Eigenvalue Spectra and their Sparsest Matrix Representations

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Abstract

In this paper, we attempt to find the sparsest matrix for any given spectrum of eigenvalues with specified algebraic and geometric multiplicities. We prove that the Jordan Normal Form can be used to construct the sparsest representation for any 2×2 or 3×3 spectrum and provide a partial classification of the 4×4 case. We show that the Jordan Normal Form, a tempting initial answer, is not the sparsest representation for all spectra, and provide a sequence of $n \times n$ matrices that are sparser than their Jordan Normal Form. Finally, we provide methods to construct sparse matrices for more spectra, and briefly address the case where zero is an eigenvalue.

Contents

1 Introduction

The eigenvalue spectra of many classes of matrices, such as the adjacency matrices of graphs, and the matrices of Markov Chains, have been thoroughly studied in the literature. However, there has been little dedicated study to the eigenvalue spectra of sparse matrices. In this paper, we begin such an exploration by trying to find the sparsest matrix representation for any given eigenvalue spectrum. We find that for many of the 2×2, 3×3, and 4×4 cases, the Jordan Normal Form of a matrix with a given eigenvalue spectrum is the sparsest representation of that spectrum, but that this is not true in general. We then provide examples of larger matrices that are sparser than their Jordan-Normal form and methods for constructing sparse matrices given an eigenvalue spectrum. In order to summarize our results, the following definitions are needed.

Definition 1.1. For any matrix $A \in \mathbb{C}^{n\times n}$, we define $N(A) = \#\{\text{zero entries in A}\}$. Similarly, for any eigenvalue spectrum S, we define

$$N(S) = \max\{N(A) \mid A \in \mathbb{C}^{n\times n} \text{ has spectrum S}\}$$

Definition 1.2. A matrix $A \in \mathbb{C}^{n\times n}$ is jordan-sparse if it has less zero entries than its Jordan Normal Form.

1.1 Summary of Results and Methods. Note that we will be disregarding eigenvalue spectra that are diagonalizable or have 0 as one of their eigenvalues. The diagonalizable case is uninteresting, since the diagonal matrix will be the sparsest matrix representation. Spectra with 0 eigenvalues appear to also be uninteresting, but this may not be true in general. This is further discussed in Section 5. Lastly, when referring to the Jordan-Normal Form of an eigenvalue spectrum, we mean this to be the Jordan-Normal Form of any matrix representation of said spectrum.

Theorem 1.0.1. Let $A \in \mathbb{C}^{n\times n}$ be non-diagonalizable and $\ker(A) = \{0\}$. Then $N(A) < n(n-1)$.

Theorem 1.0.2. Let $A \in \mathbb{C}^{2\times 2}$ or let $A \in \mathbb{C}^{3\times 3}$. Let A be non-diagonalizable and $\ker(A) = \{0\}$. Then A is not jordan-sparse.

Theorem 1.0.2 summarizes our classification of 2×2 and 3×3 matrices. The 4×4 case is more complicated. While most spectra still have the Jordan-Normal form as their sparsest representation, a few spectra admit jordan-sparse matrices. In Section 4, we look for more jordan-sparse matrices.

Theorem 1.0.3. Let $A = \begin{bmatrix} 0 & \lambda \\ \lambda & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ for $\lambda \neq 0$. Then the sequence of matrices $(G_n)_{n\geq 1}$ such that

$$G_1 = A \qquad G_2 = \begin{bmatrix} A & B \\ 0 & A \end{bmatrix} \qquad G_n = \begin{bmatrix} A & B & & & \\ & A & B & & \\ & & A & \ddots & \\ & & & \ddots & B \\ & & & & A \end{bmatrix} \text{ for } n \geq 3$$

contains jordan-sparse matrices for $n \geq 2$.

Notice the placement of A on the block-diagonal and B on the super block-diagonal. The matrices A and B chosen for the G_n sequence are not the only choices that result in a jordan-sparse matrix. For certain matrices $A \in \mathbb{C}^{n\times n}$, a matrix $B \in \mathbb{C}^{n\times n}$ can be constructed to create a similar sequence of jordan-sparse matrices.

2 Notation

We will introduce some notation to help us talk about eigenvalue spectra, which don't have standardized notation.

Notation 2.1. A spectrum S is any finite subset of $\{(a,b,c) : a \in \mathbb{C},\ b,c \in \mathbb{N},\ b \geq c\}$. These 3-tuples are intended to represent the parameters (eigenvalue, algebraic multiplicity, geometric multiplicity).

Notation 2.2. Given a spectrum S, we will let $J(S)$ denote the Jordan Normal Form of S.

Notation 2.3. Let S be a spectrum. $|S|_a$ will denote the sum of the algebraic multiplicities of the eigenvalues of S. $|S|_g$ will denote the sum of the geometric multiplicities of the eigenvalues of S.

Notation 2.4. We say a spectrum S is non-diagonalizable if $|S|_a \neq |S|_g$. We also say S is non-zero if S does not contain zero eigenvalues.

Notation 2.5. For $A \in \mathbb{C}^{n\times n}$, we will let $J(A) \in \mathbb{C}^{n\times n}$ denote the Jordan Normal Form of A.

Notation 2.6. For $A \in \mathbb{C}^{n\times n}$, we will let $s(A)$ denote the spectrum of A.

Example 2.7. If $A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$, then $s(A) = \{(3,2,1)\}$. A has eigenvalue 3 with $\text{almu}(3) = 2$ and $\text{gemu}(3) = 1$.

