METRIC LINES AND SYMPLECTIC REDUCTIONS OF THE SPECIAL EUCLIDEAN GROUP ON THE PLANE

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ABSTRACT. We consider the Special Euclidean group on the plane $\mathrm{SE}(2)$ endowed with a left-invariant sub-Riemannian structure. In this article, we classify the geodesics over $\mathrm{SE}(2)$ according to their reduced dynamics. As a consequence, we characterize all globally minimizing geodesics, or metric lines, and periodic geodesics over $\mathrm{SE}(2)$. In addition, we study the symplectic reductions of $T^*\,\mathrm{SE}(2)$ by the action groups $\mathbb{R}^2$ and $\mathrm{SE}(2)$. We show that there exists a symplectomorphism between the two reductions $T^*\,\mathrm{SE}(2)\,//_{\mu}\mathbb{R}^2$ and $T^*\,\mathrm{SE}(2)\,//_{\overline{\mu}}\,\mathrm{SE}(2)$ for $\mu \neq 0$ and $\overline{\mu} := (\mu_\theta, \mu)$.

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1. Introduction

The distance minimizing properties of geodesics over many settings is an extensively studied topic and active area of research. It is a common fact that geodesics over the Euclidean space are lines, and that for any segment of the geodesic, it minimizes the distance between the two endpoints. In general this fact fails to hold true. For example, Myer's Theorem ([15, Theorem 12.24]) states that if (M, g) is a complete and connected Riemannian manifold of dimension d whose Ricci curvature satisfies $Ric(v, v) \geq (d-1)/r^2$ for all unit vectors v and some positive constant r. Then any geodesic of length greater than πr is not globally minimizing.

A method of studying the minimization properties of geodesics is to study the variation fields of geodesics connecting two points. This theory leads us to classical tools such as Jacobi fields and conjugate points, and the extremely important fact that geodesics do not minimize past conjugate points. For the formal definition of these two objects, see [11, sub-Chapter 1.6] or [15, Chapter 10] .

Similar to Riemannian geometry, one can formalize the definition of geodesics to the sub-Riemannian geometry setting. For the formal definition of a sub-Riemannian geodesic, see [1, sub-sub-Chapter 4.7.2] or [20, sub-Chapter 1.4]. We are especially interested in sub-Riemannian geodesics that minimize globally. This leads to the following definition.

Definition 1.1. *Let M be a sub-Riemannian manifold, $dist_M(\cdot, \cdot)$ be the sub-Riemannian distance on M, and $|\cdot| : \mathbb{R} \to [0, \infty)$ be the absolute value. We say that a curve $\gamma : \mathbb{R} \to M$ is a metric line if it is a globally minimizing geodesic, i.e.,*

$$|a - b| = dist_M(\gamma(a), \gamma(b)) \quad \textit{for all compact intervals} \quad [a, b] \subset \mathbb{R}.$$

An alternative term for "metric lines" are "globally minimizing geodesics", "isometric embeddings of the real line", or "infinite-geodesics."

The Special Euclidean group on the plane $\mathrm{SE}(2)$ is a Lie Group consisting of the space of rotations and translations. It has a semidirect group structure given by $\mathrm{SE}(2) = \mathrm{SO}(2) \ltimes \mathbb{R}^2$, where $\mathrm{SO}(2)$ is the Special Orthogonal group on the plane. For a discussion of the left invariant sub-Riemannian structure over $\mathrm{SE}(2)$, see sub-Section 3.1. In [19], I. Moiseev and Y. Sachkov used optimal synthesis to study the length-minimizing times of the sub-Riemannian geodesics over $\mathrm{SE}(2)$. Through their work, a family of metric lines was provided. One of the primary goals of this paper is to classify the metric lines on $\mathrm{SE}(2)$ through the Hamilton-Jacobi theory and give an alternative proof of I. Moiseev and Y. Sachkov's result.

The symplectic reduction of the sub-Riemannian geodesic flow by $\mathrm{SE}(2)$ is a common tool in studying the geodesics over $\mathrm{SE}(2)$, where the semidirect group structure of $\mathrm{SE}(2)$ plays a main role, consult [16, Chapter 1] and [18] for the general theory of symplectic reductions and [17] for the theory in the case of semidirect products. However, we will use an alternative method; the group $\mathrm{SE}(2)$ has the structure of a metabelian Carnot group , i.e., the commutator group $[\mathrm{SE}(2), \mathrm{SE}(2)] \simeq \mathbb{R}^2$ is abelian. Thus, we will consider the action of $\mathbb{R}^2$ and perform the symplectic reduction of the sub-Riemannian geodesic flow by $\mathbb{R}^2$, where the reduced space $T^* \mathrm{SE}(2) //_\mu \mathbb{R}^2$ is equivalent to $T^* \mathrm{SO}(2)$ as a symplectic manifold, refer to [7] for a detailed discussion of symplectic reductions in the case of metabelian groups.

This symplectic reduction gives us a bijection between sub-Riemannian geodesics in $\mathrm{SE}(2)$ and curves α_μ in $T^*\,\mathrm{SO}(2)$. Let $T^*\,\mathrm{SO}(2)$ be given the coordinate system (p_θ, θ), we define the curves in α_μ by the equation

$$\alpha_\mu := \{(p_\theta, \theta) \in T^*\,\mathrm{SO}(2) \mid 1 = p_\theta^2 + R^2 \cos^2(\theta - \delta)\}, \tag{1}$$

where $\mu = (R, \delta)$ is in $(\mathbb{R}^2)^* \simeq \mathbb{R}^2$.

In sub-sub-Section 3.2.2, we provide a prescription to build a curve in $\mathrm{SE}(2)$ given the curve α_μ, the **Background Theorem** gives the prescription to sub-Riemannian geodesics in $\mathrm{SE}(2)$ parameterized by arc-length, Conversely, the **Background Theorem** states that every arc-length parameterized geodesic in $\mathrm{SE}(2)$ can be achieved by the prescription applied to some curve α_μ.

We will follow the approach of A. Bravo-Doddoli and R. Montgomery in [6, 8] to use the metableian structure of $\mathrm{SE}(2)$ to give a classification of the metric lines. In sub-sub-Section 3.2.1 we compute the reduced Hamiltonian H_μ and classify the sub-Riemannian geodesics in $\mathrm{SE}(2)$ according to its reduced dynamics. Through this method, we can classify sub-Riemannian geodesics on $\mathrm{SE}(2)$ as only one of the following types: lines, θ-periodic and heteroclinic geodesics, refer to sub-sub-Section 3.2.1 for their formal definitions.

Our first main theorem gives a precise classification of metric lines over $\mathrm{SE}(2)$.

Theorem A. *The metric lines in the* $\mathrm{SE}(2)$ *are precisely the geodesics of the type line and heteroclinic.*

The proof to Theorem A consists of two parts; proving that the geodesics of the type heteroclinic and line are metric lines and showing that the geodesics of the type θ-periodic are not globally minimizing.

Our method to prove the first part is to find a calibration function, refer to [2, sub-Chapter 9.47] for the general Hamilton-Jacobi theory, for calibration functions see [12, sub-Chapter 2.8] and to [8] for the theory in the context of the sub-Riemannian geodesics. The second part is proven in Proposition 5.2, which states that geodesics of type periodic do not minimize pass its θ-period.

The second main theorem classifies the periodic geodesics over $\mathrm{SE}(2)$.

Theorem B. *A geodesic of the type θ-periodic is periodic if and only if $R = 0$, where R defines the curve α_μ from equation* (1)*.*

The proof of Theorem B lies in deriving formulas for a geodesic's θ period, then calculating the sub-Riemannian distance spanned by the x and y coordinates over this period. Following a simple calculation and contradiction argument, we prove that the only periodic geodesics over $\mathrm{SE}(2)$ are those of which $R = 0$ according to its reduced dynamics.

It is well known that a generic co-adjoint orbit has dimension two and is equivalent to $T^*\,\mathrm{SE}(2)\,//_{\overline{\mu}}\,\mathrm{SE}(2)$ as a symplectic manifold. The dimension of the space $T^*\,\mathrm{SE}(2)\,//_{\mu}\mathbb{R}^2$ is also two. The last main theorem shows that these spaces are equivalent.

Theorem C. *Let us identify $(\mathbb{R}^2)^*$ and $\mathfrak{se}(2)^*$ with $\mathbb{R}^2$ and $\mathfrak{so}(2)\times\mathbb{R}^2$, respectively. If μ is a nonzero element in $\mathbb{R}^2$ and $\overline{\mu} = (\mu_\theta, \mu)$ is in $\mathfrak{se}(2)^*$, then there exists a symplectomorphism from $T^*\,\mathrm{SE}(2)\,//_{\overline{\mu}}\,\mathrm{SE}(2)$ to $T^*\,\mathrm{SE}(2)\,//_{\mu}\mathbb{R}^2$.*

Although, when we consider the action of the whole group $\mathrm{SE}(2)$ we obtain three integrals of motion, one more than when we consider the action of the subgroup $\mathbb{R}^2$. The dimension of the co-isotropic group $SE(2)_\mu$ is one less than the one of $\mathbb{R}^2_\mu$. In this way the dimension of the reduced spaces are the same.

If $\mu = 0$, the statement is not true. Though $T^* \operatorname{SE}(2) //_\mu \mathbb{R}^2$ will still be isomorphic to $T^* \operatorname{SO}(2)$, $T^* \operatorname{SE}(2) //_{\overline{\mu}} \operatorname{SE}(2)$ becomes a single point.

1.1. **Organization of the paper.** The contents of this article is split into two parts: Part 1: Metric Lines and Part 2: Symplectic Reduction. In Section 2 we introduce the basic concepts in sub-Riemannian geometry, Lie groups, and sub-Riemannian geodesics. In Section 3 we apply this discussion on the setting of $\operatorname{SE}(2)$ as well as finish the proof to Theorem B. In order to finish the proof to Theorem A, we introduce the theory of Hamilton-Jacobi and calibration functions in Section 4 and complete the proof in Section 5.

We dedicate Part 2 into proving Theorem C. In sub-Sections 6.1 and 6.2, we introduce concepts in the adjoint action on $\operatorname{SE}(2)$ and symplectic geometry. In Section 7 we split the proof to Theorem C into four parts. In Proposition 7.1 we discuss the momentum maps of the action groups $\mathbb{R}^2$ and $\operatorname{SE}(2)$ on $T^* \operatorname{SE}(2)$. In Propositions 7.2 and 7.3, we calculate the symplectic reductions of of these actions before finishing the proof in sub-Section 7.1.

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Part 1. **Metric Lines**

2. Preliminaries

2.1. **Sub-Riemannian Geometry.**

Definition 2.1. *A sub-Riemannian geometry on a manifold M consists of a distribution, i.e. a sub-bundle $\mathcal{D} \subset TM$, equipped with a fiber inner-product $\langle \cdot, \cdot \rangle$.*

For further reading on sub-Riemannian geometry, see [1, Chapter 3] and [21, Chapter 1]. $\mathcal{D}$ is also called the **horizontal distribution**, we say an object on M is **horizontal** if it is tangent to $\mathcal{D}$.

