THE REALIZABILITY PROBLEM FOR LINEAR ISOMETRIES AND STEINER OPERADS OVER CYCLIC GROUPS

DAVID CHENG, CARL GUO, HARAN MOULI

Abstract. We study the problem of classifying N_∞-operads realized by linear isometries operads and N_∞-operad pairs realized by linear isometries and Steiner operads over complex G-universes for cyclic groups G. We prove that if $\gcd(|G|, 6) < 6$, then every saturated G-transfer system $\mathcal{R}$ is realized by $\mathcal{L}(U)$ for some complex G-universe U. Furthermore, when $\gcd(|G|, 6) = 1$, we determine all compatible pairs $(\mathcal{R}_m, \mathcal{R}_a)$ of G-transfer systems realized by $(\mathcal{L}(U), \mathcal{K}(U))$ for some complex G-universe U. We also provide analogous results for real G-universes.

Contents

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\begin{tikzcd}
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\end{tikzcd}
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1. Introduction

For a G-universe U, the Steiner operad $\mathcal{K}(U)$ and linear isometries operad $\mathcal{L}(U)$ capture the additive and multiplicative structures on G-spectra. In [BH1], Blumberg and Hill pose the question of which homotopy types of N_∞-operads are realized by G-equivariant Steiner and linear isometries operads. As explained by Rubin in [Rub], the data of an N_∞-operad can be encoded in a simple fashion using a transfer system on G:

Definition 1.1. A *transfer system* $\mathcal{R}$ on a group G is a partial order $\to_{\mathcal{R}}$ on $\mathrm{Sub}(G)$, the set of subgroups of G, that refines subgroup inclusion and satisfies:

- *Restriction*: If $K \to_{\mathcal{R}} H$ and $L \leqslant H$, then $K \cap L \to_{\mathcal{R}} L$.
- *Conjugation*: $\mathcal{R}$ is G-invariant under the conjugation action of G on $\mathrm{Sub}(G)$.

In this language, for a G-universe U, the Steiner operad $\mathcal{K}(U)$ and linear isometry operad $\mathcal{L}(U)$ realize the transfer systems $\mathcal{R}_a$ and $\mathcal{R}_m$ respectively, where:

- $\mathcal{R}_a$ is the transfer system with $K \to_{\mathcal{R}_a} H$ if and only if $K \leqslant H \leqslant G$ and K is the stabilizer of some vector $v \in R^G_H U$.
- $\mathcal{R}_m$ is the transfer system with $K \to_{\mathcal{R}_m} H$ if and only if $K \leqslant H \leqslant G$ and $I^H_K R^G_K U = R^G_H U$, where I and R are induction and restriction.

We explain in Appendix A why these relations are indeed transfer systems. It turns out that the transfer systems coming from Steiner and linear isometries operads enjoy certain special properties.

Definition 1.2. Given a transfer system $\mathcal{R}$ on G, we say that:

- $\mathcal{R}$ is *cosaturated* if it is generated by arrows of the form $H \to_{\mathcal{R}} G$.
- $\mathcal{R}$ is *saturated* if for all subgroups $K \leqslant M \leqslant H$ with $K \to_{\mathcal{R}} H$, we also have $K \to_{\mathcal{R}} M$ and $M \to_{\mathcal{R}} H$.

We show in Appendix A that any transfer system realized by a Steiner (resp. linear isometries) operad must always be cosaturated (resp. saturated). For abelian groups G, one can see, as shown in Theorem 4.11 in [Rub], that a cosaturated transfer system $\mathcal{R}$ is realizable by a Steiner operad if and only if it is generated by arrows $H \to_{\mathcal{R}} G$ with G/H cyclic. In particular, for cyclic groups G, all cosaturated transfer systems can be realized by $\mathcal{K}(U)$ for some G-universe U.

The realizability of linear isometries operads is trickier. As discussed in [Rub], it is not true for all abelian groups G that all saturated transfer systems are realized by $\mathcal{L}(U)$ for some G-universe U. MacBrough proved in [Mac] that this is true for cyclic groups of order relatively prime to 6 (Theorem 3.5) and certain abelian groups of rank 2 (Theorem 3.14), but false for abelian groups of higher rank.

Furthermore, one can ask which pairs $(\mathcal{R}_m, \mathcal{R}_a)$ of saturated and cosatured transfer systems can be realized by a G-universe U using $\mathcal{L}(U)$ and $\mathcal{K}(U)$ respectively. A necessary constraint is that the pair $(\mathcal{R}_m, \mathcal{R}_a)$ must be compatible. In [BH2], Blumberg and Hill define the notion of compatible pairs; its translation in the language of transfer systems is included in [Ch] as Definition 4.6:

Definition 1.3. Let $\mathcal{R}_m$ and $\mathcal{R}_a$ be transfer systems on G. We say $(\mathcal{R}_m, \mathcal{R}_a)$ is a *compatible pair* if for all subgroups $K, L \leqslant H$ in G, if $K \to_{\mathcal{R}_m} H$ and $K \cap L \to_{\mathcal{R}_a} K$, then $L \to_{\mathcal{R}_a} H$.

$$
\begin{tikzcd}
 & H & \\
L & & K \arrow[ul, "\mathcal{R}_m"'] \\
 & K \cap L \arrow[ur, dashed, "\mathcal{R}_a"'] &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
 & H & \\
L \arrow[ur, dashed, "\mathcal{R}_a"] & & K \arrow[ul, "\mathcal{R}_m"'] \\
 & K \cap L \arrow[ul, "\mathcal{R}_m"] \arrow[ur, dashed, "\mathcal{R}_a"'] &
\end{tikzcd}
$$

The compatibility condition says that every configuration identical to the picture on the left must be part of a diamond identical to the picture on the right. Note that we have $K \cap L \to_{\mathcal{R}_m} L$ by restricting $K \to_{\mathcal{R}_m} H$.

Remark 1.1. In Definition 4.6 of [Ch], Chan includes the condition $\mathcal{R}_m \subseteq \mathcal{R}_a$. This is simply a special case of the compatibility axiom with $L = K$.

In Appendix A, we explain why the transfer system pair realized by Steiner and linear isometries operads must always be compatible; this is Proposition 6.16 in [BH2]. Thus, one can ask which compatible pairs $(\mathcal{R}_m, \mathcal{R}_a)$ with $\mathcal{R}_m$ saturated and $\mathcal{R}_a$ cosaturated are realized using $(\mathcal{L}(U), \mathcal{K}(U))$.

Unlike in the literature, throughout this report, **we shall work with complex G-universes unless specified otherwise** for the sake of simplicity. However, all the results over $\mathbb{C}$ will have appropriate counterparts over $\mathbb{R}$ which can be obtained easily with a slight modification to incorporate the action of Galois; see Remark 3.2.

- In Section 2, we prove some auxillary results on the structure of cosaturated/saturated transfer systems and study their relationship with their fibrant/cofibrant subgroups respectively.
- In Section 3, we push the tight pair argument in [Mac] to prove the realizability of certain saturated transfer systems for even cyclic groups (in the real case, odd cyclic groups with order divisible by 3).
- In Section 4, we give a new proof that for odd cyclic G, all saturated transfer systems are realizable from linear isometries operads. The counterpart in the real case is Theorem 3.5 in [Mac].
- In Section 5, we present our conjecture for which saturated transfer sytsems are realizable for even cyclic G. We prove the conjecture when $3 \nmid |G|$ and give our progress in the case where $6 \mid |G|$.
- In Section 6, we explain how the condition in the conjecture from Section 5 generalizes to necessary (but not sufficient) local constraints for realizing saturated transfer systems over any cyclic group G and subfield k of $\mathbb{C}$.
- In Section 7, we prove that for cyclic groups G with order not divisible by 2 and 3 (in the real case, 2, 3, and 5), all compatible pairs are realizable from Steiner and linear isometries operads.
- In Appendix A, we prove the previously mentioned facts about the transfer systems realized by Steiner and linear isometries operads; this has been included with the aim of keeping our report accessible to those without a background in equivariant stable homotopy theory.
- In Appendix B, we study the set of transfer systems on finite groups using category theory, partially inspired by the results of Section 2. This will not be important to the rest of the report.
- In Appendix C, we mention some basic facts about realizable compatible pairs for $G = \mathbb{F}_p^n$ and $k = \mathbb{Q}$, with emphasis on $n = 3$.

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2. Cosaturated/Saturated Transfer Systems

The dual notions of fibrancy and cofibrancy of subgroups was first introduced by MacBrough in Definition 2.7 in [Mac]:

Definition 2.1. Let G be a group, $\mathcal{R}$ be a transfer system, and $L \leqslant G$:

- We say that L is $\mathcal{R}$-*fibrant* if $L \to_{\mathcal{R}} G$.
- We say that L is $\mathcal{R}$-*cofibrant* if $M \to_{\mathcal{R}} L$ implies $M = L$.

We shall denote the set of fibrant and cofibrant subgroups of a transfer system by $\mathrm{fib}(\mathcal{R})$ and $\mathrm{cof}(\mathcal{R})$ respectively. The following lemmas give equivalent definitions for fibrancy and cofibrancy and establish some properties of $\mathrm{fib}(\mathcal{R})$ and $\mathrm{cof}(\mathcal{R})$.

Lemma 2.1. *Let G be a group, $\mathcal{R}$ be a transfer system, and $L \leqslant G$. The following statements are equivalent:*

(1) L is $\mathcal{R}$-fibrant.
(2) For all $H \leqslant G$, we have $H \cap L \to_{\mathcal{R}} H$.

Proof. For (1) $\Rightarrow$ (2), use the restriction axiom of $\mathcal{R}$ on $L \to_{\mathcal{R}} G$ with $H \leqslant G$. For (1) $\Leftarrow$ (2), simply choose $H = G$. □

Corollary 2.1. *The set* $\mathrm{fib}(\mathcal{R})$ *is closed under intersection and conjugation.*

Proof. If L_1 and L_2 are $\mathcal{R}$-fibrant, then for all subgroups $H \leqslant G$, we have the arrows $H \cap L_1 \to_{\mathcal{R}} H$ and $(H \cap L_1) \cap L_2 \to_{\mathcal{R}} H \cap L_1$. By transitivity, it follows that $H \cap (L_1 \cap L_2) \to_{\mathcal{R}} H$, and since H was arbitrary, $L_1 \cap L_2$ is $\mathcal{R}$-fibrant. The closure of $\mathrm{fib}(\mathcal{R})$ under conjugation follows from the conjugation axiom on $\mathcal{R}$. □

Lemma 2.2. *Let G be a group, $\mathcal{R}$ be a transfer system, and $L \leqslant G$. The following statements are equivalent:*

(1) L is $\mathcal{R}$-cofibrant.
(2) For all arrows $K \to_{\mathcal{R}} H$, we have $L \leqslant H \Leftrightarrow L \leqslant K$.

Proof. For (1) $\Rightarrow$ (2), if $K \to_{\mathcal{R}} H$ and $L \leqslant H$, then by the restriction axiom of $\mathcal{R}$, we have $K \cap L \to_{\mathcal{R}} L$, implying that $K \cap L = L$ and $L \leqslant K$. Clearly, $L \leqslant K$ implies $L \leqslant H$ since $K \leqslant H$. For (2) $\Rightarrow$ (1), taking $H = L$, we see that for all arrows $K \to_{\mathcal{R}} L$, we have $L \leqslant K$, forcing $K = L$. □

Corollary 2.2. *The set* $\mathrm{cof}(\mathcal{R})$ *is closed under compositum and conjugation.*

Proof. If L_1 and L_2 are $\mathcal{R}$-cofibrant, then for all subgroups $H \leqslant G$, the condition $\langle L_1, L_2 \rangle \leqslant H$ is equivalent to $L_1 \leqslant H$ and $L_2 \leqslant H$. Thus, for all arrows $K \to_{\mathcal{R}} H$, we have $\langle L_1, L_2 \rangle \leqslant H \Leftrightarrow \langle L_1, L_2 \rangle \leqslant K$, implying that $\langle L_1, L_2 \rangle$ is $\mathcal{R}$-cofibrant. The closure of $\mathrm{cof}(\mathcal{R})$ under conjugation follows from the conjugation axiom on $\mathcal{R}$. □

Remark 2.1. For experts on the empty set, note that any collection of subgroups of $\mathrm{Sub}(G)$ closed under intersection (resp. compositum) must contain G (resp. the trivial subgroup e). For any transfer system $\mathcal{R}$, one may immediately check that G is $\mathcal{R}$-fibrant and e is $\mathcal{R}$-cofibrant.

Fibrancy and cofibrancy are important for cosaturated and saturated transfer systems due to the following two propositions, the latter being Lemma 2.9 in [Mac].

Proposition 2.1. *Let $\mathcal{R}$ and $\mathcal{R}'$ be transfer systems and $\mathcal{R}'$ be cosaturated. The inclusions $\mathcal{R}' \subseteq \mathcal{R}$ and $\mathrm{fib}(\mathcal{R}') \subseteq \mathrm{fib}(\mathcal{R})$ are equivalent.*

Proof. The forward implication is clear. For the other direction, since $\mathcal{R}'$ is cosaturated, it is the minimal transfer system with all subgroups in $\mathrm{fib}(\mathcal{R}')$ being fibrant. Thus, $\mathrm{fib}(\mathcal{R}') \subseteq \mathrm{fib}(\mathcal{R})$ implies that $\mathcal{R}' \subseteq \mathcal{R}$. □

Proposition 2.2. *Let $\mathcal{R}$ and $\mathcal{R}'$ be transfer systems and $\mathcal{R}'$ be saturated. The inclusions $\mathcal{R}' \supseteq \mathcal{R}$ and $\mathrm{cof}(\mathcal{R}') \subseteq \mathrm{cof}(\mathcal{R})$ are equivalent.*

Proof. The forward implication is clear. For the other direction, let H be a subgroup and $K < H$ be a proper subgroup with $K \to_{\mathcal{R}} H$. In particular, H is not $\mathcal{R}$-cofibrant, so it is also not $\mathcal{R}'$-cofibrant.

Let L be a minimal subgroup of G with $L \to_{\mathcal{R}'} H$; by minimality and transitivity, we see that L must be $\mathcal{R}'$-cofibrant, and hence $\mathcal{R}$-cofibrant. Since $L \leqslant H$, we have $L \leqslant K \leqslant H$. Since $\mathcal{R}'$ is saturated and $L \to_{\mathcal{R}'} H$, we have $K \to_{\mathcal{R}'} H$. Since we chose $K \to_{\mathcal{R}} H$ arbitrarily, it follows that $\mathcal{R}' \supseteq \mathcal{R}$. □

We denote the cosaturated/saturated transfer systems on G by $\mathbf{CTS}(G)/\mathbf{STS}(G)$ respectively. We also denote the set of subsets of $\mathrm{Sub}(G)$ which are closed under intersection/compositum and conjugation by $\mathbf{Fib}(G)/\mathbf{Cof}(G)$ respectively.

Theorem 2.1. *The map $\mathrm{fib} : \mathbf{CTS}(G) \to \mathbf{Fib}(G)$ is an inclusion-preserving bijective correspondence.*

Proof. By Proposition 2.1, we know that fib is inclusion-preserving and injective. It suffices to show that every $S \in \mathbf{Fib}(G)$ can be realized from $\mathbf{CTS}(G)$. Let $\mathcal{R}$ be the relation on $\mathrm{Sub}(G)$ with $K \to_{\mathcal{R}} H$ if and only if there exists $L \in S$ with $H \cap L = K$. We claim that $\mathcal{R}$ is the required cosaturated transfer system.

- *Partial Order Refining Inclusion*: Follows by definition.
- *Transitivity*: If $K \to_{\mathcal{R}} M$ and $M \to_{\mathcal{R}} H$, then there exist $L_1, L_2 \in S$ such that $H \cap L_1 = M$ and $M \cap L_2 = K$. We then have $H \cap (L_1 \cap L_2) = K$, and since $L_1 \cap L_2 \in S$, it follows that $K \to_{\mathcal{R}} H$.
- *Restriction*: If $K \to_{\mathcal{R}} H$ and $M \leqslant H$, then there exists $L \in S$ such that $H \cap L = K$. Intersecting with M yields $M \cap L = K \cap M$, so $K \cap M \to_{\mathcal{R}} M$.
- *Conjugation*: Follows from S being closed under conjugation.

Finally, we check that $\mathrm{fib}(\mathcal{R}) = S$. Indeed, for any $H \leqslant G$, we have $H \to_{\mathcal{R}} G$ if and only if there exists $L \in S$ such that $H = G \cap L = L$, as required. □

Theorem 2.2. *The map $\mathrm{cof} : \mathbf{STS}(G) \to \mathbf{Cof}(G)$ is an inclusion-reversing bijective correspondence.*

Proof. By Proposition 2.2, we know that cof is inclusion-reversing and injective. It suffices to show that every $S \in \mathbf{Cof}(G)$ can be realized from $\mathbf{STS}(G)$. Let $\mathcal{R}$ be the relation on $\mathrm{Sub}(G)$ with $K \to_{\mathcal{R}} H$ if and only if $K \leqslant H$ and for all $L \in S$, we have $L \leqslant H \Leftrightarrow L \leqslant K$. We claim that $\mathcal{R}$ is the required saturated transfer system.

- *Partial Order Refining Inclusion*: Follows by definition.
- *Transitivity*: If $K \to_{\mathcal{R}} M$ and $M \to_{\mathcal{R}} H$, then for all $L \in S$, we have $L \leqslant K \Leftrightarrow L \leqslant M \Leftrightarrow L \leqslant H$, so $K \to_{\mathcal{R}} H$.
- *Restriction*: If $K \to_{\mathcal{R}} H$ and $M \leqslant H$, then for all $L \leqslant M$, we have $L \leqslant H$, and since $K \to_{\mathcal{R}} H$, we also have $L \leqslant K$. It follows that $L \leqslant M \Leftrightarrow L \leqslant K \cap M$ and hence $K \cap M \to_{\mathcal{R}} M$.
- *Conjugation*: Follows from S being closed under conjugation.

Finally, we check that $\mathrm{cof}(\mathcal{R}) = S$. By Lemma 2.2, we know that $\mathrm{cof}(\mathcal{R}) \supseteq S$. If H is $\mathcal{R}$-cofibrant, let M be the composite of all $L \in S$ with $L \leqslant H$; we have $M \in S$ since S is closed under compositum. By construction, we have $M \to_{\mathcal{R}} H$, forcing $M = H$ and proving that $\mathrm{cof}(\mathcal{R}) \subseteq S$, as required. □

In Appendix B, we interpret Theorem 2.1 and Theorem 2.2 through the lens of category theory. The categorical interpretation will naturally uncover some properties of cosaturated and saturated transfer systems.

3. Weak Tight Pairs

The notions of subinductors and diagrams are introduced in [Mac]. We recall that for a subinductor J on a finite group G and a subgroup $H \leqslant G$, the residue at H is defined as $\mathrm{res}_J(H) := \bigcup_{K<H} J_K^H(\hat{K})$. The main idea in [Mac] is to use the following theorem (Theorem 2.12) to realize saturated transfer systems:

Theorem 3.1 (MacBrough)**.** *Let G be a finite abelian group, J be a G-subinductor, and D be an R-stable G-diagram. If $\mathcal{R}$ is a saturated transfer system such that for all $\mathcal{R}$-cofibrant H and for all proper subgroups $K < H$, we have:*

$$I_K^H(D(H)) \not\subseteq D(K) \cup \mathrm{res}_J(H)$$

then $\mathcal{R}$ is realized by $\mathcal{L}(U)$ for some G-universe U.

Inspired by this condition, one can attempt to realize all saturated transfer system $\mathcal{R}$ on G using an appropriate pair (D, J).

Definition 3.1. Let G be a finite abelian group. A *tight pair* is a pair (D, J) where D is an R-stable diagram and J is a sub-inductor, such that

(1) For all $K < H \leqslant G$, we have $I_K^H(D(K)) \not\subseteq D(H) \cup \mathrm{res}_J(H)$.
(2) For all $H \leqslant G$, $D(H) \not\subseteq \mathrm{res}_J(H)$.

Remark 3.1. Although we only require Axiom (1) is required for Theorem 3.1, Axiom (2) allows us to use localization: if G and G' are abelian groups of relatively prime order with tight pairs (D, J) and (D', J'), then the group $G \times G'$ admits the tight pair $(D \otimes D', J \otimes J')$. Thus, in order to find tight pairs for cyclic groups G, we can reduce to the case of cyclic p-groups.

3.1. **Single-Valued Subinductors.** For every fixed prime p and non-negative integer k, we have the isomorphism $\hat{C}_{p^k} \cong \mathbb{Z}_p/p^k\mathbb{Z}_p$, where $\mathbb{Z}_p$ denotes the p-adic integers. Expressing the elements of $\mathbb{Z}_p$ in base p, we see that the elements of $\hat{C}_{p^k}$ can be expressed as k-digit strings in base p.

- Let $\pi_k : \mathbb{Z}_p \to \hat{C}_{p^k}$ be the standard projection homomorphism.
- Let $s_k : \hat{C}_{p^k} \to \mathbb{Z}_p$ be the set-theoretic section to π_k given by adding leading zeros in base p.

Definition 3.2. For a cyclic p-group G, the *standard single-valued subinductor* J is defined as follows: for $K \leqslant H$, if $\hat{H} = \hat{C}_{p^h}$ and $\hat{K} = \hat{C}_{p^k}$, then $J_K^H = \pi_h \circ s_k$.