Notation 2.8. For $A \in \mathbb{C}^{n\times n}$, $\chi_A(x)$ will denote the characteristic polynomial of A.

3 Small Matrices

Throughout Sections 3 and 4, we are only considering matrices which are non-diagonalizable and have trivial kernel. We will begin by proving bounds on the sparseness of certain spectra. We then examine spectra of smaller matrices, determining if they can have matrix representations which are sparser than their Jordan Normal Form.

3.1 A Bound on Sparseness

We begin with a bound on on how sparse non-diagonalizable matrices with trivial kernel can be, and some immediate consequences. This will help us examine small matrices.

Theorem 3.1.1. Let $A \in \mathbb{C}^{n\times n}$ be non-diagonalizable and $\ker(A) = \{0\}$. Then $N(A) < n(n-1)$.

Proof. To begin, assume for contradiction that $N(A) > n(n-1)$. There are $n^2 - N(A)$ non-zero terms in A. Note

$$n^2 - N(A) < n^2 - n(n-1) = n$$

A has n columns, so there must be a column j containing only zeroes for $1 \leq j \leq n$. However, letting e_j denote the j^{th} standard basis vector in $\mathbb{C}^n$, we have $e_j \in \ker(A)$ which is a contradiction. Thus $N(A) \leq n(n-1)$.

Now suppose $N(A) = n(n-1)$. There are exactly $n^2 - N(A) = n$ non-zero terms in A. By the same argument above, each column of A must contain one non-zero term. Suppose for contradiction that two columns of A, Ae_i and Ae_j, both contain non-zero values $a_i, a_j \in \mathbb{C}$ respectively in the k^{th} coordinate. Then

$$A(a_j e_i - a_i e_j) = a_j A e_i - a_i A e_j = a_j a_i e_k - a_i a_j e_k = 0$$

Thus $a_j e_i - a_i e_j \in \ker(A)$ which is a contradiction. So A is a generalized permutation matrix. It follows that A can be uniquely decomposed as $A = DP$ for $D \in \mathbb{C}^{n\times n}$ a diagonal matrix and $P \in \mathbb{C}^{n\times n}$ a permutation matrix. We change basis to a permuted standard basis $\mathcal{B}$ such that

$$[P]_{\mathcal{B}} = \begin{bmatrix} C_1 & & & \\ & C_2 & & \\ & & \ddots & \\ & & & C_k \end{bmatrix}$$

where each C_i is a block corresponding a ℓ_i-cycle for some $1 \leq \ell_i \leq n$ [LHW23]. Notice that the change of basis matrix $S_{\mathcal{E}\to\mathcal{B}}$ is a permutation matrix. It is known that the normalizer of diagonal matrices in $\mathrm{GL}_n(\mathbb{C})$ is the set of generalized permutation matrices [Kol85]. So $[D]_{\mathcal{B}}$ is a diagonal matrix. Additionally change of basis preserves eigenvalues, which are exactly the diagonal values in D. It follows that

$$D = \begin{bmatrix} a_1 & 0 & \dots & 0 \\ 0 & a_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_n \end{bmatrix} \implies [D]_{\mathcal{B}} = \begin{bmatrix} a_{\sigma(1)} & 0 & \dots & 0 \\ 0 & a_{\sigma(2)} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{\sigma(n)} \end{bmatrix}$$

For some $\sigma \in S_n$. By matrix multiplication, we have that

$$[A]_{\mathcal{B}} = \begin{bmatrix} C_1' & & & \\ & C_2' & & \\ & & \ddots & \\ & & & C_k' \end{bmatrix}$$

where each C_i' is a block corresponding a generalized ℓ_i-cycle for the same $1 \leq \ell_i \leq n$ as above, with elements in $\{a_1, a_2, \dots, a_n\}$. The characteristic polynomial of a block C_i' is given

$$\chi_{C_i'}(x) = x^{\ell_i} - (-1)^{\ell_i} a_{j_1} \cdots a_{j_{\ell_i}}$$

where $a_{j_1}, \dots, a_{j_{\ell_i}}$ are the non-zero elements in C_i' [GM15]. Because each element in $\{a_1, a_2, \dots, a_n\}$ is non-zero, each C_i' has distinct eigenvalues over $\mathbb{C}$ and is thus diagonalizable. Thus $[A]_{\mathcal{B}}$ and A are diagonalizable. This contradicts A being non-diagonalizable. It follows that $N(A) < n(n-1)$. □

Corollary 3.1.2. If S is a non-diagonalizable spectrum, then $N(S) < n\,(n-1)$ where $n = |S|_a$.

Proof. By definition $N(S) = \max\{N(A) | A \in \mathbb{C}^{n\times n} \text{ and } s(A) = S\}$. Notice that for all $A \in \mathbb{C}^{n\times n}$ such that $s(A) = S$, clearly A is non-diagonalizable. It follows from Theorem 3.1.1 that

$$N(A) < n(n-1)$$

for all $A \in \mathbb{C}^{n\times n}$ such that $s(A) = S$. Thus $N(S) < n(n-1)$. □

Proposition 3.1.3. If S is a non-zero spectrum, then $N(J(S)) = n^2 - 2n + m$ where $n = |S|_a$, $m = |S|_g$.

Proof. Let $n = |S|_a$ and $m = |S|_g$. We have that $J(S) \in \mathbb{C}^{n\times n}$. Additionally $J(S)$ has all non-zero eigenvalues on the diagonal. The number of ones on the superdiagonal is exactly $n - m$. Thus

$$N(J(S)) = n^2 - n - (n - m) = n^2 - 2n + m$$

□

Corollary 3.1.4. Let S be a non-zero spectrum such that $|S|_a = |S|_g + 1$, then $N(J(S)) = N(S)$.