Let γ be a smooth horizontal curve, we define the length of γ by

$$\ell(\gamma) = \int \|\dot{\gamma}\| dt,$$

where $\|\dot{\gamma}\| = \sqrt{\langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle}$ is computed by the inner product on the horizontal tangent space $\mathcal{D}_{\gamma(t)}$ and integrated over the domain of the curve. We define the **sub-Riemmanian distance** between the two points A and B in M as

$$d(A, B) = \inf\ \ell(\gamma)$$

where the infimum is taken over all smooth horizontal curves connecting A and B. If there are no curves connecting A and B, we say the distance is infinite.

We say that a distribution $\mathcal{D} \subset TM$ is **involutive** if for any two horizontal vector fields X and Y, their vector field bracket $[X, Y] := XY - YX$ is also horizontal. Then the Frobenius theorem [5, Theorem 8.3] tells us that the set of horizontal paths through a fixed point A sweeps out a smooth immersed sub-manifold whose dimension equals to the rank of the distribution. Let $N \subset M$ be an immersed submanifold of M, we say that N is an **integral manifold** of D if $T_pN = D_p$ for all $p \in N$. We say D is **integrable** if at each point of M there exists an integral manifold of D.

On the other hand, we say that a distribution $\mathcal{D}$ is **bracket-generating** if for any point q on M and any element X in T_qM, X can be generated by the Lie brackets of elements in $\mathcal{D}_q$. Chow's theorem ([20, Theorem 2.2]) states that if a distribution $\mathcal{D} \subset TM$ is bracket-generating, then the set of points that can be horizontally connected to a point A in M is the component of M containing A.

Therefore, if M is connected and the distribution is bracket-generating, any two points on M can be horizontally connected. For such a manifold, we can discuss minimizing curves.

Definition 2.2. *We say that a smooth horizontal path is a* ***minimizing geodesic*** *if it realizes the distance between two points.*

2.2. **Lie groups.** A Lie group consists of a group G with a smooth manifold structure such that the multiplication and inverse maps are smooth. For a detailed discussion on Lie Groups, see [3, 9, 23] or [14, Chapter 7].

Let us denote by $\mathfrak{g}$ the tangent space of the identity element. This vector space has the structure of a **Lie algebra** defined as follows: Let V be a real vector space, we define the **Lie Bracket** as the bilinear map $V \times V \to V$ such that it is antisymmetric and satisfies the Jacobi identity. That is, for all $X, Y, Z \in V$, it satisfies

- $[X, Y] = -[Y, X]$, and
- $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$.

For instance, on the space of n by n matrices, it follows that the commutator bracket $[A, B] = AB - BA$ for matrices A, B satisfies the definition of the Lie bracket.

Let $\gamma : \mathbb{R} \to G$ be an integral curve of a vector field $X \in \mathfrak{g}$, that is

$$\dot{\gamma}(t_0) = X_{\gamma(t_0)}$$

for all $t_0 \in \mathbb{R}$. We denote the set of all such γ's the **one parameter subgroup generated by** X. By general theory, there is a one-to-one correspondence between one parameter subgroups of G and T_eG (see [14, Chapter 20]).

2.3. **Hamilton equations.** Every manifold M possesses a contravariant tensor called the **cometric**, which is a section of the bundle $S^2(TM) \subset TM \otimes TM$, where $S^2(TM)$ denotes the space of symmetric bilinear forms on T^*M. In turn, this cometric defines a fiber-bilinear form $((\cdot, \cdot)) : T^*M \otimes T^*M \to \mathbb{R}$, for further discussion on cometrics, see [20, Page 7]. We define the **Hamiltonian function** $H : T^*M \to \mathbb{R}$ by

$$H(q, p) = \frac{1}{2}(p, p)_q,$$

where $p \in T^*M$ and $q \in M$.

2.3.1. *Sub-Riemannian geodesic flow.* On the cotangent bundle T^*M, we can define a a natural 1-form p, called the **tautological 1-form**, (consult [2, Section 37] or [14, Chapter 22] for the formal definition of the tautological 1-form). If we consider the canonical coordinates $\{p_1, ..., p_n, x_1, ..., x_n\}$ on the cotangent bundle T^*M, then it is given by the expression

$$p = p_{x_1} dx_1 + \cdots + p_{x_n} dx_n.$$

Let X be a vector field, we say that $P_X : T^*M \to \mathbb{R}$ is the **momentum function** associated to X if $P_X(p, x) = p(X(q))$, where p is the tautological one-form. On a sub-Riemannian manifold, let P_{X_i} be the associated momentum functions for an orthonormal frame $\{X_i\}$ over the manifold. Then the sub-Riemannian geodesic flow is governed by the Hamiltonian $H = \frac{1}{2} \sum P_{X_i}^2$ ([20, Proposition 1.5.5]).

In terms of canonical coordinates $\{p_1, ..., p_n, x_1, ..., x_n\}$ over M, the geodesic flow is governed by the Hamilton equations

$$\dot{x}^i(t) = \frac{\partial H}{\partial p^i}(x(t), p(t)) \text{ and } \dot{p}_i(t) = -\frac{\partial H}{\partial x^i}(x(t), p(t)). \tag{2}$$

for $i = 1, ..., n$. For an alternative derivation of the Hamilto equations, see [16, sub-Chapter 2.2]. Next, we discuss how a manifold can be endowed with a Poisson structure.

Definition 2.3. *Let P be a smooth manifold. A **Poisson bracket** on a manifold P is a bilinear operation $\{\cdot, \cdot\} : C^\infty(P) \times C^\infty(P) \to \mathbb{R}$ such that:*

- *$(C^\infty(P), \{\cdot, \cdot\})$ is a Lie algebra.*
- *$\{\cdot, \cdot\}$ is a derivation in each factor, i.e., for all F, G and $H \in C^\infty(P)$*

$$\{FG, H\} = \{F, H\}G + F\{G, H\}.$$

We say that a smooth manifold P is a Poisson manifold if P possesses a Poisson bracket.

In terms of canonical coordinates on T^*M, the Poisson bracket of two functions f, g is given by

$$\{f, g\} = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial x^i}.$$

In sub-Section 6.2, we expand further on our discussion of Poisson structures through the frame of symplectic geometry. With the Poisson Bracket defined, we have that the Hamiltonian equations (2) are equivalent as writing

$$\frac{df}{dt} = \{f, H\}.$$

for all $f \in C^\infty(T^*M)$ (see [20]sub-Section 1.7.

3. The Euclidean group as a sub-Riemannian manifold and the sub-Riemannian geodesic flow

3.1. $\mathrm{SE}(2)$ as a Lie group.

The special Euclidean group is a three-dimensional Lie group. By definition, it is the semidirect product $\mathrm{SE}(2) := \mathrm{SO}(2) \ltimes \mathbb{R}^2$, where $\mathrm{SO}(2)$ is the group of rotations in $\mathbb{R}^2$ and is diffeomorphic to the one-dimensional circle $\mathbf{S}^1$.

Let $(\theta, x, y) \in (0, 2\pi) \times \mathbb{R}^2$ be the coordinates over $\mathrm{SE}(2)$, then a point g in $\mathrm{SE}(2)$ has a matrix representation given by

$$g = \begin{pmatrix} R_\theta & \mathbf{x} \\ 0 & 1 \end{pmatrix}, \text{ where } R_\theta = \begin{pmatrix} cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \text{ and } \mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

If $R_\theta, R_{\theta'}$ are rotational two by two matrices and $\mathbf{x}, \mathbf{y}$ are column vectors in $\mathbb{R}^2$, then multiplication over $\mathrm{SE}(2)$ is given by

$$\begin{pmatrix} R_{\theta'} & \mathbf{x} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} R_\theta & \mathbf{y} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} R_{\theta'+\theta} & \mathbf{x} + R_{\theta'}\mathbf{y} \\ 0 & 1 \end{pmatrix}.$$

The Lie algebra $\mathfrak{se}(2)$ is spanned by the vectors $\{E_\theta, E_u, E_v\}$ given by

$$E_\theta = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_u = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{and} \quad E_v = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix};$$

with the Lie bracket relations

(3) $$[E_\theta, E_u] = E_v, \quad [E_\theta, E_v] = -E_u, \quad \text{and} \quad [E_u, E_v] = 0.$$

By equation (3), we have that $[\mathfrak{se}(2), \mathfrak{se}(2)]$ is spanned by the vectors $\{E_u, E_v\}$. Therefore, $[\mathfrak{se}(2), \mathfrak{se}(2)]$ is an abelian ideal making $\mathrm{SE}(2)$ a metabelian Lie group.

For the rest of the paper, we will write the left-invariant vector fields in terms of the operators $\frac{\partial}{\partial\theta}$, $\frac{\partial}{\partial x}$, and $\frac{\partial}{\partial y}$ rather than their matrix representations. For the definition of left-invariant vector fields, see [9, Chapter 7]. These vector fields are given by

(4) $$X_\theta = \frac{\partial}{\partial\theta}, \quad X_u = \cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y} \quad \text{and} \quad X_v = -\sin\theta\frac{\partial}{\partial x} + \cos\theta\frac{\partial}{\partial y}.$$

The frame $\{X_\theta, X_u\}$ forms a non-integrable distribution $\mathcal{D}$ over $\mathrm{SE}(2)$. The sub-Riemannian structure over $\mathrm{SE}(2)$ is constructed by declaring this frame to be orthonormal.

3.2. The cotangent bundle $T^*\mathrm{SE}(2)$.

Consider the cotangent bundle $T^*\mathrm{SE}(2)$ with the canonical coordinates $(p_\theta, p_x, p_y, \theta, x, y)$. The momentum functions associated to the left-invariant vector fields given in equation (4) are

$$P_\theta = p_\theta, \quad P_u = p_x\cos\theta + p_y\sin\theta, \quad \text{and} \quad P_v = -p_x\sin\theta + p_y\cos\theta.$$

As $\{X_\theta, X_u\}$ forms an orthonormal basis over the distribution on $\mathrm{SE}(2)$, the Hamiltonian function governing the sub-Riemannian geodesic flow is

(5) $$H(p,g) = \frac{1}{2}\left((P_\theta^2 + P_u^2) = \frac{1}{2}(p_\theta^2 + (p_x\cos\theta + p_y\sin\theta)^2\right)$$

For the formal definition of the sub-Riemannian geodesic flow, refer to [1, Chapter 3] or [20, Chapter 1].

If $(p(t), \gamma(t)) \in T^*\mathrm{SE}(2)$ is a solution to the Hamiltonian system (5) for $H(p,g)$, then the set of Hamilton equations

$$\dot\theta = \frac{\partial H}{\partial p_\theta}, \quad \dot x = \frac{\partial H}{\partial p_x}, \quad \text{and} \quad \dot y = \frac{\partial H}{\partial p_y}$$

implies that $\dot\gamma(t)$ is tangent to the distribution $\mathcal{D}$, since

(6) $$\dot\gamma(t) = P_\theta X_\theta(\gamma(t)) + P_u X_u(\gamma(t)).$$

When we choose the energy level $H(p,g) = \frac{1}{2}$, the corresponding geodesic $\gamma(t)$ is parameterized by arc-length.