Definition 3.3. Let G be any cyclic group; say $G = C_{p_1^{\alpha_1}} \times \cdots \times C_{p_l^{\alpha_l}}$:

- For any $H \leqslant G$ of order $\prod p_i^{\beta_i}$ and $\chi \in \hat{H}$, the *base* p_i*-part* of χ is the image of χ under the projection $\hat{H} \to \hat{C}_{p_i^{\beta_i}}$ as a β_i-digit string in base p_i.
- The *standard single-valued subinductor* J of G is the tensor product of the standard single-valued subinductors on $C_{p_i^{\alpha_i}}$ for $1 \leqslant i \leqslant l$.

Thus, for all $K \leqslant H \leqslant G$, the map $J_K^H : \hat{K} \to \hat{H}$ adds $\nu_p(|H|) - \nu_p(|K|)$ leading zeros in the base p part for each prime p; here, the quantity $\nu_p(x)$ denotes the largest non-negative integer k such that $p^k \mid x$. One may think of the standard single-valued subinductor J as coming from compatible set-theoretic section maps to the standard projections corresponding to the limit $\hat{\mathbb{Z}} = \varprojlim C_n$.

Definition 3.4. Let $G = C_{p^k}$. The finite tree $T(G)$ is the rooted tree of degree p and height k such that:

- For all $0 \leqslant d \leqslant k$, the nodes at depth d denote the elements of $\hat{H}$ where $H \leqslant G$ is the subgroup of order p^d.
- For all $0 \leqslant d \leqslant d' \leqslant k$, the depth d ancestor of any depth d' node is given by its image under $\iota^* : \hat{C}_{p^{d'}} \to \hat{C}_{p^d}$ where ι^* is the dual map to the subgroup inclusion $\iota : C_{p^d} \to C_{p^{d'}}$ in G.

For any G-universe U, the R-stable diagram corresponding to it is given by $D(H) = R_H^G U$. We may interpret U as a non-empty subset of leaves in $T(G)$ and D as the collection of all ancestors of the leaves in U. Through this interpretation:

- The saturated transfer system $\mathcal{R}$ realized by $\mathcal{L}(U)$ has $K \to_{\mathcal{R}} H$ if and only if every K-node in D has all its H-descendants in D.
- A single-valued subinductor J on G is given by a choice of a child for every non-leaf node in $T(G)$.

We immediately see that changing U upto the action of $\mathrm{Aut}(T(G))$ on $T(G)$ does not change $\mathcal{R}$. Technically, this may leave us with a non-empty G-semi-universe that does not contain the trivial representation, but this is unimportant since any non-empty U can be changed into a universe by translation.

Thus, the single-valued subinductors J are only important upto the automorphisms of $T(G)$. However, $\mathrm{Aut}(T(G))$ clearly acts transitively on the set of all single-valued G-subinductors, so every single-valued subinductor on the cyclic p-group G is "isomorphic" to the standard single-valued subinductor.

Furthermore, if G is any cyclic group with any single-valued subinductor J, then J must be the tensor product of subinductors on the Sylow p-subgroups of G. Indeed, from Mackey's formula, we see that the maps J_K^H must act independently on each coordinate. Thus, every single-valued subinductor on G is "isomorphic" to the standard single-valued subinductor on G.

3.2. **Construction of Tight Pairs.**

We remind the reader once again that we shall mainly focus on universes over $\mathbb{C}$, unlike in the literature. For clarity, we repeat the $\mathbb{C}$-version of the proof in [Mac] for the existence of tight pairs:

Lemma 3.1. *Let $G = C_{p^\alpha}$ be a cyclic p-group with $p \geq 3$. Then G has a tight pair.*

Proof. Let J be the standard single-valued subinductor. Let D be the G-diagram such that for all $H \leqslant G$:

$$D(H) = \begin{cases} \{0\} & \text{if } H = e \\ \{0, p^{h-1}\} & \text{if } H = C_{p^h} \text{ where } h > 0 \end{cases}$$

We claim that (D, J) form a tight pair:

- *D is R-stable*: For all $H \leqslant G$ and proper $K < H$, we have $R^H_K D(H) = \{0\} \subseteq D(K)$ since the non-trivial element in $D(H)$ in base p has all non-leading digits equal to 0.
- *Axiom* (1): For all $H \leqslant G$ and proper $K < H$, we have $2p^{h-1} \in I^H_K(D(K))$, where $H = C_{p^h}$. This element is not in $D(H)$ by construction and it is not in $\mathrm{res}_J(H)$ since the leading digit in base p is non-zero.
- *Axiom* (2): For all $H \leqslant G$, we have $D(H) \not\subseteq \mathrm{res}_J(H)$. This is trivial for $H = e$ since $\mathrm{res}_J(e) = \emptyset$. For non-trivial H, the non-zero element in $D(H)$ is not in $\mathrm{res}_J(H)$ because its leading digit in base p is non-zero.

Thus, $G = C_{p^\alpha}$ has a tight pair. □

Corollary 3.1. *Let G be a cyclic group of odd order. Every saturated transfer system on G can be realized by $\mathcal{L}(U)$ for some G-universe U.*

Remark 3.2 (Digit-Gluing)**.** All the results in this paper over $\mathbb{C}$ have appropriate analogues over $\mathbb{R}$ that can be obtained as follows:

- For each odd prime p, we can write the elements of $\mathbb{Z}_p$ in *weird base* p by using the digits $\{-\frac{p-1}{2}, \cdots, \frac{p-1}{2}\}$.
- We can then consider the equivalence relation on $\mathbb{Z}_p$ where two numbers are equivalent if their corresponding digits are equal or opposites.
- We then obtain an equivalence relation on $\hat{G}$ for all odd cyclic groups G by taking the equivalence relation mentioned above in each weird base p part.

The equivalence classes in $\hat{G}$ are Galois-invariant. The set of equivalence classes behave in weird base p like standard base $\frac{p+1}{2}$. Thus, for any claim that we make over $\mathbb{C}$ requiring the bound $p > N$ for some N, the analogous bound over $\mathbb{R}$ becomes $\frac{p+1}{2} > N$. For instance, the analogue of Lemma 3.1 over $\mathbb{R}$ requires the bound $\frac{p+1}{2} > 2$, i.e. $p > 3$, which is indeed the original result in [Mac].

3.3. **Weak Tight Pairs.** Next, we try to extend the tight pair approach to even cyclic groups. However, we need to make a few modifications:

- We cannot include Axiom (2) in general since it is too restrictive on the diagram D. This comes at the price of localization.
- We relax Axiom (1) for $H = C_2$. In Section 5, we shall show that certain saturated transfer systems are not realizable over even cyclic groups (Proposition 5.2). We can reduce the problem to looking at those $\mathcal{R}$ with $e \to_{\mathcal{R}} C_2$, so Axiom (1) is not necessary for $H = C_2$.

Definition 3.5. Let G be a finite abelian group. A *weak tight pair* is a pair (D, J) where D is an R-stable diagram and J is a sub-inductor such that (D, J) satisfy Axiom (1) for all $H \in \mathrm{Sub}(G) \setminus S$ for some specified subset $S \subseteq \mathrm{Sub}(G)$. An *almost tight pair* is a weak tight pair satisfying Axiom (2).

Lemma 3.2. *Let $G = C_{2^\alpha}$ be a cyclic 2-group with $\alpha \geqslant 1$. Then, G has an almost tight pair which fails Axiom* (1) *for $S = \{C_2\}$.*

Proof. Take (D, J) where J is the single-valued subinductor and D is the diagram from Lemma 3.1. The same argument shows R-stability and Axiom (2).

- For all non-trivial $H \leqslant G$ with $H \neq C_2$ and proper $K < H$, we have $2^{h-1} + 2^{h-2} \in I_K^H D(K)$, where $H = C_{2^h}$. This element is not in $D(H)$ by construction and it is not in $\mathrm{res}_J(H)$ since the leading digit is non-zero, so Axiom (1) holds for H larger than C_2.
- However, Axiom 1 fails for $H = C_2$ with $K = e$. Indeed, we have $I_K^H D(K) \subseteq D(H) \cup \mathrm{res}_J(H)$ since $D(H) = \hat{H}$.

Thus, (D, J) is an almost tight pair which fails Axiom (1) for $S = \{C_2\}$. □

Lemma 3.3. *Let G and G' be cyclic groups of relatively prime order. Let (D, J) be an almost tight pair for G failing Axiom (1) for $S \subseteq \mathrm{Sub}(G)$. Let (D', J') be a tight pair for G'. Then, $(D \otimes D', J \otimes J')$ is an almost tight pair of $G \times G'$ failing Axiom (1) at $S \times \mathrm{Sub}(G')$.*

Proof. The same proof in Lemma 3.3 of [Mac] applies. □

Theorem 3.2. *Let G be an even cyclic group and $\mathcal{R}$ be a saturated transfer system with no $\mathcal{R}$-cofibrant subgroups H such that $\nu_2(|H|) = 1$. Then, there exists a G-universe U realizing $\mathcal{R}$.*

Proof. The result follows from Lemma 3.1, Lemma 3.2, and Lemma 3.3. □

Remark 3.3. The condition on $\mathcal{R}$ in Theorem 3.2 is equivalent to the statement $C_m \to C_{2m}$ for all odd $m \mid |G|$. In the notation of Definition 5.1, this means that the 2^1-partition on $\mathcal{R}_2^1$ has all subgroups below it. Indeed, this comes from Proposition 5.1; any subgroup that is minimal beyond the partition must be $\mathcal{R}$-cofibrant, so no such subgroups must exist.

Theorem 3.2 is the limit to how far one can extend the localization approach with almost tight pairs. Indeed, the group $G = C_2$ does not have tight pairs (D, J) since Axiom (1) requires $D(C_2) \subseteq \mathrm{res}_J(C_2)$, which violates Axiom (2). For the remainder of this section, we shall work with weak tight pairs.

Lemma 3.4. *Let $G = C_{2^\alpha}$ be a cyclic 2-group with $\alpha \geqslant 1$. Then, G has a weak tight pair which fails Axiom (1) for $S = \emptyset$.*

Proof. Take (D, J) where J is the single-valued subinductor and D is the diagram with $D(H) = \{0\}$ for all subgroups $H \leqslant G$. For all $K < H$, the set $I_K^H D(K)$ contains elements with leading digit 1, but $D(H) \cup \mathrm{res}_J(H)$ does not. □

Corollary 3.2. *Let $G = C_{2^\alpha}$ be a cyclic 2-group. All saturated transfer systems over G are realizable by G-universes.*

As previously mentioned, we shall henceforth relax Axiom (1) for $H = C_2$. We wish to understand which cyclic groups G have weak tight pairs with $S = \{C_2\}$.

Example 3.1. Let $G = C_6 = C_2 \times C_3$, let J be the standard single-valued subinductor on G, and let D be the G-diagram below:

$D(C_2) = \{(0,*), (1,*)\}$	$D(C_6) = \{(0,0), (1,0), (0,1)\}$
$D(e) = \{(*,*)\}$	$D(C_3) = \{(*,0), (*,1)\}$

The first and second coordinates denote the base 2 and 3 parts respectively, where $*$ denotes the empty string. The pair (D, J) is a weak tight pair with $S = \{C_2\}$.

Lemma 3.5. *Let $G = C_{2p^\alpha}$ where p is an odd prime. Then, G has a weak tight pair which fails at $S = \{C_2\}$.*

Proof. Let J be the standard single-valued subinductor on G. Let D be defined as below where $0 < k \leqslant \alpha$; the first and second coordinates denote the base 2 and p parts respectively.

$D(C_2) = \{(0,*), (1,*)\}$	$D(C_{2p^k}) = \{(0,0), (1,0)\}$
$D(C_1) = \{(*,*)\}$	$D(C_{p^k}) = \{(*,0), (*,p^{k-1})\}$

It is clear that D is R-stable. It suffices to check Axiom (1) for non-trivial $H \neq C_2$:

- For $H = C_{p^k}$, Axiom (1) holds due to the same argument in Lemma 3.1.
- For $H = C_{2p^k}$ and proper subgroup $K < H$, $I_K^H D(K)$ contains $(1, p^{k-1})$, unlike $D(H) \cup \mathrm{res}_J(H)$ does not, so $I_K^H D(K) \not\subseteq D(H) \cup \mathrm{res}_J(H)$.

Thus, (D, J) is a weak tight pair failing at $S = \{C_2\}$. □

Corollary 3.3. *Let $G = C_{2p^\alpha}$ for an odd prime p and let $\mathcal{R}$ be a saturated transfer systems with $e \to_{\mathcal{R}} C_2$. Then, $\mathcal{R}$ is realized by some G-universe U.*

Lemma 3.6. *Let $G = C_{2^\alpha p}$ where p is an odd prime. Then G has a weak tight pair which fails at $S = \{C_2\}$.*

Proof. Let J be the standard single-valued subinductor on G. Let D be defined as below where $0 < k \leqslant \alpha$; the first and second coordinates denote the base 2 and p parts respectively.

$D(C_p) = \{(*,0), (*,1)\}$	$D(C_{2^k p}) = \{(0,0), (0,1)\}$
$D(C_1) = \{(*,*)\}$	$D(C_{2^k}) = \{(0,*), (2^{k-1},*)\}$

The same argument in Lemma 3.5 proves that (D, J) is a weak tight pair failing precisely at $S = \{C_2\}$. □

Corollary 3.4. *Let $G = C_{2^\alpha p}$ for an odd prime p and let $\mathcal{R}$ be a saturated transfer systems with $e \to_{\mathcal{R}} C_2$. Then, $\mathcal{R}$ is realized by some G-universe U.*

3.4. **Hunting for Weak Tight Pairs.** The goal of this subsection is to find weak tight pairs for cyclic groups whose orders are divisible by $6p$ for some prime $p > 3$ which fail Axiom (1) at $S = \{C_2\}$. The weakest version of this problem is to ask for weak tight pairs for $G = C_{6p}$.

Let $G = C_{6p} = C_2 \times C_3 \times C_p$. For all $H \leqslant G$, we shall express the elements of $\hat{H}$ with three coordinates, representing the base 2, base 3, and base p parts in the mentioned order. Assume that (D, J) is some weak tight pair for G; we can make the following simplifications:

- We may change $D(C_2)$ to $\hat{C}_2$ if necessary; this does not affect R-stability and is allowed since Axiom (1) is allowed to fail at $H = C_2$.
- Let D' be the diagram given by $D'(H) = \bigcup_{K \leqslant H} J_K^H D(K)$. Then, (D', J) is also a weak tight pair for which Axiom (1) only fails only at $S = \{C_2\}$; note that D' is R-stable by Mackey's formula. Thus, we may replace D by D' in order to assume that D is J-stable. In particular, note that for all $K \leqslant H$, we have $R_K^H D(H) = D(K)$ by Frobenius reciprocity.
- We know that $\{(*,0,*)\} \subseteq D(C_3) \subsetneq \hat{C}_3$, where the strict inclusion comes from Axiom (1) on $H = \hat{C}_3$. If $D(C_3) = \{(*,0,*)\}$, then we must have $D(C_6) = \{(0,0,*), (1,0,*)\}$ since $R_{C_2}^{C_6} D(C_6) = \hat{C}_2$ and $R_{C_3}^{C_6} D(C_6) = D(C_3)$.

However, this would imply that $I_{C_3}^{C_6} D(C_3) \subseteq D(C_6)$, which contradicts Axiom (1) at $H = C_6$. Thus, assume WLOG that $D(C_3) = \{(*, 0, *), (*, 1, *)\}$.

Lemma 3.7. *We have* $|J_{C_2}^{C_6} D(C_2) \cup J_{C_3}^{C_6} D(C_3)| = |D(C_6)| = 3$.

Proof. First, observe that $|J_{C_2}^{C_6} D(C_2) \cup J_{C_3}^{C_6} D(C_3)| \leqslant |D(C_6)| \leqslant 3$. The first inequality comes from the J-stability of D and the second inequality comes from $D(C_6) \subsetneq I_{C_3}^{C_6} D(C_3)$ by Axiom (1) with $H = C_6$ and $K = C_3$.

It suffices to show that $|J_{C_2}^{C_6} D(C_2) \cup J_{C_3}^{C_6} D(C_3)| = 3$. Assume the contrary; by Frobenius reciprocity, we know that $|J_{C_2}^{C_6} D(C_2)| \geqslant 2$ and $|J_{C_3}^{C_6} D(C_3)| \geqslant 2$, so $J_{C_2}^{C_6} D(C_2) = J_{C_3}^{C_6} D(C_3)$ with both $J_{C_2}^{C_6}$ and $J_{C_3}^{C_6}$ being single-valued.

By Frobenius reciprocity, this implies that the map $R_{C_2}^{C_6} J_{C_3}^{C_6} : D(C_3) \to D(C_2)$ is a bijection. By Mackey's formula, this is identical to $J_e^{C_2} R_e^{C_3} : D(C_3) \to D(C_2)$. However, this map cannot be a bijection since it factors through $D(e)$, which has strictly smaller cardinality than $D(C_2)$ and $D(C_3)$. Thus, we have the required contradiction, proving that $|J_{C_2}^{C_6} D(C_2) \cup J_{C_3}^{C_6} D(C_3)| = |D(C_6)| = 3$. $\square$

Lemma 3.8. *There exists some* $\chi \in D(C_6)$ *such that we have* $\{\chi\} \subsetneq J_P^{C_6} R_P^{C_6} \chi$ *for both* $P = C_2$ *and* $P = C_3$.

Proof. Assume the contrary. Let $\chi \in D(C_6)$ be arbitrary; pick $P \in \{C_2, C_3\}$ such that $J_P^{C_6} R_P^{C_6} \chi$ does not properly contain $\{\chi\}$. By Lemma 3.7, we have $J_P^{C_6} R_P^{C_6} \chi \supseteq \{\chi\}$, so we must have $J_P^{C_6} R_P^{C_6} \chi = \{\chi\}$.

Denote $\psi := R_P^{C_6} \chi$. Let $K = C_6$, $L = P \times C_p$, and $H = K \times C_p$ be subgroups of G. By Mackey's formula, the map $J_L^H : I_P^L \psi \to I_K^H \chi$ is a bijection. Since $\chi \in D(C_6)$ was arbitrary, $I_K^H D(K) \subseteq J_L^H \hat{L}$, which contradicts Axiom (1), as required. $\square$

Assume that $D(C_6) = \{(0, 0, *), (1, 0, *), (0, 1, *)\}$. By Frobenius reciprocity, we must have $J_{C_2}^{C_6}(1, *, *) = \{(1, 0, *)\}$ and $J_{C_3}^{C_6}(*, 1, *) = \{(0, 1, *)\}$. By Lemma 3.8, we have $J_{C_2}^{C_6}(0, *, *) = \{(0, 0, *), (0, 1, *)\}$ and $J_{C_3}^{C_6}(*, 0, *) = \{(0, 0, *), (1, 0, *)\}$. The other three possibilites for $D(C_6)$ determine J similarly.

4. Linear Isometries Operads over Odd Cyclic Groups

Throughout this section, we shall work with the standard single-valued subinductor J defined in Definition 3.3. Given a G-saturated transfer system $\mathcal{R}$, we shall algorithmically construct a G-diagram D that corresponds to some G-universe U, i.e. a diagram D such that for all $K \leqslant H \leqslant G$:

- We have $D(K) = R_K^H D(H)$, i.e. D corresponds to $U = D(G)$.
- We have $D(H) = I_K^H D(K)$ if and only if $K \to_{\mathcal{R}} H$.

The section reproves (the $\mathbb{C}$-analogue of) the result in [Mac] that all saturated transfer systems over odd cyclic groups G are realizable by G-universes. The key idea in the proof is identical to MacBrough's approach, but the details are relatively simpler. Furthermore, the algorithmic approach will generalize quite well in Section 5 and Section 7.

4.1. A Naive Approach.

We begin with an intuitive first attempt at solving the realizability problem for linear isometries operads over cyclic groups. Spoiler alert: it doesn't work!

Algorithm 1. *Let G be a cyclic group.*

- ***Input***: *A saturated G-transfer system $\mathcal{R}$.*
- ***Output***: *A G-diagram D.*

We construct D on subgroups $H \leqslant G$ inductively by iterating through $\mathrm{Sub}(G)$ in any order completing its partial order. For the base case, we let $D(e) = \hat{e}$. For non-trivial subgroups H:

$$D(H) = \left(\bigcup_{K \to_{\mathcal{R}} H} I_K^H D(K) \right) \cup \left(\bigcup_{K \not\to_{\mathcal{R}} H} J_K^H D(K) \right)$$

where K iterates through the proper subgroups of H.

Remark 4.1. It suffices to iterate through the maximal subgroups $K < H$ by the transitivity of (sub)-induction and the saturation of $\mathcal{R}$.