Proof. By Proposition 3.1.3 we know $N(J(S)) = n^2 - 2n + (n-1) = n(n-1) - 1$. Then by Corollary 3.1.2 we have

$$N(J(S)) \leq N(S) < n(n-1) \implies n(n-1) - 1 \leq N(S) < n(n-1)$$

Because $N(S) \in \mathbb{Z}$, it follows that $N(S) = n(n-1) - 1$. Thus $N(S) = N(J(S))$. □

This corollary gives us a set of spectra for which we know the Jordan Normal Form is the sparsest matrix representation. In particular, for any spectrum such that the sum of algebraic multiplicities and sum of geometric multiplicities differ by one, the Jordan Normal Form is the sparsest matrix representation of that spectrum.

3.2 The 2×2 and 3×3 Cases

We will show any matrix in $\mathbb{C}^{2\times 2}$ or $\mathbb{C}^{3\times 3}$ cannot be sparser than it's Jordan Normal Form.

Theorem 3.2.1. Let $A \in \mathbb{C}^{2\times 2}$ such that A is non-diagonalizable and $\ker(A) \neq \{0\}$. Then A is not jordan-sparse.

Proof. We must have $s(A) = \{(\lambda, 2, 1)\}$ for $\lambda \in \mathbb{C} - \{0\}$. By Theorem 3.1.1 we know $N(A) < 2$. By Proposition 3.1.3 we know $N(J(A)) = 1$. Thus $N(A) \leq N(J(A))$, so A is not jordan-sparse. □

Corollary 3.2.2. Let S be a non-diagonalizable, non-zero spectrum such that $|S|_a = 2$. Then $N(S) = N(J(S))$.

Theorem 3.2.3. Let $A \in \mathbb{C}^{3\times 3}$ such that A is non-diagonalizable and $\ker(A) \neq \{0\}$. Then A is not jordan-sparse.

Proof. We will proceed with cases. Assume that $\lambda_1, \lambda_2 \in \mathbb{C} - \{0\}$ and $\lambda_1 \neq \lambda_2$.

(i) $s(A) = \{(\lambda_1, 2, 1), (\lambda_2, 1, 1)\}$. By Proposition 3.1.3 we know $N(J(A)) = 5$. By Theorem 3.1.1 we know $N(A) < 6$. Thus $N(A) \leq N(J(A))$, so A is not jordan-sparse.

(ii) $s(A) = \{(\lambda_1, 3, 2)\}$. By Proposition 3.1.3 we know $N(J(A)) = 5$. By Theorem 3.1.1 we know $N(A) < 6$. Thus $N(A) \leq N(J(A))$, so A is not jordan-sparse.

(iii) $s(A) = \{(\lambda_1, 3, 1)\}$. By Proposition 3.1.3 we know $N(J(A)) = 4$. By Theorem 3.1.1 we know $N(A) < 6$. We must prove that there does not exists $B \in \mathbb{C}^{3\times 3}$ such that $N(B) = 5$, $\ker(B) = \{0\}$, and $s(B) = S$. We will do this by way of contradiction. Once again, we will proceed with cases.

(a) WLOG assume e_1 is a λ_1-eigenvector. Notice that e_2 and e_3 cannot be λ_1-eigenvectors. Thus,

$$B = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & 0 & b \\ 0 & a & c \end{bmatrix} \qquad \text{or} \qquad B = \begin{bmatrix} \lambda_1 & 0 & b \\ 0 & 0 & c \\ 0 & a & 0 \end{bmatrix}$$

for $a, b, c \in \mathbb{C} - \{0\}$. Any other placement of 3 non-zero values will result in B having non-trivial kernel or too many eigenvectors. In the left case, we get $\operatorname{tr}(B) = \lambda_1 + c \implies c = 2\lambda_1$. Knowing this, $\chi_B(x) = (x - \lambda_1)(x^2 - 2\lambda_1 x - ab) \implies ab = -\lambda_1^2$. But then

$$B(\lambda_1 e_2 - a e_3) = a\lambda_1 e_3 - (ab e_2 - 2a\lambda_1 e_3) = \lambda_1^2 e_2 - a\lambda_1 e_3 = \lambda_1(\lambda_1 e_2 - a e_3)$$

which contradicts B having one eigenvector. In the right case, notice $\operatorname{tr}(B) = \lambda_1$ and thus B does not have the correct spectrum.

(b) e_1, e_2, and e_3 are not eigenvectors. We have a few possibilities.

(1) $B = \begin{bmatrix} 0 & b & 0 \\ a & 0 & c \\ 0 & 0 & d \end{bmatrix}$. Note $\operatorname{tr}(B) = d \implies d = 3\lambda_1$. Then $\chi_B(x) = (x - 3\lambda_1)(x^2 - ab)$. So B does not have the correct spectrum.

(2) $B = \begin{bmatrix} 0 & b & c \\ a & 0 & 0 \\ 0 & 0 & d \end{bmatrix}$. Note $\operatorname{tr}(B) = d \implies d = 3\lambda_1$. Then $\chi_B(x) = (x - 3\lambda_1)(x^2 - ab)$. So B does not have the correct spectrum.

(3) $B = \begin{bmatrix} 0 & 0 & c \\ a & 0 & d \\ 0 & b & 0 \end{bmatrix}$. Note $\operatorname{tr}(B) = 0$. B does not have the correct spectrum.

