Theorem 1.1. *The orbital equilibrium for $r = z = 0$ has stable z oscillations and instability in the radial direction at arbitrary spin rate.*

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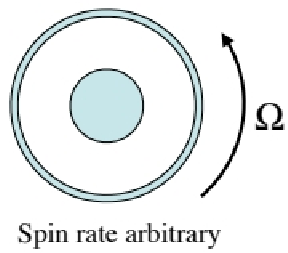

FIGURE 2. Scenario $r = z = 0$

The second part of the theorem 1.1 shows the correctness of Maxwell's statement. On the other hand, we apply the same method to the following two scenarios to find new orbital equilibria for $r = 0, \theta = \frac{\pi}{2}, z > 0$ and $r > R, \theta = 0, z = 0$ (i.e. planar external equilibria). See the Figure 3 and 4 below. We will also show the following theorem:

Theorem 1.2. *The orbital equilibria for $r = 0, \theta = \pi/2, z > 0$ is stable in both z and θ for a large enough z at the spin rate $\Omega^2 = \frac{G(M_R+M_S)}{[R^2+z^2]^{\frac{3}{2}}}$. However, it is always unstable in the combined variations of r,θ.*

The scenario for $r > R, \theta = 0, z = 0$ is much more interesting because there is a bifurcation happening in r direction and we can show

Theorem 1.3. *The orbital equilibria for $r > R, \theta = 0, z = 0$ bifurcates into existence at $r^* > R$ for large enough H^2. It is stable if $r > r^*$ and unstable if $r < r^*$, where $H^2 = \frac{G(M_R+M_S)I_H(r)^2}{\pi r^2}\left[\frac{K(k)}{r+R} + \frac{E(k)}{r-R}\right]$, $k^2 = \frac{4rR}{(r+R)^2}$.*

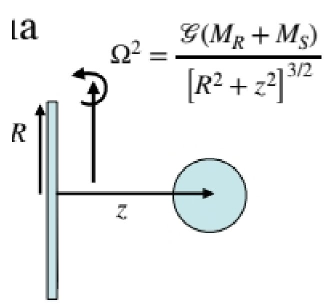

FIGURE 3. Scenario $r = 0, \theta = \frac{\pi}{2}, z > 0$

The amended potential method can be widely applied in many areas such as astronomy and space science. It can be used in energy optimization for planetary pair to determine tidal equilibrium states, see [1]. The problem of control of rings of satellites can also be treated, see [4]

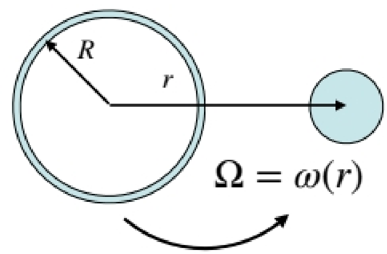

FIGURE 4. Scenario $r > R, \theta = 0, z = 0$

2. Background

Definition 2.1 (Amended Potential)**.** The "Angular Momentum Relative" Amended Potential $\mathscr{E}$ is used, which allows one to evaluate equilibria and their energetic stability, and given by:

$$\mathscr{E}(Q) = \frac{H^2}{2I_H(Q)} + \mathscr{U}(Q) \leq E \tag{1}$$

where H is the total angular momentum, E is the total system energy, Q is the set of minimal coordinates of relative position and and attitude needed to specify the system configuration, I_H is the total moment of inertia about the angular momentum direction, $\mathscr{U}$ is the gravitational potential energy. We also define $\frac{H}{I_H} = \dot{\theta}$ as the spin rate.

Definition 2.2 (Moment of Inertia)**.** The total moment of inertia about the angular momentum direction I_H is defined by the following tensor product (i.e. quadratic form):

$$I_H = \hat{H} \cdot I \cdot \hat{H} \tag{2}$$

where I is the system's total instantaneous moment of inertia.

Remark 2.3. We can represent the system's total instantaneous moment of inertia I using total inertia dyadic of N-body system defined as the following:

$$I = \int_{\mathscr{B}} -\tilde{r} \cdot \tilde{r} dm \tag{3}$$

where $\tilde{r}$ obeys the following rule for cross product dyadic:

$$\vec{a} \times \vec{b} = \tilde{a} \cdot \vec{b} = \vec{a} \cdot \tilde{b} \tag{4}$$

Remark 2.4. There are two ways to derive the Amended Potential $\mathscr{E}$. One requires more physics background, using the reduced Lagrangian, see [9]. This report will present a math-tasted way using Sundman's Inequality. For more details, the readers may refer to [7].

The total angular momentum is defined as the following:

$$H = \int_{\mathscr{B}} \vec{r} \times \vec{v} dm \tag{5}$$

In addition, the kinetic energy of the system is defined as the following:

$$T = \frac{1}{2} \int_{\mathscr{B}} v^2 dm \tag{6}$$

where $\mathscr{B}$ refers to the spacial distribution of the rigid body. Apply Cauchy-Schwartz inequality, we get:

$$\begin{aligned}
H^2 &= (\hat{H} \cdot H)^2 \\
&= \left(\int_{\mathscr{B}} \hat{H} \cdot \vec{r} \times \vec{v} dm \right)^2 = \left(\int_{\mathscr{B}} \vec{v} \cdot \hat{H} \times \vec{r} dm \right)^2 \\
&\leq \left(\int_{\mathscr{B}} |\vec{r}||\hat{H} \times \vec{v}| dm \right)^2 \leq \left(\int_{\mathscr{B}} v^2 dm \right) \left(\int_{\mathscr{B}} |\hat{H} \times \vec{r}|^2 dm \right) \\
&\leq 2T \int_{\mathscr{B}} |\hat{H} \times \vec{r}|^2 dm
\end{aligned}$$