To compute the differential equations for P_θ, P_u and P_v, we use the relation $\{P_X, P_y\} = P_{[X,Y]}$ between the Poisson bracket and vector field bracket given by for vectors X and Y, we compute that their time derivatives as

(7) $$\dot P_\theta = P_u P_v, \quad \dot P_u = -P_\theta P_v, \quad \text{and} \quad \dot P_v = P_\theta P_u.$$

Since the Hamiltonian function $H(p, g)$ does not depend on the variables x and y, p_x and p_y have derivative zero and thus are constant motions. Equivalently, the Hamiltonian function $H(p, g)$ is invariant under the action of $\mathbb{R}^2$ by left-multiplication. Therefore, the momentum map associated to the action is given by

$$J(p, g) = (p_x, p_y) = \mu \in \mathbb{R}^2,$$

where we identify $(\mathbb{R}^2)^*$ with $\mathbb{R}^2$ itself. For the formal definition of the momentum map, see sub-sub-Section 6.2.1. If $(p(t), \gamma(t))$ is a solution of the sub-Riemannian geodesic flow, then we say that a geodesic $\gamma(t)$ has **momentum** μ if $J(p(t), \gamma(t)) = \mu$.

3.2.1. *Reduced dynamics.* Let us consider the level set $\mu = (a, b)$, then the inverse image $J^{-1}(a, b)$ is diffeomorphic to $T^* \operatorname{SO}(2) \times \mathbb{R}^2 \times \mu$. We obtain the reduced Hamiltonian given by

$$H_\mu(p_\theta, \theta) = \frac{1}{2}(p_\theta^2 + R^2 \cos^2(\theta - \delta)), \tag{8}$$

where the bijection between (a, b) and (R, δ) is given by $(R \cos \delta, R \sin \delta) = (a, b)$. The reduced Hamilton equations are

$$\dot{p}_\theta = R^2 \cos(\theta - \delta) \sin(\theta - \delta) \quad \text{and} \quad \dot{\theta} = p_\theta. \tag{9}$$

We think of a point (p_θ, θ) in $T^* \operatorname{SO}(2)$ as a point in the cylinder $\mathbb{R} \times \operatorname{SO}(2) \simeq \mathbb{R} \times \mathbf{S}^1$. The level set $H_\mu^{-1}(\frac{1}{2})$ is the curve α_μ given by the equation 1.

$$\alpha_\mu := \{(p_\theta, \theta) \in \mathbb{R} \times \mathbf{S}^1 : \ 1 = p_\theta^2 + R^2 \cos^2(\theta - \delta)\} \tag{10}$$

3.2.2. *Background Theorem.* This sub-Section is dedicated to present the method to build sub-Riemannian geodesics and prove the **Background Theorem**. Let us prescribe the method as follows: consider the initial value problem given by the Hamilton equations (9) and the initial conditions $\alpha(t_0)$ in α_μ. Having found the solution $(p_\theta(t), \theta(t))$, we define the geodesic $\gamma(t)$ by the differential equation

$$\dot{\gamma}(t) = p_\theta(t) X_\theta(\gamma(t)) + R \cos(\theta(t) - \delta) X_u(\gamma(t)). \tag{11}$$

The **Background Theorem** states that γ is a geodesic with momentum μ

Background Theorem. *The above prescription yields a sub-Riemannian geodesic in* $\operatorname{SE}(2)$ *with momentum* μ *parameterized by arc-length. Conversely, every geodesic in* $\operatorname{SE}(2)$ *parameterized by arc-length with momentum* μ *can be achieved by this prescription applied to the curve* α_μ.

Proof. Let γ be a curve in $\operatorname{SE}(2)$ defined by equation (11) for a fixed value of μ. By construction, the curve γ is tangent to the distribution and by comparing equations (11) with (6), we conclude that it is enough to prove that the restriction of the left-invariant momentum functions $P_\theta(t)$, $P_u(t)$ and $P_v(t)$ restricted to the level set $J^{-1}(\mu)$ are equal to the functions

$$F_\theta(t) = \dot{\theta}(t), \quad F_u(t) = R \cos(\theta(t) - \delta), \quad \text{and} \quad F_v(t) = R \sin(\theta(t) - \delta),$$

respectively. Thus, we must prove that $F_\theta(t)$, $F_u(t)$, and $F_\mu(t)$ satisfy the equations given by (7). Using the reduced Hamilton equations given by equation (9), we have

$$\begin{aligned} \dot{P}_\theta(t) &= R^2 \cos(\theta(t) - \delta) \sin(\theta - \delta) = F_u(t) F_v(t), \\ \dot{P}_u(t) &= -R \sin(\theta - \delta) \dot{\theta} = -P_\theta(t) F_v(t), \\ \dot{F}_v(t) &= R \cos(\theta - \delta) \dot{\theta} = F_\theta(t) F_u(t). \end{aligned} \tag{12}$$

Therefore, the equations in (12) are identical to those from (7). We conclude that γ is a sub-Riemannian geodesic in $\mathrm{SE}(2)$ which by construction has momentum μ.

Conversely, let γ be an arbitrary geodesic in $\mathrm{SE}(2)$ parameterized by arc-length with momentum μ. The restriction of the Hamiltonian H to the level set $J^{-1}(\mu)$ is by definition the reduced Hamiltonian H_μ, the coordinates p_θ and θ satisfies the reduced Hamiltonian equations (9). In addition, the momentum functions $P_\theta(t)$ and $P_u(t)$ restricted to the level set $J^{-1}(\mu)$ have the form $P_\theta(t) = \dot\theta$ and $P_u(t) = R\cos(\theta(t) - \delta)$, the equation (6) is identical to (11). Thus, γ is achieved by the prescription applied to the curve α_μ. □

Let us describe the curve α_μ to understand the symmetries of the geodesic flow. The following proposition classifies the level set $H_\mu^{-1} = \alpha_\mu$.

Lemma 3.1. *The level set α_μ consists precisely of the following cruves:*

- *If $R > 1, \alpha_\mu$ consist precisely of two contractible, simple and closed smooth curves. The first curve is given by*

$$\alpha_\mu^1 = \{(p_\theta, \theta) = (\pm\sqrt{1 - R^2\cos^2(\theta - \delta)}, \theta) \mid \theta \in [\theta^1_{min}, \theta^1_{max}]\},$$

  where θ^1_{min} and θ^1_{max} are given by the solutions to the equation $\theta = \operatorname{arccos}(\frac{1}{R}) + \delta$ using the principal branch of arccos, then $[\theta^1_{min}, \theta^1_{max}] \subset [\delta, \pi + \delta]$.
  The second curve is given by

$$\alpha_\mu^2 = \{(p_\theta, \theta) = (\pm\sqrt{1 - R^2\cos^2(\theta - \delta)}, \theta) \mid \theta \in [\theta^2_{min}, \theta^2_{max}]\},$$

  where θ^2_{min} and θ^2_{max} are given the solution to the equation $\theta = \operatorname{arccos}(\frac{1}{R}) + \delta + \pi$ using the principal branch, then $[\theta^2_{min}, \theta^2_{max}] \subset [\delta + \pi, \delta + 2\pi]$.
- *If $R = 1, \alpha_\mu$ consists of precisely one non-contractible, non-simple and closed curve.*
- *If $0 < R < 1, \alpha_\mu$ consists of precisely two non-contractible, simple, and closed smooth curves given by*

$$\alpha_\mu^\pm = \{(p_\theta, \theta) = (\pm\sqrt{1 - R^2\cos^2(\theta - \delta)}, \theta) \mid \theta \in \mathbf{S}^1\}$$

Proof. By the inverse value theorem α_μ is smooth if and only if

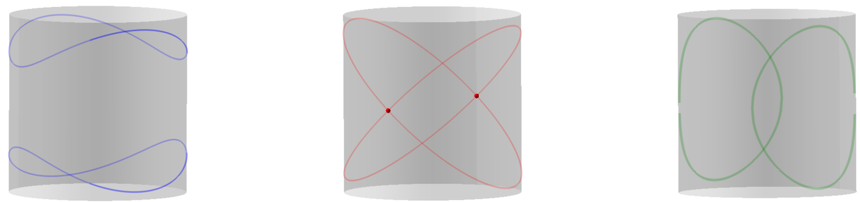
$$dH_\mu\big|_{\alpha_\mu} = (p_\theta, -R^2\cos(\theta - \delta)\sin(\theta - \delta)) \neq 0.$$

If $p_\theta \neq 0$ then $dH_\mu\big|_{\alpha_\mu} \neq 0$. Thus, it is enough to focus on the case $p_\theta = 0$. The condition $p_\theta = 0$ implies $R^2\cos^2(\theta - \delta) = 1$, thus $dH_\mu\big|_{\alpha_\mu} = 0$ if and only if $(p_\theta, \theta) = (0, \delta)$ or $(p_\theta, \theta) = (0, \pi + \delta)$. The conditions $\theta = \delta$ or $\theta = \pi + \delta$ imply $R = 1$. Therefore, α_μ is a smooth curve if $R \neq 1$.

If $R > 1$, then the level set α_μ is well defined when $0 \leq 1 - R^2\cos^2(\theta - \delta)$ and θ has two disjoint intervals where this inequality holds. When we parameterize the curves by θ this inequality yields the domains $[\theta^1_{min}, \theta^1_{max}]$ and $[\theta^2_{min}, \theta^2_{max}]$ as defined by Lemma 3.1. When $p_\theta = 0$ the positive and negative roots coincide, making each curve a simple closed curve.

If $R = 1$, then the level set α_μ is well defined for all θ. In addition, we can parameterize the level set by the expression $p_\theta = \pm|\sin(\theta - \delta)|$. When $p_\theta = 0$ the positive and negative root coincide, making the level set a non-simple and closed curve.

If $R > 1$, the level set α_μ is well defined for all θ. In addition, we can parameterize the curves by θ, the fact that $p_\theta \neq 0$ implies that the positive and negative root never coincide, making the level set consists of two non-contractible, simple, and closed curves. □



(A) $R > 1$ (B) $R = 1$ (C) $R < 1$

FIGURE 3.1. The panels show the curve α_μ for the three different cases

Remark 3.2. *We notice that the definition of the curves α_μ^1 and α_μ^2 for the $R > 1$ case can be extended to the case $R = 1$ when $[\theta^1_{min}, \theta^1_{max}] = [0, \pi]$, and $[\theta^2_{min}, \theta^2_{max}] = [\pi, 2\pi]$ are the domains of the curves, respectively. Similarly, the definition of the curves α_μ^+ and α_μ^- for the $0 < R < 1$ case can be extended to the case $R = 1$.*

The following lemma describe the symmetries of the reduced Hamiltonian flow, which helps us study the symmetries of the geodesic flow.

