HILBERT–KUNZ MULTIPLICITY IN A TWO-PARAMETER FAMILY

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Abstract. We compute the Hilbert–Kunz multiplicity of the Segre product of Monsky's point-S_4 and line-S_4 quartic hypersurfaces, showing that the Hilbert–Kunz multiplicity may vary in a two-parameter family. Appealing to a result of Trivedi, our computation requires an analysis of the Hilbert–Kunz density functions of the aforementioned hypersurfaces.

1. Introduction

Let $(R, \mathfrak{m})$ denote either a Noetherian local ring or a standard graded ring over a field with unique (resp. homogeneous) maximal ideal $\mathfrak{m}$. Further assume that R has prime characteristic $p > 0$, and denote $d = \dim(R)$. The *Hilbert–Kunz function of* R, denoted $e_n(R)$, is defined to be

$$n \mapsto \ell_R(R/\mathfrak{m}^{[p^n]})$$

where $\mathfrak{m}^{[p^n]}$ denotes the ideal generated by elements of the form r^{p^n} for $r \in \mathfrak{m}$. The main result of [Mon83] says that $e_n(R) = e_{\mathrm{HK}}(R) \cdot p^{nd} + O(p^{(d-1)n})$, where the leading term $e_{\mathrm{HK}}(R)$ is called the *Hilbert–Kunz multiplicity* of R. The goal of the present article is to give an explicit formula for the Hilbert–Kunz multiplicity of the fibers of a certain two parameter family $\Bbbk[s,t] \to R$. This family is inspired by two well known one-parameter families considered in [Mon98a; Mon98b], and in fact our ring R is obtained as the Segre product of the rings considered in *op. cit.*.

Let $\Bbbk$ be an algebraically closed field of characteristic 2 and let $\alpha \in \Bbbk^*$. In the sequel, we will consider the irreducible quartic polynomials

$$g_\alpha = \alpha x^2 y^2 + x^3 z + y^3 z + z^4 + xyz^2$$
$$h_\alpha = \alpha z^4 + (x^2 + yz)(y^2 + xz)$$

in $\Bbbk[x,y,z]$. The Hilbert–Kunz theory of these polynomials demonstrates a jump in severity between the generic fiber and a special fiber, according to the algebraic (resp. dynamical) complexity of the parameter α. More specifically, the Hilbert–Kunz function of g_α depends on the degree of the field extension $[\mathbb{F}_2(\lambda) : \mathbb{F}_2] =: m(\alpha)$ where $\alpha = \lambda^2 + \lambda$, whereas that of h_α depends on the stopping time (denoted in the sequel by $l(\alpha)$) of a certain dynamical system attached to α — see Settings 2.3 and 3.1 respectively for a summary of the results of [Mon98a; Mon98b]. These hypersurfaces are also notable for exhibiting tight closure's failure to localize [BM10; Bor+24].

The primary contribution of this article is a closed formula for the Hilbert–Kunz multiplicity of the Segre products of the above quartic hypersurfaces. For $\gamma \in \Bbbk^*$, we denote $R_\gamma := \Bbbk[x,y,z]/(g_\alpha)$ and $S_\gamma := \Bbbk[x,y,z]/(h_\alpha)$.

Theorem 1.1. *Let $\alpha,\beta \in \Bbbk^*$. Then*

$$e_{\mathrm{HK}}(R_\alpha \# R_\beta) = 12 - 2\cdot 2^{-m(\alpha)-2m(\beta)} + 6(4^{-m(\alpha)} + 4^{-m(\beta)}) - \frac{2}{3}\cdot 8^{-m(\alpha)} \text{ if } m(\alpha) \leq m(\beta)$$

$$e_{\mathrm{HK}}(S_\alpha \# S_\beta) = 12 - 2\cdot 2^{-2l(\alpha)-4l(\beta)} + 6(4^{-2l(\alpha)} + 4^{-2l(\beta)}) - \frac{2}{3}\cdot 8^{-2l(\alpha)} \text{ if } l(\alpha) \leq l(\beta)$$

$$e_{\mathrm{HK}}(R_\alpha \# S_\beta) = \begin{cases} 12 - 2\cdot 2^{-m(\alpha)-4l(\beta)} + 6(4^{-m(\alpha)} + 4^{-2l(\beta)}) - \frac{2}{3}\cdot 8^{-m(\alpha)} : & m(\alpha) \leq 2l(\beta) \\ 12 - 2\cdot 2^{-2l(\beta)-2m(\beta)} + 6(4^{-2l(\beta)} + 4^{-m(\alpha)}) - \frac{2}{3}\cdot 8^{-2l(\beta)} : & 2l(\beta) \leq m(\alpha) \end{cases}$$

Our method for computing the above multiplicities involves understanding the *Hilbert–Kunz density functions* of the g_α and h_α, as introduced by Trivedi in [Tri18]. These are compactly supported continuous real-valued functions which allow one to recover the Hilbert–Kunz multiplicity by integration. Moreover, a tool emerging from *op. cit.* is that the density function of the Segre product may be realized as a certain convolution of the density functions of each factor. We summarize Trivedi's results in Section 2.