Lemma 4.1. *The diagram D obtained from Algorithm 1 satisfies $R_K^H D(H) = D(K)$ for all $K \leqslant H \leqslant G$.*

Proof. Clearly, we have $R_K^H D(H) \supseteq R_K^H J_K^H D(K) = D(K)$, so it suffices to show that D is R-stable. Let K, L, H be subgroups of G with $K < H$ and $L < H$. We must first show that $R_L^H J_K^H D(K) \subseteq D(L)$. We may prove this by induction on $|H|$ using the Mackey formula axiom for sub-inductors:

$$R_L^H J_K^H D(K) \subseteq J_{K\cap L}^L R_{K\cap L}^K D(K) \subseteq J_{K\cap L}^L D(K \cap L) \subseteq D(L)$$

Next, we must show that if $K \to_{\mathcal{R}} H$, then $R_L^H I_K^H D(K) \subseteq D(L)$. The same argument applies after noting that we have $K \cap L \to_{\mathcal{R}} L$ by the restriction axiom on $\mathcal{R}$ and using Mackey's formula for I. □

Corollary 4.1. *The diagram D from Algorithm 1 comes from the G-universe $U = D(G)$. If $K \to_{\mathcal{R}} H$, then $D(H) = I_K^H D(K)$.*

Thus, we see that the saturated transfer system $\mathcal{R}_D$ generated by D contains $\mathcal{R}$. However, it may contain accidental arrows as the following example shows:

Example 4.1. Let $G = C_{15}$ and $\mathcal{R}$ be the saturated transfer system below:

$$\begin{tikzcd} C_5 & C_{15} \\ e \arrow[r] & C_3 \end{tikzcd}$$

$D(C_5) = \{(*, 0)\}$	$D(C_{15}) = \{(d, 0) : d \in C_3\}$
$D(e) = \{(*, *)\}$	$D(C_3) = \{(d, *) : d \in C_3\}$

The G-diagram D generated by Algorithm 1 is given above. To clarify notation, we have expressed elements in $C_{3^i 5^j}$ using the isomorphism $C_{3^i 5^j} \cong C_{3^i} \times C_{5^j}$. We have also used $*$ to denote the unique element of C_1. Although $\mathcal{R}_D$ contains $e \to C_3$, it also contains the "accidental arrow" $C_5 \to C_{15}$. Thus, $\mathcal{R}_D \supsetneq \mathcal{R}$.

4.2. **Misteaks are Good.** The reason behind the failure of Algorithm 1 in the above example is that $D(C_5)$ does not have any "new" elements when compared to $D(e)$. We can tackle this problem by modifying our algorithm appropriately.

Definition 4.1. An element in $\hat{H}$ is *new* if it is not contained in $J_K^H \hat{K}$ for any proper subgroup (equivalently, maximal subgroup) $K < H$.

Algorithm N. *Let G be a cyclic group.*

- ***Input***: *A saturated G-transfer system $\mathcal{R}$ and a G-diagram N.*
- ***Output***: *A G-diagram D.*

We construct D on subgroups $H \leqslant G$ by induction:

$$D(H) = \left(\bigcup_{K \to_{\mathcal{R}} H} I_K^H D(K) \right) \cup \left(\bigcup_{K \not\to_{\mathcal{R}} H} J_K^H D(K) \right) \cup N(H)$$

where K iterates through the maximal subgroups of H.

Remark 4.2. The idea behind Algorithm N is to modify Algorithm 1 by making our diagram D necessarily contain N. Typically, $N(H)$ will be a set of new elements in $\hat{H}$. One may think of Algorithm 1 as a special case of Algorithm N with $N(e) = \hat{e}$ and $N(H) = \emptyset$ for non-trivial subgroups H.

Algorithm 2. *Let G be a cyclic group.*

- ***Input***: *A saturated G-transfer system $\mathcal{R}$.*
- ***Output***: *A G-diagram D.*

Let N be the new element G-diagram defined as follows:

$$N(H) = \begin{cases} \{(p_1^{\beta_1 - 1}, p_2^{\beta_2 - 1}, \dots, p_l^{\beta_l - 1})\} & \text{if } H \text{ is } \mathcal{R}\text{-cofibrant} \\ \emptyset & \text{otherwise} \end{cases}$$

where $\hat{H} \cong \prod C_{p_i^{\beta_i}}$ and $p^{-1} = *$ in all coordinates with $\beta_i = 0$. Return the diagram D obtained by executing Algorithm N with $\mathcal{R}$ and N.

Lemma 4.2. *In the diagram D obtained from Algorithm 2, for all $\mathcal{R}$-cofibrant H and $K < H$, we have $R_K^H N(H) \subseteq D(K)$.*

Proof. We show by induction that for every subgroup L of order $\prod p_i^{\gamma_i}$, the set $D(L)$ contains all elements of the form $(\chi_1, \dots, \chi_l)$ where each χ_i can be either $p_i^{\gamma_i - 1}$ ($*$ if $\gamma_i = 0$) or 0. First, the claim is true if L is not cofibrant, since $D(L) \supseteq I_M^L D(M)$ for any $M < L$ with $M \to_{\mathcal{R}} L$ using the induction hypothesis on M.

On the other hand, if L is cofibrant, the claim still holds true if any coordinate is not equal to $p_i^{\gamma_i - 1}$ since if $[L : M] = p_i$, then it follows from $D(L) \supseteq J_M^L D(M)$ using the induction hypothesis on M. Finally, the claim is true if all coordinates are equal to $p_i^{\gamma_i - 1}$ and L is cofibrant from $D(L) \supseteq N(L)$. Now, the lemma follows from noting that $R_K^H N(H)$ is an element of aforementioned form in $\hat{K}$ for $L = K$. $\square$

Corollary 4.2. *The diagram D obtained from Algorithm 2 is non-empty and satisfies $R_K^H D(H) = D(K)$ for all $K \leqslant H \leqslant G$. Thus, it comes from the G-universe $U = D(G)$. Furthermore, if $K \to_{\mathcal{R}} H$, then $D(H) = I_K^H D(K)$.*

Proof. Use Lemma 4.2 and the arguments in Lemma 4.1. $\square$

Theorem 4.1. *For every odd cyclic G and saturated transfer system $\mathcal{R}$, the diagram D from Algorithm 2 realizes $\mathcal{R}$. Thus, all saturated transfer systems $\mathcal{R}$ over G are realizable by G-universes.*

Proof. Let $\mathcal{R}_D$ be the saturated transfer system realized by D. We know that $\mathcal{R}_D \supseteq \mathcal{R}$, and by Proposition 2.2, it suffices to prove that every $\mathcal{R}$-cofibrant H is also $\mathcal{R}_D$-cofibrant. Observe that by construction, $D(H)$ has precisely 1 new element. On the other hand, for any maximal $K < H$, we know that $D(K)$ contains new elements (from the proof of Lemma 4.2), so $I_K^H D(K)$ contains at least $p - 1$

new elements, where $p = [H : K]$, by adding any non-zero leading digit in the p-coordinate. Thus, $I_K^H D(K) \not\subseteq D(H)$, as required. $\square$

Example 4.2. Let $G = C_{15}$ and $\mathcal{R}$ be the saturated transfer system below:

$$\begin{tikzcd} C_5 & C_{15} \\ \\ e \arrow[r] & C_3 \end{tikzcd}$$

$\{(*, 0), (*, 1)\}$	$\{(d, 0) : d \in C_3\} \cup \{(0, 1), (1, 1)\}$
$\{(*, *)\}$	$\{(d, *) : d \in C_3\}$

The G-diagram D generated by Algorithm 2 is given above. In contrast to Algorithm 1, we see that there no longer an accidental arrow $C_5 \to C_{15}$.

Remark 4.3. One can verify that Algorithm 1 in fact works for cyclic p-groups since for $K < H$ with $K \not\to_{\mathcal{R}} H$, we have $|D(H)| = |D(K)| < |I_K^H D(K)|$. In general, the accidental arrows created in Algorithm 1 are always parallel to necessary arrows, but since the subgroup lattice for C_{p^k} is a chain, there are no distinct parallel arrows.

Remark 4.4. One can also use the new element $(\frac{1}{1-p_1}, \dots, \frac{1}{1-p_l})$ to yield a variant of Algorithm 2. The corresponding version of Lemma 4.2 would be to show that this new element exists in every subgroup and the same proof applies. We shall find use for this when constructing algorithms for even cyclic groups.

Remark 4.5. The analogous result over $\mathbb{R}$ (rf. Remark 3.2) is that for all cyclic groups G with order divisible by neither 2 nor 3, all saturated transfer systems $\mathcal{R}$ over G are realizable.

5. Linear Isometries Operads over Even Cyclic Groups

When we move to even cyclic groups, it is no longer true that every saturated transfer system is realizable. Before we discuss why this is the case, we shall introduce some new notation regarding saturated transfer systems:

5.1. Decomposing Saturated Transfer Systems.

Let G be a cyclic group and p be a prime with $G = C_{p^\alpha} \times G'$ where $p \nmid |G'|$. For each $0 \leqslant k \leqslant \alpha$, the subgroups $H \leqslant G$ with $\nu_p(|H|) = k$ form a sublattice isomorphic to $\mathrm{Sub}(G')$.

Definition 5.1. Let $\mathcal{R}$ be a saturated transfer system on G and $0 \leqslant k \leqslant \alpha$:

- The p^k-*layer* of $\mathcal{R}$, denoted $\mathcal{R}_p^k$, is the saturated transfer system on the lattice isomorphic to $\mathrm{Sub}(G')$ obtained from $\mathcal{R}$ by restricting to the subgroups $H \leqslant G$ with $\nu_p(|H|) = k$.
- An arrow $K \to_{\mathcal{R}} H$ is a p-*arrow* if and only if $[H : K] = p$.
- For $k > 0$, the p^k-*partition* is the imaginary partition on $\mathcal{R}_p^k$ separating the subgroups which do and do not receive p-arrows.

Example 5.1. Let $G = C_6$ and $\mathcal{R}$ be the saturated transfer system below:

$$\begin{tikzcd} C_2 & C_6 \\ \\ e \arrow[r] & C_3 \end{tikzcd}$$

- For the prime $p = 2$:
  - $\mathcal{R}_2^0$ is the complete transfer system on $\mathrm{Sub}(C_3)$ and $\mathcal{R}_2^1$ is the empty transfer system on $\mathrm{Sub}(C_3)$.
  - There are no 2-arrows in the saturated transfer system.
  - The 2^1-partition on $\mathcal{R}_2^1$ has no subgroups below it.
- For the prime $p = 3$:
  - $\mathcal{R}_3^0$ and $\mathcal{R}_3^1$ are both the empty transfer system on $\mathrm{Sub}(C_2)$.
  - The arrow $e \to_{\mathcal{R}} C_3$ is a 3-arrow from $\mathcal{R}_3^0$ to $\mathcal{R}_3^1$.
  - The 3^1-partition on $\mathcal{R}_3^1$ has C_3 below it and C_6 above it.
- For the prime $p = 5$, we have $\mathcal{R}_5^0 = \mathcal{R}$ and there are no 5-arrows.

For any prime p, the data of $\mathcal{R}$ can be provided using its p^k-layers and the p-arrows between consecutive layers. However, not every choice of layers and p-arrows yields a valid saturated transfer system. The following proposition tells us the required compatibility constraints:

Proposition 5.1. *The data of layers $\mathcal{R}_p^k$ for $0 \leqslant k \leqslant \alpha$ and p-arrows between $\mathcal{R}_p^{k-1}$ and $\mathcal{R}_p^k$ (equivalently, p^k-partitions) for $0 < k \leqslant \alpha$ yields a well-defined saturated transfer system $\mathcal{R}$ on G if and only if the following conditions hold for $k > 0$:*

(1) $\mathcal{R}_p^{k-1} \supseteq \mathcal{R}_p^k$, i.e. higher layers are finer.
(2) The set of subgroups on $\mathcal{R}_p^k$ receiving p-arrows is downward-closed (which we refer to as the set of subgroups below the p^k-partition).
(3) No arrows on $\mathcal{R}_p^k$ cut across the p^k-partition, i.e. go from a subgroup below the p^k-partition to a subgroup above the partition.
(4) If $K \leqslant H$ lie below the partition on $\mathcal{R}_p^k$ and their index p subgroups are $K' \leqslant H'$ on $\mathcal{R}_p^{k-1}$, then $K' \to_{\mathcal{R}_p^{k-1}} H'$ if and only if $K \to_{\mathcal{R}_p^k} H$. Thus, $\mathcal{R}_p^{k-1}$ and $\mathcal{R}_p^k$ are identical at least till the p^k-partition.

Proof. We can obtain a relation $\mathcal{R}$ on $\mathrm{Sub}(G)$ by gluing the data of layers and p-arrows and closing it under the saturation axiom. Since G is abelian, the conjugation axiom is trivial. The constraints (1) and (2) come from the restriction axiom on non-p-arrows and p-arrows respectively.

It remains to check transitivity. We already have transitivity on the layers since they are saturated transfer systems. It suffices to show that for $K \leqslant H$ on $\mathcal{R}_p^k$ with index p subgroups $K' \leqslant H'$ on $\mathcal{R}_p^{k-1}$:

- If $K' \to K$ and $K \to H$,then $H' \to H$.
- If $K' \to H'$ and $H' \to H$, then $K \to H$.

These are constraints (3) and (4) respectively. Thus, the given compatibility constraints are equivalent to the saturated transfer system axioms on $\mathcal{R}$. □

Lemma 5.1. *Let $\mathcal{R}$ be a saturated transfer system on G. Let p be a prime and k be an integer with $0 < k \leqslant \nu_p(|G|)$. The following are equivalent:*

(1) There are no subgroups below the p^k-partition on $\mathcal{R}_p^k$.
(2) There are no p-arrows from $\mathcal{R}_p^{k-1}$ to $\mathcal{R}_p^k$.
(3) We have $C_{p^{k-1}} \not\to_{\mathcal{R}} C_{p^k}$.

Proof. The equivalence (1) $\Leftrightarrow$ (2) follows by the definition of the p^k-partition. We also clearly have (2) $\Rightarrow$ (3). The implication (2) $\Leftarrow$ (3) follows from the contrapositive of the restriction axiom since $C_{p^{k-1}} \not\to_{\mathcal{R}} C_{p^k}$ implies $C_{p^{k-1}m} \not\to_{\mathcal{R}} C_{p^k m}$ for all divisors $m \mid |G|$ relatively prime to p. □

Definition 5.2. We say that the p^k-partition on $\mathcal{R}$ is *trivial* if any of the equivalent conditions in Lemma 5.1 hold.

5.2. **Obstruction to Realizability.** We present the smallest example of a saturated transfer system over a cyclic group that is not realizable by a G-universe:

Example 5.2. Let $G = C_6 = C_3 \times C_2$ and $\mathcal{R}$ be as below:

$$\begin{tikzcd} C_2 & C_6 \\ e \arrow[r] & C_3 \end{tikzcd}$$

$\{(*,0),(*,1)\}$	$\{(d,0) : d \in C_3\} \cup \{(0,1),(1,1)\}$
$\{(*,*)\}$	$\{(d,*) : d \in C_3\}$

First, note that the diagram from Algorithm 2 does not work since we have an arrow $e \to C_2$ (because the bound $p-1 > 1$ does not hold). If we try to take $D(C_2) = \{(*,0)\}$, then we must use the diagram from Algorithm 1, which creates the accidental arrow $C_2 \to C_6$. Thus, $\mathcal{R}$ is not realizable by a G-universe.

The following proposition imposes a necessary condition for realizability that explains why the saturated transfer system in Example 5.2 is not realizable.

Proposition 5.2. *Let G be a cyclic group and $\mathcal{R}$ be a saturated transfer system. Let $\beta \leqslant \nu_2(|G|)$ be maximal such that for all $0 < k \leqslant \beta$, the 2^k-partition on $\mathcal{R}$ is trivial. Then, $\mathcal{R}$ is realizable only if $\mathcal{R}_2^0 = \cdots = \mathcal{R}_2^\beta$.*

Proof. Assume that $\mathcal{R}$ is a saturated transfer system realized by a G-universe U. We are given that $C_{2^{k-1}} \not\to_{\mathcal{R}} C_{2^k}$ for all $0 < k \leqslant \beta$. Start by assuming that $e \not\to C_2$; this implies that $D(C_2) = \{0\}$. For all odd divisors $m \mid |G|$, we must have $D(C_{2m}) = J_m^{2m} D(C_m)$. Since each element in $D(C_m)$ is lifted uniquely with the same base 2 part to $D(C_{2m})$, we have $\mathcal{R}_2^0 = \mathcal{R}_2^1$. In general, $D(C_{2^k}) = \{0\}$ for all $0 < k \leqslant \beta$, and the same argument implies that $\mathcal{R}_2^0 = \cdots = \mathcal{R}_2^\beta$. $\square$

Proposition 5.2 is an example of a *local obstruction*. In general, for any subfield $k \subseteq \mathbb{C}$ and rational integer prime p, we have a local constraint for realizability. In the case of $k = \mathbb{C}$, the constraint is trivial for $p > 2$ and Proposition 5.2 for $p = 2$. In the case of $k = \mathbb{R}$, the constraint is trivial for $p > 3$, identical to Proposition 5.2 for $p = 3$, but stronger for $p = 2$. We shall expand on this in Section 6.

Conjecture 5.1. *Let G be a cyclic group and $\mathcal{R}$ be a saturated transfer system. Then, $\mathcal{R}$ is realizable if and only if it is not excluded by Proposition 5.2.*

One may think of Conjecture 5.1 as a local-global statement. We shall see in Section 6 that the more general local-global conjecture is not true for all subfields $k \subseteq \mathbb{C}$. We shall provide a counterexample for $k = \mathbb{Q}$, and consequently infinitely many other subfields of $\mathbb{C}$, including all number fields.

Lemma 5.2. *Let G be an even cyclic group and $\mathcal{R}$ be a saturated transfer system on G satisfying the necessary condition for realizability in Proposition 5.2. Let $\pi : G \to G''$ be the quotient map with kernel C_{2^β} and let $\mathcal{R}''$ be the saturated transfer system on G'' defined as follows:*

- *For all $0 \leqslant k \leqslant \nu_2(|G''|)$, we have $(\mathcal{R}'')_2^k = \mathcal{R}_2^{\beta+k}$.*
- *For all $0 < k \leqslant \nu_2(|G''|)$, the 2^k-partition on $(\mathcal{R}'')_2^k$ identical to the $2^{\beta+k}$-partition on $\mathcal{R}_2^{\beta+k}$.*

Equivalently, for all $K'' \leqslant H'' \leqslant G''$, we have $K'' \to_{\mathcal{R}''} H''$ if and only if $K \to_{\mathcal{R}} H$ where $K = \pi^{-1}(K'')$ and $H = \pi^{-1}(H'')$. Then, $\mathcal{R}$ is realizable by a G-universe if and only if $\mathcal{R}''$ is realizable by a G''-universe.

Proof. From the proof in Conjecture 5.1, we know that if the G-universe U realizes $\mathcal{R}$, then we must have $D(C_{2^\beta}) = 0$ in the corresponding diagram D. This means $U = \pi^*(U'')$ for some G''-universe U'', where $\pi^* : \hat{G}'' \to \hat{G}$ is the dual to π. We can then see that U'' realizes $\mathcal{R}''$. Conversely, if U'' realizes $\mathcal{R}''$, then the G-universe $U = \pi^*(U'')$ realizes $\mathcal{R}$. □

Conjecture 5.2. *Let G be an even cyclic group and $\mathcal{R}$ be a saturated transfer system with $e \to_{\mathcal{R}} C_2$. Then, $\mathcal{R}$ is realizable.*

Lemma 5.3. *Conjecture 5.1 and Conjecture 5.2 are equivalent.*

Proof. The forward implication is clear since we have $\beta = 0$, where the chain of equality of layers is trivial. For the reverse implication, let $\mathcal{R}$ satisfy the necessary condition in Conjecture 5.1. We can use Lemma 5.2 to reduce to the case where $\beta = 0$. We already know from Theorem 4.1 that the conjecture is true for odd cyclic G, so we may further assume that $2 \mid |G|$. We thus have the arrow $e \to_{\mathcal{R}} C_2$, so Conjecture 5.2 implies realizability. Thus, the two conjectures are equivalent. □

Due to the equivalence of Conjecture 5.1 and Conjecture 5.2, for the remainder of this section, we may focus our attention towards saturated transfer systems $\mathcal{R}$ on even cyclic groups G with $e \to_{\mathcal{R}} C_2$.

5.3. **The Case $G = C_{2p^{\alpha'}}$.** We start by looking at the even cyclic group $G = C_{2p^{\alpha'}}$ with odd prime p. A G-universe U has representations which are 0 and 1 modulo 2. The elements in each class modulo 2 can be identified with elements of $\hat{C}_{p^{\alpha'}}$ under projection. Thus, we can specify U using the $C_{p^{\alpha'}}$-universes U'_0 and U'_1 corresponding to the 0 mod 2 and 1 mod 2 classes in U respectively. (Technically, it may be possible for U'_1 to be the empty semi-universe on G').

For $i = 0, 1$, let the $C_{p^{\alpha'}}$-universe U'_i have corresponding diagram D'_i and realize the saturated transfer system $\mathcal{R}'_i$. The saturated transfer system $\mathcal{R}$ realized by U is given by the following data:

- $\mathcal{R}_2^0$ is realized by $U'_0 \cup U'_1$ (or $D'_0 \cup D'_1$).
- $\mathcal{R}_2^1 = \mathcal{R}'_0 \cap \mathcal{R}'_1$.
- The subgroups C_{2m} below the 2^1-partition on $\mathcal{R}_2^1$ are the subgroups for which $D'_0(C_m) = D'_1(C_m)$, where m denotes any odd divisor of $|G|$.