Lemma 3.3. *The reduced Hamiltonian has the following asymmetries:*

- *If $R \geq 1$ and $(p_\theta(t), \theta(t))$ is a solution laying in α_μ^1, then $(p_\theta(t), \theta(t) + \pi)$ is a solution laying in α_μ^2.*
- *If $R \leq 1$ and $(p_\theta(t), \theta(t))$ is a solution laying in α_μ^+, then $(-p_\theta(-t), \theta(-t))$ is a solution laying in α_μ^-.*

Proof. If $R \geq 1$, we notice that if (p_θ, θ) is a point in α_μ^1, then $(p_\theta, \theta + \pi)$ is a point in α_μ^2. Moreover, if $(p_\theta(t), \theta(t))$ is a solution, then it enough to prove that $(\tilde{p}_\theta, \tilde{\theta}) = (p_\theta(t), \theta(t) + \pi)$ is also a solution. Indeed, $(\tilde{p}_\theta, \tilde{\theta})$ satisfies the differential equations

$$\dot{\tilde{\theta}} = \dot{\theta} = p_\theta = \tilde{p}_\theta \quad \text{and}$$
$$\dot{\tilde{p}}_\theta = \dot{p}_\theta = R^2 \cos(\theta - \delta)\sin(\theta - \delta) = R^2 \cos(\tilde{\theta} - \delta)\sin(\tilde{\theta} - \delta).$$

If $R \leq 1$, we notice that if (p_θ, θ) is a point in α_μ^+, then $(-p_\theta, \theta)$ is a point in α_μ^-. The time reversibility of the reduced Hamiltonian system implies the second part of the lemma, i.e., if $(p_x(t), \theta(t))$ is a solution to the reduced Hamiltonian system (9), then $(-p_x(-t), \theta(-t))$ lays in α_μ^-. $\square$

3.2.3. *Classification of sub-Riemannian geodesics.* In this sub-sub-Section, we classify the sub-Riemannian geodesics according to their reduced dynamics. A geodesic γ can be classified as one and only one of the following three types:

(**Line**) We say a geodesic γ is of the type line if $\dot{\theta} = 0$. A geodesic is of the type line if and only if its reduced dynamics is trivial, i.e., if $R = 1$ and $\theta = \delta$ or $\theta = \delta + \pi$.

(**Heteroclinic**) We say a geodesic γ is of the type heteroclinic if its reduced dynamics is heteroclinic. The reduced dynamics is heteroclinic if and only if $R = 1$ and $\dot{\theta} \neq 0$.

(θ-**periodic**) We say a geodesic γ is of the type θ-periodic if its reduced dynamics is periodic. The reduced dynamics is periodic if and only if $R \neq 1$.

We further classify geodesics of the type θ-periodic as the two following types:

(**Libration**) We say a geodesic γ is of the type libration if its reduced dynamics has a libration solution. If θ is L-periodic, then the reduced dynamics has a libration solution, i.e., $\theta([t_0, t_0 + L]) \subset \mathrm{SO}(2)$, if and only if $R > 1$.

(**Oscillatory**) We say a geodesic γ is oscillatory if its reduced dynamics has a oscillatory solution. If θ is L-periodic, then the reduced dynamics has a oscillatory solution, i.e., $\theta([t_0, t_0 + L]) = \mathrm{SO}(2)$, if and only if $R < 1$.

Let $\pi : T^* \mathrm{SO}(2) \to \mathrm{SO}(2)$ be the canonical projection $\pi(p_\theta, \theta) = \theta$, we also denote β_μ as the projection of the curve α_μ by π within one period of θ. For geodesics of the type θ-periodic, the following proposition gives explicit formulas for the θ-period, as well as the distances travelled by the x and y coordinates during this period.

Proposition 3.4. *Let γ be a sub-Riemannian geodesic of the type θ-periodic corresponding to one of curves defined in Lemma 3.1 and β_μ be the projection of α_μ by the canonical projection $\pi : T^* \mathrm{SO}(2) \to \mathrm{SO}(2)$. Then, the θ-period is given by*

$$L(\mu) := \int_{\beta_\mu} \frac{d\theta}{\sqrt{1 - R^2 \cos^2(\theta - \delta)}}$$

The changes $\Delta x(\mu)$ and $\Delta y(\mu)$ perform by the coordinates x and y after the geodesic travels a period $L(\mu)$ are given by

$$\Delta x(\mu) := \int_{\beta_\mu} \frac{R \cos(\theta - \delta) \cos(\theta) d\theta}{\sqrt{1 - R^2 \cos^2(\theta - \delta)}} \quad \textit{and} \quad \Delta y(\mu) := \int_{\beta_\mu} \frac{R \cos(\theta - \delta) \sin(\theta) d\theta}{\sqrt{1 - R^2 \cos^2(\theta - \delta)}}.$$

In addition, the changes $\Delta x(\mu)$ and $\Delta y(\mu)$ are independent of the initial point.

To prove Proposition 3.4, we need to employ the calibration function which we will introduce in Section 4. Therefore, we will delay the proof until sub-Section 4.1. However, the above proposition allows us to prove Theorem B directly.

3.3. **Proof of Theorem B.**

Proof. From Proposition 3.4, we know that a geodesics γ of the type θ-periodic is periodic if and only if

$$\Delta x(\mu) = 0 \ \text{ and } \ \Delta y(\mu) = 0.$$

Let us proceed by contradiction; assuming $R \neq 0$, we will show that $\Delta x(\mu)$ and $\Delta y(\mu)$ cannot vanish simultaneously. Proposition 3.4 and the cosine addition formula implies

$$a \Delta x(\mu) + b \Delta y(\mu) = R^2 \int_{\beta_\mu} \frac{\cos^2(\theta - \delta) d\theta}{\sqrt{1 - R^2 \cos^2(\theta - \delta)}} > 0$$

Therefore, if $R \neq 0$, the geodesic γ is not periodic.

When $R = 0$, x and y are constant and θ is periodic, so the geodesic is periodic. □

4. Hamilton-Jacobi Equation and Calibration Functions

In this Section, Hamilton-Jacobi theory will be the main tool in proving Proposition 3.4. We will utilize Hamilton-Jacobi theory to build a calibration function. For more on the Hamilton-Jacobi theory, refer to [13] and to [22] on calibration functions.

Let us first introduce the proper definitions. Given a sub-Riemannian manifold M and a Hamiltonian function $H : T^*M \to \mathbb{R}$, the time-independent Hamilton Jacobi equation is a partial differential equation in $S : M \to \mathbb{R}$ given by

$$H(dS, q) = const, \tag{13}$$

where dS is the differential of S.

We can rewrite the Hamilton-Jacobi equation in the following way. First, let us introduce horizontal gradients.

Definition 4.1. *Let M be a sub-Riemannian manifold with distribution $\mathcal{D}$. For a function $S : M \to \mathbb{R}$, the horizontal gradient of S, denoted as $\nabla_{hor} S$, is the unique horizontal vector field satisfying, for every $q \in M$.*

$$\langle \nabla_{hor} S, v \rangle_q = dS(v),$$

for all $v \in \mathcal{D}_q$.

When the Hamiltonian H is purely kinetic (see [2, Chapter 2]), the Hamilton-Jacobi equation is also known as the **Eikonal equation** and we can rewrite 13 as $||\nabla_{hor} S|| = 1$. A solution S to the Eikonal equation has the property that it measure the oriented distance from the point q to a given sub-manifold. For a detailed discussion, see [20, sub-Chapter 1.5].

From our Hamiltonian function in equation (5), we see that the Hamilton Jacobi equation is a partial differential equation in S given by

$$1 = (\frac{\partial S_\mu}{\partial \theta})^2 + (\cos\theta \frac{\partial S_\mu}{\partial x} + \sin\theta \frac{\partial S_\mu}{\partial y})^2. \tag{14}$$

Let us solve the sub-Riemannian Eikonal equation by reducing it to an ordinary differential equation. Consider the ansatz

$$S_\mu(\theta, x, y) = f(\theta) + R\cos(\delta)x + R\sin(\delta)y. \tag{15}$$

Substituting into equation (14) we find that

$$1 = (f'(\theta))^2 + R^2\cos^2(\theta - \delta).$$

Therefore $f(\theta)$ is a solution to the differential equation

$$f'(\theta) = \pm\sqrt{1 - R^2\cos^2(\theta - \delta)},$$

and the solution S_μ is given by

$$S_\mu(\theta, x, y) = \pm \int \sqrt{1 - R^2\cos^2(\theta - \delta)} d\theta + R\cos(\delta)x + R\sin(\delta)y. \tag{16}$$

Proposition 4.2. *Let S_μ be a solution to the Eikonal equation given by equation* (16)*, and γ be a geodesic with momentum μ, then $\nabla_{hor} S_\mu = \dot{\gamma}$.*

Before we make the proof to Proposition 4.2, let us introduce the co-frame of left-invariant one forms:

$$\Theta_\theta = d\theta, \quad \Theta_u = \cos\theta dx + \sin\theta dy, \quad \text{and} \quad \Theta_v = -\sin\theta dx + \cos\theta dy.$$

Remark 4.3. *An alternative way to define the non-integrable distribution $\mathcal{D}$ over* $\mathrm{SE}(2)$ *is as the kernel of Θ_v, since $\Theta_v(X_\theta) = \Theta_v(X_u) = 0$.*

Proof. Let γ be a geodesic of momentum μ, without loss of generality let us consider the positive root to the partial differential equation (14) and assume that $\dot{\theta}$ is positive; then we can write the differential of S_μ in terms of the co-frame

$$\begin{aligned} dS_\mu &= \sqrt{1 - R^2\cos^2(\theta-\delta)}d\theta + R\cos(\delta)dx + R\sin(\delta)dy \\ &= \sqrt{1 - R^2\cos^2(\theta-\delta)}d\theta + R\cos(\theta-\delta)d\Theta_u + R\sin(\theta-\delta)d\Theta_v. \end{aligned}$$

Definition 4.1 gives us the horizontal gradient

$$\nabla_{hor} S_\mu = \sqrt{1 - R^2\cos^2(\theta-\delta)}X_\theta + R\cos(\theta-\delta)X_u.$$

By equation (11), we conclude that $\nabla_{hor} S_\mu = \dot{\gamma}(t)$.

□

Let us now introduce the definition of calibration functions.