2. Density Function: Point-S_4 Quartics

In this section we compute the Hilbert–Kunz density function for Monsky's point-S_4 quartic polynomial. First, we recall the facts from [Tri18] that will be pertinent to us.

2.1. Trivedi's density function.

Theorem 2.1 ([Tri18]). *Let $(R,\mathfrak{m})$ be a Noetherian ring which is standard graded over a perfect field of prime characteristic $p > 0$ with $d = \dim(R) \geq 2$. Then there exists a uniformly convergent sequence $\{h_n : \mathbb{R} \to \mathbb{R}\}_{n\in\mathbb{N}}$ of compactly supported piecewise linear continuous functions such that*

(1) $\mathrm{HKD}(R)(w) := \lim_{n\to\infty} h_n(w)$ *is a compactly supported function with*

$$e_{\mathrm{HK}}(R) = \int_{\mathbb{R}} \mathrm{HKD}(R)(w)\, \mathrm{d}w;$$

(2) $\mathrm{HKD}(R)(-)$ *may be computed via* $\mathrm{HKD}(w) = \lim_{n\to\infty} f_n(w)$ *where*

$$f_n(w) := \frac{\ell_R\left((R/\mathfrak{m}^{[p^n]})_{\lfloor wp^n \rfloor}\right)}{p^{n(d-1)}}.$$

The fact we will leverage is the multiplicative nature of the density function under Segre products, which is the key ingredient in proving Theorem 1.1. We recall this result at the required level of generality for us, which is much more restrictive than what is proven in *op. cit.*

Theorem 2.2 ([Tri18, Proposition 2.17]). *Let $R_1,\dots,R_T$ be Noetherian rings of $\dim(R_i) = 2$ which are standard graded over a common algebraically closed field of prime characteristic $p > 0$. Further assume that the R_i all have the same Hilbert–Samuel multiplicity e_{HS}. Then:*

$$\mathrm{HKD}(R_1 \# \cdots \# R_T)(w) = e_{HS}^T w^T - \prod_{i=1}^{T} (e_{HS} w - \mathrm{HKD}(R_i)(w))$$

$$= \sum_{i=1}^{T} (-1)^{i+1} e_{HS}^{T-i} w^{T-i} e_i(\mathrm{HKD}(R_1)(w), \dots, \mathrm{HKD}(R_T)(w))$$

where e_i is the i-th elementary symmetric polynomial.

2.2. **Point-S_4 quartics.** We assume the following setup for the remainder of this section.

Setting 2.3. For an element $\alpha \in \Bbbk^*$ which is algebraic over $\mathbb{F}_2$ write $\alpha = \lambda^2 + \lambda$. Then $m(\alpha)$ is defined to be the degree of the field extension $\mathbb{F}_2(\lambda) \supseteq \mathbb{F}_2$ (for $\alpha \in \Bbbk^*$ transcendental over $\mathbb{F}_2$, set $m(\alpha) = \infty$). For ease of notation, we let $g = g_\alpha = \alpha x^2y^2 + x^3z + y^3z + z^4 + xyz^2$, $m = m(\alpha)$, and $q = 2^{m+r}$ for $r \geq 1$. Let $\mathbb{O} = \Bbbk[x,y,z]/\mathfrak{m}^{[q]}$ is the artinian graded ring where $\mathfrak{m} = (x,y,z)$. Then $\varphi_i : \mathbb{O}_i \to \mathbb{O}_{i+4}$ denotes multiplication by g, with kernel and cokernel N_i and C_i, respectively. If $0 \neq u \in \mathbb{O}_t$ is written in descending powers of z, i.e. $u = A_l(x,y)z^l + \cdots + A_1(x,y)z + A_0(x,y)$, we define the *$z$-degree* to be $\deg_z(u) := l$.

We provide a dossier for the facts from [Mon98a] that will be used in this section.

Fact 2.4. Let $n \geq 2$.