In particular, if we take $U'_0 \supseteq U'_1$, then $\mathcal{R}_2^0$ and $\mathcal{R}_2^1$ will be the saturated transfer systems realized by U'_0 and U'_1 respectively. We shall analyze two natural choices for such universes U'_0 and U'_1:

- We may take U'_0 and U'_1 as given by Algorithm 2. In this case, the 2^1-partition divides $\mathcal{R}_2^1$ at the point where $\mathcal{R}_2^0$ and $\mathcal{R}_2^1$ are no longer identical. By Proposition 5.1, this is the largest possible 2^1-partition.
- We may also take U'_0 as given by Algorithm 2 and U'_1 as given by Algorithm 1. By Remark 4.3, U'_1 indeed realizes $\mathcal{R}_2^1$. In this case, the 2^1-partition divides $\mathcal{R}_2^1$ at the first point where $\mathcal{R}_2^1$ does not have an arrow. By Proposition 5.1, this is the smallest possible 2^1-partition given the 2-arrow $e \to_{\mathcal{R}} C_2$.

- If the 2^1-partition is trivial, then we must have $\mathcal{R}_2^0 = \mathcal{R}_2^1$ for realizability. We may then take U_0' as given by Algorithm 2 and U_1' to be the empty semi-universe to realize the corresponding $\mathcal{R}$.

This hints that the above choices are the extremes in a spectrum of algorithms one may obtain by mixing Algorithm 1 and Algorithm 2 appropriately.

Algorithm $\langle \mathbf{2} \mid \mathbf{1}\rangle_2^k$. *Let $G = C_{2^\alpha} \times G'$ where G' is an odd cyclic group.*

- ***Input****: A saturated G'-transfer system $\mathcal{R}'$, a saturated G-transfer system $\mathcal{R}$, and an integer k with $0 < k \leqslant \alpha$. We package this data as $(\mathcal{R}', \mathcal{R}, k)$.*
- ***Output****: A G'-diagram D'.*

If the 2^k-partition on $\mathcal{R}_2^k$ has no subgroups below it, then we return the empty diagram. Otherwise, let N_{Alg1} and N_{Alg2} denote the new element G'-diagrams with respect to $\mathcal{R}_2^k$ corresponding to Algorithm 1 and Algorithm 2 respectively. Let N_2^k be the new element G'-diagram where for all $H' \leqslant G'$:

$$N_2^k(H') = \begin{cases} N_{\mathrm{Alg2}}(H') & \textit{if } H \textit{ is below the } 2^k\textit{-partition on } \mathcal{R}_2^k \\ N_{\mathrm{Alg1}}(H') & \textit{if } H \textit{ is above the } 2^k\textit{-partition on } \mathcal{R}_2^k \end{cases}$$

where $H \leqslant G$ is the subgroup on $\mathcal{R}_2^k$ with $[H : H'] = 2^k$. Return the G'-diagram D' obtained by executing Algorithm N with $\mathcal{R}'$ and N_2^k.

Remark 5.1. Algorithm $\langle 2 \mid 1\rangle_2^k$ does not depend fully on $\mathcal{R}$, but only the 2^k-partition on $\mathcal{R}_2^k$.

Proposition 5.3. *With the notation in Algorithm $\langle 2 \mid 1\rangle_2^k$:*

1. *For all subgroups $K' \leqslant H' \leqslant G'$, the diagram D' obtained from Algorithm $\langle 2 \mid 1\rangle_2^k$ satisfies $R_{K'}^{H'} D'(H') = D'(K')$.*
2. *The diagram D' is empty if the 2^k-partition is trivial and comes from the G'-universe $U' = D'(G')$ otherwise.*
3. *For all subgroups $K' \leqslant H' \leqslant G'$, if $K' \to_{\mathcal{R}'} H'$, then $D'(H') = I_{K'}^{H'} D'(K')$.*

Proof. All the statements above follow from ideas in Section 4.

1. Use the arguments in Lemma 4.1 and Lemma 4.2. Note that it is important that we apply Algorithm 2 before the partition and Algorithm 2 after since $N_{\mathrm{Alg2}}(H') \supseteq N_{\mathrm{Alg1}}(H')$.
2. It is easy to see that D' is empty if and only if the 2^k-partition is trivial. It follows from (1) that D' comes from U' when the partition is non-trivial, similar to Corollary 4.1.
3. Follows from (1), similar to Corollary 4.1.

Thus, the proposition is proven. □

Proposition 5.4. *Let $G = C_{2^\alpha p^{\alpha'}} = C_{2^\alpha} \times G'$ where $G' = C_{p^{\alpha'}}$ and let $\mathcal{R}$ be a saturated transfer system on G with non-trivial 2^k-partition. For all saturated G' transfer systems $\mathcal{R}'$, the diagram D' from Algorithm $\langle 2 \mid 1\rangle_2^k$ realizes $\mathcal{R}'$.*

Proof. We know from Proposition 5.3 that if the diagram D' comes from the universe U', then $\mathcal{L}(U')$ realizes a saturated transfer system containing $\mathcal{R}'$. By Proposition 2.2, it suffices to show that for every non-trivial $\mathcal{R}'$-cofibrant $H' \leqslant G'$ with maximal subgroup K', we have $D'(H') \neq I_{K'}^{H'} D'(K')$. Indeed, the new elements in $D'(H')$ come from $N_2^k(H')$, which all have leading digit 1 in their base p part.

On the other hand, $I_{K'}^{H'} D'(K')$ has elements with leading digits other than 0 and 1 since $p-1>1$. Thus, $\mathcal{L}(U')=\mathcal{R}'$. $\square$

Algorithm E1. *Let $G = C_{2p^{\alpha'}} = C_2 \times G'$ where $G' = C_{p^{\alpha'}}$.*

- ***Input**: A saturated G-transfer system $\mathcal{R}$ with $e \to_{\mathcal{R}} C_2$.*
- ***Output**: A G-universe U.*

We define the G-universe U using G'-universes U_0' and U_1' corresponding to the elements in U which are $0 \bmod 2$ and $1 \bmod 2$ respectively.

- *Let U_0' be the G'-universe corresponding to the G'-diagram obtained from Algorithm 2 with input $\mathcal{R}_2^0$.*
- *Let U_1' be the G'-universe corresponding to the G'-diagram obtained from Algorithm $\langle 2 \mid 1\rangle_2^k$ with input $(\mathcal{R}_2^1, \mathcal{R}, 1)$.*

Theorem 5.1. *For all saturated transfer systems $\mathcal{R}$ on $G = C_{2p^{\alpha'}}$ with $e \to_{\mathcal{R}} C_2$, the universe U from Algorithm E1 realizes $\mathcal{R}$. Thus, all saturated transfer systems on G not excluded by Proposition 5.2 are realizable by G-universes.*

Proof. We know from Theorem 4.1 that U_0' realizes $\mathcal{R}_2^0$. First, we show that the saturated transfer system realized by U has the correct 2^k-layers:

- *The 2^0-layer is $\mathcal{R}_2^0$*: It suffices to make sure that the projection of U to $\hat{C}_{p^{\alpha'}}$ is equal to U_0', or equivalently, $U_0' \supseteq U_1'$. This follows from $N_{\mathrm{Alg2}} \supseteq N_2^1$ (which in turn follows from $N_{\mathrm{Alg2}} \supseteq N_{\mathrm{Alg1}}$).
- *The 2^1-layer is $\mathcal{R}_2^1$*: The 2^1-layer is the intersection of the saturated transfer systems realized by U_0' and U_1'. The former realizes $\mathcal{R}_2^0$, and by Proposition 5.4, the latter realizes $\mathcal{R}_2^1$ since the 2^1-partition on $\mathcal{R}$ is non-trivial. By (1) in Proposition 5.1, it follows that the 2^1-layer is equal to $\mathcal{R}_2^0 \cap \mathcal{R}_2^1 = \mathcal{R}_2^1$.

We are left to prove that the 2^1-partition is correct. Let the G'-diagrams corresponding to U_0' and U_1' be equal to D_0' and D_1' respectively. Draw the 2^1-partition of $\mathcal{R}_2^1$ on $\mathrm{Sub}(G')$:

- For all $H' \leqslant G'$ below the 2^1-partition, we have $D_0'(H') = D_1'(H')$ since both diagrams are obtained using Algorithm 2 until the partition.
- For all $H' \leqslant G'$ above the 2^1-partition, we have $D_0'(H') \supsetneq D_1'(H')$. We already have inclusion, and by R-stability, it suffices to prove inequality for H' which are minimal above the partition. By (3) in Proposition 5.1, all such H' are $\mathcal{R}_2^1$-cofibrant. The new elements in $D_0'(H')$ and $D_1'(H')$ are $N_{\mathrm{Alg2}}(H')$ and $N_{\mathrm{Alg1}}(H')$ respectively. We indeed have $N_{\mathrm{Alg2}}(H') \supsetneq N_{\mathrm{Alg1}}(H')$ for all non-trivial $\mathcal{R}_2^1$-cofibrant H'.

Since the subgroups H below the 2^1-partition on the saturated transfer system of U are precisely those subgroups with $D_0'(H') = D_1'(H')$ (where $[H : H'] = 2$), we have the correct 2^1-partition. $\square$

5.4. **The Case $G = C_{2^\alpha p^{\alpha'}}$.** We extend Algorithm E1 to $G = C_{2^\alpha p^{\alpha'}} = C_{2^\alpha} \times G'$ by specifying a G-universe U in terms of the G'-universes $U_0', \dots, U_{2^\alpha-1}'$ corresponding to the elements of U which are $0, \dots, (2^\alpha - 1) \bmod 2^\alpha$ respectively. In addition to mixing algorithms, we will also need to mix layers.

Definition 5.3. Let $G = C_{2^\alpha} \times G'$ where G' is an odd cyclic group and let $\mathcal{R}$ be a saturated transfer system on G. For $0 \leqslant i < j \leqslant \alpha$, the saturated transfer system $\mathcal{R}' = \langle \mathcal{R}_2^i \mid \mathcal{R}_2^j \rangle$ on G' is defined as follows: for subgroups $K' \leqslant H' \leqslant G'$:

- If both K' and H' are below the 2^j-partition on $\mathrm{Sub}(G')$, then $K' \to_{\mathcal{R}'} H'$ if and only if the corresponding arrow exists in $\mathcal{R}_2^i$.
- If both K' and H' are above the 2^j-partition on $\mathrm{Sub}(G')$, then $K' \to_{\mathcal{R}'} H'$ if and only if the corresponding arrow exists in $\mathcal{R}_2^j$.
- If K' is below the 2^j-partition and H' is above the 2^j-partition, then we have $K' \not\to_{\mathcal{R}'} H'$.

Simply put, $\mathcal{R}' = \langle \mathcal{R}_2^i \mid \mathcal{R}_2^j \rangle$ is obtained from $\mathcal{R}_2^j$ by replacing the part below the 2^j-partition with the same part in $\mathcal{R}_2^i$.

Proposition 5.5. *With notation as in Definition 5.3, $\langle \mathcal{R}_2^i \mid \mathcal{R}_2^j \rangle$ is a saturated transfer system on G'.*

Proof. First, we exhibit the restriction axiom:

- When we restrict an arrow below (resp. above) the 2^j-partition to a potential arrow below (resp, above) the 2^j-partition, the restriction axiom follows from the same axiom for $\mathcal{R}_2^i$ (resp. $\mathcal{R}_2^j$).
- When we restrict an arrow above the 2^j-partition to a potential arrow below the 2^j-partition, the restriction axiom follows from the same axiom for $\mathcal{R}_2^j$ combined with $\mathcal{R}_2^j \subseteq \mathcal{R}_2^i$ (which comes from (1) in Proposition 5.1).

Since there are no arrows which cut across the 2^j-partition in $\langle \mathcal{R}_2^i \mid \mathcal{R}_2^j \rangle$, the transitivity and saturation axioms follow from the same axioms for $\mathcal{R}_2^i$ and $\mathcal{R}_2^j$. Finally, the conjugation axiom is trivial for abelian groups. □

Remark 5.2. Constraint (4) in Proposition 5.1 is equivalent to saying that for all $0 < k \leqslant \alpha$, we have $\langle \mathcal{R}_2^{k-1} \mid \mathcal{R}_2^k \rangle = \mathcal{R}_2^k$.

Algorithm E2. *Let $G = C_{2^\alpha p^{\alpha'}} = C_{2^\alpha} \times G'$ where $G' = C_{p^{\alpha'}}$.*

- ***Input**: A saturated G-transfer system $\mathcal{R}$ with $e \to C_2$.*
- ***Output**: A G-universe U.*

We define the G-universe U using G'-universes U_i' for $0 \leqslant i < 2^\alpha$. The elements of U_i' correspond to the elements of U which are $i \bmod 2^\alpha$. Construct the diagram D_i' corresponding to U_i' as given below:

***Value of** i*	***Algorithm***	***Input***
0	*Algorithm 2*	$\mathcal{R}_2^0$
1	*Algorithm $\langle 2 \mid 1 \rangle_2^k$*	$(\mathcal{R}_2^1, \mathcal{R}, 1)$
2^{j-1}	*Algorithm $\langle 2 \mid 1 \rangle_2^k$*	$(\langle \mathcal{R}_2^0 \mid \mathcal{R}_2^j \rangle, \mathcal{R}, j)$
$2^{j-1} < i < 2^j - 1$	*Return Input*	$D'_{i-2^{j-1}}$
$2^j - 1$	*Algorithm $\langle 2 \mid 1 \rangle_2^k$*	$(\mathcal{R}_2^j, \mathcal{R}, 1)$

where j runs over all integers satisfying $2 \leqslant j \leqslant \alpha$.

Remark 5.3. Technically, the case $i = 1$ follows from $j = 1$. We simultaneously have $i = 2^{j-1}$ and $i = 2^j - 1$; both give the right answer, the former by Remark 5.2.

Theorem 5.2. *For all saturated transfer systems $\mathcal{R}$ on $G = C_{2^\alpha p^{\alpha'}}$ with $e \to_{\mathcal{R}} C_2$, the universe U from Algorithm E2 realizes $\mathcal{R}$. Thus, all saturated transfer systems on G not excluded by Proposition 5.2 are realizable by G-universes.*

Proof. The idea is to generalize the argument in the proof of Theorem 5.1. We first show that for all $0 < j \leqslant \alpha$ and $2^{j-1} \leqslant i \leqslant 2^j - 1$, we have $D_i' \subseteq D'_{i-2^{j-1}}$. By construction, we have equality for $2^{j-1} < i < 2^j - 1$, so it suffices to deal with the two extreme cases:

- $i = 2^{j-1}$: We have $\langle \mathcal{R}_2^0 \mid \mathcal{R}_2^j \rangle \subseteq \mathcal{R}_2^0$ by (1) in Remark 5.2. Furthermore, we also have $N_2^j \subseteq N_2$. It follows that $D'_{2^{j-1}} \subseteq D_0$.
- $i = 2^j - 1$: We may assume $j > 1$ since we have already dealt with $i = 1$ in the previous case. Since $\mathcal{R}_2^j \subseteq \mathcal{R}_2^{j-1}$, it follows that $D'_{2^j-1} \subseteq D'_{2^{j-1}-1}$.

Now, for all $0 \leqslant j \leqslant \alpha$, the G'-universes corresponding to the residue classes $0, \ldots, (2^j - 1) \bmod 2^j$ in the projection of U to $\hat{C}_{2^j p^{\alpha'}}$ are precisely $U'_0, \ldots, U'_{2^j-1}$. Thus, we must show that the intersection of the saturated transfer systems realized by these G'-universes is precisely $\mathcal{R}_2^j$. We prove by induction; the base case $j = 0$ follows from Theorem 4.1. For $j > 0$:

- By the induction hypothesis, the intersection of the saturated transfer systems realized by $U'_0, \ldots, U'_{2^{j-1}-1}$ is $\mathcal{R}_2^{j-1}$.
- Since the G'-universes $U'_{2^{j-1}+1}, \ldots, U'_{2^j-2}$ are simply copies of previous G'-universes, they do not affect our calculation.
- By Proposition 5.3, $U'_{2^{j-1}}$ is the empty G'-semi-universe if the 2^j-partition is trivial, in which case it realizes the complete transfer system on G'. Otherwise, it realizes $\langle \mathcal{R}_2^0 \mid \mathcal{R}_2^j \rangle$ by Proposition 5.4.
- By Proposition 5.4, U'_{2^j-1} realizes $\mathcal{R}_2^j$. Note that we are using $e \to C_2$. We indeed get $\mathcal{R}_2^{j-1} \cap \langle \mathcal{R}_2^0 \mid \mathcal{R}_2^j \rangle \cap \mathcal{R}_2^j = \mathcal{R}_2^j$ (or $\mathcal{R}_2^{j-1} \cap \mathcal{R}_2^j = \mathcal{R}_2^j$) as required.

It remains to prove that for all $0 < j \leqslant \alpha$, the 2^j-partitions are realized correctly. A subgroup $H' \leqslant G'$ lies below the 2^j-partition in the saturated transfer system realized by U (technically, the subgroup $C_{2^j} \times H' \leqslant G$) if and only if for all $2^{j-1} \leqslant i \leqslant (2^j - 1)$, we have $D'_i(H') = D'_{i-2^{j-1}}(H')$. We must prove that these are precisely the subgroups below the 2^j-partition on $\mathcal{R}_2^j$:

- For $2^{j-1} < i < (2^j - 1)$, we have $D'_i(H') = D'_{i-2^{j-1}}(H')$ for every subgroup $H' \leqslant G'$ since $D'_i = D'_{i-2^{j-1}}$ by construction.
- For $i = 2^{j-1}$, if the 2^j-partition is trivial, $D'_{2^{j-1}}(H') = \emptyset \subsetneq D'_0(H')$ for all $H' \leqslant G'$. For a non-trivial partition, we have $D'_{2^{j-1}}(H') = D'_0(H')$ for H' below the partition since the layers $\mathcal{R}_2^0$ and $\langle \mathcal{R}_2^0 \mid \mathcal{R}_2^j \rangle$ are identical below the partition and we are applying Algorithm 2 till the partition. We have $D'_{2^{j-1}}(H') \subsetneq D'_0(H')$ for all H' which are minimal after the partition (and hence, all H' after the partition) since $N_{\mathrm{Alg2}}(H') \supsetneq N_{\mathrm{Alg1}}(H')$ for all non-trivial $\mathcal{R}_2^j$-cofibrant H'.
- For $i = 2^j - 1$, we assume $j > 1$ since for $j = 1$, we coincide with $i = 2^{j-1}$. Both D'_{2^j-1} and $D'_{2^{j-1}-1}$ are obtained using Algorithm $\langle 2 \mid 1 \rangle_2^k$ with $\mathcal{R}$ and $k = 1$. Since $\mathcal{R}_2^{j-1}$ and $\mathcal{R}_2^j$ are identical till the 2^j-partition, we have $D'_{2^j-1}(H') = D_{2^{j-1}-1}(H')$ for all H' below the 2^j-partition, and perhaps some subgroups H' above the 2^j-partition.

Thus, for $2^{j-1} < i \leqslant (2^j - 1)$, we have $D'_i(H') = D'_{i-2^{j-1}}(H')$ for all H' below the 2^j-partition, and for $i = 2^{j-1}$, we have equality if and only if H' is below the 2^j-partition. It follows that the 2^j-partition is realized correctly. In conclusion, all the layers and partitions in the saturated transfer system realized by U match with $\mathcal{R}$, proving that U realizes $\mathcal{R}$. $\square$

Remark 5.4. The task of precisely realizing $\mathcal{R}_2^j$ is delegated to U'_{2^j-1} and the task of precisely realizing the 2^j-partition on $\mathcal{R}_2^j$ is delegated to U'_{2^j-1}. For $j = 1$, we can (and must) do this with the same G'-universe because we have the compatibility condition $\langle \mathcal{R}_2^0 \mid \mathcal{R}_2^1 \rangle = \mathcal{R}_2^1$ from Remark 5.2.

5.5. **The General Case.** We shall now generalize Algorithm E2 for general even cyclic groups. Let $G = C_{2^\alpha} \times G'$ where G' is an odd cyclic group. Previously, when G' was a cyclic p-group, the properties we used about Algorithm 2 and Algorithm 1 in the proof of Algorithm E2 to realize a given saturated transfer system $\mathcal{R}$ on G were the following:

- Both Algorithm 2 and Algorithm 1 realize all saturated transfer systems on G' (by Theorem 4.1 and Remark 4.3 respectively).
- For any $\mathcal{R}$-cofibrant H with $\nu_2(|H|) = k > 0$, if $[H : H'] = 2^k$ and H' is non-trivial, then we have $N_{\mathrm{Alg3}}(H') \supsetneq N_{\mathrm{Alg2}}(H')$.

However, for a general odd cyclic group G', Algorithm 1 no longer realizes all saturated transfer systems on G', as shown in Example 4.1. We can try using a new pair of Algorithm N type algorithms that satisfy both the properties above for as many odd cyclic groups G' as possible. We shall use a modified version of the new element we eluded to in Remark 4.4 combined with the one in Algorithm 2 to construct a new algorithm to realize saturated transfer systems on G':

Algorithm 3. *Let G' be an odd cyclic group.*

- ***Input**: A saturated G'-transfer system $\mathcal{R}'$.*
- ***Output**: A G'-diagram D'.*

Let N be the new element G'-diagram defined as follows:

$$N(H) = \begin{cases} \{(p_1^{\beta_1-1}, p_2^{\beta_2-1}, \ldots, p_l^{\beta_l-1}), (\frac{p_1-2}{1-p_1}, \ldots, \frac{p_l-2}{1-p_l})\} & \text{if } H' \text{ is } \mathcal{R}\text{-cofibrant} \\ \emptyset & \text{otherwise} \end{cases}$$

where $\hat{H}' \cong \prod C_{p_i^{\beta_i}}$ and $p^{-1} = *$ in all coordinates with $\beta_i = 0$. Return the diagram D' obtained by executing Algorithm N with $\mathcal{R}'$ and N.