Definition 4.4. *Let M be a sub-Riemannian manifold with distribution $\mathcal{D}$, we say that a function $S : M \to \mathbb{R}$ is a calibration function for the geodesic γ if the following conditions hold:*

- *$dS(\gamma'(t)) = 1$ for all t.*
- *$|dS(v)| \leq \|v\|$ for all vectors v in $\mathcal{D}$, where $\|\cdot\|$ is the sub-Riemannian norm given by $\|v\| := \sqrt{\langle v, v\rangle}$ for all vectors v in D.*

Lemma 4.5. *Let γ be a sub-Riemannian geodesic with momentum μ, then the function $S_\mu(\theta, x, y)$ given by* (16) *is a calibration function for γ.*

Proof. Let us prove the first condition from Definition 4.4. Proposition 4.2 implies $\dot{\gamma}(t) = \nabla_{hor} S_\mu$, by using Definition 4.1 we get

$$dS_\mu(\dot{\gamma}(t)) = \langle \dot{\gamma}(t), \dot{\gamma}(t)\rangle = 1.$$

To prove the second condition, let v be an arbitrary vector in $\mathcal{D}$, we have that

$$|dS_\mu(v)| = |\langle \nabla_{hor} S_\mu, v\rangle| \leq \|\nabla_{hor} S_\mu\|\|v\| = \|v\|$$

by the Cauchy–Schwarz inequality. Therefore, such S is a calibration function for γ. □

Having introduced calibration functions and its related results, we are now ready to prove Proposition 3.4.

4.1. **Proof of Proposition 3.4.**

Proof. From equations (9) and (10), since (θ, p_θ) lies in α_μ, we know that the time derivative of θ is given by

$$\frac{d\theta}{dt} = \pm\sqrt{1 - R^2\cos^2(\theta-\delta)}.$$

By the inverse function theorem, locally we can write t as a function of θ by

$$\frac{dt}{d\theta} = \pm\frac{1}{\sqrt{1 - R^2\cos^2(\theta-\delta)}}.$$

If $R < 1$, the geodesic's θ coordinates oscillates from 0 to 2π. The θ-period $L(\mu)$ satisfies that for any $t_0 \in \mathbb{R}$, we have $\theta(t_0 + L(\mu)) = \theta(t_0) + 2\pi$. Therefore the sign of $\frac{d\theta}{dt}$ doesn't change and we have

$$L(\mu) = \int_{t_0}^{t_0+L(\mu)} dt = \int_{\theta(t_0)}^{\theta(t_0+L(\mu))} \frac{dt}{d\theta}d\theta = \int_{\beta_\mu} \frac{d\theta}{\sqrt{1 - R^2\cos^2(\theta-\delta)}}.$$

If $R > 1$, the geodesic librates. So the sign of $\frac{d\theta}{dt}$ changes when $\theta = \delta \pm \arccos(\frac{1}{R})$ and $\theta = \delta \pm \arccos(\frac{1}{R}) + \pi$ with the principle branch of $\arccos$. If $\dot{\theta}(t^*) > 0$, then there exists $\epsilon > 0$ such that $\dot{\theta} > 0$ when $t \in [t^* - \epsilon, t* + \epsilon]$. Therefore,$[\theta(t^* - \epsilon), \theta(t^* + \epsilon)]$ is positively determined. Similarly, when $\dot{\theta}(t) < 0$, there will be a small interval where θ is negatively oriented. The negative part of $\frac{dt}{d\theta}$ integrated over a negatively oriented interval results in a positive integral. Therefore, we can write

$$L(\mu) = \int_{\theta(t_0)}^{\theta(t_0+L(\mu))} |\frac{dt}{d\theta}| d\theta = \int_{\beta_\mu} \frac{d\theta}{\sqrt{1 - R^2\cos^2(\theta - \delta)}}.$$

To determine the distance travelled by x and y, let us make use of the calibration function constructed in Lemma 4.5 and write

$$\begin{aligned} L(\mu) &= \int_{t_0}^{t_0+L(\mu)} 1 dt \\ &= \int_{t_0}^{t_0+L(\mu)} dS_\mu(\dot{\gamma}(t)) dt \\ &= \int_{\beta_\mu} \sqrt{1 - (a\cos\theta + b\sin\theta)^2} d\theta + a\Delta x(\mu) + b\Delta y(\mu) \end{aligned}$$

Therefore, from the two directions, we obtain the equation

$$\int_{\beta_\mu} \frac{d\theta}{\sqrt{1 - R^2\cos^2(\theta - \delta)}} = \int_{\beta_\mu} \sqrt{1 - (a\cos\theta + b\sin\theta)^2} d\theta + a\Delta x(\mu) + b\Delta y(\mu) \tag{17}$$

By taking the derivative with respect to a on equation (17), upon rearranging we obtain the relation

$$\Delta x(\mu) = \int_{\beta_\mu} \frac{(a\cos\theta + b\sin\theta)\cos\theta d\theta}{\sqrt{1 - (a\cos\theta + b\sin\theta)^2}} = \int_{\beta_\mu} \frac{R\cos(\theta - \delta)\cos(\theta) d\theta}{\sqrt{1 - R^2\cos^2(\theta - \delta)}}.$$

A similar calculation gives us the expression for $\Delta y(\mu)$. Since β_μ and δ only depends on α_μ, the changes are independent of the initial point as long as they remain on the same α_μ. □

4.2. **Minimizing Method.** We will once again utilize the calibration function constructed in Lemma 4.5 to study the metric lines on SE(2). The following proposition exemplifies this.

Proposition 4.6. *Let M be a sub-Riemannian Manifold and $S : M \to \mathbb{R}$ be a C^2 global solution of the Eikonal equation, then the integral curves of its horizontal gradient flow given by $\dot{\gamma}(t) = \nabla_{hor} S(\gamma(t))$ are metric lines.*

Proof. As S is a C^2 global solution on M, dS is an exact 1-form. By Stoke's theorem, for two arbitrary curves γ and $\widetilde{\gamma}$ sharing the same end points A and B in M, we have that

$$\int_\gamma dS = \int_{\widetilde{\gamma}} dS = S(A) - S(B)$$

Furthermore, for any smooth curve $\widetilde{\gamma}$ in a simply connected domain Ω, we have

$$\int_{\widetilde{\gamma}} dS = \int \langle \nabla S, \dot{\widetilde{\gamma}} \rangle dt \leq \int \|\dot{\widetilde{\gamma}}(t)\| \|\nabla_{hor} S(\widetilde{\gamma}(t))\| dt = \int_{\widetilde{\gamma}} \|\dot{\widetilde{\gamma}}\| dt = \ell(\widetilde{\gamma})$$

where ℓ is the sub-Riemannian length.

From the Cauchy–Schwarz inequality we have that the equality of the above inequality holds if and only if $\dot{\widetilde{\gamma}}(t) = f\nabla_{hor}S(\widetilde{\gamma}(t))$ for some scalar function f. That is, γ is a reparameterization of an integral curve of $\nabla_{hor}S$. Since any γ satisfying $\dot{\gamma}(t) = \nabla_{hor}S(\gamma(t))$ is an integral curve, therefore

$$dS(\dot{\gamma}) = \langle \nabla_{hor}S, \dot{\gamma}\rangle = \langle \nabla_{hor}S, \nabla_{hor}S\rangle = 1.$$

For any other curve $\widetilde{\gamma}$ in Ω, we have

$$dS(\dot{\widetilde{\gamma}}) = \langle \nabla_{hor}S, \dot{\widetilde{\gamma}}\rangle < \|\dot{\widetilde{\gamma}}\|$$

The inequality above is strict since $\widetilde{\gamma}$ is different from γ on at least an open set, because the two curves are smooth. Therefore, the inequality becomes

$$\ell(\gamma) = S(A) - S(B) = \int_{\widetilde{\gamma}} dS < \ell(\widetilde{\gamma}).$$

This completes the proof. □

In view of Proposition 4.6, in order to classify the metric lines on $\mathrm{SE}(2)$ we need to study when the calibration function defined in equation (16) is globally defined and C^2.

Proposition 4.7. *The calibration function given by equation* (16) *is not globally defined for any value of R. However, the calibration function given by*

$$S_\mu(\theta, x, y) = \pm cos(\theta - \delta) + x\cos(\delta) + y\sin(\delta)$$

is a globally defined and smooth calibration function for geodesics of the type heteroclinic and line corresponding to α_μ, where $R = 1$.

Proof. When $R = 1$, equation (16) gives

$$S_\mu(\theta, x, y) = \pm \int |\sin(\theta - \delta)| d\theta + x\cos(\delta) + y\sin(\delta).$$

This is not a globally smooth calibration function. However, the corresponding geodesic is either of the type heteroclinic or line. Therefore, θ is either constrained in $(\delta, \delta + \pi)$, $(\delta - \pi, \delta)$, or fixed at δ. Notice that the sign of $\sin(\theta - \delta)$ is fixed in each of the three regions. Therefore we can ignore the absolute value and write

$$S_\mu(\theta, x, y) = \pm \int \sin(\theta - \delta) d\theta + \cos(\delta)x + \sin(\delta)y = \mp\cos(\theta - \delta) + x\cos(\delta) + y\sin(\delta)$$

By an argument similar to the proof of Lemma 4.5, we can check that it is also a calibration function for our geodesic. □

By Propositions 4.6 and 4.7, the following result follows directly.

Proposition 4.8. *Line and heteroclinic geodesics corresponding to the curves defined in Lemma 3.1 are globally minimizing.*

5. Proof of Theorem A

Having completed Proposition 4.8 we have proved the first part of Theorem A. To complete the proof, it remains to show that geodesics of the type θ-periodic fail to qualify as metric lines. To do so, we will prove that such geodesics fails to minimize past its θ-period. The proof relies on the basic theory of Jacobi fields and conjugate points. For readers unfamiliar with the subject, refer to [10, sub-Chapter 4.8], [11, sub-Chapter 1.6], or [15, Chapter 10].

5.1. **Cut time in** $\mathrm{SE}(2)$**.** Let us formalize the definition of the cut time.

Definition 5.1. *Let γ be a sub-Riemannian geodesic parameterized by arc-length, we define the cut time of γ as*

$$t_{cut}(\gamma) = \sup\{t > 0 \mid \gamma|_{[0,t]} \text{ is length minimizing}\}.$$

If geodesics have a finite cut-time, then by definition it fails to be a metric line. The following proposition demonstrates that this is the case for geodesics of the type θ-periodic.

Proposition 5.2. *If a geodesic is of the type θ-periodic with period L, then L is an upper-bound for the cut time.*

Proof. Let $\gamma(t)$ be an L periodic sub-Riemannian geodesic with momentum μ and initial condition $\gamma(0) = (\theta_0, x_0, y_0)$, we will consider two cases when $\dot{\theta}_0 \neq 0$ and $\dot{\theta}_0 = 0$.

We first remark that the second case only corresponds to geodesics of the type libration ($R > 1$). First, we observe that since at every half period $L/2$, $\theta(t)$ reverses direction, we have

$$\dot{\theta}(t) = 0 \;\text{ for }\; t = \frac{kL}{2} \;\text{ and }\; k \in \mathbb{Z}.$$

By the **Background Theorem** and $\dot{\theta} = p_\theta$, we have that

$$\cos(\theta(\frac{kL}{2})) = \frac{1}{R^2},$$

which exists a solution if and only if $R > 1$, and we have that θ_0 is a solution to $\theta = \arccos(1/R^2)$.