(4) $B = \begin{bmatrix} 0 & 0 & c \\ a & 0 & 0 \\ 0 & b & d \end{bmatrix}$. Note $\chi_B(x) = x^3 - dx^2 - abc \neq (x - \lambda_1)^3$. B does not have the correct spectrum.

(5) $B = \begin{bmatrix} 0 & b & c \\ 0 & 0 & d \\ a & 0 & 0 \end{bmatrix}$. Note $\text{tr}(B) = 0$. B does not have the correct spectrum.

(6) $B = \begin{bmatrix} 0 & b & 0 \\ 0 & 0 & c \\ a & 0 & d \end{bmatrix}$. Note $\chi_B(x) = x^3 - dx^2 - abc \neq (x - \lambda_1)^3$. B does not have the correct spectrum.

In any case, we find that such a matrix B cannot exist. Thus $N(A) \leq N(J(A))$.

For each spectrum we get $N(A) \leq N(J(A))$, so any non-diagonalizable $A \in \mathbb{C}^{3\times3}$ with trivial kernel is not jordan-sparse. □

Corollary 3.2.4. Let S be a non-zero, non-diagonalizable spectrum such that $|S|_a = 3$. Then $N(S) = N(J(S))$.

3.3 The 4×4 Case

In the 4×4 case, it is no longer true that the sparsest matrix representation of all spectra is the Jordan Normal Form. We provide an example of this below, as well as a summary of the sparsest matrix representation for all spectra where the sum of the algebraic multiplicities is 4.

Eigenvalue Spectrum	Sparsest Matrix	Reason
$\{(\lambda, 4, 3)\}$	JNF	Thm 3.1.1
$\{(\lambda, 4, 2)\}$	JNF Suspected	Code (See Section 6)
$\{(\lambda, 4, 1)\}$	JNF	Thm 3.3.2
$\{(\lambda_1, 3, 2), (\lambda_2, 1, 1)\}$	JNF	Thm 3.1.1
$\{(\lambda_1, 3, 2), (\lambda_2, 1, 1)\}$	JNF	Thm 3.1.1
$\{(\lambda_1, 3, 1), (\lambda_2, 1, 1)\}$	Unsure	
$\{(\lambda_1, 2, 2), (\lambda_2, 2, 1)\}$	JNF	Thm 3.1.1
$\{(\lambda_1, 2, 1), (\lambda_2, 2, 1)\}$	Unsure (See Example 3.3.1)	
$\{(\lambda_1, 2, 1), (\lambda_2, 1, 1), (\lambda_3, 1, 1)\}$	JNF	Thm 3.1.1

A table with our results from the 4×4 case. Note that the Jordan Normal Form is the sparsest matrix representation for all but 3 spectra.

Example 3.3.1. Consider the matrix $A \in \mathbb{C}^{4\times4}$ below.

$$A = \begin{bmatrix} & \lambda & & 1 \\ \lambda & & & \\ & & & \lambda \\ & & \lambda & \end{bmatrix}$$

Clearly $N(A) = 11$. By a few straightforward computations, we get that $\chi_A(x) = (x - \lambda)^2(x + \lambda)^2$. Additionally,

$$\dim(\ker(A - \lambda I_4)) = 1 \qquad \dim(\ker(A + \lambda I_4)) = 1$$

It follows that

$$J(A) = \begin{bmatrix} \lambda & 1 & & \\ & \lambda & & \\ & & -\lambda & 1 \\ & & & -\lambda \end{bmatrix}$$

Clearly $N(J(A)) = 10$. We have found a matrix A which is jordan-sparse.

Theorem 3.3.2. If $A \in \mathbb{C}^{4\times 4}$ and $s(A) = (\lambda, 4, 1)$ for arbitrary $\lambda \in \mathbb{C}$, then A is not jordan-sparse.

Proof. This proof involved casework similar to Theorem 3.2.3. In total there were 152 cases of matrices to check. We will provide a small example of some of these cases below.

We want to show there does not exist $B \in \mathbb{C}^{4\times 4}$ such that $s(B) = \{(\lambda, 4, 1)\}$ and $N(B) > 9$. Suppose $B \in \mathbb{C}^{4\times 4}$ and $N(B) = 10$ for contradiction. WLOG assume e_1 is a λ-eigenvector. Assume the 2^{nd} and 3^{rd} columns of B have 1 non-zero value, and the 4^{th} column has 3 non-zero values. One such matrix we have to check is

$$B = \begin{bmatrix} \lambda & 0 & 0 & c \\ 0 & 0 & 0 & d \\ 0 & a & 0 & 0 \\ 0 & 0 & b & e \end{bmatrix}$$

For $a, b, c, d, e \in \mathbb{C}$. Notice that $\chi_B(x) = (x - \lambda)\left(x^2(x - e) - abd\right) \neq (x - \lambda)^4$. It follows that $s(B) \neq \{(\lambda, 4, 1)\}$. We must check every case of a basis vector being an eigenvector or not, and every case of a column having a different number of non-zero elements up to permutation. $\square$

Corollary 3.3.3. Suppose $n \in \mathbb{Z}$ and $n \geq 4$. Then $\exists B \in \mathbb{C}^{n\times n}$ such that B is jordan-sparse.