Lemma 5.4. *In the diagram D' obtained from Algorithm 3, for all $\mathcal{R}'$-cofibrant H' and $K' < H'$, we have $R_{K'}^{H'} N(H') \subseteq D(K')$.*

Proof. Use the argument in Lemma 4.2 to show that for all $L' \leqslant G'$ of order $\prod p_i^{\gamma_i}$, the set $D'(L')$ contains all elements of the form $(\chi_1, \ldots, \chi_l)$ where each χ_i can be either $p_i^{\gamma_i-1}$ ($*$ if $\gamma_i = 0$) or 0. The exact same induction argument shows that L' also contains $(\frac{p_1-2}{1-p_1}, \ldots, \frac{p_l-2}{1-p_l})$, the element with all base p_i parts equal to the repunit of the digit $p_i - 2$. The lemma follows from these facts. $\square$

Corollary 5.1. *The diagram D' obtained from Algorithm 3 is non-empty and satisfies $R_{K'}^{H'} D'(H') = D'(K')$ for all $K' \leqslant H' \leqslant G'$. Thus, it comes from the G'-universe $U' = D'(G')$. Furthermore, if $K' \to_{\mathcal{R}'} H'$, then $D'(H') = I_{K'}^{H'} D(K')$.*

Proof. Repeat the argument in Corollary 4.2. $\square$

Corollary 5.2. *For every odd cyclic G' and saturated transfer system $\mathcal{R}'$, the diagram D' from Algorithm 3 realizes $\mathcal{R}'$. Thus, all saturated transfer systems $\mathcal{R}'$ over G' are realizable by G'-universes.*

Proof. Repeat the argument in Proposition 5.4 with the observation that none of the new elements in N_{Alg3} have leading digit $p_i - 1$ in their base p_i part for all primes $p_i \mid |G'|$. □

Algorithm $\langle \mathbf{3} \mid \mathbf{2}\rangle_2^k$. *Let $G = C_{2^\alpha} \times G'$ where G' is an odd cyclic group.*

- ***Input****: A saturated G'-transfer system $\mathcal{R}'$, a saturated G-transfer system $\mathcal{R}$, and an integer k with $0 < k \leqslant \alpha$. We package this data as $(\mathcal{R}', \mathcal{R}, k)$.*
- ***Output****: A G'-diagram D'.*

If the 2^k-partition on $\mathcal{R}_2^k$ has no subgroups below it, then we return the empty diagram. Otherwise, let N_{Alg2} and N_{Alg3} denote the new element G'-diagrams with respect to $\mathcal{R}_2^k$ corresponding to Algorithm 2 and Algorithm 3 respectively. Let N_2^k be the new element G'-diagram where for all $H' \leqslant G'$:

$$N_2^k(H') = \begin{cases} N_{\mathrm{Alg3}}(H') & \text{if } H \text{ is below the } 2^k\text{-partition on } \mathcal{R}_2^k \\ N_{\mathrm{Alg2}}(H') & \text{if } H \text{ is above the } 2^k\text{-partition on } \mathcal{R}_2^k \end{cases}$$

where $H \leqslant G$ is the subgroup on $\mathcal{R}_2^k$ with $[H : H'] = 2^k$. Return the G'-diagram D' obtained by executing Algorithm N with $\mathcal{R}'$ and N_2^k.

Proposition 5.6. *With the notation in Algorithm $\langle 3 \mid 2\rangle_2^k$:*

1. *(1) For all subgroups $K' \leqslant H' \leqslant G'$, the diagram D' obtained from Algorithm $\langle 3 \mid 2\rangle_2^k$ satisfies $R_{K'}^{H'} D'(H') = D'(K')$.*
2. *(2) The diagram D' is empty if the 2^k-partition is trivial and comes from the G'-universe $U' = D'(G')$ otherwise.*
3. *(3) For all subgroups $K' \leqslant H' \leqslant G'$, if $K' \to_{\mathcal{R}'} H'$, then $D'(H') = I_{K'}^{H'} D'(K')$.*

Proof. Repeat the argument in Proposition 5.6. □

Proposition 5.7. *Let $G = C_{2^\alpha} \times G'$ where G' is an odd cyclic group and let $\mathcal{R}$ be a saturated transfer system on G with non-trivial 2^k-partition. For all saturated G' transfer systems $\mathcal{R}'$, the diagram D' from Algorithm $\langle 3 \mid 2\rangle_2^k$ realizes $\mathcal{R}'$.*

Proof. Repeat the argument in Proposition 5.4. □

Lemma 5.5. *Let $G = C_{2^\alpha} \times G'$ where G' is an odd cyclic group and let $\mathcal{R}$ be a saturated transfer system on G. Let $H \leqslant G$ be $\mathcal{R}$-subgroup with $\nu_2(|H|) = k > 0$. Let the subgroup $H' < H$ with $[H : H'] = 2^k$ is non-trivial. We have $N_{\mathrm{Alg3}}(H') \supsetneq N_{\mathrm{Alg2}}(H')$ if and only if $H' \neq C_3$.*

Proof. With $H' < H \leqslant G$ defined as above, we have $N_{\mathrm{Alg3}}(H') \supsetneq N_{\mathrm{Alg2}}(H')$ if and only $(p_1^{\beta_1 - 1}, p_2^{\beta_2 - 1}, \ldots, p_l^{\beta_l - 1}) \neq (\frac{p_1 - 2}{1 - p_1}, \ldots, \frac{p_l - 2}{1 - p_l})$ in $\hat{H}'$.

- We have strict inclusion if $p \mid |H'|$ for some $p > 3$; the leading digits in the base p parts of the two new elements are 1 and $p - 2$, which are unequal.
- We have strict inclusion if $\nu_3(|H'|) > 1$; the last digit in the base 3 parts of the two new elements are 0 and 1, which are unequal.

If we are not in either of the above cases, then $H' = C_3$. In this case, both the new elements are given by 1 in the base 3 part and $*$ in the base p part for every other prime $p \mid |G'|$, so we have $N_{\mathrm{Alg3}}(H') = N_{\mathrm{Alg2}}(H')$. □

Algorithm E3. *Let $G = C_{2^\alpha p^{\alpha'}} = C_{2^\alpha} \times G'$ where $G' = C_{p^{\alpha'}}$.*

- ***Input****: A saturated G-transfer system $\mathcal{R}$ with $e \to C_2$.*
- ***Output****: A G-universe U.*

We define the G-universe U using G'-universes U'_i for $0 \leqslant i < 2^\alpha$. The elements of U'_i correspond to the elements of U which are $i \bmod 2^\alpha$. Construct the diagram D'_i corresponding to U'_i as given below:

Value of i	***Algorithm***	***Input***
0	*Algorithm 3*	$\mathcal{R}_2^0$
1	*Algorithm* $\langle 3 \mid 2\rangle_2^k$	$(\mathcal{R}_2^1, \mathcal{R}, 1)$
2^{j-1}	*Algorithm* $\langle 3 \mid 2\rangle_2^k$	$(\langle \mathcal{R}_2^0 \mid \mathcal{R}_2^j\rangle, \mathcal{R}, j)$
$2^{j-1} < i < 2^j - 1$	*Return Input*	$D'_{i-2^{j-1}}$
$2^j - 1$	*Algorithm* $\langle 3 \mid 2\rangle_2^k$	$(\mathcal{R}_2^j, \mathcal{R}, 1)$

where j runs over all integers satisfying $2 \leqslant j \leqslant \alpha$.

Theorem 5.3. *Let $G = C_{2^\alpha} \times G'$ with G' an odd cyclic group. Let $\mathcal{R}$ be a saturated transfer system on G with $e \to_{\mathcal{R}} C_2$. The universe U from Algorithm E3 realizes $\mathcal{R}$ if and only if at least one of the following hold true:*

(1) The order of G (equivalently, G') is not divisible by 3.
(2) For all $0 < k \leqslant \alpha$, the subgroup $H' = C_3$ is below the 2^k-partition on $\mathcal{R}_2^k$, i.e. we have $C_{2^{k-1}.3} \to_{\mathcal{R}} C_{2^k.3}$.
(3) The subgroup $H = C_3$ is not $\mathcal{R}$-cofibrant, i.e. $e \to_{\mathcal{R}} C_3$.

Proof. The same argument from Theorem 5.2 applies with the exception of showing that $D'_{2^{j-1}}$ realizes the 2^j-partition accurately. For a non-trivial 2^j-partition and $H' \leqslant G'$ minimal above the 2^j-partition, we need $D'_{2^{j-1}}(H') \subsetneq D'_0(H')$. By Lemma 5.5, this is true for $H' \neq C_3$.

Furthermore, if $H' = C_3$, we have $D'_{2^{j-1}}(C_3) = \{0, 1\} \subseteq \hat{C}_3$, so we still have strict inclusion if $D'_0(C_3) = \hat{C}_3$. Since D'_0 realises $\mathcal{R}_2^0$ by construction, we have $D'_0(C_3) = \hat{C}_3$ if and only if $e \to_{\mathcal{R}} C_3$. Thus, Algorithm $\langle 3 \mid 2\rangle_2^k$ realizes $\mathcal{R}$ if and only if one of the given conditions hold. □

Corollary 5.3. *For all cyclic groups G with order not divisible by 3, all saturated transfer systems on G are realizable by linear isometries operads of G-universes.*

Remark 5.5. We cannot make Algorithm $\langle 3 \mid 2\rangle_2^k$ work for a larger class of cyclic groups by replacing Algorithm 3 and Algorithm 2 with a different pair of algorithms. The obstruction at C_3 exists because $N(C_3)$ must contain at least 1 element for the smaller algorithm, hence at least 2 elements for the bigger algorithm, but $\varphi(3) = 2$.

Remark 5.6. The analogous results over $\mathbb{R}$ (rf. Remark 3.2) are as follows. Let G be a cyclic group with order divisible by 3.

- The necessary condition Proposition 5.2 with 2 replaced by 3 is required for $\mathcal{R}$ to be realizable.
- For all cyclic groups G with order divisible by neither 2 nor 5, all saturated transfer systems over G satisfying the necessary condition are realizable.
- Furthermore, for all odd cyclic groups G, if $\mathcal{R}$ is a G-saturated transfer system with $e \to_{\mathcal{R}} C_5$, then $\mathcal{R}$ is realizable.

6. Local Obstructions

Yet to complete

7. Compatible Pairs for Cyclic Groups

Throughout this report, we shall use the term *compatible pairs* to refer to pairs $(\mathcal{R}_m, \mathcal{R}_a)$ satisfying the compatibility axiom in Definition 1.3 such that $\mathcal{R}_m$ **and** $\mathcal{R}_a$ **are saturated and cosaturated respectively**. Indeed, by Appendix A, any pair of transfer systems realized by $(\mathcal{L}(U), \mathcal{K}(U))$ must have this form.

Lemma 7.1. *Let G be an abelian group, U be a G-universe with corresponding diagram D, and $\mathcal{R}_a$ be the cosaturated transfer system realized by $\mathcal{K}(U)$. For any $H \leqslant G$, the subgroups $K \leqslant H$ with $K \to_{\mathcal{R}_a} H$ are precisely those of the form $\cap_{\chi\in S} \ker\chi$ for some $S \subseteq D(H)$.*

Proof. Pick a basis for U such that each basis element spans a 1-dimensional G-subrepresentation. Any vector $v \in R^G_H U$ can be written as a non-trivial linear combination of basis elements. Let $S \subseteq D(H)$ be the set of $\chi \in D(H)$ for which v has a non-zero component on a basis element which spans a copy of χ. We can see that $\mathrm{Stab}(v) = \cap_{\chi\in S} \ker\chi$. Since any subset $S \subseteq D(H)$ can be obtained by some appropriate choice of $v \in R^G_H U$, the lemma follows. $\square$

7.1. **Cyclic p-groups.** We start with the simple case of $G = C_{p^\alpha}$. It is almost true that all compatible pairs are realizable, but there is a tiny caveat.

Example 7.1. Let $G = C_2$ and consider the compatible pair $(\mathcal{R}_m, \mathcal{R}_a)$ with $\mathcal{R}_m$ empty and $\mathcal{R}_a$ complete. The former forces $U = \{0\}$ but the latter forces $U = \hat{C}_2$, so the pair is not realizable.

Proposition 7.1. *Let $G = C_{2^\alpha}$ and let $(\mathcal{R}_m, \mathcal{R}_a)$ be a compatible pair on G. Let β be minimal such that $K = C_{2^\beta}$ is $\mathcal{R}_a$-fibrant. If $\beta < \alpha$ and $H = C_{2^{\beta+1}}$ is $\mathcal{R}_m$-cofibrant, then $(\mathcal{R}_m, \mathcal{R}_a)$ is not realizable.*

Proof. Assume for the sake of contradiction that some G-universe U realizes the given compatible pair. Let the corresponding diagram be D. By the minimality of K, no proper subgroup of K has an arrow to K in $\mathcal{R}_a$. By Lemma 7.1, we must have $D(K) = \{0\}$. Since H is $\mathcal{R}_m$-cofibrant, we must have $D(H) = \{0\}$. By Lemma 7.1, this would imply that $K \not\to_{\mathcal{R}_a} H$. However, this is a contradiction since $K \to_{\mathcal{R}_a} H$ holds by the restriction axiom. $\square$

We shall construct an algorithm which realizes all compatible pairs not excluded by Proposition 7.1. The idea is to use Algorithm N with a new element diagram tailored to realize the required cosaturated transfer system.

Algorithm P1. *Let $G = C_{p^\alpha}$ be a cyclic p-group.*

- ***Input****: A G-compatible pair $(\mathcal{R}_m, \mathcal{R}_a)$.*
- ***Output****: A G-diagram D.*

Define the new element G-diagram N by:

$$N(H) = \begin{cases} \{*\} & \text{if } H = e \\ \{p^{\beta-1}\} & \text{if } H \text{ is } \mathcal{R}_m\text{-cofibrant and } K \to_{\mathcal{R}_a} H \\ \emptyset & \text{otherwise} \end{cases}$$

where $H = C_{p^\beta}$ and $K = C_{p^{\beta-1}}$. Return the diagram D obtained by executing Algorithm N with $\mathcal{R}_m$ and N.

Theorem 7.1. *Let $G = C_{p^\alpha}$ and $(\mathcal{R}_m, \mathcal{R}_a)$ be any compatible pair. The pair is realized by Algorithm P1 if and only if it is not excluded by Proposition 7.1.*

Proof. The diagram D satisfies $R^H_K D(H) = D(K)$ by Frobenius reciprocity on J, so it comes from a G-universe U. Let $(\mathcal{L}(U), \mathcal{K}(U))$ realize the compatible pair $(\mathcal{R}'_m, \mathcal{R}'_a)$, we wish to show this is equal to $(\mathcal{R}_m, \mathcal{R}_a)$.

First, we show that $\mathcal{R}'_a = \mathcal{R}_a$; our strategy will be to prove for all subgroups H that $(- \to_{\mathcal{R}_a} H) = (- \to_{\mathcal{R}'_a} H)$ by induction. The base case $H = e$ is trivial. For non-trivial H, let K be the maximal subgroup of H. For any cosaturated $\mathcal{R}''_a$ and proper subgroup $M < K$, the statements $M \to_{\mathcal{R}''_a} K$, $M \to_{\mathcal{R}''_a} H$, and $M \to_{\mathcal{R}''_a} G$ are equivalent. Indeed, this follows from the construction of $\mathcal{R}''_a$ from its fibrant subgroups in the proof of Theorem 2.1. Taking $\mathcal{R}''_a = \mathcal{R}_a, \mathcal{R}'_a$, by the induction hypothesis, it suffices to check that $K \to_{\mathcal{R}_a} H$ if and only if $K \to_{\mathcal{R}'_a} H$.

If $K \to_{\mathcal{R}_a} H$, then we have $p^{\beta-1} \in D(H)$ whether or not H is $\mathcal{R}_m$-cofibrant; in the former case, because $p^{\beta-1} \in N(H)$, and in the latter case, since $p^{\beta-1} \in I^H_K(0)$. By Lemma 7.1, we get $K \to_{\mathcal{R}'_a} H$. On the other hand, if $K \not\to_{\mathcal{R}_a} H$, then $D(H)$ has no new elements and the only representation in $D(H)$ vanishing on K must be 0, which has kernel H. Thus, $K \not\to_{\mathcal{R}'_a} H$, proving that $\mathcal{R}'_a = \mathcal{R}_a$.

Next, we show by induction that $\mathcal{R}'_m = \mathcal{R}_m$. By construction, we already know that $\mathcal{R}_m \subseteq \mathcal{R}'_m$, so by Proposition 2.2, it suffices to show that every $\mathcal{R}_m$-cofibrant H is also $\mathcal{R}'_m$-cofibrant. Let L be the minimal $\mathcal{R}_a$-fibrant subgroup; by the definition of N, for all $H \leqslant L$, we have $D(H) = \{0\}$, so any such H is $\mathcal{R}'_m$-cofibrant.

If $H > L$ is $\mathcal{R}_m$-cofibrant with order p^α and maximal subgroup K, then $D(H) = J^H_K D(K) \cup \{p^{\beta-1}\}$ has exactly 1 new element. We know $I^H_K D(K)$ has $(p-1)|D(K)|$ new elements, and this is greater than 1 (and H is $\mathcal{R}'_m$-cofibrant) unless $p = 2$ and $D(K) = \{0\}$. Since $L \to_{\mathcal{R}_a} K$, this forces $L = K$, leading us to the conditions of Proposition 7.1. Thus, Algorithm P1 works away from the compatible pairs ruled out by Proposition 7.1. $\square$

7.2. **Partial Subinductors.** In general, it is difficult to use the standard subinductor J to realize a compatible pair $(\mathcal{R}_m, \mathcal{R}_a)$ because $\mathcal{R}_a$ imposes restrictions on the representations we may use. We shall get around this difficulty by instead using partial subinductors:

Definition 7.1. Let $\mathfrak{D}$ be an R-stable G-diagram. A *partial subinductor* $\mathfrak{J}$ with respect to the diagram $\mathfrak{D}$ consists of the data of a join-semilattice homomorphisms $\mathfrak{J}^H_K : \mathcal{P}(\mathfrak{D}(K)) \to \mathcal{P}(\mathfrak{D}(H))$ for each $K \leqslant H \leqslant G$ satisfying:

- *Transitivity*: For all $K \leqslant M \leqslant H$, we have $\mathfrak{J}^H_M \mathfrak{J}^M_K = \mathfrak{J}^H_K$.
- *Identity*: For all $K \leqslant H$, we have $0_H \in \mathfrak{J}^H_K(0_K)$.
- *Frobenius Reciprocity*: For all $K \leqslant H$, we have $R^H_K \mathfrak{J}^H_K = \mathrm{id}_{\mathcal{P}(\mathfrak{D}(K))}$.
- *Mackey's Formula*: For all $K, L \leqslant H$ and all $S \subseteq \mathfrak{D}(K)$, we have $R^H_L \mathfrak{J}^H_K S \subseteq \mathfrak{J}^L_{K\cap L} R^K_{K\cap L} S$.

Thus, a subinductor on G is a partial subinductor with $\mathfrak{D}$ equal to the complete diagram on G. Given a cyclic group G with order divisible by neither 2 nor 3 and a cosaturated transfer system $\mathcal{R}_a$, we shall use the following partial subinductor:

Definition 7.2. Let $\mathcal{R}_a$ be a cosaturated transfer system:

- The G-diagram $\mathfrak{D}$ corresponding to $\mathcal{R}_a$ is defined as follows: for any $H \leqslant G$, we have $\mathfrak{D}(H) = \{\chi : \ker \chi \to_{\mathcal{R}_a} H\}$.

- The partial subinductor $\mathfrak{J}$ corresponding to $\mathcal{R}_a$ (with respect to $\mathfrak{D}$) is defined as follows: for any $K \leqslant H \leqslant G$ and $\psi \in \mathfrak{D}(K)$, the set $\mathfrak{J}_K^H\psi$ contains all $\chi \in I_K^H\chi \cap \mathfrak{D}(H)$ such that for all primes $p \mid |H|$, all the new digits in the base p part of χ (the leading $\nu_p(|H|) - \nu_p(|K|)$ digits) are 0's and 1's.