On the other hand, using Remark 2.3, we can furtherly simplify the result:

$$\begin{aligned}
\int_{\mathscr{B}} |\hat{H} \times \vec{r}|^2 dm = \int_{\mathscr{B}} (\hat{H} \times \vec{r}) \cdot (\hat{H} \times \vec{r}) dm &= \int_{\mathscr{B}} -(\hat{H} \times \vec{r}) \cdot (\vec{r} \times \hat{H}) dm \\
&= \int_{\mathscr{B}} -\hat{H} \cdot (\tilde{r} \cdot \tilde{r}) \cdot \hat{H} dm = \hat{H} \cdot \int_{\mathscr{B}} -\tilde{r} \cdot \tilde{r} dm \cdot \hat{H} \\
&= \hat{H} \cdot I \cdot \hat{H} = I_H
\end{aligned}$$

Hence, we have $H^2 \leq 2TI_H$. Recall $T = E - \mathscr{U}$. Then we get $H^2 \leq 2(E - \mathscr{U})I_H$. It follows that $\mathscr{E} = \frac{H^2}{2I_H} + \mathscr{U} \leq E$.

Remark 2.5. Since we are working in a rotating frame, besides the gravitational potential energy caused by attraction, there is also a centrifugal potential due to repulsion. That leads to the extra term $\frac{H^2}{2I_H}$.

In the Ring-Sphere formulation shown in Figure 1, we are working in the frame of the ring. Accordingly, the mutual potential is given by

$$\mathscr{U}(r, z) = -\frac{GM_R M_S}{\sqrt{(R+r)^2 + z^2}} K(k) \tag{7}$$

where $K(k)$ is the complete Elliptic Integral of the 1st kind defined as the following:

$$K(k) = \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}} \tag{8}$$

k is called modulus and $k < 1$. Here we have

$$k^2 = \frac{4rR}{(r+R)^2 + z^2} \tag{9}$$

The readers may refer to [2] to see the derivation of the potential between the ring and the sphere. The result given by [2] is:

$$U = \frac{2GM_R}{p\pi} \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}} = \frac{2GM_R}{p\pi} K(k^2) \tag{10}$$

where

$$k^2 = 1 - \frac{q^2}{p^2}$$
$$q^2 = (x - R)^2 + z^2$$
$$p^2 = (x + R)^2 + z^2$$

To get Equ.7, we need to do some modifications by replacing x with r to match our situation and normalizing the boundary of integration of Elliptic integral by $\frac{\pi}{2}$. Eventually, we can treat the sphere as a point mass to obtain the exact form as Equ.7. The similar idea can be found in [10]. On the other hand, the moment of inertia about the angular momentum direction is given by:

$$I_H(r, z, \theta.\phi) = \hat{H} \cdot I_R \cdot \hat{H} + \hat{H} \cdot I_S \cdot \hat{H} + \hat{H} \cdot I_{COM} \cdot \hat{H} \tag{11}$$

where I_R, I_S and I_{COM} represent the moment of inertia tensor of the ring, the sphere and the center of mass of the ring-sphere system respectively. It is also known that:

$$I_R = M_R R^2 Diag\left(\frac{1}{2}, \frac{1}{2}, 1\right) \tag{12}$$

$$I_{COM} = \frac{M_R M_S}{M_R + M_S}[r^2 + z^2] \tag{13}$$

There are few remarks on how to derive Equ.18. Denote the angle between the dash line representing the projection of $\hat{H}$ on x-y plane and x-axis as ψ. Then we have:

$$\hat{H} = \sin\theta\cos\psi\hat{x} + \sin\theta\sin\psi\hat{y} + \cos\theta\hat{z} \tag{14}$$

It follows that:

$$\begin{aligned}
\hat{H} \cdot I_R \cdot \hat{H} &= M_R R^2 \begin{pmatrix}\sin\theta\cos\psi & \sin\theta\sin\psi & \cos\theta\end{pmatrix} \begin{pmatrix}\frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1\end{pmatrix} \begin{pmatrix}\sin\theta\cos\psi \\ \sin\theta\sin\psi \\ \cos\theta\end{pmatrix} \\
&= \frac{1}{2} M_R R^2 \sin^2\theta + M_R R^2 \cos^2\theta \\
&= \frac{1}{2} M_R R^2 (1 + \cos^2\theta)
\end{aligned}$$

Since the sphere is symmetric, so its moment of inertial along any direction is the same. That implies $\hat{H} \cdot I_S \cdot \hat{H} = I_S$. To compute the third term in Equ.18, we need to know the rejection of the distance between the ring and the sphere with respect to $\hat{H}$. Denote the vector starting from the center of the ring pointing to the center of the sphere as $\vec{d}$. We also established a new coordinate system $\{r,\, r^{\perp}, z\}$. Now, under this new coordinate system, we can get the following expressions:

$$\vec{d} = r\hat{r} + z\hat{z} \tag{15}$$

$$\hat{H} = \sin\theta\cos\phi\hat{r} + \sin\theta\sin\phi\hat{r}^{\perp} + \cos\theta\hat{z} \tag{16}$$

Thus, the length of the projection can be calculated by:

$$\begin{aligned}|proj_{\hat{H}}\vec{d}| &= \vec{d}\cdot\hat{H}\\ &= r\sin\theta\cos\phi + z\cos\theta\end{aligned}$$