Case $\dot{\theta}_0 \neq 0$: there are exactly two geodesics with initial condition $\gamma(0)$ and momentum μ, namely $\gamma(t)$ and $\tilde{\gamma(t)}$, the latter of which is defined as the one whose reduced dynamics are solution to the Hamiltonian H_μ and have the initial condition $(\tilde{p}_\theta(0), \tilde{\theta}(0)) = (-\dot{\theta}_0, \theta_0)$. The time reversibility (see Lemma 3.3) implies $\theta(t) = \tilde{\theta}(-t)$ for all t, and the periodicity gives $\theta(L) = \tilde{\theta}(L)$.

Moreover, we claim that $\gamma(L) = \tilde{\gamma}(L)$. This follows by Prop 3.4, which states the difference in $\tilde{x}$ and $\tilde{y}$ over the period L is the same as the difference for $\tilde{x}$ and $\tilde{y}$. That is to say

$$\gamma(L) = \gamma(0) + (0, \Delta x(\mu), \Delta y(\mu)) = \tilde{\gamma}(L)$$

Therefore, we have constructed two distinct geodesics meeting at time L, implying that any geodesics of the type θ-periodic fails to minimize past its period L.

Let μ be such that $R > 1$, then if $\dot{\theta}(0) = 0$, we have $\dot{\theta} = 0$ at the point $\theta \in \{\theta^1_{min}, \theta^1_{max}, \theta^1_{min}, \theta^2_{max}\}$ as defined in Lemma 3.1, we will show that $\gamma(L)$ is conjugate to $\gamma(0)$ along γ, thus γ is not minimizing past $\gamma(L)$. To do so, we will construct a killing vector field, which by general theory is a Jacobi field when restricted to a geodesic. By [20, page 8], a vector field K is a killing vector field if and only if its momentum function P_K commutes with the Hamiltonian with respect to the Poisson bracket, i.e, $\{H, P_K\} = 0$. It follows that $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ are killing vector fields, since

$$\dot{P}_{\frac{\partial}{\partial x}} = \{H, P_{\frac{\partial}{\partial x}}\} = 0 \quad\text{and}\quad \dot{P}_{\frac{\partial}{\partial y}} = \{H, P_{\frac{\partial}{\partial y}}\} = 0,$$

(see sub-sub-Section 2.3.1).

We are now ready to prove the second case of Proposition 5.2. Consider the two Jacobi fields on γ:

$$W_1(t) = \cos(\theta_0)\frac{\partial}{\partial x} + \sin(\theta_0)\frac{\partial}{\partial y} \quad \text{restricted to } \gamma, \text{ and}$$
$$W_2(t) = \dot{\gamma}(t).$$

We see that at $t = kL$, we have $W_2(kL) = X_u(\gamma(kL)) = \cos(\theta_0)\frac{\partial}{\partial x} + \sin(\theta_0)\frac{\partial}{\partial y}$. Therefore, $W_1(0) = W_1(L) = W_2(0) = W_2(L)$. Thus the Jacobi Field $J = W_1 - W_2$ vanishes at $t = 0$ and $t = L$. Moreover, J is not trivial since $\dot{\theta}(t) \neq 0$ along $0 < t < L/2$ and $L/2 < 0 < L$. At the time $t = L/2$, we have that

$$W_2(L/2) = \cos(\theta(L/2))\frac{\partial}{\partial x} + \sin(\theta(L/2))\frac{\partial}{\partial y} \neq W_1(L/2).$$

Showing that $\gamma(L)$ is a conjugate point and fails to minimize beyond $t = L$. □

5.2. **Proof of Theorem A.**

Proof. Proposition 4.8 shows that the geodesics of the type line and heteroclinic are metric lines. Proposition 5.2 implies that geodesics of type θ-periodic do not qualify as metric lines, since the period L is an upper-bound for its cut time.

Therefore, we conclude that the metric lines in $\mathrm{SE}(2)$ are precisely the geodesics of the type line and heteroclinic. □

Part 2. **Symplectic Reduction**

6. Preliminaries

In this section, we introduce the basic theories of group action, symplectic geometry, and reduction. For readers unfamiliar with these subjects, refer to [4, sub-Chapters 3.1-3.10] or [18].

6.1. **Group Action and the Action of $\mathbb{R}^2$ on $\mathrm{SE}(2)$.**

We say G is a group acting on a manifold M if each g in G induces a map from M to itself given by $(g, p) \mapsto g \cdot p$, where p, $g \cdot p \in M$. We say that an action is **free** if the only element in G which fixes every element in M is the identity element. We define the **orbit** of $p \in M$ as the set $\{g \cdot p \mid g \in G\}$. The space of all such orbits, denoted by M/G, with the quotient topology is called the **orbit space**.

Consider a continuous left action of a Lie group G on a manifold M. We say that this action is **proper** if the mapping $(g, p) \mapsto (g \cdot p, p)$ is a proper map for all $g \in G$ and $p \in M$. We recall that a map $F : X \to Y$ between two topological spaces X and Y is called a **proper map** if $F^{-1}(K)$ is compact for every compact set $K \subset Y$.

Let us present the following theorem, which allows us to study $\mathrm{SE}(2)/\mathbb{R}^2$ with the structure of a manifold.

Theorem 6.1. *(Quotient Manifold Theorem) Let G be a Lie group action on a manifold M acting smoothly, freely, and properly. Then the orbit space M/G is a manifold of dimension $\dim M - \dim G$, and the quotient map $\pi : M \to M/G$ is a smooth submersion.*

For a proof to Theorem 6.1, see [14, Chapter 21], and consult [24, Chapter 7] for a more extensive discussion on quotient topology theory.

Consider the left action of $\mathbb{R}^2$ on $\mathrm{SE}(2)$ given by

$$\begin{pmatrix} \mathbb{I} & \mathbf{x} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} R_\theta & \mathbf{y} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} R_\theta & \mathbf{x}+\mathbf{y} \\ 0 & 1 \end{pmatrix}$$

for column vectors $\mathbf{x}, \mathbf{y}$ in $\mathbb{R}^2$ and R_θ the two by two rotational matrix. It follows directly that this is a free action. It is also proper since $\mathbb{R}^2$ is a closed subgroup of $\mathrm{SE}(2)$, that is, the product and inverse operation on $\mathbb{R}^2$ is smooth.

By the Quotient Manifold Theorem 6.1, $\mathrm{SE}(2)/\mathbb{R}^2$ is a manifold of dimension 1. Moreover, by definition of the semi-direct product we can identify $\mathrm{SO}(2)$ with $\mathrm{SE}(2)/\mathbb{R}^2$.

6.1.1. *Adjoint action.* Let G be a Lie group, we define the conjugation map by $g \in G$ as the map $I_g : G \to G$ defined by $I_g(h) = ghg^{-1}$. We can then define the **adjoint action** $Ad_g : \mathfrak{g} \to \mathfrak{g}$ by $Ad_g = (I_g)_*$, where $(I_g)_*$ denotes the pushforward of I_g. For more details on the pullbacks and pushforwards of maps, see [15, Chapters 8 and 11]. The adjoint action induces the **co-adjoint action** between dual spaces as $Ad^*_{g^{-1}} : \mathfrak{g}^* \to \mathfrak{g}^*$ through the formula

$$(Ad^*_{g^{-1}}(\mu), \xi) = (\mu, Ad_{g^{-1}}(\xi))$$

for all $\mu \in \mathfrak{g}^*$ and $\xi \in \mathfrak{g}$. Here $(\cdot,\cdot)$ stands for the pairing for dual spaces, where $(\mu, \xi) = \mu(\xi)$. We define the **co-adjoint orbit** of $\mu \in \mathfrak{g}^*$ as the set

$$Orb(\mu) = \{Ad^*_{g^{-1}}(\mu) \mid g \in G\}.$$

Finally, for each $\mu \in \mathfrak{g}^*$, we define the **co-isotropic group** of a group G as

$$G_\mu = \{g \in G \mid Ad^*_g(\mu) = \mu\}.$$

Let us express the adjoint action on $\mathrm{SE}(2)$ with matrix representations. Let $g = (R_\theta, \mathbf{x})$ be in $\mathrm{SE}(2)$, we can calculate that the adjoint map Ad_g acting on the element

$$(\alpha, \mathbf{v}) = \begin{pmatrix} \alpha\mathbb{J} & \mathbf{v} \\ 0 & 0 \end{pmatrix} \in \mathfrak{se}(2)$$

, where $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, as

$$Ad_g(\alpha, \mathbf{v}) = \begin{pmatrix} -\alpha\mathbb{J} & \alpha\mathbb{J}\mathbf{x} + R_\theta\mathbf{v} \\ 0 & 0 \end{pmatrix}$$

Let $\mu = (\mu_\theta, \mathbf{u}) \in \mathfrak{se}(2)^*$ with the inner pairing of $(\mu, \mathbf{u})$ and $(\alpha, \mathbf{v})$ given by

$$((\mu_\theta, \mathbf{u}), (\alpha, \mathbf{v})) = \mu_\theta\alpha + \mathbf{u}\cdot\mathbf{v}$$

where $(\cdot)$ is the dot product, we find that the co-adjoint action is given by

$$Ad^*_g(\mu_\theta, \mathbf{u}) = (\mu_\theta + \mathbf{u}\cdot\mathbb{J}\mathbf{x}, R_\theta\mathbf{u}).$$

6.1.2. $\mathrm{SE}(2)$ *as a group action on* $T^*\,\mathrm{SE}(2)$. In the statement to Theorem C, we consider $\mathrm{SE}(2)$ as a group acting on its cotangent bundle $T^*\,\mathrm{SE}(2)$. Each element h in $\mathrm{SE}(2)$ induces a **left translation** map $L_h : \mathrm{SE}(2) \to \mathrm{SE}(2)$ by $L_h(g) = h \cdot g$ for $g \in \mathrm{SE}(2)$ and $(\cdot)$ is the group multiplication over $\mathrm{SE}(2)$. The left translation map induces a group action on $T^*\,\mathrm{SE}(2)$ in the following way.

Let (p, g) be an element in $T^*\,\mathrm{SE}(2)$ and h be an element in $\mathrm{SE}(2)$. We define the group action of $\mathrm{SE}(2)$ on $T^*\,\mathrm{SE}(2)$ to be the mapping

$$(h, (p, g)) = ((L_{h^{-1}})_* p, L_h g).$$

6.2. **Symplectic Geometry.** We say that a two-form ω is **symplectic** if ω is closed and non-degenerate. A manifold is **symplectic** if it is equipped with a symplectic form. As discussed in sub-sub-Section 2.3.1, every cotangent bundle T^*M posses a natural tautological 1-form:

$$p = p_1 dx_1 + \cdots + p_n dx_n$$

in canonical coordinates. This 1-form induces a symplectic structure on T^*M with the symplectic form $\omega = dp = \sum_i dp_i \wedge dx_i$, making T^*M a symplectic manifold. For more details, see [4, Chapter 5].