Example 7.2. Let $G = C_{36} = C_{2^2} \times C_{3^2}$ and $\mathcal{R}_a$ be the cosaturated transfer system on G with fibrant subgroups $e, C_2, C_3, C_6, C_9, C_{12}, C_{36}$. For the partial subinductor $\mathfrak{J}$ corresponding to $\mathcal{R}_a$:

- If $K = C_6$ and $H = C_{36}$, then $\mathfrak{J}_K^H((0,0)) = \{(00,00),(00,10),(10,10)\}$ (equivalently, $\mathfrak{J}_K^H(0) = \{0,12,30\}$). Note that we do not include $(10,00)$ because its kernel is $C_2 \times C_{3^2} = C_{18}$, which is not fibrant.
- If $K = C_9$ and $H = C_{36}$, then $(*,21) \in \mathfrak{D}(K)$ since its kernel is e. We have $\mathfrak{J}_K^H((*,21)) = \{(10,21),(01,21),(11,21)\}$. Note that we do not include $(00,21)$ because its kernel is $C_{2^2} = C_4$, which is not fibrant.
- If $K = e$ and $H = C_4$, then $\mathfrak{J}_K^H((*,*)) = \{(00,*),(10,*),(01,*),(11,*)\}$ since all subgroups $L \leqslant H$ satisfy $L \to_{\mathcal{R}_a} H$.

Lemma 7.2. *The diagram $\mathfrak{D}$ corresponding to $\mathcal{R}_a$ is R-stable.*

Proof. For any $K \leqslant H$, let $\chi \in \mathfrak{D}(H)$ be arbitrary and let $L = \ker\chi$. By the construction of $\mathfrak{D}$, we have $L \to_{\mathcal{R}_a} H$. By restriction, we have $P \to_{\mathcal{R}_a} K$, where $P = K \cap L$. Define $\psi = R_K^H\psi$; we also know that $\ker\psi = P$, so $\psi \in \mathfrak{D}(K)$. Thus, the diagram $\mathfrak{D}$ is R-stable. $\square$

We are yet to prove that $\mathfrak{J}$ is indeed a partial subinductor. In order to do so, we shall first need the following important lemma:

Lemma 7.3. *Let $\mathcal{R}_a$ be a cosaturated transfer system with corresponding diagram $\mathfrak{D}$ and partial subinductor $\mathfrak{J}$. Let $K \leqslant H$ be any subgroups and $\psi \in \mathfrak{D}(K)$ be any element; say $\ker\psi = P$. Let $L \leqslant H$ be any subgroup with $L \to_{\mathcal{R}_a} H$ and $K \cap L = P$. Then, there exists some $\chi \in \mathfrak{J}_K^H\psi$ with $\ker\chi = L$.*

Proof. We construct the required χ explicitly by specifying the new digits in its base p part for each prime $p \mid |H|$.

- If the base p part of ψ is zero (possibly empty), we have $\nu_p(|P|) = \nu_p(|H|)$. In this case, we add the digits $11\cdots1100\cdots00$ where the number of 0's is equal to $\nu_p(|L|) - \nu_p(|P|)$ and the number of 1's is equal to $\nu_p(|H|) - \nu_p(|L|)$.
- If the base p part of ψ is non-zero, we add any combination of 0's and 1's. For instance, we may just add all $\nu_p(|H|) - \nu_p(|K|)$ digits as 0's.

We need to make sure that $\nu_p(|\ker\chi|) = \nu_p(|L|)$. In the first case, the number of trailing 0's in χ is equal to $\nu_p(|L|) - \nu_p(|P|) + \nu_p(|K|) = \nu_p(|L|)$. In the second case, the number of trailing 0's in χ is equal to $\nu_p(|P|) = \min(\nu_p(|K|), \nu_p(|L|)) = \nu_p(|L|)$; note that $\nu_p(|P|) < \nu_p(|L|)$ since the base p part of ψ is non-zero. $\square$

Proposition 7.2. *The $\mathfrak{J}$ corresponding to $\mathcal{R}_a$ is a partial subinductor.*

Proof. We must verify the following details about our construction:

- *Frobenius Reciprocity*: Let $K \leqslant H$. For all $\psi \in \mathfrak{D}(K)$ with $\ker\psi = P$, we must find some $\chi \in \mathfrak{D}(H)$ with $\ker\chi = L$ satisfying $K \cap L = P$. By Lemma 7.3, it suffices to find $L \leqslant H$ with $L \to_{\mathcal{R}_a} H$ and $K \cap L = P$. By the construction in Theorem 2.1, there exists some $M \leqslant G$ with $M \to_{\mathcal{R}_a} G$ and $M \cap K = P$, we may choose $L = M \cap H$ by the restriction axiom.

- *Identity*: For $K \leqslant H$, we have $0_K \in \mathfrak{D}(K)$ and $0_H \in \mathfrak{D}(H)$ since every subgroup has an $\mathcal{R}_a$-arrow to itself. It is also clear that $0_H \in \mathfrak{I}_K^H 0_K$ since every new digit in each base p part is 0.
- *Transitivity*: Let $K \leqslant M \leqslant H$ and $\psi \in \mathfrak{D}(K)$. Clearly, $\mathfrak{I}_M^H \mathfrak{I}_K^M \psi \subseteq \mathfrak{I}_K^H \psi$. For the reverse inclusion, we must show $R_M^H \mathfrak{I}_K^H \psi \subseteq \mathfrak{I}_K^M \chi$. If $\chi \in \mathfrak{I}_K^H \psi$, then $\ker \chi \to_{\mathcal{R}_a} H$ implies that $\ker R_M^H \chi \to_{\mathcal{R}_a} M$ by the restriction axiom. The condition that all new digits are 0's and 1's also holds.
- *Mackey's Formula*: For $K, L \leqslant H$ and $\psi \in \mathfrak{D}(H)$, any $\rho \in R_L^H \mathfrak{I}_K^H \psi$ satisfies the following properties:
  - By Mackey's formula for I, we have $\rho \in I_{K\cap L}^L R_{K\cap L}^K \psi$
  - By Lemma 7.2, we have $\rho \in R_L^H \mathfrak{D}(H) \subseteq \mathfrak{D}(L)$
  - For any $p \mid |L|$, if $\nu_p(|L|) > \nu_p(|K \cap L|)$, then $\nu_p(|K \cap L|) = \nu_p(|K|)$. Any element in $\mathfrak{I}_K^H \psi$ has its first $\nu_p(|H|) - \nu_p(|K|)$ digits in the base p part as 0's and 1's, so same holds for the first $\nu_p(|L|) - \nu_p(|K \cap L|)$ digits in the base p part of ρ.

  Combining these three facts, we see that $\rho \in \mathfrak{I}_{K\cap L}^L R_{K\cap L}^K \chi$.

Thus, $\mathfrak{I}$ is a partial subinductor with respect to $\mathfrak{D}$. $\square$

Lemma 7.4. *Let $(\mathcal{R}_m, \mathcal{R}_a)$ be a compatible pair. Let $\mathfrak{D}$ and $\mathfrak{I}$ be the diagram and partial subinductor corresponding to $\mathcal{R}_a$. If $K \to_{\mathcal{R}_m} H$, then $\mathfrak{D}(H) = I_K^H \mathfrak{D}(K)$.*

Proof. We already know that $\mathfrak{D}(H) \subseteq I_K^H \mathfrak{D}(K)$ by R-stability. For the reverse inclusion, assume that $\psi \in \mathfrak{D}(K)$ and $\chi \in I_K^H \psi$. If $\ker \psi = P$ and $\ker \chi = L$, then we have $K \cap L = P$. We know that $K \to_{\mathcal{R}_m} H$ and $K \cap L \to_{\mathcal{R}_a} K$. By compatibility, it follows that $L \to_{\mathcal{R}_a} H$, so $\chi \in \mathfrak{D}(H)$. $\square$

7.3. **Obstruction to Realizability.** When G was a cyclic p-group for $p > 2$, we saw that all compatible pairs on G are realizable using Steiner and linear isometries operads. However, this is not true for general cyclic groups G, even if the prime divisors of $|G|$ are taken arbtirarily large.

Example 7.3. Consider the following compatible pair on $G = C_{pq} = C_p \times C_q$:

$$\begin{tikzcd} C_q & C_{pq} \\ e \arrow[r, "\mathcal{R}_m"] & C_p \end{tikzcd} \qquad \begin{tikzcd} C_q \arrow[r, dashed, "\mathcal{R}_a"] & C_{pq} \\ e \arrow[r, dashed, "\mathcal{R}_a"] & C_p \end{tikzcd}$$

Assume for the sake of contradiction that $(\mathcal{R}_m, \mathcal{R}_a)$ can be realized by some universe U with corresponding diagram D. Starting with $D(e) = \{(*, *)\}$:

- Since $e \to_{\mathcal{R}_m} C_p$, we have $D(C_p) = \{(d, *) : d \in C_p\}$.
- Since $e \not\to_{\mathcal{R}_a} C_q$, all $\chi \in D(C_q)$ have $\ker \chi = C_q$, so $D(C_q) = \{(*, 0)\}$.
- From above, we must have $D(C_{pq}) = \{(d, 0) : d \in C_p\}$.

However, this leads us to a contradiction since the given choice of D would also yield an arrow $C_q \to_{\mathcal{R}_m} C_{pq}$, which is not actually present in $\mathcal{R}_m$.

The obstruction we are facing is similar to what went wrong in Example 4.1, but this time, our hands were tied on our choice for $D(C_q)$ because of the cosaturated transfer system $\mathcal{R}_a$. This obstruction generalizes as follows:

Proposition 7.3. *Let G be a cyclic group and $(\mathcal{R}_m, \mathcal{R}_a)$ be a compatible pair. Let L be the minimal $\mathcal{R}_a$-fibrant subgroup of G; such a subgroup exists since $\mathrm{fib}(\mathcal{R}_a)$ is closed under intersections. Let p be any prime and $\beta = \nu_p(|L|)$:*

(1) For all $0 < k \leqslant \beta$, the p^k-partition in $\mathcal{R}_m$ is trivial.
(2) Furthermore, if $(\mathcal{R}_m, \mathcal{R}_a)$ is realizable, then $(\mathcal{R}_m)_p^0 = \cdots = (\mathcal{R}_m)_p^\beta$.

Proof. For (1), assume for the sake of contradiction that we have $C_{p^{k-1}} \to_{\mathcal{R}_m} C_{p^k}$ for some $0 < k \leqslant \beta$. By Remark 1.1, we also have $C_{p^{k-1}} \to_{\mathcal{R}_a} C_{p^k}$. By the construction in Theorem 2.1, we know that $C_{p^{k-1}} = C_{p^k} \cap M$ for some $\mathcal{R}_a$-fibrant M. However, this is a contradiction since $C_{p^k} \leqslant L \leqslant M$.

For (2), let U be a G-universe with $(\mathcal{L}(U), \mathcal{K}(U))$ realizing the pair $(\mathcal{R}_m, \mathcal{R}_a)$. Let the G-diagram corresponding to U be D. For all $0 \leqslant k \leqslant \beta$, no proper subgroup of C_{p^k} has an $\mathcal{R}_a$-arrow to C_{p^k} by the argument above, so $D(C_{p^k}) = \{0\}$. By the same argument in Proposition 5.2, we must have $(\mathcal{R}_m)_p^0 = \cdots = (\mathcal{R}_m)_p^\beta$. □

Lemma 7.5. *Let G be a cyclic group and $(\mathcal{R}_m, \mathcal{R}_a)$ be a G-compatible pair satisfying the necessary condition for realizability in Proposition 7.3. Let L be the minimal $\mathcal{R}_a$-fibrant subgroup of G. Let $\pi : G \to G''$ be the quotient map with kernel L and let $(\mathcal{R}''_m, \mathcal{R}''_a)$ be the G''-compatible pair defined such that for all $K'' \leqslant H'' \leqslant G''$ with $K = \pi^{-1}(K'')$ and $H = \pi^{-1}(H'')$:*

- *$K'' \to_{\mathcal{R}''_m} H''$ if and only if $K \to_{\mathcal{R}_m} H$.*
- *$K'' \to_{\mathcal{R}''_a} H''$ if and only if $K \to_{\mathcal{R}_a} H$.*

The G-compatible pair $(\mathcal{R}_m, \mathcal{R}_a)$ is realizable by a G-universe if and only if the G''-compatible pair $(\mathcal{R}''_m, \mathcal{R}''_a)$ is realizable by a G''-universe.

Proof. We use the same argument in Lemma 5.2. If a G-universe U realizes the pair $(\mathcal{R}_m, \mathcal{R}_a)$ and has corresponding diagram D, then $D(L) = \{0\}$. Then, we have $U = \pi^*(U'')$ for some G''-universe U''; the pair $(\mathcal{R}''_m, \mathcal{R}''_a)$ is realized by U''. Conversely, if U'' realizes $(\mathcal{R}''_m, \mathcal{R}''_a)$, then $U = \pi^*(U'')$ realizes $(\mathcal{R}_m, \mathcal{R}_a)$. □

Using Lemma 7.5, it suffices to consider compatible pairs on cyclic groups where the trivial subgroup e is $\mathcal{R}_a$-fibrant. Since $L = e$, the necessary condition in Proposition 7.3 is trivial. Furthermore, this allows us to use Algorithm N-type algorithms using diagrams N consisting of representations with trivial kernel.

7.4. **Cyclic Groups away from 2 and 3.** Let G be a cyclic group with $|G|$ divisible by neither 2 nor 3. We shall prove that all compatible pairs over G satisfying the necessary condition in Proposition 7.3 are realizable. As mentioned above, we may assume that e is $\mathcal{R}_a$-fibrant.

Algorithm P2. *Let G be a cyclic group with order divisible by neither 2 nor 3.*

- ***Input**: A G-compatible pair $(\mathcal{R}_m, \mathcal{R}_a)$ with $e \to_{\mathcal{R}_a} G$.*
- ***Output**: A G-diagram D.*

Let N be the G-diagram given by:

$$N(H) = \begin{cases} \{(\frac{2}{1-p_1}, \cdots, \frac{2}{1-p_l})\} & \text{if } H \text{ is } \mathcal{R}_m\text{-cofibrant} \\ \emptyset & \text{otherwise} \end{cases}$$

Return the G-diagram D obtained by executing Algorithm N with input $\mathcal{R}_m$ after replacing the subinductor J with the partial subinductor $\mathfrak{J}$ corresponding to $\mathcal{R}_a$.

Theorem 7.2. *The diagram D obtained from Algorithm P2 comes from a G-universe U and the pair $(\mathcal{R}_m, \mathcal{R}_a)$ is realized by $(\mathcal{L}(U), \mathcal{K}(U))$.*

Proof. First, we shall prove inductively that for all $H \leqslant G$, the subgroups of H which are kernels of elements in $D(H)$ are precisely the subgroups $L \leqslant H$ with $L \to_{\mathcal{R}_a} H$. This will simultaenously prove that Algorithm P2 is well-defined since $D(H) \subseteq \mathfrak{D}(H)$ and that the diagram D realizes $\mathcal{R}_a$. The base case is trivial. For any non-trivial subgroup H, let $K < H$ be any maximal subgroup:

- If $K \not\to_{\mathcal{R}_m} H$, then we clearly have $\mathfrak{J}_K^H D(K) \subseteq \mathfrak{J}_K^H \mathfrak{D}(K) \subseteq \mathfrak{D}(H)$. Else, if $K \to_{\mathcal{R}_m} H$, then we have $I_K^H D(K) \subseteq I_K^H \mathfrak{D}(K) = \mathfrak{D}(H)$ by Lemma 7.4. Finally, we also have $N(H) \subseteq \mathfrak{D}(H)$ since all elements in $N(H)$ have kernel equal to e, which satisfies $e \to_{\mathcal{R}_a} H$ by restriction. Thus, $D(H) \subseteq \mathfrak{D}(H)$.
- On the other hand, let $L \leqslant H$ with $L \to_{\mathcal{R}_a} H$. If $P = K \cap L$, we have $P \to_{\mathcal{R}_a} K$ by the restriction axiom. By the induction hypothesis, there exists $\psi \in D(K)$ with $\ker \psi = P$. Thus, by Lemma 7.3, there exists some $\chi \in \mathfrak{J}_K^H D(K) \subseteq D(H)$ such that $\ker \chi = L$.

Next, we prove that the diagram D corresponds to a universe U by using the same idea outlined in Remark 4.4. For any subgroup $H \leqslant G$, it suffices to show that $(\frac{2}{1-p_1}, \dots, \frac{2}{1-p_l}) \in D(H)$. We may do so by induction.

- The claim is trivially true for $\mathcal{R}_m$-cofibrant H since the set $N(H)$ contains the required element and $N(H) \subseteq D(H)$.
- For non-cofibrant H, we have some maximal $K < H$ with $K \to_{\mathcal{R}_m} H$. By the induction hypothesis, the required element lies in $I_K^H D(K) \subseteq D(H)$.

We have already shown that $\mathcal{K}(U)$ realizes $\mathcal{R}_a$, so it remains to show that $\mathcal{L}(U)$ realizes $\mathcal{R}_m$. Assume that $\mathcal{L}(U)$ realizes $\mathcal{R}'_m$. It is clear that $\mathcal{R}'_m \supseteq \mathcal{R}_m$ by construction. For the reverse inclusion, by Proposition 2.2, it suffices to prove that every $\mathcal{R}_m$-cofibrant subgroup is $\mathcal{R}'_m$-cofibrant.

Indeed, assume for the sake of contradiction that H is $\mathcal{R}_m$-cofibrant and there exists maximal $K < H$ with $K \to_{\mathcal{R}'_m} H$. The element $(\frac{2}{1-p_1}, \dots, \frac{2}{1-p_l}) \in D(H)$ has all digits equal to 2 in each base p part. Consider the prime $p = [H : K]$; since $D(H) = I_K^H D(K)$, changing the leading digit of the aforementioned element in the base p part to 3 should still give an element in $D(H)$. Note that 3 is a digit since $p > 3$. However, this element does not come from $N(H)$ and it cannot come from $\mathfrak{J}$ since none of its leading digits are 0 or 1, yielding the required contradiction. □

Remark 7.1. The analogous result over $\mathbb{R}$ (rf. Remark 3.2) is that for all cyclic groups G with order divisible by none of 2, 3, and 5, all G-compatible pairs satisfying the necessary condition in Proposition 7.3 are realizable.

7.5. Odd Cyclic Groups with Order Divisible by 3.

We can slightly modify Algorithm P2 in order to realize certain compatible pairs for cyclic groups with order not divisible by 2 but divisible by 3. We denote the primes dividing $|G|$ by $p_0, p_1, \dots, p_l$ where $p_0 = 3$.

Algorithm P3. *Let G be an odd cyclic group with order divisible by* 3.

- ***Input***: *A G-compatible pair $(\mathcal{R}_m, \mathcal{R}_a)$ with $e \to_{\mathcal{R}_a} G$.*
- ***Output***: *A G-diagram D.*

Let N be the G-diagram given by:

$$N(H) = \begin{cases} \{(\frac{2}{1-p_0}, \frac{2}{1-p_1}, \cdots, \frac{2}{1-p_l}), (\frac{1}{1-p_0}, \frac{3}{1-p_1}, \cdots, \frac{3}{1-p_l})\} & \text{if } H \text{ is } \mathcal{R}_m\text{-cofibrant} \\ \emptyset & \text{otherwise} \end{cases}$$

Return the G-diagram D obtained by executing Algorithm N with input $\mathcal{R}_m$ after replacing the subinductor J with the partial subinductor $\mathfrak{J}$ corresponding to $\mathcal{R}_a$.

Theorem 7.3. *The diagram D obtained from Algorithm P3 comes from a G-universe U. The universe U realizes $(\mathcal{R}_m, \mathcal{R}_a)$ if and only if $e \to_{\mathcal{R}_m} C_3$.*

Proof. The proof in Theorem 7.2 applies with obvious modifications regarding $N(H)$ until the step where we are left to show that every $\mathcal{R}_m$-cofibrant H is also $\mathcal{R}'_m$-cofibrant. First, H cannot receive a p-arrow for $p \neq 3$ since changing the leading digit in the base p-part of $(\frac{2}{1-p_0}, \frac{2}{1-p_1}, \cdots, \frac{2}{1-p_l}) \in D(H)$ to 4 gives an element that is not in $N(H)$ and that does not come from $\mathfrak{J}$. Note that 4 is a valid digit in base p since $p > 4$.

Now, assume that H receives a 3-arrow. If we change the leading digit in the base 3-part of $(\frac{1}{1-p_0}, \frac{3}{1-p_1}, \cdots, \frac{3}{1-p_l})$ to 2, then the element χ formed cannot come from $\mathfrak{J}$ since none of its leading digits are 0 or 1; it must thus come from $N(H)$. Since the leading digit in the base 3 part is 2, we must have $\chi = (\frac{2}{1-p_0}, \frac{2}{1-p_1}, \cdots, \frac{2}{1-p_l})$.

- Comparing base 3-parts forces $\nu_3(|H|) = 1$.
- Comparing base p-parts for $p \neq 3$ forces $\nu_p(|H|) = 0$.

Thus, we must have $H = C_3$. This would be a contradiction if $e \to_{\mathcal{R}_m} C_3$. Otherwise, $\mathcal{R}_m$ is indeed not realized since $D(C_3) = \hat{C}_3$. □

Remark 7.2. The analogous result over $\mathbb{R}$ (rf. Remark 3.2) is that for all cyclic groups G with order divisible by none of 2, 3, and 7, all G-compatible pairs with $e \to_{\mathcal{R}_a} G$ and $e \to_{\mathcal{R}_m} C_5$ are realizable. We must exclude the prime 7 since the argument in Theorem 7.3 uses the bound $p > 4$ for primes $p \neq 3$, and after digit-gluing, this turns into the bound $p > 7$ for primes $p \neq 5$.

7.6. **Examples.** Consider the group $G = C_{15}$; we prove that all compatible pairs satisfying Proposition 7.3 are realizable.