Simply by Pythagorean Theorem, we can obtain:

$$\hat{H}\cdot I_{COM}\cdot\hat{H} = \frac{M_R M_S}{M_R + M_S}[r^2 + z^2 - (r\sin\theta\cos\phi + z\cos\theta)^2] \tag{17}$$

In the last step, we get the explicit formula:

(18)

$$I_H(r,z,\theta,\phi) = \frac{1}{2}M_R R^2(1+\cos^2\theta)+I_S+\frac{M_R M_S}{M_R + M_S}[r^2+z^2-(r\sin\theta\cos\phi+z\cos\theta)^2]$$

By Definition 2.1, we know the amended potential of the ring-sphere formulation is

$$\mathscr{E}(r,z,\theta,\phi) = \frac{H^2}{2I_H(r,z,\theta,\phi)} + \mathscr{U}(r,z) \tag{19}$$

Definition 2.6 (Relative Equilibria)**.** Formally, when the system spins around a given axis with a fixed spin rate, we say the system is in a relative equilibrium. More generally, a relative equilibrium for a mechanical system is an orbit which is given by the action of a suitable group. For more details, see [5].

Remark 2.7. Relative equilibria and energetic stability for orbiting or resting N rigid body systems can be found using only the amended potential.

Theorem 2.8 (Stability Condition, also see [8])**.** *Given the configuration of the system $Q = \{\vec{r_{ij}}|r_{ij} \geq d\}$, then the system obtains the relative equilibrium provided that the following variational condition holds for all admissible δQ:*

$$\delta\mathscr{E} = \frac{\partial\mathscr{E}}{\partial Q}\cdot\delta Q = 0 \tag{20}$$

The relative equilibrium is stable provided that

$$\delta^2\mathscr{E} = \delta Q\cdot\frac{\partial^2\mathscr{E}}{\partial Q^2}\cdot\delta Q \geq 0 \tag{21}$$

Remark 2.9. Suppose q is one of the admissible coordinates in the configuration, one can fatherly rewrite the stability condition in terms of the derivatives of $\mathscr{E}$ as the following:

$$\mathscr{E}_q = -\frac{H^2}{2I_H^2}\frac{\partial I_H}{\partial q} + \frac{\partial\mathscr{U}}{\partial q} \tag{22}$$

$$\mathscr{E}_{qq} = \frac{H^2}{I_H^3}\left(\frac{\partial I_H}{\partial q}\right)^2 - \frac{H^2}{2I_H^2}\frac{\partial^2 I_H}{\partial q^2} + \frac{\partial^2 \mathscr{U}}{\partial q^2} \tag{23}$$

Generally, if we have combined variations, then we need to check the definiteness of the Hessian matrix. If it is positive definite, then the relative equilibrium is stable. This form of stability condition is more useful and convenient in the computations in the next part.

3. Main Results

The basic idea of this part is to use Remark 2.9 to find the relative equilibria in each scenario and make the energetic stability analysis corresponding to each equilibrium.

3.1. **Proof of Theorem 1.1.** We need to show $r = z = 0$ is the orbital equilibrium at arbitrary spin rate. The following proposition is introduced:

Proposition 3.1. [3, Section 19.5] *If $|k| < 1$, the complete Elliptic Integral of first kind $K(k)$ admits the following Maclaurin Expansion:*

$$K(k) = \frac{\pi}{2}\sum_{m=0}^{\infty}\frac{\left(\frac{1}{2}\right)_m\left(\frac{1}{2}\right)_m}{m!m!}k^{2m} \tag{24}$$

where $(a)_m$ refers to Pochhammer's symbol[3, Section 5.2(iii)].

To take derivatives easily, we rewrite Proposition 3.1 for the normalized version as:

$$K(k^2) = \sum_{m=0}^{\infty}\frac{((2m-1)!!)^2}{2^{2m}(m!)^2}k^{2m} \tag{25}$$

$$= \sum_{m=0}^{\infty}\left(\frac{(2m-1)!!}{(2m)!!}\right)^2 k^{2m} \tag{26}$$

$$= 1 + \left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 k^4 + \dots \tag{27}$$

Recall Equ.9 implies $k = 0$ when $r = z = 0$. Hence, it follows that $K(0) = 1$ and

$$\frac{dK(k^2)}{dk^2}\Big|_{k=0} = \frac{1}{4} + 2\left(\frac{1\cdot 3}{2\cdot 4}\right)^2 k^2 + \dots\Big|_{k=0} = \frac{1}{4} \tag{28}$$

We are easy to see:

$$2k\frac{\partial k}{\partial z}\Big|_{r=z=0} = \frac{-8rRz}{[(r+R)^2+z^2]^2}\Big|_{r=z=0} = 0 \tag{29}$$

and

$$2k\frac{\partial k}{\partial r}\Big|_{r=z=0} = \frac{4R[(r+R)^2+z^2] - 8rR(r+R)}{[(r+R)^2+z^2]^2}\Big|_{r=z=0} = \frac{4}{R} \tag{30}$$

It is also clear that:

$$\frac{\partial \mathscr{U}}{\partial \theta}\Big|_{r=z=0} = \frac{\partial \mathscr{U}}{\partial \phi}\Big|_{r=z=0} = \frac{\partial I_H}{\partial \theta}\Big|_{r=z=0} = \frac{\partial I_H}{\partial \phi}\Big|_{r=z=0} = 0 \tag{31}$$