Proof. Let A be the same matrix from Example 3.3.1, and consider the block diagonal matrix

$$B = \begin{bmatrix} A & \\ & I_{n-4} \end{bmatrix}$$

Notice that

$$J(B) = \begin{bmatrix} J(A) & \\ & I_{n-4} \end{bmatrix}$$

From Example 3.3.1 we know

$$N(B) = n^2 - 5 - (n-4) = n^2 - n - 1 \qquad \text{and} \qquad N(J(B)) = n^2 - 6 - (n-4) = n^2 - n - 2$$

Clearly $N(B) > N(J(B))$ for all $n \geq 4$, and thus B is jordan-sparse. $\square$

Equivalently, we have shown that for any $n \geq 4$ there exists a spectrum S such that $|S|_a = n$ and $J(S)$ is not the sparsest matrix representation of S.

4 Examples

Before we continue, another reminder that throughout Section 4 we are only considering spectra which are non-diagonalizable and non-zero. In other words, we are only considering matrices which are non-diagonalizable and have trivial kernel. We will now explore more examples of jordan-sparse matrices other than Example 3.3.1.

4.1 A Sequence of Jordan-Sparse Matrices

It is now clear that there exist matrices $A \in \mathbb{C}^{n\times n}$ such that $N(A) > N(J(A))$ for $n \geq 4$. We will examine how large the gap $N(A) - N(J(A))$ can be for particular matrices A by attempting to further generalize Example 3.3.1.

Definition 4.1.1. Let $A = \begin{bmatrix} 0 & \lambda \\ \lambda & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ for $\lambda \neq 0$. We will define the sequence of matrices $(G_n)_{n\geq 1}$ such that

$$G_1 = A \qquad G_2 = \begin{bmatrix} A & B \\ 0 & A \end{bmatrix} \qquad G_n = \begin{bmatrix} A & B & & & \\ & A & B & & \\ & & A & \ddots & \\ & & & \ddots & B \\ & & & & A \end{bmatrix} \text{ for } n \geq 3$$

Lemma 4.1.2. Consider the sequence of matrices $(G_n)_{n\geq 1}$. Then $\chi_{G_n}(x) = (x-\lambda)^n(x+\lambda)^n$.

Proof. Notice that for all $n \in \mathbb{Z}_{\geq 1}$, G_n is block upper-triangular with n copies of A along the block diagonal. It follows that $\chi_{G_n}(x) = \chi_A(x)^n$. We know that $\chi_A(x) = (x-\lambda)(x+\lambda)$. Thus, $\chi_{G_n}(x) = (x-\lambda)^n(x+\lambda)^n$. □

Lemma 4.1.3. $s(G_n) = \{(\lambda, n, 1), (-\lambda, n, 1)\}$

Proof. We already know that $\text{almu}(\lambda) = n = \text{almu}(-\lambda)$ from Lemma 4.1.2. It suffices to show that $\text{gemu}(\lambda) = 1 = \text{gemu}(-\lambda)$. This is equivalent to showing

$$\dim(\ker(G_n - \lambda I_{2n})) = 1 = \dim(\ker(G_n + \lambda I_{2n}))$$

for $n \geq 2$. Let $x = \begin{bmatrix} x_1 & x_2 & \cdots & x_{2n} \end{bmatrix}^\top$ and suppose that $(G_n - \lambda I_{2n})x = 0$. Then we have

$$\begin{bmatrix} -\lambda x_1 + \lambda x_2 + x_4 \\ \lambda x_1 - \lambda x_2 \\ -\lambda x_3 + \lambda x_4 + x_6 \\ \lambda x_3 - \lambda x_4 \\ \vdots \\ -\lambda x_{2n-3} + \lambda x_{2n-2} + x_{2n} \\ \lambda x_{2n-3} - \lambda x_{2n-2} \\ -\lambda x_{2n-1} + \lambda x_{2n-1} \\ \lambda x_{2n-1} - \lambda x_{2n} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Because $\lambda \neq 0$, we must have that $x_i = x_{i+1}$ for all odd i such that $1 \leq i \leq 2n$. Considering the 1^{st} coordinate of our vector we must have that $x_4 = 0$. Similarly, considering the 3^{rd} coordinate of our vector we must have that $x_6 = 0$. By repeating this process, we get that

$$x_4 = 0 \implies x_6 = 0 \implies x_8 = 0 \implies \cdots \implies x_{2n} = 0$$

Thus $x_i = 0$ for all $3 \leq i \leq 2n$. Because $x_1 = x_2$ and there are no restrictions on x_1's value, we have that $x \in \text{span}(e_1 + e_2)$. Clearly if $y \in \text{span}(e_1 + e_2)$ then $(G_n - \lambda I_{2n})y = 0$. We have shown that

$$\ker(G_n - \lambda I_{2n}) = \text{span}(e_1 + e_2)$$

It follows that $\dim(\ker(G_n - \lambda I_{2n})) = 1$.

A very similar argument as above works to show $\dim(\ker(G_n + \lambda I_{2n})) = 1$. The only alteration being $x_i = -x_{i+1}$ for all odd i such that $1 \leq i \leq 2n$, and thus $\ker(G_n + \lambda I_{2n}) = \text{span}(e_1 - e_2)$. □

Theorem 4.1.4. Consider the sequence of matrices $(G_n)_{n\geq 1}$. Then $N(G_n) - N(J(G_n)) = n - 1$.

Proof. We know that $N(G_n) = (2n)^2 - 2n - (n-1)$ by counting. It follows from Lemma 4.1.3 and Proposition 3.1.3 that $N(J(G_n)) = (2n)^2 - 4n + 2$. Thus

$$N(G_n) - N(J(G_n)) = \big((2n)^2 - 3n + 1\big) - \big((2n)^2 - 4n + 2\big) = n - 1$$

□

Corollary 4.1.5.