- By Lemma 7.5, we may assume that e is $\mathcal{R}_a$-fibrant.
- By Theorem 7.3, it suffices to consider the examples with $e \not\to_{\mathcal{R}_m} C_3$.

$$\begin{tikzcd} C_5 & C_{15} \\ e & C_3 \end{tikzcd} \qquad \begin{tikzcd} C_5 & C_{15} \\ e \arrow[u] & C_3 \end{tikzcd} \qquad \begin{tikzcd} C_5 & C_{15} \\ e \arrow[u] & C_3 \arrow[u] \end{tikzcd}$$

The above three diagrams include all possibilities for $\mathcal{R}_m$. In the first two cases, $\mathcal{R}_a$ has all 4 possibilities: each of C_3 and C_5 can either be $\mathcal{R}_a$-fibrant or not. In the last case, $\mathcal{R}_a$ must be the complete cosaturated transfer system by the compatibility axioms. We shall exhaust all 9 cases. Throughout our examples, we denote the representations in $\hat{C}_{15}$ as elements in $\hat{C}_3 \times \hat{C}_5$.

Case 1: Let $\mathcal{R}_m$ be the empty saturated transfer system. Construct the G-universe U as follows. Start with $U = \{(0,0), (1,1)\}$:

- Include $(0,1)$ in U if and only if C_3 is $\mathcal{R}_a$-fibrant.
- Include $(1,0)$ in U if and only if C_5 is $\mathcal{R}_a$-fibrant.

Then, U realizes $(\mathcal{R}_m, \mathcal{R}_a)$.

Case 2: Let $\mathcal{R}_m$ be the saturated transfer system with $e \to_{\mathcal{R}_m} C_5$ only. Construct the G-universe U as follows. Start with $U = \{(0,0),(1,1),(1,2),(1,3),(1,4)\}$:

- Include $(0,1)$ in U if and only if C_3 is $\mathcal{R}_a$-fibrant.
- Include $(1,0)$ in U if and only if C_5 is $\mathcal{R}_a$-fibrant.

Then, U realizes $(\mathcal{R}_m, \mathcal{R}_a)$.

Case 3: Let $\mathcal{R}_m$ be the saturated transfer system with the arrows $e \to_{\mathcal{R}_m} C_5$ and $C_3 \to_{\mathcal{R}_m} C_{15}$. We must have $\mathcal{R}_a$ being the complete cosaturated transfer system. We may choose $U = \{(0,d),(1,d) : d \in \hat{C}_5\}$.

Appendix A. Preliminaries

We prove the claims from Section 1 about the transfer systems coming from Steiner and linear isometries operads. Let G be a finite group and U be a G-universe. We shall use K and H to denote subgroups of G. Define the relations:

- $K \to_{\mathcal{R}_a} H$ iff K is the maximal subgroup of H fixing some vector in $R^G_H U$.
- $K \to_{\mathcal{R}_m} H$ iff $K \leqslant H \leqslant G$ and $I^H_K R^G_K U = R^G_H U$.

Lemma A.1. *Let $\mathcal{R}'_a$ be the relation $K \to_{\mathcal{R}'_a} H$ iff K is the maximal subgroup of H fixing some subspace in $R^G_H U$. Then, $\mathcal{R}_a = \mathcal{R}'_a$.*

Proof. First, we have $\mathcal{R}_a \subseteq \mathcal{R}'_a$ since the fixing subgroup of a vector is the same as the subspace spanned by it. For the reverse direction, it suffices to show that if W is a subspace in $R^G_H U$ with K as its maximal fixing subgroup in H, then some vector $v \in W$ also has maximal fixing subgroup K in H.

For any subgroup M of H properly containing K, the subspace of W fixed by M is a proper subspace. There are finitely many choices for M, so the corresponding subspaces cannot cover W; any vector $v \in W$ not covered by the proper subspaces has maximal fixing subgroup K in H. □

Proposition A.1. *$\mathcal{R}_a$ is a cosaturated transfer system.*

Proof. We verify the necessary conditions:

- *Transitivity*: If $K \to_{\mathcal{R}_a} M$ and $M \to_{\mathcal{R}_a} H$, then we have subspaces W_1 and W_2 of U with fixing subgroups L_1 and L_2 in G such that $L_1 \cap M = K$ and $L_2 \cap H = M$. The fixing subspace of $W_1 + W_2$ is $L_1 \cap L_2$ and we have $(L_1 \cap L_2) = K$. Transitivity follows by Lemma A.1.
- *Restriction and Cosaturation*: The arrows in the relation are precisely the restrictions of the arrows to G.
- *Conjugation*: Apply conjugation on U.

Thus, $\mathcal{R}_a$ is a cosaturated transfer system. □

Proposition A.2. *$\mathcal{R}_m$ is a saturated transfer system.*

Proof. We verify the necessary conditions:

- *Transitivity*: If $K \to_{\mathcal{R}_m} M$ and $M \to_{\mathcal{R}_m} H$, then $K \to_{\mathcal{R}_m} H$ since:

$$I^H_K R^G_K U = I^H_M I^M_K R^G_K U = I^H_M R^G_M U = R^G_H U$$

where the first equality holds true by the transitivity of induction.

- *Restriction*: If $K \to_{\mathcal{R}_m} H$ and $L \leqslant H$, then $K \cap L \to_{\mathcal{R}_m} L$ since:

$$I^L_{K\cap L} R^G_{K\cap L} U = I^L_{K\cap L} R^K_{K\cap L} R^G_K U \hookrightarrow R^H_L I^H_K R^G_K U = R^H_L R^G_H U = R^G_L U$$

  where the inclusion holds true by Mackey's formula. The reverse inclusion $I^L_{K\cap L} R^G_{K\cap L} U \supseteq R^G_L U$ holds true by Frobenius reciprocity.
- *Saturation*: If $K \to_{\mathcal{R}_m} H$ and $K \leqslant M \leqslant H$, then we have $K \to_{\mathcal{R}_m} M$ by restriction. We also have $M \to_{\mathcal{R}_m} H$ since:

$$I^H_M R^G_M U \hookrightarrow I^H_M I^M_K R^M_K R^G_M U = I^H_K R^G_K U = R^G_H U$$

  where the inclusion holds true by Frobenius reciprocity. The reverse inclusion $I^H_M R^G_M U \supseteq R^G_H U$ also holds true by Frobenius reciprocity.
- *Conjugation*: Apply conjugation on U.

Thus, $\mathcal{R}_m$ is a saturated transfer system. □

Proposition A.3. *$(\mathcal{R}_m, \mathcal{R}_a)$ is a compatible pair.*

Proof. Yet to complete □

Appendix B. Categorical Properties of Transfer Systems

For any finite group G, let $\mathbf{TS}(G)$ be the set of transfer systems on G. The sets $\mathbf{TS}(G)$, $\mathbf{CTS}(G)$, $\mathbf{STS}(G)$, $\mathbf{Fib}(G)$, $\mathbf{Cof}(G)$ are partially ordered under inclusion. Consequently, we may interpret them as categories with objects as elements and morphisms as inclusions. The categories $\mathbf{CTS}(G)$ and $\mathbf{STS}(G)$ are full subcategories of $\mathbf{TS}(G)$. The main results of Section 2 can be summarized as:

- Theorem 2.1: $\mathrm{fib} : \mathbf{CTS}(G) \to \mathbf{Fib}(G)$ is an equivalence of categories.
- Theorem 2.2: $\mathrm{cof} : \mathbf{STS}(G) \to \mathbf{Cof}(G)^{\mathrm{op}}$ is an equivalence of categories.

B.1. Products/Coproducts.

In a poset, all limits (reps. colimits) are products (resp. coproducts). The products and coproducts are given by meet and join respectively. We begin by analyzing the initial and final objects in these categories, which correspond to the empty coproduct and product.

- **Initial Object**: The common initial object in $\mathbf{TS}(G)$, $\mathbf{CTS}(G)$, $\mathbf{STS}(G)$ is the empty transfer system $\mathcal{R}_{\text{initial}}$ with $K \to_{\mathcal{R}} H$ if and only if $K = H$. We have $\mathrm{fib}(\mathcal{R}_{\text{initial}}) = \{G\}$ being the initial object in $\mathbf{Fib}(G)$ and $\mathrm{cof}(\mathcal{R}_{\text{initial}}) = \mathrm{Sub}(G)$ being the final object in $\mathbf{Cof}(G)$.
- **Final Object**: The common final object in $\mathbf{TS}(G)$, $\mathbf{CTS}(G)$, $\mathbf{STS}(G)$ is the complet transfer system $\mathcal{R}_{\text{final}}$ with $K \to_{\mathcal{R}} H$ for all $K \leqslant H$. We have $\mathrm{fib}(\mathcal{R}_{\text{final}}) = \mathrm{Sub}(G)$ being the final object in $\mathbf{Fib}(G)$ and $\mathrm{cof}(\mathcal{R}_{\text{final}}) = \{G\}$ being the initial object in $\mathbf{Cof}(G)$.

Lemma B.1. *The product in $\mathbf{TS}(G)$ and $\mathbf{STS}(G)$ is given by intersection.*

Proof. Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be transfer systems on G. We must show that $\mathcal{R}_1 \cap \mathcal{R}_2$ is also a transfer system on G. One may easily check that the transitivity, restriction, and conjugation axioms for $\mathcal{R}_1 \cap \mathcal{R}_2$ follow from the same axioms for $\mathcal{R}_1$ and $\mathcal{R}_2$. Furthermore, if $\mathcal{R}_1$ and $\mathcal{R}_2$ are both saturated, then one may also verify that $\mathcal{R}_1 \cap \mathcal{R}_2$ is saturated. □

Remark B.1. The product in $\mathbf{CTS}(G)$ is *not* intersection.

$$\begin{tikzcd} C_2 & C_6 \\ e \arrow[u] \arrow[r] \arrow[ur] & C_3 \end{tikzcd} \qquad \begin{tikzcd} C_2 & C_6 \\ e \arrow[u] & C_3 \arrow[u] \end{tikzcd} \qquad \begin{tikzcd} C_2 & C_6 \\ e \arrow[u] & C_3 \end{tikzcd}$$

Consider the three transfer systems above for $G = C_6$. The first and second are cosaturated; the former is generated by $e \to C_6$ and the latter is generated by $C_3 \to C_6$. However, the last transfer system is the intersection of the first two, but it is not cosaturated (it is non-empty despite having no proper fibrant subgroups).

Recall that the transfer system generated by a set of arrows is the product (intersection) of all transfer systems containing the given set of arrows; this is well-defined since $\mathbf{TS}(G)$ has products and a final object.

Definition B.1. For a set $\{\mathcal{R}_i\}$ of transfer systems over G, the sum is defined as the transfer system generated by $\cup \mathcal{R}_i$.

Lemma B.2. *The coproduct in* $\mathbf{TS}(G)$ *and* $\mathbf{CTS}(G)$ *is given by summation.*

Proof. It is clear from construction that the coproduct in $\mathbf{TS}(G)$ is sum. It suffices to verify that the sum of cosaturated transfer systems is cosaturated. Indeed, since cosaturated transfer systems are generated by arrows to G, so are their sums, which are consequently cosaturated. □

Remark B.2. The coproduct in $\mathbf{STS}(G)$ is *not* summation.

$$\begin{tikzcd} C_2 & C_6 \\ e \arrow[u] \arrow[r] & C_3 \end{tikzcd} \qquad \begin{tikzcd} C_2 \arrow[r] & C_6 \\ e \arrow[r] & C_3 \end{tikzcd} \qquad \begin{tikzcd} C_2 \arrow[r] & C_6 \\ e \arrow[u] \arrow[r] \arrow[ur] & C_3 \end{tikzcd}$$

Consider the three transfer systems above for $G = C_6$. The first and second are saturated. Their summation is given by the last transfer system, which has the extra arrow $e \to C_6$ by the transitivity axiom. However, the last transfer system is not saturated since $e \to C_6$ and $C_3 \not\to C_6$.

Although the product in $\mathbf{TS}(G)$ is not the product in $\mathbf{CTS}(G)$, we do know that $\mathbf{CTS}(G)$ has products since every join-semilattice with a unique minimal element is a lattice. By Theorem 2.1, we may interpret the product in $\mathbf{CTS}(G)$ as the product in $\mathbf{Fib} * G$.

Dually, although the coproduct in $\mathbf{TS}(G)$ is not the coproduct in $\mathbf{STS}(G)$, we do know that $\mathbf{STS}(G)$ has coproducts since every meet-semilattie with a unique maximal element is a lattice. By Theorem 2.2, we may interpret the coproduct in $\mathbf{STS}(G)$ as the product in $\mathbf{Cof}(G)$.

Lemma B.3. *The products in* $\mathbf{Fib}(G)$ *and* $\mathbf{Cof}(G)$ *are given by intersection.*

Proof. For any collection of subsets $\{S_i\}$ of $\mathrm{Sub}(G)$ which are closed under intersection (resp. compositum) and conjugation, one may easily verify that $\bigcap\{S_i\}$ also has the same property. □

B.2. **Cosaturation/Saturation Functors.** We saw earlier that the product in $\mathbf{CTS}(G)$ does not match the product in $\mathbf{TS}(G)$. Dually, the coproduct in $\mathbf{STS}(G)$ does not match the coproduct in $\mathbf{TS}(G)$. We can fix this using the cosaturation and saturation functors respectively.

Definition B.2. Let $\iota_C : \mathbf{CTS}(G) \to \mathbf{TS}(G)$ and $\iota_S : \mathbf{STS}(G) \to \mathbf{TS}(G)$ denote obvious forgetful functors.

- The *cosaturation functor* $s_C : \mathbf{TS}(G) \to \mathbf{CTS}(G)$ is the following section to ι_C: for any transfer system $\mathcal{R}$, the cosaturated transfer system $s_C(\mathcal{R})$ is the coproduct of all cosaturated transfer systems contained in $\mathcal{R}$.
- The *saturation functor* $s_S : \mathbf{TS}(G) \to \mathbf{STS}(G)$ is the following section to ι_S: for any transfer system $\mathcal{R}$, the saturated transfer system $s_S(\mathcal{R})$ is the product of all saturated transfer systems containing $\mathcal{R}$.

Proposition B.1. *With the notation above, (ι_C, s_C) and (s_S, ι_S) are adjoint pairs.*

Proof. Immediate from definition. □

Lemma B.4. *The product in $\mathbf{CTS}(G)$ is the cosaturation of the product in $\mathbf{TS}(G)$. Dually, the coproduct in $\mathbf{STS}(G)$ is the saturation of the coproduct in $\mathbf{TS}(G)$.*

Proof. Follows from Proposition B.1 □

Remark B.3. Recall that left adjoints commute with colimits and right adjoints commute with limits. Thus, Proposition B.1 explains why the coproduct in $\mathbf{CTS}(G)$ matches the coproduct in $\mathbf{TS}(G)$ and why the product in $\mathbf{STS}(G)$ matches the product in $\mathbf{TS}(G)$.

Lemma B.5. *Let $\mathcal{R}$ be a transfer system on G. We have:*

- $\mathrm{fib}(s_C(\mathcal{R})) = \mathrm{fib}(\mathcal{R})$
- $\mathrm{cof}(s_S(\mathcal{R})) = \mathrm{cof}(\mathcal{R})$

Proof. For the former, use Proposition 2.1 and Theorem 2.1. For the latter, use Proposition 2.2, and Theorem 2.2. □

Using Lemma B.5, Theorem 2.1, and Theorem 2.2, we have the following description of the cosaturation and saturation functors in terms of other functors:

- The functor s_C is obtained by composing $\mathrm{fib} : \mathbf{TS}(G) \to \mathbf{Fib}(G)$ with $\mathrm{fib}^{-1} : \mathbf{Fib}(G) \to \mathbf{CTS}(G)$.
- The functor s_S is obtained by composing $\mathrm{cof} : \mathbf{TS}(G) \to \mathbf{Cof}(G)^{\mathrm{op}}$ with $\mathrm{cof}^{-1} : \mathbf{Cof}(G)^{\mathrm{op}} \to \mathbf{STS}(G)$.

We shall now describe the cosaturation and saturation functors more explicitly.

Theorem B.1. *Let $\mathcal{R}$ be a transfer system on G. We have $K \to_{s_C(\mathcal{R})} H$ if and only if there exists $L \leqslant G$ with $L \to_{\mathcal{R}} G$ and $K = H \cap L$.*

Proof. This follows from the description of $s_C(\mathcal{R})$ in Lemma B.5 and the explicit construction of $\mathrm{fib}^{-1} : \mathbf{Fib}(G) \to \mathbf{CTS}(G)$ in Theorem 2.1. □

Theorem B.2. *Let $\mathcal{R}$ be a transfer system on G. We have $K \to_{s_S(\mathcal{R})} H$ if and only if $P \to_{\mathcal{R}} H$ for some $P \leqslant K \leqslant H$.*

Proof. Consider the relation $\mathcal{R}'$ with $K \to_{\mathcal{R}'} H$ if and only if $P \to_{\mathcal{R}} H$ for some $P \leqslant K \leqslant H$. By the saturation axiom on $s_S(\mathcal{R})$, it is clear that $\mathcal{R}' \subseteq s_S(\mathcal{R})$, so it suffices to prove that $\mathcal{R}'$ is a saturated transfer system.

- *Transitivity*: Let $K \to_{\mathcal{R}'} M$ and $M \to_{\mathcal{R}'} H$. By construction, there exists some $K' \leqslant K$ with $K' \to_{\mathcal{R}} M$ and some $M' \leqslant M$ with $M' \to_{\mathcal{R}} H$. By the restriction axiom, since $K' \to_{\mathcal{R}} M$ and $M' \leqslant M$, we have $K' \cap M' \to_{\mathcal{R}} M'$. We also have $M' \to_{\mathcal{R}} H$, so by transitivity, $K' \cap M' \to_{\mathcal{R}} H$. Since $K' \cap M' \leqslant K \leqslant H$, it follows by construction that $K \to_{\mathcal{R}'} H$.

$$\begin{tikzcd}
K' \cap M' \arrow[d, dash, dashed] \arrow[dr, "\mathcal{R}"] & & \\
K' \arrow[d, dash, dashed] \arrow[dr, "\mathcal{R}"] & M' \arrow[d, dash, dashed] \arrow[dr, "\mathcal{R}"] & \\
K \arrow[r, "\mathcal{R}'"'] & M \arrow[r, "\mathcal{R}'"'] & H
\end{tikzcd}$$

- *Restriction*: Let $K \to_{\mathcal{R}'} H$ and $L \leqslant H$. By construction, there exists some $K' \leqslant K$ with $K' \to_{\mathcal{R}} H$. By the restriction axiom on $\mathcal{R}$, we have $K' \cap L \to_{\mathcal{R}} L$. Since $K' \cap L \leqslant K \cap L \leqslant L$, we have $K \cap L \to_{\mathcal{R}'} L$.
- *Conjugation*: Follows from $\mathcal{R}$ being conjugation invariant.
- *Saturation*: Follows from construction since the arrows in $\mathcal{R}'$ are precisely obtained by saturating arrows in $\mathcal{R}$.

Thus, we have $\mathcal{R}' = s_S(\mathcal{R})$, as required. $\square$

Alter. Let $H \leqslant G$ be any subgroup and let M be a minimal subgroup satisfying the property $M \to_{\mathcal{R}} H$. By transitivity, M is $\mathcal{R}$-cofibrant. Furthermore, we know that from Lemma 2.2 that every $\mathcal{R}$-cofibrant subgroup of H must be contained in M. Thus, M is the compositum of all $\mathcal{R}$-cofibrant subgroups contained in H, i.e. the unique maximal $\mathcal{R}$-cofibrant subgroup of H.

Now, $\mathcal{R}'$ is precisely the relation given by $K \to_{\mathcal{R}'} H$ if and only if $M \leqslant K$, or equivalently, for all $\mathcal{R}$-cofibrant $L \leqslant H$, we have $L \leqslant K$. It follows that $\mathcal{R}' = s_S(\mathcal{R})$ from the description of $s_S(\mathcal{R})$ in Lemma B.5 and the explicit construction of $\mathrm{cof}^{-1} : \mathbf{Cof}(G)^{\mathrm{op}} \to \mathbf{STS}(G)$ in Theorem 2.2. $\square$

B.3. **Restriction.** Let $\varphi : G \to G'$ be a homomorphism of finite groups.

Definition B.3. The *restriction* functor $\mathrm{Res}(\varphi) : \mathbf{TS}(G') \to \mathbf{TS}(G)$ is defined by $\mathrm{Res}(\varphi) : \mathcal{R}' \mapsto \mathcal{R}$, where $\mathcal{R}$ is the G-transfer system generated by the set of arrows $\{\varphi^{-1}(K') \to_{\mathcal{R}} \varphi^{-1}(H') : K' \to_{\mathcal{R}'} H'\}$.

Note that $\mathrm{Res}(\varphi)$ is a functor since larger $\mathcal{R}'$ yield larger $\mathcal{R}$. We shall provide an explicit description for $\mathrm{Res}(\varphi)$. First, we shall require some helpful lemmas. Let $N \trianglelefteq G$ be the normal subgroup $\ker \varphi$:

Lemma B.6. *For $K \leqslant H \leqslant G$, the following are equivalent:*

(1) $K \cap N = H \cap N$.
(2) $[\varphi(H) : \varphi(K)] = [H : K]$.
(3) $H \cap KN = K$.