Here we compute other ingredients:

$$\frac{\partial \mathscr{U}}{\partial r}\Big|_{r=z=0} = -GM_RM_S\frac{\partial[(R+r)^2+z^2]^{-\frac{1}{2}}K(k^2)}{\partial r}\Big|_{r=z=0} \tag{32}$$

$$= -GM_RM_S\left[-[(R+r)^2+z^2]^{-\frac{3}{2}}(R+r)K(k^2) + [(R+r)^2+z^2]^{-\frac{1}{2}}\frac{dK(k^2)}{dk^2}2k\frac{\partial k}{\partial r}\right]\Big|_{r=z=0} \tag{33}$$

$$= -R^{-2} + \frac{1}{4}R^{-1}\frac{4}{R} = 0 \tag{34}$$

$$\frac{\partial \mathscr{U}}{\partial z}\Big|_{r=z=0} = -GM_RM_S\frac{\partial[(R+r)^2+z^2]^{-\frac{1}{2}}K(k^2)}{\partial z}\Big|_{r=z=0} \tag{35}$$

$$= -GM_RM_S\left[-[(R+r)^2+z^2]^{-\frac{3}{2}}zK(k^2) + [(R+r)^2+z^2]^{-\frac{1}{2}}\frac{dK(k^2)}{dk^2}2k\frac{\partial k}{\partial z}\right]\Big|_{r=z=0} = 0 \tag{36}$$

$$\frac{\partial I_H}{\partial r}\Big|_{r=z=0} = \frac{M_RM_S}{M_R+M_S}\left[2r - 2(r\sin\theta\cos\phi + z\cos\theta)\sin\theta\cos\phi\right]\Big|_{r=z=0} = 0 \tag{37}$$

$$\frac{\partial I_H}{\partial z}\Big|_{r=z=0} = \frac{M_RM_S}{M_R+M_S}\left[2z - 2(r\sin\theta\cos\phi + z\cos\theta)\cos\theta\right]\Big|_{r=z=0} = 0 \tag{38}$$

Therefore, we have $\frac{\partial I_H}{\partial q} = \frac{\partial \mathscr{U}}{\partial q} = 0$ for $r = z = 0$. Thus the orbital equilibrium exists for all spin rate $\Omega = \frac{H}{I_H}$. Next, we will sequentially do the energetic stability analysis in z and r directions separately and compute the second derivative of $\mathscr{E}$. Notice that since $\frac{\partial I_H}{\partial r}\Big|_{r=z=0} = \frac{\partial I_H}{\partial z}\Big|_{r=z=0} = 0$, Equ.59 will simply become:

$$\mathscr{E}_{zz} = -\frac{H^2}{2I_H^2}\frac{\partial^2 I_H}{\partial z^2} + \frac{\partial^2 \mathscr{U}}{\partial z^2} \tag{39}$$

$$\mathscr{E}_{rr} = -\frac{H^2}{2I_H^2}\frac{\partial^2 I_H}{\partial r^2} + \frac{\partial^2 \mathscr{U}}{\partial r^2} \tag{40}$$

Also notice that geometrically, when $r = z = 0$, the direction of $\hat{H}$ coincides with z-axis, so $\theta = 0$. We compute the ingredients for $\mathscr{E}_{zz}$ and get:

$$\frac{\partial^2 I_H}{\partial z^2}\Big|_{r=z=0} = \frac{2M_RM_S}{M_R+M_S}\sin^2\theta\Big|_{\theta=0} = 0 \tag{41}$$

$$\frac{\partial}{\partial z}\left(2k\frac{\partial k}{\partial z}\right)\Big|_{r=z=0} = \frac{\partial}{\partial z}\left(\frac{-8rRz}{[(r+R)^2+z^2]^2}\right)\Big|_{r=z=0} = 0 \tag{42}$$

(43)
$$\frac{\partial^2 \mathscr{U}}{\partial z^2}\Big|_{r=z=0} = -GM_RM_S\left[-[(R+r)^2+z^2]^{-\frac{3}{2}}K(k^2) - [(R+r)^2+z^2]^{-\frac{1}{2}}\frac{dK(k^2)}{dk^2}\frac{\partial}{\partial z}\left(2k\frac{\partial k}{\partial z}\right)\right]\Big|_{r=z=0}$$
(44)
$$= \frac{GM_RM_S}{R^3}$$

It implies that $\mathscr{E}_{zz} > 0$, so the z-oscillations are stable. Additionally, We compute the ingredients for $\mathscr{E}_{rr}$ and get:

$$\frac{\partial^2 I_H}{\partial r^2}\Big|_{r=z=0} = \frac{2M_RM_S}{M_R+M_S}(1-\sin^2\theta\cos^2\phi)\Big|_{\theta=0} = \frac{2M_RM_S}{M_R+M_S} \tag{45}$$

$$\frac{\partial}{\partial r}\left(2k\frac{\partial k}{\partial r}\right)\Big|_{r=z=0} = 4R\frac{\partial}{\partial r}\left[\frac{-r^2+z^2+R^2}{[(r+R)^2+z^2]^2}\right]\Big|_{r=z=0} = -\frac{16}{R^2} \tag{46}$$

$$\frac{\partial}{\partial r}\left(\frac{dK(k^2)}{dk^2}\right)\Big|_{r=z=0} = \frac{d^2K(k^2)}{d(k^2)^2}2k\frac{\partial k}{\partial r}\Big|_{r=z=0} = 2\left(\frac{1\cdot 3}{2\cdot 4}\right)^2\frac{4}{R} = \frac{9}{8R} \tag{47}$$