(a) For $n \in \mathbb{Z}_{>1}$, G_n is jordan-sparse.

(b) $\lim_{n\to\infty} N(G_n) - N(J(G_n)) = \infty$.

(c) For any $A \in \mathbb{C}^{n\times n}$, there exists $m \in \mathbb{Z}$ such that $B = \begin{bmatrix} A & \\ & G_m \end{bmatrix}$ is jordan-sparse.

Proof.

(a) Immediate from Theorem 4.1.4.

(b) Immediate from Theorem 4.1.4.

(c) If A is jordan-sparse, the statement is trivial. Suppose A is not jordan-sparse, and let $m = N(J(A)) - N(A)$. Notice that

$$B = \begin{bmatrix} A & \\ & G_{m+2} \end{bmatrix}$$

is jordan-sparse.

□

Corollary 4.1.5(a) tells us that for the spectrum $S = \{(\lambda, n, 1), (-\lambda, n, 1)\}$ with $n \geq 2$, the Jordan Normal Form is not the sparsest matrix representation. Corollary 4.1.5(b) indicates that there exist matrices which are arbitrarily sparser than their Jordan Normal Form, and that the Jordan Normal Form isn't always as accurate of a guess as we previously thought. Finally, Corollary 4.1.5(c) tells us that for any spectrum S, we may extend it to a larger spectrum $S \subset S'$ such that the Jordan Normal Form of S' is not the sparsest matrix representation.

Now, we wish to further generalize the $(G_n)_{n\geq 1}$ sequence of matrices to obtain more jordan-sparse matrices with different spectra.

Definition 4.1.6. Let $k \in \mathbb{Z}_{>1}$, $P_k \in \mathbb{C}^{k\times k}$ the standard matrix representation of a k-cycle. Also let $B \in \mathbb{C}^{k\times k}$ be a matrix with all zeros except a 1 in the $(1, k)$ entry. We will define the sequence of matrices $(G_n^k)_{n\geq 1}$ such that

$$G_1^k = P_k \qquad G_2^k = \begin{bmatrix} P_k & B \\ 0 & P_k \end{bmatrix} \qquad G_n^k = \begin{bmatrix} P_k & B & & & \\ & P_k & B & & \\ & & P_k & \ddots & \\ & & & \ddots & B \\ & & & & P_k \end{bmatrix} \text{ for } n \geq 3$$

Notice that (G_n) from Definition 4.1.1 is the same sequence as (G_n^2) from Definition 4.1.6.

Theorem 4.1.7. $N(G_n^k) - N(J(G_n^k)) = (k - 1)(n - 1)$

Proof. Through the same reasoning as Lemma 4.1.2, it's true that

$$\chi_{G_n^k}(x) = \chi_{P_k}(x)^n = (x^k - 1)^n$$

Now, we will show that $\dim(\ker(G_n^k - \lambda I_{nk})) = 1$ for λ a k^{th} root of unity and $n \geq 2$. Let $x = \begin{bmatrix} x_1 & x_2 & \cdots & x_{nk} \end{bmatrix}^\top$ and suppose that $(G_n^k - \lambda I_{nk})x = 0$. Then we have

$$\begin{bmatrix} -\lambda x_1 + x_k + x_{2k} \\ x_1 - \lambda x_2 \\ x_2 - \lambda x_3 \\ \vdots \\ x_{k-1} - \lambda x_k \\ -\lambda x_{k+1} + x_{2k} + x_{3k} \\ x_{k+1} - \lambda x_{k+2} \\ x_{k+2} - \lambda x_{k+3} \\ \vdots \\ x_{nk-1} - \lambda x_{nk} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Notice that we have

$$x_1 = \lambda x_2 = \lambda^2 x_3 = \cdots = \lambda^{k-1} x_k$$

The same relation holds for $x_{k+1}, \cdots, x_{2k}$ and so on. Considering the 1^{st} coordinate, we get

$$-\lambda(\lambda^{k-1}x_k) + x_k + x_{2k} = 0 \implies -x_k + x_k + x_{2k} = 0 \implies x_{2k} = 0$$

Through the same argument we get $x_{3k} = \cdots = x_{nk} = 0$. It follows that

$$\ker(G_n^k - \lambda I_{nk}) = \operatorname{span}(e_1 + \lambda e_2 + \cdots + \lambda^{k-1} e_k)$$

Thus $\dim(\ker(G_n^k - \lambda I_{nk})) = 1$. By counting we get $N(G_n^k) = (nk)^2 - nk - (n-1)$. By Proposition 3.1.3 we know $N(J(G_n^k)) = (nk)^2 - 2nk + k$. It follows that

$$N(G_n^k) - N(J(G_n^k)) = \big((nk)^2 - nk - (n-1)\big) - \big((nk)^2 - 2nk + k\big) = nk - n - k + 1 = (k-1)(n-1)$$

□

This may *seem* like a better gap at first glance, but in reality the $k-1$ factor accounts for the fact that these G_n^k matrices are growing faster in size than our original G_n construction.

4.2 The Block Diagonal Trick

When constructing the G_n matrices (Section 4.1), we placed specific block matrices on the diagonal and block super-diagonal. We now generalize this idea, providing conditions under which a matrix $A \in \mathbb{C}^{n\times n}$ and some matrix $B \in C^{n\times n}$ can be placed on the block diagonal and super-diagonal, respectively, to create a larger matrix that has the same eigenvalues and corresponding geometric multiplicities as A. The Block-Diagonal Trick can be used to find jordan-sparse matrices for a variety of eigenvalue spectra.