Proof. We prove that (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1):

- (1) $\Rightarrow$ (2): We have $|\varphi(K)| = [K : K \cap N]$ and $|\varphi(H)| = [H : H \cap N]$. The ratio of these equations gives $[\varphi(H) : \varphi(K)] = [H : K]$.

- (2) $\Rightarrow$ (3): Using the fact that $[HN : KN] = [H : K]$, we get:
$$|K| = \frac{|H| \cdot |KN|}{|HN|} = \frac{|HKN| \cdot |H \cap KN|}{|HN|} = |H \cap KN|$$
It is clear that $K \leqslant H \cap KN$, so we must have equality.
- (3) $\Rightarrow$ (1): We have $K \cap N = (H \cap KN) \cap N = H \cap (KN \cap N) = H \cap N$.

Thus, the given statements are equivalent. □

Lemma B.7. *For $K, L \leqslant G$, we have $\varphi(KN \cap L) = \varphi(K) \cap \varphi(L)$.*

Proof. Since $\varphi(KN \cap L) \subseteq \varphi(KN) \subseteq \varphi(K)$ and $\varphi(KN \cap L) \subseteq \varphi(L)$, we have the inclusion $\varphi(KN \cap L) \subseteq \varphi(K) \cap \varphi(L)$. For the reverse inclusion, assume that $g' = \varphi(\ell)$ and $g' \in \varphi(K)$. It follows that $\varphi^{-1}(g') \subseteq KN$, and in particular, $\ell \in KN$. Thus, $g' \in \varphi(KN \cap L)$ and $\varphi(KN \cap L) \supseteq \varphi(K) \cap \varphi(L)$. □

Theorem B.3. *Let $\mathcal{R}'$ be a G'-transfer system and $\mathcal{R}$ be the relation on* $\mathrm{Sub}(G)$ *given by $K \to_{\mathcal{R}} H$ if and only if $K \leqslant H$, $K \cap N = H \cap N$, and $\varphi(K) \to_{\mathcal{R}'} \varphi(H)$. We have $\mathcal{R} = \mathrm{Res}(\varphi)(\mathcal{R}')$.*

Before we prove the above theorem, we give an equivalent description of the above relation $\mathcal{R}$ that we shall find insightful:

Lemma B.8. *Let $\mathcal{R}$ be the relation above; we have $K \to_{\mathcal{R}} H$ if and only if it comes from restricting $\varphi^{-1}(K') \to_{\mathcal{R}} \varphi^{-1}(H')$ where $K' = \varphi(K)$ and $H' = \varphi(H)$.*

Proof. ($\Rightarrow$) Assume that $K \to_{\mathcal{R}} H$; this means that $K \leqslant H$, $K \cap N = H \cap N$, and $K' \to_{\mathcal{R}'} H'$. We know that $\varphi^{-1}(K') = KN$ and $\varphi^{-1}(H') = HN$, and since $KN \leqslant HN$, $KN \cap N = HN \cap N = N$, and $K' \to_{\mathcal{R}'} H'$, we also get the arrow $KN \to_{\mathcal{R}} HN$. Finally, $K \to_{\mathcal{R}} H$ is obtained from restricting $KN \to_{\mathcal{R}} HN$ by Lemma B.6 since $H \cap KN = K$.

($\Leftarrow$) Let $L \leqslant HN$ be any subgroup; we wish to show that $KN \cap L \to_{\mathcal{R}} L$. It is clear that $KN \cap L \leqslant L$ and $(KN \cap L) \cap N = L \cap N$, so it remains to show that $\varphi(KN \cap L) \to_{\mathcal{R}'} \varphi(L)$. By Lemma B.7, we know that $\varphi(KN \cap L) = \varphi(K) \cap \varphi(L)$, and the arrow $\varphi(K) \cap \varphi(L) \to_{\mathcal{R}'} \varphi(L)$ comes from restricting $K' \to H'$. □

Proof of Theorem B.3. It is clear that the relation $\mathcal{R}$ contains the generating arrows of $\mathrm{Res}(\varphi)(\mathcal{R}')$. Furthermore, by Lemma B.8, we also have $\mathcal{R} \subseteq \mathrm{Res}(\varphi)(\mathcal{R}')$. It suffices to show that $\mathcal{R}$ is a transfer system:

- *Transitivity*: Follows from transitivity in $\mathcal{R}'$.
- *Restriction*: Follows from Lemma B.8.
- *Conjugation*: Follows from conjugation in $\mathcal{R}'$ and normality of N.

We conclude that $\mathcal{R} = \mathrm{Res}(\varphi)(\mathcal{R}')$. □

Definition B.4. Let **FinGrp** denote the category of finite groups. Let **TS** denote the 2-category with objects as $\mathbf{TS}(G)$ for finite groups G and morphisms as functors. The functor $\mathrm{Res} : \mathbf{FinGrp}^{\mathrm{op}} \to \mathbf{TS}$ is defined as follows:

- On objects, $\mathrm{Res}(G) = \mathbf{TS}(G)$.
- On morphisms, $\mathrm{Res}(\varphi)$ is the restriction functor.

We are yet to prove that Res satisfies functoriality, i.e. for finite groups G, G', G'' with maps $\varphi : G \to G'$ and $\psi : G' \to G''$, we need $\mathrm{Res}(\varphi) \circ \mathrm{Res}(\psi) = \mathrm{Res}(\psi \circ \varphi)$. One may prove this using Theorem B.3 with the condition $K \cap N = H \cap N$ being rewritten as (2) in Lemma B.6. We shall prove this more directly with the following useful proposition about the interaction between restriction and generation:

Proposition B.2. *Let $S = \{K' \to_{\mathcal{R}'} H'\}$ be a G'-conjugation invariant set of arrows that generate $\mathcal{R}'$. Then, the arrows in $\varphi^{-1}(S) = \{\varphi^{-1}(K') \to_{\mathcal{R}} \varphi^{-1}(H')\}$ generate the transfer system $\mathcal{R} = \mathrm{Res}(\varphi)(\mathcal{R}')$.*

Proof. Since S is G'-conjugation invariant, we can build $\mathcal{R}'$ using transitivity and restriction alone. It suffices to show that the same can be done in $\mathcal{R}$ so that we get the preimage of all the arrows in $\mathcal{R}'$.

- *Transitivity*: Let $K' \to_{\mathcal{R}'} M'$ and $M' \to_{\mathcal{R}'} H'$. We can introduce the arrow $K' \to_{\mathcal{R}'} H'$ in $\mathcal{R}'$. Correspondingly, we can obtain $\varphi^{-1}(K') \to_{\mathcal{R}} \varphi^{-1}(H')$ by combining $\varphi^{-1}(K') \to_{\mathcal{R}} \varphi^{-1}(M')$ and $\varphi^{-1}(M') \to_{\mathcal{R}} \varphi^{-1}(H')$.
- *Restriction*: Let $K' \to_{\mathcal{R}'} H'$ and $L' \leqslant H'$. We can introduce the arrow $K' \cap L' \to_{\mathcal{R}'} L'$ in $\mathcal{R}'$. Correspondingly, we can obtain $\varphi^{-1}(K' \cap L') \to_{\mathcal{R}} \varphi^{-1}(L')$ by restricting $\varphi^{-1}(K') \to_{\mathcal{R}} \varphi^{-1}(H')$ with $\varphi^{-1}(L') \leqslant \varphi^{-1}(H')$. Note that $\varphi^{-1}(K' \cap L') = \varphi^{-1}(K') \cap \varphi^{-1}(L')$.

Thus, $\mathcal{R} = \mathrm{Res}(\varphi)(\mathcal{R}')$ is generated by $\varphi^{-1}(S)$. □

Remark B.4. Proposition B.2 would not hold if we drop the conjugation-invariance hypothesis. For example, let $G = C_2 \times C_2$, let $G' = G \rtimes \mathrm{Aut}(G)$, and let $\varphi : G \to G'$ be the obvious inclusion map. Denote the index 2 subgroups of G by A, B, C. These subgroups are conjugate in G' but not in G. The G'-transfer system $\mathcal{R}'$ with arrows $e \to_{\mathcal{R}'} A$, $e \to_{\mathcal{R}'} B$, $e \to_{\mathcal{R}'} C$ is generated by $e \to_{\mathcal{R}'} A$. However, $\mathcal{R} = \mathrm{Res}(\varphi)(\mathcal{R}')$ has the same arrows and is not generated by $e \to_{\mathcal{R}} A$.

Corollary B.1. *The map* $\mathrm{Res} : \mathbf{FinGrp}^{\mathrm{op}} \to \mathbf{TS}$ *is a functor.*

Proof. Let $\varphi : G \to G'$ and $\psi : G' \to G''$ be group homomorphisms. Let $\mathcal{R}''$ be a G''-transfer system, let $\mathcal{R}' = \mathrm{Res}(\psi)(\mathcal{R}'')$, and let $\mathcal{R} = \mathrm{Res}(\varphi)(\mathcal{R}')$. By definition, $\mathcal{R}'$ is generated by $\{\psi^{-1}(K'') \to_{\mathcal{R}'} \psi^{-1}(H'') : K'' \to_{\mathcal{R}''} H''\}$. Next, by Proposition B.2, $\mathcal{R}$ is generated by $\{\varphi^{-1}(\psi^{-1}(K'')) \to_{\mathcal{R}} \varphi^{-1}(\psi^{-1}(H'')) : K'' \to_{\mathcal{R}''} H''\}$. Thus, $\mathcal{R} = \mathrm{Res}(\psi \circ \varphi)(\mathcal{R}'')$, as required. □

B.4. **Induction and Coinduction.** The functor $\mathrm{Res}(\varphi)$ behaves quite well with limits and colimits in $\mathbf{TS}(G)$ and $\mathbf{TS}(G')$:

Lemma B.9. *The functor* $\mathrm{Res}(\varphi)$ *commutes with non-empty products. Furthermore, it sends the final object in* $\mathbf{TS}(G')$ *to the final object in* $\mathbf{TS}(G)$ *if and only if* φ *is injective, i.e.* $N = 1$.

Proof. Let $\mathcal{R}'_1$ and $\mathcal{R}'_2$ be G'-transfer systems with restrictions $\mathcal{R}_1$ and $\mathcal{R}_2$. Let $\mathcal{R}' = \mathcal{R}'_1 \cap \mathcal{R}'_2$ and $\mathcal{R} = \mathcal{R}_1 \cap \mathcal{R}_2$. By Theorem B.3:

$$K \to_{\mathcal{R}} H \Leftrightarrow \begin{cases} K \to_{\mathcal{R}_1} H \\ K \to_{\mathcal{R}_2} H \end{cases} \Leftrightarrow \begin{cases} K \leqslant H \\ K \cap N = H \cap N \\ \varphi(K) \to_{\mathcal{R}'_1} \varphi(H) \\ \varphi(K) \to_{\mathcal{R}'_2} \varphi(H) \end{cases} \Leftrightarrow \begin{cases} K \leqslant H \\ K \cap N = H \cap N \\ \varphi(K) \to_{\mathcal{R}'} \varphi(H) \end{cases}$$

Thus, $\mathrm{Res}(\varphi)$ commutes with non-empty products. Now, let $\mathcal{R} = \mathrm{Res}(\varphi)(\mathcal{R}'_{\mathrm{final}})$. For subgroups $K \leqslant H \leqslant G$, we have $K \to_{\mathcal{R}} H$ if and only if $K \cap N = H \cap N$. This holds true for all $K \leqslant H$ if and only if $N = 1$. □

Lemma B.10. *The functor* $\mathrm{Res}(\varphi)$ *commutes with coproducts.*

Proof. It follows from Proposition B.2 that the restriction of the sum of two transfer systems is identical to the sum of the restrictions. Finally, let $\mathcal{R} = \mathrm{Res}(\varphi)(\mathcal{R}'_{\mathrm{initial}})$, we have $K \to_{\mathcal{R}} H$ if and only if $K \leqslant H$, $K \cap N = H \cap N$, and $\varphi(K) \to_{\mathcal{R}'_{\mathrm{initial}}} \varphi(H)$. The second condition gives $[\varphi(H) : \varphi(K)] = [H : K]$ by Lemma B.6 and the third condition gives $\varphi(K) = \varphi(H)$. Thus, $K = H$, proving that $\mathcal{R} = \mathcal{R}_{\mathrm{initial}}$. □

Definition B.5. Let $\varphi : G \to G'$ be a homomorphism of finite groups:

- The *coinduction functor* $\mathrm{Coind}(\varphi) : \mathbf{TS}(G) \to \mathbf{TS}(G')$ is the right adjoint functor to $\mathrm{Res}(\varphi) : \mathbf{TS}(G') \to \mathbf{TS}(G)$.
- The *induction functor* $\mathrm{Ind}(\varphi) : \mathbf{TS}(G) \to \mathbf{TS}(G')$ is the left adjoint functor to $\mathrm{Res}(\varphi) : \mathbf{TS}(G') \to \mathbf{TS}(G)$.

Proposition B.3. *The functor* $\mathrm{Coind}(\varphi)$ *exists for all* $\varphi : G \to G'$. *The functor* $\mathrm{Ind}(\varphi)$ *exists if and only if* φ *is injective.*

Proof. By Lemma B.10 and Lemma B.9, the proposition can be rewritten as saying that $\mathrm{Res}(\varphi)$ is right (resp. left) adjoint if and only if it commutes with products (resp. coproducts). Necessity is clear since products and coproducts are limits and colimits respectively.

To prove sufficience, assume that $\mathrm{Res}(\varphi)$ commutes with coproducts. For all G-transfer systems $\mathcal{R}$, define $\mathrm{Coind}(\varphi)(\mathcal{R})$ to be the coproduct of all G'-transfer systems $\mathcal{R}'$ satisfying $\mathrm{Res}(\varphi)(\mathcal{R}') \subseteq \mathcal{R}$. The functoriality of $\mathrm{Coind}(\varphi)$ follows from the functoriality of $\mathrm{Res}(\varphi)$. We must show that:

$$\mathrm{Hom}_{\mathbf{TS}(G)}(\mathrm{Res}(\varphi)(\mathcal{R}'), \mathcal{R}) = \mathrm{Hom}_{\mathbf{TS}(G')}(\mathcal{R}', \mathrm{Coind}(\varphi)(\mathcal{R}'))$$

This is equivalent to saying that $\mathrm{Res}(\varphi)(\mathcal{R}') \subseteq \mathcal{R}$ if and only if $\mathcal{R}' \subseteq \mathrm{Coind}(\varphi)(\mathcal{R}')$. The forward implication follows by construction, and by functoriality, the reverse implication holds if $\mathrm{Res}(\varphi)\mathrm{Coind}(\varphi)(\mathcal{R}) \subseteq \mathcal{R}$. Indeed, this follows immediately from $\mathrm{Res}(\varphi)$ commuting with coproducts and the definition of $\mathrm{Coind}(\varphi)$. The construction of $\mathrm{Ind}(\varphi)$ is dual. □

By Corollary B.1 and adjointness, we can package coinduction into a functor $\mathrm{Coind} : \mathbf{FinGrp} \to \mathbf{TS}$ such that $(\mathrm{Res}, \mathrm{Coind})$ are adjoint. Similarly, if we denote the subcategory of $\mathbf{FinGrp}$ consisting of all injections by $\mathbf{InjFinGrp}$, then we can restrict Res to a functor $\mathrm{Res} : \mathbf{InjFinGrp}^{\mathrm{op}} \to \mathbf{TS}$ and package induction into a functor $\mathrm{Ind} : \mathbf{InjFinGrp} \to \mathbf{TS}$ such that $(\mathrm{Ind}, \mathrm{Res})$ are adjoint.

Theorem B.4. *Let* $\mathcal{R}$ *be a* G*-transfer system. Let* $\mathcal{R}'$ *be the relation on* $\mathrm{Sub}(G')$ *with* $K' \to_{\mathcal{R}'} H'$ *if and only if* $K' \leqslant H'$ *and* $\varphi^{-1}(g'K'(g')^{-1}) \to_{\mathcal{R}} \varphi^{-1}(g'H'(g')^{-1})$ *for all* $g' \in G'$. *Then,* $\mathcal{R}' = \mathrm{Coind}(\varphi)(\mathcal{R})$.

Proof. We start by proving that $\mathcal{R}'$ is a G'-transfer system:

- *Transitivity*: Follows from transitivity in $\mathcal{R}$.
- *Restriction*: If $K' \to_{\mathcal{R}'} H'$ and $L' \leqslant H'$, we require $K' \cap L' \to_{\mathcal{R}'} L'$. The inclusion of subgroups is clear. Since $\varphi^{-1}((K')^{g'}) \to_{\mathcal{R}} \varphi^{-1}((H')^{g'})$, the restriction axiom on $\mathcal{R}$ gives $\varphi^{-1}((K')^{g'}) \cap \varphi^{-1}((L')^{g'}) \to_{\mathcal{R}} \varphi^{-1}((L')^{g'})$. Since $\varphi^{-1}((K')^{g'}) \cap \varphi^{-1}((L')^{g'}) = \varphi^{-1}((K' \cap L')^{g'})$, we get $K' \cap L' \to_{\mathcal{R}'} L'$.
- *Conjugation*: Follows from construction.

Next, we show that $\mathcal{R}'$ is equal to $\mathrm{Coind}(\varphi)(\mathcal{R})$:

- $\mathcal{R}' \subseteq \mathrm{Coind}(\varphi)(\mathcal{R})$: By adjointness, this is equivalent to $\mathrm{Res}(\varphi)(\mathcal{R}') \subseteq \mathcal{R}$. By Proposition B.2, it suffices to show that for all $K' \to_{\mathcal{R}'} H'$ in our generating set, we have $\varphi^{-1}(K') \to_{\mathcal{R}} \varphi^{-1}(H')$, which follows from $g' = 1$.
- $\mathcal{R}' \supseteq \mathrm{Coind}(\varphi)(\mathcal{R})$: By adjointness, this is equivalent to saying that every G'-transfer system $\mathcal{R}'_1$ with $\mathrm{Res}(\varphi)(\mathcal{R}'_1) \subseteq \mathcal{R}$ satisfies $\mathcal{R}'_1 \subseteq \mathcal{R}'$. Indeed, if $K' \to_{\mathcal{R}'_1} H'$, by the conjugation axiom for $\mathcal{R}'_1$, we have $(K')^{g'} \to_{\mathcal{R}'_1} (H')^{g'}$ for all $g' \in G'$. By definition, $\varphi^{-1}((K')^{g'}) \to_{\mathrm{Res}(\varphi)(\mathcal{R}'_1)} \varphi^{-1}((H')^{g'})$, so the same arrow also exists in $\mathcal{R}$. Thus, $K' \to_{\mathcal{R}'} H'$.

We conclude that $\mathcal{R}' = \mathrm{Coind}(\varphi)(\mathcal{R})$. □

Theorem B.5. *Let $\varphi : G \to G'$ be an injective homomorphism and let $\mathcal{R}$ be a G-transfer system. Let $\mathcal{R}'$ be the G'-transfer system generated by the arrows in $\mathcal{R}$ in the sense of the embedding $\varphi : \mathrm{Sub}(G) \to \mathrm{Sub}(G')$. Then, $\mathcal{R}' = \mathrm{Ind}(\varphi)(\mathcal{R})$.*

Proof. $\mathcal{R}'$ is generated by the arrows necessary to make $\mathcal{R} \subseteq \mathrm{Res}(\varphi)(\mathcal{R}')$ true, so by adjointness, we have $\mathcal{R}' = \mathrm{Ind}(\varphi)(\mathcal{R})$. □

B.5. **Restriction in CTS and STS.** Yet to complete

Lemma B.11. *If $\mathcal{R}'$ is a cosaturated transfer system, then $\mathcal{R} = \mathrm{Res}(\varphi)(\mathcal{R}')$ is also a cosaturated transfer system.*

Proof. By Proposition B.2, $\mathcal{R}$ is generated by arrows $\varphi^{-1}(L') \to G$, where L' ranges through the $\mathcal{R}'$-cofibrant subgroups of G'. Thus, $\mathcal{R}$ is cosaturated. □

Lemma B.12. *If $\mathcal{R}'$ is a saturated transfer system, then $\mathcal{R} = \mathrm{Res}(\varphi)(\mathcal{R}')$ is also a saturated transfer system.*

Proof. We use Theorem B.3. If $K \leqslant M \leqslant H$ and $K \to_{\mathcal{R}} H$, then we have $K \cap N = H \cap N$ and $\varphi(K) \to_{\mathcal{R}'} \varphi(H)$. We then also have $M \cap N = H \cap N$ and $\varphi(M) \to_{\mathcal{R}'} \varphi(H)$, the latter from saturation in $\mathcal{R}'$. Thus, it follows that $M \to_{\mathcal{R}} H$, proving that $\mathcal{R}$ is saturated. □

Consequently, the functor $\mathrm{Res}(\varphi) : \mathbf{TS}(G') \to \mathbf{TS}(G)$ can be restricted to the functors $\mathrm{Res}(\varphi) : \mathbf{CTS}(G') \to \mathbf{CTS}(G)$ and $\mathrm{Res}(\varphi) : \mathbf{STS}(G') \to \mathbf{STS}(G)$. We may then ask for an explicit description for the restriction functors on $\mathbf{Fib}(G)$ and $\mathbf{Cof}(G)$ after transfer of structure using fib and cof.

Appendix C. Compatible Pairs for Vector Spaces

Yet to complete

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