(48)
$$\frac{\partial^2 \mathscr{U}}{\partial r^2}\Big|_{r=z=0} = -GM_RM_S\left(\frac{3(R+r)^2K(k^2)}{[(R+r)^2+z^2]^{\frac{5}{2}}} - \frac{K(k^2)+2(R+r)\frac{dK(k^2)}{dk^2}2k\frac{\partial k}{\partial r}}{[(R+r)^2+z^2]^{\frac{3}{2}}} + \frac{\frac{\partial}{\partial r}\left(\frac{dK(k^2)}{dk^2}\right)2k\frac{\partial k}{\partial r} + \frac{dK(k^2)}{dk^2}\frac{\partial}{\partial r}\left(2k\frac{\partial k}{\partial r}\right)}{[(R+r)^2+z^2]^{\frac{1}{2}}}\right)$$
(49)
$$\Big|_{r=z=0} = -\frac{GM_RM_S}{2R^3}$$

It implies that $\mathscr{E}_{rr} < 0$, so the orbital equilibrium is unstable in the radial direction. Physically, given a small perturbation on the sphere along r direction, it just escapes from the original position and breaks the ring, which verified Maxwell's statement.

3.2. **Proof of Theorem 1.2.** We basically follow the same steps in the previous part and show $r = 0, \theta = \frac{\pi}{2}, z > 0$ is an orbital equilibrium. Firstly, we focus on z direction:

(50)
$$\frac{\partial I_H}{\partial z}\Big|_{r=0,\theta=\frac{\pi}{2},z>0} = \frac{M_RM_S}{M_R+M_S}\left[2z - 2(r\sin\theta\cos\phi + z\cos\theta)\cos\theta\right]\Big|_{r=0,\theta=\frac{\pi}{2},z>0} = \frac{2M_RM_S}{M_R+M_S}z$$

(51)
$$\frac{\partial \mathscr{U}}{\partial z}\Big|_{r=0,\theta=\frac{\pi}{2},z>0} = -GM_R M_S \frac{\partial[(R+r)^2+z^2]^{-\frac{1}{2}}K(k^2)}{\partial z}\Big|_{r=0,\theta=\frac{\pi}{2},z>0}$$
(52)
$$= -GM_R M_S\left[-[(R+r)^2+z^2]^{-\frac{3}{2}}zK(k^2)+[(R+r)^2+z^2]^{-\frac{1}{2}}\frac{dK(k^2)}{dk^2}2k\frac{\partial k}{\partial z}\right]$$
(53)
$$\Big|_{r=0,\theta=\frac{\pi}{2},z>0} = \frac{GM_S M_R z}{[R^2+z^2]^{\frac{3}{2}}}$$

When the spin rate is given by $\Omega^2 = \frac{G(M_R+M_S)}{[R^2+z^2]^{\frac{3}{2}}}$, we have:

$$\mathscr{E}_z = -\frac{H^2}{2I_H^2}\frac{\partial I_H}{\partial z} + \frac{\partial \mathscr{U}}{\partial z}\Big|_{r=0,\theta=\frac{\pi}{2},z>0} \tag{54}$$

$$= -\frac{G(M_R+M_S)}{2[R^2+z^2]^{\frac{3}{2}}}\cdot\frac{2M_R M_S}{M_R+M_S}z + \frac{GM_S M_R z}{[R^2+z^2]^{\frac{3}{2}}} = 0 \tag{55}$$

Thus, it is an orbital equilibrium along z direction. Also notice that in this case, by Equ. 18, we have:

$$I_H = \frac{1}{2}M_R R^2 + I_S + \frac{M_R M_S}{M_R+M_S}z^2 \tag{56}$$

Let's see the second derivatives:

$$\frac{\partial^2 I_H}{\partial z^2}\Big|_{r=0,\theta=\frac{\pi}{2},z>0} = \frac{2M_R M_S}{M_R+M_S}\sin^2\theta\Big|_{\theta=\frac{\pi}{2}} = \frac{2M_R M_S}{M_R+M_S} \tag{57}$$

$$\frac{\partial^2 \mathscr{U}}{\partial z^2}\Big|_{r=0,\theta=\frac{\pi}{2},z>0} = GM_R M_S\left[\frac{1}{[r^2+z^2]^{\frac{3}{2}}} - \frac{3z^2}{[r^2+z^2]^{\frac{5}{2}}}\right] \tag{58}$$

(59)
$$\mathscr{E}_{zz} = \frac{H^2}{I_H^3}\left(\frac{\partial I_H}{\partial z}\right)^2 - \frac{H^2}{2I_H^2}\frac{\partial^2 I_H}{\partial z^2} + \frac{\partial^2 \mathscr{U}}{\partial z^2}\Big|_{r=0,\theta=\frac{\pi}{2},z>0}$$

$$= \frac{GM_S M_R z^2[2M_R M_S z^2 + 5M_S M_R R^2 - 3M_R^2 R^2 - 6I_S(M_S+M_R)]}{[R^2+z^2]^{\frac{5}{2}}[M_R R^2(M_R+M_S)+2I_S(M_R+M_S)+2M_R M_S z^2]} \tag{60}$$

It is obvious that if z is large enough, then $\mathscr{E}_{zz} > 0$ and the orbital equilibrium is stable in z direction.