Let $H \in C^\infty(T^*M)$, we define its **Hamiltonian vector field** in terms of canonical coordinates to be

$$X_F = (\frac{\partial F}{\partial x_1}, ..., \frac{\partial F}{\partial x_n}, -\frac{\partial F}{\partial p_1}, ..., -\frac{\partial F}{\partial p_n}).$$

There is a connection between the symplectic and Poisson structure (see sub-sub-Section 2.3.1) in the following way: Every cotangent bundle T^*M has the structure of Poisson manifold, since its symplectic form induces a Poisson bracket by

$$\{F, G\} = \omega(X_F, X_G),$$

for all F, G in $C^\infty(T^*M)$.

Conversely, given a Poisson structure on a manifold, we can define the Hamiltonian vector field via $X_F = \{F, \cdot\}$ and define the bilinear form by $\omega(X_F, X_G) = \{F, G\}$. However, this ω is not defined everywhere, let alone closed and non-degenerate, so it is not a symplectic form. Therefore, we can treat symplectic geometry as a special case of Poisson geometry. For more details about Poisson geometry, see [4, Chapter 11]

6.2.1. *Momentum map and symplectic reduction.*

Definition 6.2. *Let (M, ω_M) and (N, ω_N) be two symplectic manifolds, we say that a map $\phi : M \to N$ is* ***symplectic*** *if it preserves the symplectic form, i.e., $\phi^*\omega_N = \omega_M$. In addition, we say that ϕ is a* ***symplectomorphism*** *if it is also an isomorphism. We say that (M, ω_M) and (N, ω_N) are* ***symplectomorphic*** *if there exists a symplectomorphism between them.*

Let (M, ω) be a symplectic manifold, consider the action of a Lie group G on M. Every $g \in G$ induces a left action map from M to itself. We say such group action is **symplectic** if every $g \in G$ induces a symplectic left action map. We define the **infinitesimal generator** of $\xi \in \mathfrak{g}$ to be a vector field σ_ξ on M by

$$\sigma_\xi(q) \coloneqq \frac{d}{dt} \exp(t\xi) \cdot q \Big|_{t=0},$$

for $q \in M$. See [23, Chapter 4] for the definition of the exponential map. Let us next define the momentum map.

Definition 6.3 (Momentum Map)**.** *Let a Lie group G act symplecticly on a manifold M. Suppose there is a linear map $P : \mathfrak{g} \to C^\infty(M)$ such that*

$$X_{P(\xi)} = \sigma_\xi, \quad \textit{for all } \xi \in \mathfrak{g}.$$

Then we call the map $J : M \to \mathfrak{g}^*$ the ***momentum map*** *of the action group G if*

$$\langle J(p), \xi \rangle = P_\xi(p), \quad \textit{for all } \xi \in \mathfrak{g} \textit{ and } p \in M.$$

Let $J : M \to \mathfrak{g}^*$ be the momentum map of the action group G on the manifold M and let $\mu \in \mathfrak{g}^*$, we define the **symplectic reduction** of M by G by

$$M//_\mu G \coloneqq J^{-1}(\mu)/G_\mu,$$

where G_μ is the co-isotropic group.

The following theorem tells us that a reduced symplectic manifold still possess a symplectic structure and that its symplectic form is unique. Let $i_\mu : J^{-1}(\mu) \to M$ denote the inclusion map and $\pi_\mu : J^{-1}(\mu) \to M//_\mu G$ be the projection mapping. We have the following result (For more details, see [4, Section 3.10])

Theorem 6.4 (Symplectic Reduction Theorem)**.** *There exists a unique symplectic form ω_μ on $M//G_\mu$ satisfying*

$$i_\mu^* \omega = \pi_\mu^* \omega_\mu.$$

7. Proof of Theorem C

Let us break down the proof to Theorem C in four components. In Proposition 7.1 we calculate the momentum maps of the action of $\mathbb{R}^2$ and $\mathrm{SE}(2)$ on $T^* \mathrm{SE}(2)$ respectively. In Propositions 7.2 and 7.3 we find the reductions of $T^* \mathrm{SE}(2)$ by these two actions. Finally, these three propositions allows us to complete the proof to Theorem C in sub-Section 7.1.

Proposition 7.1. *Let $\mathbb{R}^2$ be acting on $T^* \mathrm{SE}(2)$, then the momentum map $J_{\mathbb{R}^2} : T^* \mathrm{SE}(2) \to \mathbb{R}^2$ is given by*

$$J_{\mathbb{R}^2}(p, g) = (p_x, p_y),$$

where (p, g) denotes an element in $T^* \mathrm{SE}(2)$ in terms of the canonical coordinates. On the other hand, let $\mathrm{SE}(2)$ be acting on $T^* \mathrm{SE}(2)$, then the momentum map $J_{\mathrm{SE}(2)} : T^* \mathrm{SE}(2) \to \mathrm{SE}(2)^*$ is given by

$$J_{\mathrm{SE}(2)}(p, g) = (p_\theta - y p_x + x p_y, p_x, p_y).$$

Proof. For an element $g \in \mathbb{R}^2$, we can treat it as an element in $\mathrm{SE}(2)$ given by the matrix representation

$$g = \begin{pmatrix} \mathbb{I} & \mathbf{x} \\ 0 & 1 \end{pmatrix}$$

where $\mathbf{x} = (x, y)^T$ is a column vector in $\mathbb{R}^2$. Next, we consider $\xi = (u, v) \in (\mathbb{R}^2)^* \simeq \mathbb{R}^2$, where we identity ξ as a row vector in $\mathbb{R}^2$. We calculate that the infinitesimal generator of ξ to be the vector field σ_ξ given by

$$\sigma_\xi = \frac{d}{dt} \exp(\gamma(t))(p, g)\Big|_{t=0} = (0, 0, 0, \xi),$$

where $\gamma(t)$ is a curve in $\mathbb{R}^2$ subject to the initial conditions

$$\gamma(0) = (0, 0) \text{ and } \dot{\gamma}(0) = \xi.$$

Next we find a linear map $P : \mathbb{R}^2 \to C^\infty(T^* \mathrm{SE}(2))$ such that it satisfies $X_{P(\xi)} = \sigma_\xi$. This is equivalent to writing

$$\frac{\partial P(\xi)}{\partial x} = \frac{\partial P(\xi)}{\partial y} = 0, \quad \frac{\partial P(\xi)}{\partial p_x} = u, \text{ and } \frac{\partial P(\xi)}{\partial p_y} = v.$$

Thus the solution is given by

$$P(\xi)(p, g) = p_x u + p_y v.$$

We obtain the momentum map $J_{\mathbb{R}^2}$ by the relation

$$\langle J_{\mathbb{R}^2}(p, g), \xi \rangle = P(\xi)(p, g) = p_x u + p_y v.$$

Giving us the expression for $J_{\mathbb{R}^2}$ as

$$J_{\mathbb{R}^2}(p, g) = (p_x, p_y).$$

A similar calculation gives us the expression for the momentum map $J_{\mathrm{SE}(2)}$. $\square$

In the next two propositions, we will show that both reductions $T^*\,\mathrm{SE}(2)\,//_{\mu}\mathbb{R}^2$ and $T^*\,\mathrm{SE}(2)\,//_{\overline{\mu}}\,\mathrm{SE}(2)$ can be identified with $T^*\,\mathrm{SO}(2)$ for $\mu \neq 0$.

Proposition 7.2. *Let μ be in $\mathbb{R}^2$, then there exist a symplectomorphism from $(T^*\,\mathrm{SE}(2)\,//_{\mu}\mathbb{R}^2, \omega_\mu)$ to $T^*\,\mathrm{SO}(2)$, where $T^*\,\mathrm{SO}(2)$ possesses the symplectic form $\omega = dp_\theta \wedge d\theta$.*

Proof. For a fixed $\mu \in \mathbb{R}^2$, since $\mathbb{R}^2$ is an Abelian group, its co-isotropic group is given by $\mathbb{R}^2_\mu = \mathbb{R}^2$. By Proposition 7.1, we find that the inverse of the momentum map $J^{-1}_{\mathbb{R}^2}(\mu)$ is given by

$$J^{-1}_{\mathbb{R}^2}(\mu) = \{(p,g) \in T^*\,\mathrm{SE}(2)\ |(p_x,p_y) = \mu\}\,.$$

We have that $J^{-1}_{\mathbb{R}^2}(\mu) \simeq T^*\,\mathrm{SO}(2) \times \mathbb{R}^2 \times \mu$, since the trivialization of $T^*\,\mathrm{SE}(2)$ implies $T^*\,\mathrm{SE}(2) \simeq T^*\,\mathrm{SO}(2) \times T^*\mathbb{R}^2$; therefore, the symplectic reduction implies

$$T^*\,\mathrm{SE}(2)\,//_{\mu}\mathbb{R}^2 = \{[(p_\theta,\mu,R_\theta,x,y)] \mid (p_\theta,\mu,R_\theta,x,y) \in J^{-1}_{\mathbb{R}^2}(\mu)\}$$

where [ ] denotes the equivalence class induced by the equivalence relation

$$(p_\theta,\mu,R_\theta,x,y) \sim (\widetilde{p_\theta},\mu,\widetilde{R_\theta},\widetilde{x},\widetilde{y}) \text{ if and only if } (p_\theta,R_\theta) = (\widetilde{p_\theta},\widetilde{R_\theta}).$$

Therefore, if we consider the basis $\{\frac{\partial}{\partial p_\theta},\frac{\partial}{\partial \theta},\frac{\partial}{\partial x},\frac{\partial}{\partial y}\}$ for $T(J^{-1}_{\mathbb{R}^2}(\mu)) \subset T(T^*\,\mathrm{SE}(2))$ and let $\mathbf{v}_1$ and $\mathbf{v}_2$ be arbitrary vectors in $T(J^{-1}_{\mathbb{R}^2}(\mu))$, where we use the short-hand notation (a_i,b_i,c_i,d_i) for $\mathbf{v}_i = a_i\frac{\partial}{\partial p_\theta} + b_i\frac{\partial}{\partial \theta} + c_i\frac{\partial}{\partial x} + d_i\frac{\partial}{\partial y}$, then $[\mathbf{v}_i]$ are vectors in $T(T^*\,\mathrm{SE}(2)\,//_{\mu}\mathbb{R}^2)$ and $\mathbf{v}_1 \sim \mathbf{v}_2$ if and only if $(a_1,b_1) = (a_2,b_2)$.