Proposition 4.2.1. Supppose $A \in \mathbb{C}^{n\times n}$ has eigenvalue λ such that $\operatorname{gemu}_A(\lambda) = 1$. There exists $B \in \mathbb{C}^{n\times n}$ such that $N(B) = n^2 - 1$,

$$C = \begin{bmatrix} A & B \\ & A \end{bmatrix}$$

and $\operatorname{gemu}_C(\lambda) = \operatorname{gemu}_A(\lambda)$

Proof. Pick non-zero $v \in \ker(A - \lambda I_n)$. Assume the k^{th} standard coordinate of v is non-zero. We will place a single 1 somewhere in the k^{th} column of B.

Let $x = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}^\top$ and consider the system of equations given by $(A - \lambda I_n)x = 0$. Because $\dim(\ker(A - \lambda I_n)) = 1$, this system of equations has infinitely many solutions. It follows that a row of $(A - \lambda I_n)x = 0$ contains a redundant equation. Assume the j^{th} row of $(A - \lambda I_n)x = 0$ contains this redundant equation. We will place a single 1 somewhere in the j^{th} row of B.

Let $B \in \mathbb{C}^{n\times n}$ be a matrix with all zeroes and a 1 in the (j, k) position. Consider

$$C = \begin{bmatrix} A & B \\ & A \end{bmatrix}$$

Clearly C has the same eigenvalues as A, so we can consider $\ker(C - \lambda I_{2n})$. Let $y = \begin{bmatrix} y_1 & y_2 & \cdots & y_{2n} \end{bmatrix}^\top$ and consider the system of equations given by $(C - \lambda I_{2n})y = 0$. Notice that by construction, the last n equations of $(C - \lambda I_{2n})y = 0$ have no solution. It follows that

$$\ker(C - \lambda I_{2n}) \simeq \ker(A - \lambda I_n)$$

and thus $\operatorname{gemu}_C(\lambda) = \operatorname{gemu}_A(\lambda)$. □

Example 4.2.2. Let

$$A = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 1 \\ 3 & 7 & 0 \end{bmatrix}$$

and notice $s(A) = \{(-2,1,1), (-1,1,1), (3,1,1)\}$. We will let $\lambda = -1$. Note that $v = 2e_1 - e_2 + e_3$ has all non-zero standard coordinates, so we may place a 1 in any column of B. Additionally,

$$(A + I_3)x = 0 \implies \begin{bmatrix} x_1 + 2x_2 \\ x_2 + x_3 \\ 3x_1 + 7x_2 + x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Notice that

$$3x_1 + 7x_2 + x_3 - 3(x_1 + 2x_2) = x_2 + x_3$$

So the 2^{nd} row of $(A + I_3)x = 0$ is redundant, and we may place a 1 in the 2^{nd} row of B. It follows that

$$B = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Letting $C = \begin{bmatrix} A & B \\ & A \end{bmatrix}$, through computation we get $\text{gemu}_C(\lambda) = \text{gemu}_A(\lambda)$.

Lemma 4.2.3. Suppose $A \in \mathbb{C}^{n \times n}$ such that $A = J(A)$. Then there exists $B \in \mathbb{C}^{n \times n}$ with zeroes and $|s(A)|_a - |s(A)|_g$ many 1's such that

$$C = \begin{bmatrix} A & B \\ & A \end{bmatrix}$$

and $\text{gemu}_A(\lambda) = \text{gemu}_C(\lambda)$ for every eigenvalue λ of A.

Proof. Let $\{e_{i_1}, \dots, e_{i_k}\}$ be the eigenvectors of A. For each index $i_1, \dots, i_k$, we will place a 1 in the corresponding columns of B. Let $\{j_1, \dots, j_k\}$ be the indices of rows of A which contain one non-zero value. We will place a 1 in the corresponding rows of B. Notice that

$$|\{i_1, \dots, i_k\}| = |\{j_1, \dots, j_k\}| = |s(A)|_a - |s(A)|_g$$

so B will contain $|s(A)|_a - |s(A)|_g$ many 1's. Now, assuming that $\{i_1, \dots, i_k\}$ and $\{j_1, \dots, j_k\}$ are written in increasing order, we will place 1's in the $(j_1, i_1), \dots, (j_k, i_k)$ positions of B. Consider

$$C = \begin{bmatrix} A & B \\ & A \end{bmatrix}$$

Through the same argument as Proposition 4.2.1, it follows that $\text{gemu}_C(\lambda) = \text{gemu}_A(\lambda)$ for each eigenvalue λ of A. □

We will illustrate Lemma 4.2.3 with an example.

Example 4.2.4. Let

$$A = \begin{bmatrix} 1 & 1 & & & \\ & 1 & 1 & & \\ & & 1 & & \\ & & & 2 & 1 \\ & & & & 2 \end{bmatrix}$$

Notice that e_1 and e_4 are eigenvectors of A, so we want to place 1's in the 1^{st} and 4^{th} columns of B. Additionally the 3^{rd} and 5^{th} rows contain one non-zero value, so we want to place 1's in the 3^{rd}

and 5^{th} rows of B. We get that

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Letting $C = \begin{bmatrix} A & B \\ & A \end{bmatrix}$, computation tells us that $\text{gemu}_C(1) = 1$ and $\text{gemu}_C(2) = 1$ as desired.