Next, we check for θ direction:

(61)
$$\frac{\partial I_H}{\partial \theta}\Big|_{r=0,\theta=\frac{\pi}{2},z>0} = -M_R R^2\cos\theta\sin\theta + \frac{M_R M_S}{M_R+M_S}[-2(r\sin\theta\cos\phi + z\cos\theta)(r\cos\phi\cos\theta - z\sin\theta)]\Big|_{r=0,\theta=\frac{\pi}{2},z>0} = 0$$

$$\frac{\partial \mathscr{U}}{\partial \theta}\Big|_{r=0,\theta=\frac{\pi}{2},z>0} = 0 \tag{62}$$

Thus, $\mathscr{E}_\theta = 0$ and it is an orbital equilibrium along θ direction. For the second derivative, we have:

$$\frac{\partial^2 I_H}{\partial\theta^2}\Big|_{r=0,\theta=\frac{\pi}{2},z>0} = -M_R R^2\cos(2\theta) - \frac{2M_R M_S}{M_R+M_S}[(r^2\cos^2\phi - z^2)\cos(2\theta) - 2zr\cos\phi\sin(2\theta)] \tag{63}$$

$$\Big|_{r=0,\theta=\frac{\pi}{2},z>0} = -\frac{2M_R M_S}{M_R+M_S}z^2 + M_R R^2 \tag{64}$$

$$\mathscr{E}_{\theta\theta}\Big|_{r=0,\theta=\frac{\pi}{2},z>0} = -\frac{H^2}{2I_H^2}\frac{\partial^2 I_H}{\partial z^2} \tag{65}$$

$$= \frac{GM_R M_S}{[R^2+z^2]^{\frac{3}{2}}}z^2 - \frac{G(M_R+M_S)M_R R^2}{2[R^2+z^2]^{\frac{3}{2}}} \tag{66}$$

When θ is sufficiently large, the second term will vanish and we will have $\mathscr{E}_{\theta\theta} > 0$. That implies the orbital equilibrium is stable in θ direction.

Finally, let's check the combined variations of θ and r. In this case we need to check it is an orbital equilibrium in radial direction and the definiteness of the Hassian matrix in terms of r, θ.Well,

$$\frac{\partial I_H}{\partial r}\Big|_{r=0,\theta=\frac{\pi}{2},z>0} = \frac{M_R M_S}{M_R+M_S}\left[2r - 2(r\sin\theta\cos\phi + z\cos\theta)\sin\theta\cos\phi\right]\Big|_{r=0,\theta=\frac{\pi}{2},z>0} = 0 \tag{67}$$

$$\frac{\partial \mathscr{U}}{\partial r}\Big|_{r=0,\theta=\frac{\pi}{2},z>0} = -GM_R M_S\frac{\partial[(R+r)^2+z^2]^{-\frac{1}{2}}K(k^2)}{\partial r}\Big|_{r=0,\theta=\frac{\pi}{2},z>0} \tag{68}$$

$$= -GM_R M_S\left[-[(R+r)^2+z^2]^{-\frac{3}{2}}(R+r)K(k^2) + [(R+r)^2+z^2]^{-\frac{1}{2}}\frac{dK(k^2)}{dk^2}2k\frac{\partial k}{\partial r}\right] \tag{69}$$

$$\Big|_{r=0,\theta=\frac{\pi}{2},z>0} = -GM_R M_S[-(R^2+z^2)^{-\frac{3}{2}}R + (R^2+z^2)^{-\frac{1}{2}}\frac{1}{4}\cdot\frac{4R}{R^2+z^2}] = 0 \tag{70}$$

Hence $\mathscr{E}_r = 0$ and that proves it is an orbital equilibrium along the radial direction.

The Hessian matrix for conbined variations of r,θ is:

$$Hess(r=,\theta=\frac{\pi}{2},z>0) = \begin{bmatrix}\mathscr{E}_{\theta\theta} & \mathscr{E}_{\theta r}\\ \mathscr{E}_{r\theta} & \mathscr{E}_{rr}\end{bmatrix}\Big|_{r=,\theta=\frac{\pi}{2},z>0} \tag{71}$$

Since $\mathscr{E}_r$ and $\mathscr{E}_\theta$ are both continuous in r and θ, so $\mathscr{E}_{\theta r}$=$\mathscr{E}_{r\theta}$. That means we only need to find $\mathscr{E}_{\theta r}$ and $\mathscr{E}_{rr}$ to evaluate the determinant of the Hessian matrix. Via some computations,we obtain:

(72)
$$\frac{\partial^2 I_H}{\partial r\partial\theta}\Big|_{r=,\theta=\frac{\pi}{2},z>0} = -\frac{2M_RM_S}{M_R+M_S}(r\cos^2\phi\sin(2\theta) - z\cos\phi\cos(2\theta))\Big|_{r=,\theta=\frac{\pi}{2},z>0}$$

(73)
$$= -\frac{2M_RM_S}{M_R+M_S}\cos\phi z$$

(74)
$$\mathscr{E}_{\theta r} = \frac{H^2}{I_H^3}\frac{\partial I_H}{\partial r}\frac{\partial I_H}{\partial\theta} - \frac{H^2}{2I_H^2}\frac{\partial^2 I_H}{\partial\theta\partial r} + \frac{\partial\mathscr{U}}{\partial\theta\partial r}\Big|_{r=,\theta=\frac{\pi}{2},z>0}$$

Since $\frac{\partial I_H}{\partial r}\Big|_{r=,\theta=\frac{\pi}{2},z>0} = \frac{\partial\mathscr{U}}{\partial\theta\partial r}\Big|_{r=,\theta=\frac{\pi}{2},z>0} = 0$, we simplify it as:

(75)
$$\mathscr{E}_{\theta r} = \frac{GM_RM_S}{[R^2+z^2]^{\frac{3}{2}}}\cos\phi z$$

Moreover,

(76)
$$\frac{\partial^2 I_H}{\partial r^2}\Big|_{r=,\theta=\frac{\pi}{2},z>0} = \frac{2M_RM_S}{M_S+M_R}(1-\sin^2\theta\cos^2\phi)\Big|_{r=0,\theta=\frac{\pi}{2},z>0} = \frac{2M_RM_S}{M_S+M_R}\sin^2\phi$$