The projection $\pi : J^{-1}_{\mathbb{R}^2}(\mu) \to T^*\,\mathrm{SE}(2)\,//_{\mu}\mathbb{R}^2$ given by the Quotient Manifold Theorem 6.1 has the form :

$$\pi_\mu(p_\theta,\mu,R_\theta,x,y) = [(p_\theta,\mu,R_\theta,x,y)]$$

There is also a natural inclusion $i_\mu : J^{-1}_{\mathbb{R}^2}(\mu) \to T^*\,\mathrm{SE}(2)$, since $J^{-1}_{\mathbb{R}^2}(\mu)$ is a sub-manifold of $T^*\,\mathrm{SE}(2)$. Hence by the Symplectic Reduction Theorem 6.4, we can find the unique reduced symplectic form ω_μ on $T^*\,\mathrm{SE}(2)\,//_{\mu}\mathbb{R}^2$ given by $\pi^*\omega_\mu = i^*\omega$, where

$$\omega = dp_\theta \wedge d\theta + dp_x \wedge dx + dp_y \wedge dy$$

is the symplectic form on $T^*\,\mathrm{SE}(2)$. By consturction, we know the symplectic form ω_μ defined by $\omega_\mu([\mathbf{v}_1],[\mathbf{v}_2]) = a_1b_2 - a_2b_1$ is a closed, non-degenerate two form on $T^*\,\mathrm{SE}(2)\,//_{\mu}\mathbb{R}^2$ and satisfies $\pi^*\omega_\mu = i^*\omega$. By the uniqueness statement of the Symplectic Reduction Theorem 6.4, this is the symplectic form we want.

Then we can build a map $\varphi : T^*\,\mathrm{SE}(2)\,//_{\mu}\mathbb{R}^2 \to T^*\,\mathrm{SO}(2)$ given by

$$\varphi_1([(p_\theta,\mu,R_\theta,x,y)]) = (p_\theta,R_\theta).$$

Since the symplectic form on $T^*\,\mathrm{SO}(2)$ is $\omega = dp_\theta \wedge d\theta$, it is easy to check that φ_1 is a symplectomorphism. $\square$

Proposition 7.3. *Let us identify $\mathfrak{se}(2)$ with $\mathfrak{so}(2) \times \mathbb{R}^2$ and let $\overline{\mu} = (\mu_\theta,\mu)$ be in $\mathfrak{se}(2)^*$.*

- *If $\mu \neq 0$, then there also exist a symplectomorphism from $(T^*\,\mathrm{SE}(2)\,//_{\overline{\mu}}\,\mathrm{SE}(2), \omega_{\overline{\mu}})$ to $T^*\,\mathrm{SO}(2)$ with the symplectic form $\omega = dp_\theta \wedge d\theta$.*
- *If $\mu = 0$, the symplectic reduction $T^*\,\mathrm{SE}(2)\,//_{\overline{\mu}}\,\mathrm{SE}(2)$ consists of one point.*

Proof. First let us consider a nonzero $\mu \in \mathbb{R}^2$. For a fixed $\overline{\mu} = (\mu_\theta, \mu) \in \mathfrak{se}(2)^*$, the Preimage Theorem implies that the inverse of the momentum map is a three dimensional manifold:

$$J^{-1}_{\mathrm{SE}(2)}(\overline{\mu}) = \{(p,g) \in T^* \mathrm{SE}(2) \mid (p,g) = (\mu_\theta + c, \mu, R_\theta, s\mu + \frac{c}{||\mu||^2_{\mathbb{R}^2}} \mathbb{J}\mu),$$

where $c, s \in \mathbb{R}$ and $R_\theta \in \mathrm{SO}(2)\}$.

Next we calculate the co-isotropic group $SE_{\overline{\mu}}(2)$. Recalling from 6.1.1, we look for all $g = (R_\theta, \mathbf{x})$ in $\mathrm{SE}(2)$ such that $Ad^*_g(\overline{\mu}) = \overline{\mu}$,i.e.,

$$(\mu_\theta + (\mu_x, \mu_y)\mathbb{J}\mathbf{x}, R_\theta \cdot (\mu_x, \mu_y)) = \overline{\mu}$$

Solving the equation above, we find that $\mathrm{SE}(2)_{\overline{\mu}}$ is given by

$$\mathrm{SE}(2)_{\overline{\mu}} = \{(\mathbb{I}, \mathbf{x}) \in \mathrm{SE}(2) \mid \mathbf{x} = s\mu \text{ where } s \in \mathbb{R}\}.$$

Therefore, with a similar approach from the Proof of Proposition 7.2, we can obtain the symplectic reduction given by

$$T^* \mathrm{SE}(2) //_{\overline{\mu}} \mathrm{SE}(2) \simeq \{(p,g) \in T^* \mathrm{SE}(2) \mid (p,g) = (\mu_\theta + c, \mu, R_\theta, \frac{c}{||\mu||^2_{\mathbb{R}^2}} \mathbb{J}\mu),$$

where $R_\theta \in \mathrm{SO}(2), c \in \mathbb{R}\}$, with the unique symplectic form

$$\omega_{\overline{\mu}} = d(\mu_\theta + c) \wedge d\theta = dc \wedge d\theta.$$

Then we can build the map $\varphi_2 : T^* \mathrm{SE}(2) //_{\overline{\mu}} \mathrm{SE}(2) \to T^* \mathrm{SO}(2)$ given by

$$\varphi_2((\mu_\theta + c, \mu, R_\theta, \frac{c}{||\mu||^2_{\mathbb{R}^2}} \mathbb{J}\mu)) = (c, R_\theta).$$

Since the symplectic form on $T^* \mathrm{SO}(2)$ is $\omega = dp_\theta \wedge d\theta$, it is easy to check that φ_2 is a symplectomorphism.

If $\mu = 0$, we can obtain the inverse of the momentum map given by

$$J^{-1}_{\mathrm{SE}(2)}(\overline{\mu}) = \{(p,g) \in T^* \mathrm{SE}(2) \mid (p,g) = (\mu_\theta, 0, 0, R_\theta, x, y),$$

where $R_\theta \in \mathrm{SO}(2)$ and $x, y \in \mathbb{R}\}$. In addition, it can be identified with $\mathrm{SE}(2)$. We also have that $\mathrm{SE}(2)_{\overline{\mu}} = \mathrm{SE}(2)$. Therefore we have the symplectic reduction is a single point since $J^{-1}_{\mathrm{SE}(2)}(\overline{\mu}) \simeq \mathrm{SE}(2) = \mathrm{SE}(2)_{\overline{\mu}}$ □

7.1. **Proof of Theorem C.**

Proof. From Propositions 7.2 and 7.3, we have that if $\mu \neq 0$, both $T^* \mathrm{SE}(2) //_\mu \mathbb{R}^2$ and $T^* \mathrm{SE}(2) //_{\overline{\mu}} \mathrm{SE}(2)$ are symplectomorphic to $T^* \mathrm{SO}(2)$. Let the corresponding symplectomorphisms be φ_1 and φ_2 from their respective proofs of Propositions 7.2 and 7.3, we can check that $\varphi := \varphi_2^{-1} \circ \varphi_1$ is the symplectomorphism between $T^* \mathrm{SE}(2) //_\mu \mathbb{R}^2$ and $T^* \mathrm{SE}(2) //_{\overline{\mu}} \mathrm{SE}(2)$. □

8. Conclusion and Future Work

We classified the geodesics on $\mathrm{SE}(2)$ with regard to their reduced dynamics. With the help of Proposition 3.4, we showed that a geodesic is periodic if and only if $R = 0$. Afterwards, we introduced calibration functions and the Hamilton-Jacobi method to show that there is a finite cut time for a θ-periodic geodesic. Then from Theorem A, we gave a full characterization of metric lines on $\mathrm{SE}(2)$. In Part 2, we introduced the symplectic reduction and showed that there exists a symplectomorphism between $T^* \mathrm{SE}(2) //_\mu \mathbb{R}^2$ and $T^* \mathrm{SE}(2) //_{\overline{\mu}} \mathrm{SE}(2)$, indicating that they have the same symplectic structure.

In the future, we hope to continue our work in the following possible directions:

(1) Extend our characterizations of the metric lines and periodic geodesics to $\mathrm{SE}(3)$, or more generally to $\mathrm{SE}(n)$.
(2) Study the eigenvalue problems and fundamental solutions of the sub-Riemannian Laplace, Heat, and Schrödinger operators on $\mathrm{SE}(2)$.

References

[1] Andrei Agrachev, Davide Barilari, and Ugo Boscain. *A Comprehensive Introduction to Sub-Riemannian Geometry*. Cambridge University Press, 2019.
[2] Vladimir Igorevich Arnold. *Mathematical methods of classical mechanics*. Springer Science.
[3] Andrew Baker. *Matrix groups: An introduction to lie group theory*. Springer International Publishing, 2012.
[4] Anthony Bloch. *Nonholonomic mechanis and control*. Springer Science.
[5] William M. Boothby. *An introduction to differentiable manifolds and Riemannian geometry*. Academic Press, 1985.
[6] Alejandro Bravo-Doddoli. Metric lines in jet space, 2023.
[7] Alejandro Bravo-Doddoli, Enrico Le Donne, and Nicola Paddeu. Sympletic reduction of the sub-riemannian geodesic flow for metabelian nilpotent groups, 2023.
[8] Alejandro Bravo-Doddoli and Richard Montgomery. Geodesics in Jet Space. *Regular and Chaotic Dynamics*, 2021.
[9] Daniel Bump. *Lie Groups*. Graduate Texts in Mathematics. Springer, 2004.
[10] Christian Bär. *Elementary Differential Geometry*. Cambridge Press, 2010.
[11] Jeff Cheeger and David G. Ebin. *Comparison theorems in Riemannian geometry*. North-Holland, 1975.
[12] Albert Fathi. *The Weak KAM Theorem in Lagrangian Dynamics*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2008.
[13] Alessio Figalli and Ludovic Rifford. Aubry sets, hamilton-jacobi equations, and the mañé conjecture. *Geometric Analysis, Mathematical Relativity, and Nonlinear Partial Differential Equations*, pages 83–104, 2013.
[14] John M. Lee. *Introduction to Smooth Manifolds*. Graduate Texts in Mathematics. Springer, 2003.
[15] John M. Lee. *Introduction to Riemannian Manifolds*. Graduate Texts in Mathematics. Springer, 2021.
[16] Jerrold Marsden. *Hamiltonian reduction by stages*. Springer Verlag, 2006.
[17] Jerrold Marsden, Tudor Ratiu, and Alan Weinstein. Semidirect product and reduction in mechanics. *Trans. Am. Math. Soc*, 2007.
[18] Jerrold Marsden and Alan Weinstein. Reduction of symplectic manifolds with symmetry. *Reports on mathematical physics*, 5(1):121–130, 1974.
[19] Igor Moiseev and Yuri L. Sachkov. Maxwell strata in sub-riemannian problem on the group of motions of a plane. *ESAIM: Control, Optimisation and Calculus of Variations*, 2010.
[20] Richard Montgomery. *A tour of subriemannian geometries, their geodesics and applications*. Number 91. American Mathematical Soc., 2002.
[21] Ludovic Rifford. *Sub-riemannian geometry and Optimal Transport*. Springer, 2014.
[22] Takashi Sakai. On riemannian manifolds admitting a function whose gradient is of constant norm. *Kodai Mathematical Journal,*, 19:39–51, 1996.
[23] John Stillwell. *Naive Lie Theory*. Undergraduate Texts in Mathematics. Springer, 2008.
[24] Loring W. Tu. *An introduction to manifolds*. Universitext. Springer, 2011.

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