Theorem 4.2.5. For any $A \in \mathbb{C}^{n\times n}$, there exists $B \in \mathbb{C}^{n\times n}$ such that

$$C = \begin{bmatrix} A & B \\ & A \end{bmatrix}$$

and $\text{gemu}_A(\lambda) = \text{gemu}_C(\lambda)$ for every eigenvalue λ of A.

Proof. Let $\mathcal{B}$ denote the Jordan Basis of A, and $S_{\mathcal{E}\to\mathcal{B}}$ the change of basis matrix. By Lemma 4.2.3, there exists $B \in \mathbb{C}^{n\times n}$ such that

$$M = \begin{bmatrix} J(A) & B \\ & J(A) \end{bmatrix}$$

satisfies $\text{gemu}_M(\lambda) = \text{gemu}_{J(A)}(\lambda) = \text{gemu}_A(\lambda)$ for every eigenvalue λ of A. Now, let

$$S = \begin{bmatrix} S_{\mathcal{E}\to\mathcal{B}} & \\ & S_{\mathcal{E}\to\mathcal{B}} \end{bmatrix}$$

Notice that

$$S^{-1}MS = \begin{bmatrix} A & B' \\ & A \end{bmatrix}$$

for some $B' \in \mathbb{C}^{n\times n}$. Letting $C = S^{-1}MS$, we get that $\text{gemu}_C(\lambda) = \text{gemu}_A(\lambda)$ for every eigenvalue λ of A. ☐

The main issue with Theorem 4.2.5 is that the matrix B is generally not very sparse.

Example 4.2.6. Let

$$A = \begin{bmatrix} 3 & -1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & 2 \end{bmatrix} \implies J(A) = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Using the same notation as Theorem 4.2.5, M will denote

$$M = \begin{bmatrix} J(A) & B \\ & J(A) \end{bmatrix}$$

After changing basis, we get that

$$S^{-1}MS = \begin{bmatrix} A & B' \\ & A \end{bmatrix} \qquad \text{where} \qquad B' = \begin{bmatrix} -2 & -1 & 5 \\ -1 & 1 & 1 \\ -1 & 0 & 2 \end{bmatrix}$$

The main reason that we began studying the Block Diagonal Trick was with the intention of this process eventually producing a jordan-sparse matrix. However, both A and B must be quite sparse to begin with in order for the Block Diagonal Trick to eventually produce a jordan-sparse matrix.

5 Generalizations and Questions

We still do not know what the sparsest matrix of any given spectrum is. We provide questions that may guide any future work in pursuit of an answer to this question. In regards to the Block Diagonal Trick, our conditions for Proposition 4.2.1 seem unnecessarily restrictive. We hope to extend Proposition 4.2.1 to larger eigenspaces than one-dimensional ones. For any matrix $A \in \mathbb{C}^{n \times n}$, we suspect we can construct a matrix B with all zeroes and at most n 1's so that the block diagonal trick works. This construction of B has succeeded for numerous examples we have attempted, but an exact process for this construction is yet to be formulated. Also of note is that many of our techniques for constructing sparse matrices involve utilizing block matrices. Below is an example of a matrix that is not block-diagonal but is jordan-sparse.

Example 5.1. $\begin{bmatrix} 0 & -2 & -1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$

Little is understood about how to construct such matrices or under what conditions they exist. Finally, we do not look at matrices for kernel throughout this paper. This is because many of the very helpful bounds we proved do not apply to matrices with kernel, and all the computer-generated examples of jordan-sparse matrices with kernel were simply matrices that were already jordan-sparse but had a 0 row or column added to them, like so:

Example 5.2. $A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$. Notice that $N(A) = 20$ and $N(J(A)) = 19$.

However, not every such jordan-sparse matrix with kernel looks like the above, and they may warrant further discussion. We provide an example with no zero rows or columns below.

Example 5.3. Let $B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ and consider the block matrix A below.

$$A = \begin{bmatrix} G_3 & \\ & B \end{bmatrix}$$

It follows that $N(A) = 54$ and $N(J(A)) = 53$.

6 Appendix A: Brute-Force Algorithm to Find Jordan-Sparse Matrices

For many eigenvalue spectra in the 4x4 case, a brute force algorithm was used to attempt to find matrices sparser than their Jordan-Normal form. This algorithm proved to be effective in finding such matrices, finding numerous jordan-sparse matrices for every spectrum known to have one. A description of the algorithm as well as a link to the Mathematica code are provided.

We first input the spectrum we want to test into the algorithm by inputting the Jordan-Normal Form of said spectrum.

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Example input into algorithm, represents spectrum $S = (1, 2, 1), (-1, 2, 1)$

The algorithm generates every possible placement of non-zero values that could result in a jordan-sparse matrix.

$$\begin{bmatrix} 0 & a_{1,2} & a_{1,3} & 0 \\ a_{2,1} & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{3,4} \\ 0 & a_{4,2} & 0 & 0 \end{bmatrix}$$

Since this matrix only has 5 non-zero values and the JNF of the spectrum has 6, it could be sparser. The algorithm generates every such matrix.

We then go through all the generated matrices, replacing the placeholders with pseudo-random non-zero values that satisfy the characteristic polynomial for the given spectrum, meaning the matrix must have the desired eigenvalues

$$. \begin{bmatrix} 0 & -2 & -1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Placeholders in matrix are replaced with pseudo-random values. Observe that the matrix has eigenvalues -1,1.

For each matrix, the geometric multiplicity of each eigenvalue is tested, and if the geometric multiplicities are correct, the matrix is returned along with any other jordan-sparse matrices.

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