(77)
$$\frac{\partial}{\partial r}\left(2k\frac{\partial k}{\partial r}\right)\Big|_{r=0,\theta=\frac{\pi}{2},z>0} = 4R\frac{\partial}{\partial r}\left[\frac{-r^2+z^2+R^2}{[(r+R)^2+z^2]^2}\right]\Big|_{r=0,\theta=\frac{\pi}{2},z>0} = -\frac{16R^2}{[R^2+z^2]^2}$$

(78)
$$\frac{\partial}{\partial r}\left(\frac{dK(k^2)}{dk^2}\right)\Big|_{r=0,\theta=\frac{\pi}{2},z>0} = \frac{d^2K(k^2)}{d(k^2)^2}2k\frac{\partial k}{\partial r}\Big|_{r=0,\theta=\frac{\pi}{2},z>0} = 2\left(\frac{1\cdot 3}{2\cdot 4}\right)^2\frac{4R}{R^2+z^2} = \frac{9R}{8[R^2+z^2]}$$

(79)
$$\frac{\partial^2\mathscr{U}}{\partial r^2}\Big|_{r=0,\theta=\frac{\pi}{2},z>0} = -GM_RM_S\left(\frac{3(R+r)^2K(k^2)}{[(R+r)^2+z^2]^{\frac{5}{2}}} - \frac{K(k^2)+2(R+r)\frac{dK(k^2)}{dk^2}2k\frac{\partial k}{\partial r}}{[(R+r)^2+z^2]^{\frac{3}{2}}} + \frac{\frac{\partial}{\partial r}\left(\frac{dK(k^2)}{dk^2}\right)2k\frac{\partial k}{\partial r} + \frac{dK(k^2)}{dk^2}\frac{\partial}{\partial r}\left(2k\frac{\partial k}{\partial r}\right)}{[(R+r)^2+z^2]^{\frac{1}{2}}}\right)$$

(80)
$$\Big|_{r=0,\theta=\frac{\pi}{2},z>0} = -GM_RM_S\left[\frac{3R^2}{2[R^2+z^2]^{\frac{5}{2}}} - \frac{1}{[R^2+z^2]^{\frac{3}{2}}}\right]$$

(81)
$$\mathscr{E}_{rr} = \frac{GM_RM_S}{[R^2+z^2]^{\frac{3}{2}}}\left[\cos^2\phi - \frac{3R^2}{2[R^2+z^2]}\right]$$

Compare $\mathscr{E}_{rr}$,$\mathscr{E}_{\theta r}$ and $\mathscr{E}_{\theta\theta}$, we realize that when $z\to\infty$, they will decay with the rate $O(\frac{1}{z^3})$,$O(\frac{1}{z^2})$ and $O(\frac{1}{z})$. Therefore, eventually the Hessian matrix will look like:

(82)
$$Hess(r=,\theta=\frac{\pi}{2},z>0) = \begin{bmatrix}\frac{GM_RM_S}{z} & \frac{GM_RM_S}{z^2}\cos\phi\\ \frac{GM_RM_S}{z^2}\cos\phi & 0\end{bmatrix}$$

The principal minor of this Hessian is

$$\det Hess^{(1)} = \frac{GM_R M_S}{z} > 0 \tag{83}$$

The determinant of this Hessian is

$$\det Hess = -\frac{GM_R M_S \cos^2\phi}{z^2} < 0 \tag{84}$$

That implies the Hessian matrix is indefinite, so the combined variation of r,θ is unstable for sufficiently large z.

3.3. **Proof of Theorem 1.3.** To show $r > R$, $\theta = 0$, $z = 0$ is an orbital equilibrium, we need to introduce the following identities:

Proposition 3.2. [3, Section 19.4(i)] *The complete Elliptic integral of the first and second kind satisfy the following differential identities:*

$$\frac{dK(k)}{dk} = \frac{E(k) - k'^2 K(k)}{kk'^2} \tag{85}$$

$$\frac{dE(k)}{dk} = \frac{E(k) - K(k)}{k} \tag{86}$$

where $E(k)$ refers to the complete Elliptic integral of the second kind defined as:

$$E(k) = \int_0^{\frac{\pi}{2}} \sqrt{1 - k^2 \sin^2\theta}\, d\theta \tag{87}$$

and k' refers to the complementary real or complex modulus such that $k^2 + k'^2 = 1$ Now we can compute the ingredients for this case. We have

(88)
$$\frac{\partial I_H}{\partial r}\Big|_{r>R,\theta=0,z=0} = \frac{M_R M_S}{M_R + M_S}\left[2r - 2(r\sin\theta\cos\phi + z\cos\theta)\sin\theta\cos\phi\right]\Big|_{r>R,\theta=0,z=0} = \frac{2M_R M_S r}{M_R + M_S}$$

$$\frac{\partial k}{\partial r}\Big|_{r>R,\theta=0,z=0} = \frac{4R(r+R)^{-2} - 8rR(r+R)^{-3}}{2k} \tag{89}$$

When the spin rate is given by

$$H^2 = \frac{G(M_R + M_S) I_H(r)^2}{\pi r^2}\left[\frac{K(k)}{r+R} + \frac{E(k)}{r-R}\right] \tag{90}$$

$$k^2 = \frac{4rR}{(r+R)^2} \tag{91}$$

, we will have

(92)
$$\frac{\partial \mathscr{U}}{\partial r}\Big|_{r>R,\theta=0,z=0} = -GM_RM_S\frac{\partial[(R+r)^2+z^2]^{-\frac{1}{2}}K(k)}{\partial r}\Big|_{r>R,\theta=0,z=0}$$

(93)
$$= -GM_RM_S\left[-[(R+r)^2+z^2]^{-\frac{3}{2}}(R+r)K(k)+[(R+r)^2+z^2]^{-\frac{1}{2}}\frac{dK(k)}{dk}\frac{\partial k}{\partial r}\right]$$

(94)
$$\Big|_{r>R,\theta=0,z=0} = -GM_RM_S\left[\frac{K(k)}{(R+r)^2}+\frac{\frac{dK(k)}{dk}\frac{\partial k}{\partial r}}{R+r}\right]\Big|_{r>R,\theta=0,z=0}$$

(95)
$$= -\frac{2GM_RM_S}{\pi}\left[\frac{E(k)(r+R)}{2r(r-R)^2}-\frac{E(k)}{(r-R)^2}-\frac{K(k)}{2r(R+r)}\right]$$

(96)
$$\mathscr{E}_r = -\frac{H^2}{2I_H^2}\frac{\partial I_H}{\partial r}+\frac{\partial \mathscr{U}}{\partial r}\Big|_{r>R,\theta=0,z=0}$$

(97)
$$= -\frac{GM_RM_S}{\pi r}\left[\frac{K(k)}{(r+R)}+\frac{E(k)}{r-R}+\frac{E(k)(r+R)}{(r-R)^2}-\frac{2E(k)r}{(r-R)^2}-\frac{K(k)}{(R+r)}\right]$$

Hence, we derived $\mathscr{E}_r = 0$, which implies it is an orbital equilibrium. With the aid of numerical plotting, we can obtain the picture H^2 v.s. r to find the bifurcation point r^* and make the stability analysis as the picture shown below: From the plotting, we see there is a pitchfork bifurcation happening when H surpasses H^*. It is stable if $r > r^*$ and unstable if $r < r^*$.

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References

[1] Fred C Adams, Konstantin Batygin, Anthony M Bloch, and Gregory Laughlin. Energy optimization in extrasolar planetary systems: the transition from peas-in-a-pod to runaway growth. *Monthly Notices of the Royal Astronomical Society*, 493(4):5520–5531, March 2020.

[2] R.A. Broucke and A. Elipe. The dynamics of orbits in a potential field of a solid circular ring. *Regular and Chaotic Dynamics*, 10(2):129, 2005.

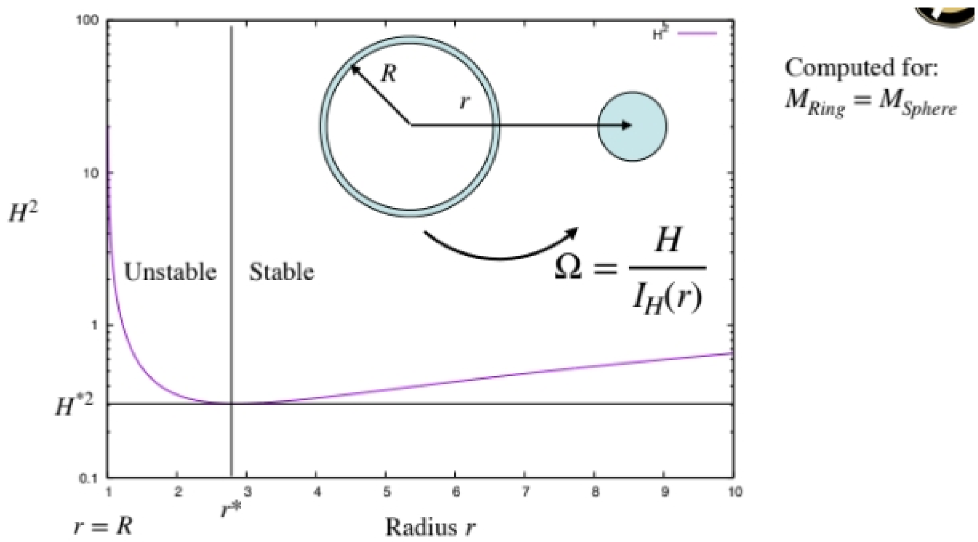

FIGURE 5. H^2-r

[3] *NIST Digital Library of Mathematical Functions* . http://dlmf.nist.gov/, Release 1.1.4 of 2022-01-15. F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller, B. V. Saunders, H. S. Cohl, and M. A. McClain, eds.

[4] P.S. Krishnaprasad. Relative equilibria and stability of rings of satellites. In *Proceedings of the 39th IEEE Conference on Decision and Control (Cat. No.00CH37187)*, volume 2, pages 1285–1288 vol.2, Dec 2000.

[5] Jerrold E. Marsden. Lectures on mechanics. 1992.

[6] James Clerk Maxwell. *On the stability of the motion of Saturn's Rings*, page 288–376. Cambridge Library Collection - Physical Sciences. Cambridge University Press, 2011.

[7] D. J. Scheeres. Minimum energy configurations in the n-body problem and the celestial mechanics of granular systems. *Celestial Mechanics and Dynamical Astronomy*, 113(3):291–320, June 2012.

[8] D. J. Scheeres. Stability of the euler resting n-body relative equlilbria, 2018.

[9] D. J. Scheeres. Derivation and properties of the angular momentum relative amended potential. *Complex Planetary Systems II Kavli-IAU Symposium 382*, July, 2023.

[10] D.J. Scheeres. Rotational fission of contact binary asteroids. *Icarus*, 189(2):370–385, 2007.