(1) [Mon98a, Theorem 3.1(2), Lemma 4.15, Lemma 4.16] When $n = m+1$, $(0) \neq N_{3\cdot 2^{n-1}-5} = \operatorname{span}_\Bbbk\{u\}$ and $N_{3\cdot 2^{n-1}-4} = \operatorname{span}_\Bbbk\{xu, yu, zu\}$. Moreover, this principal generator has $\deg_z(u) = 2^n - 4$.
(2) [Mon98a, Lemma 4.7] For every $n \geqslant m+1$, $N_{3\cdot 2^{n-1}-4-2^{n-m-1}} \neq (0)$.
(3) [Mon98a, Lemma 4.11(a)] For every $n \geqslant m+2$ and $i < 3\cdot 2^{n-1} - 4 - 2^{n-m-1}$, $N_i = (0)$.
(4) [Mon98a, Lemma 4.5] For every n and $i \leqslant 2^{n+1} - 6$, if $0 \neq v \in N_i$ and $0 \neq L(x,y,z)$ is a linear form, then $Lv \neq 0$.
(5) [Mon98a, Lemma 4.8] If $n \geqslant m+1$ and $0 \neq v \in N_{3\cdot 2^{n-1}-4-2^{n-m-1}}$ then for every $0 \leqslant s \leqslant 2^{n-m-1}$, the set $\{x^a y^b v \mid a+b = s\}$ is linearly independent.
(6) [Mon98a, Lemma 4.1] $\dim_\Bbbk(\mathbb{O}_n/g_\alpha\mathbb{O}_n) = 3\cdot 4^n - 4 + 2\sum_{i=0}^{3\cdot 2^{n-1}-4} \dim_\Bbbk N_i$.
(7) [Mon98a, Theorem 4.14 & Theorem 4.18] $\dim_\Bbbk(\mathbb{O}_n/g_\alpha\mathbb{O}_n) = 3\cdot 4^n - 4$ for $1 \leqslant n \leqslant m$, and $\dim_\Bbbk(\mathbb{O}_n/g_\alpha\mathbb{O}_n) = 3\cdot 4^n + 4^{n-m}$ otherwise.

Lemma 2.5. For each $r \geqslant 1$, there is a nonzero element $u \in N_{3q/2-4-2^{r-1}}$. Moreover, u has z-degree $q-4$.

Proof. Argue by induction on r, with the base case given by Fact 2.4(1). Given $u \in N_{3q/2-4-2^{r-1}}$ we set $v = gu^2$. As $gu \in \mathfrak{m}^{[q]}$, $g^2u^2 \in \mathfrak{m}^{[2q]}$ so indeed v is annihilated by g. By induction, u has z-degree $q-4$, and as g has z-degree 4, v has z-degree $2q-4$ as long as its leading coefficient is not in $\mathfrak{m}^{[2q]}$. Write $u = \sum_{i=0}^{q-4} P_i z^i$ with each $P_i(x,y)$ homogeneous of degree $3q/2 - 4 - 2^{r-1} - i$. Then the leading coefficient of v is P_{q-4}^2 with degree $q - 2^r$. In particular, $P_{q-4}^2 \notin \mathfrak{m}^{[2q]}$. ◻

Lemma 2.6. Let $u \in N_{3q/2-4-2^{r-1}}$ be given by Lemma 2.5. Then for each $0 \leqslant s \leqslant 2^{r-1}$, the set

$$L_s := \{x^{d_1}y^{d_2}z^{d_3}u \mid d_1 + d_2 + d_3 = s, d_3 < 4\}$$

is linearly independent in $\mathbb{O}_{3q/2-4-2^{r-1}+s}$.

Proof. If some linear combination is trivial, then $L = Q_0 + Q_1 z + Q_2 z^2 + Q_3 z^3$ annihilates u with each $Q_j(x,y)$ homogeneous of degree $s - j$. Write $u = \sum_{i=0}^{q-4} P_i z^i$ with each $P_i(x,y)$ homogeneous of degree $3q/2 - 4 - 2^{r-1} - i$. We then compute

$$0 = uL = P_{q-4}Q_3 z^{q-1} + (P_{q-5}Q_3 + P_{q-4}Q_2)z^{q-2} + (P_{q-6}Q_3 + P_{q-5}Q_2 + P_{q-4}Q_1)z^{q-3}$$

$$+ (P_{q-7}Q_3 + P_{q-6}Q_2 + P_{q-5}Q_1 + P_{q-4}Q_0)z^{q-4} + \sum_{i=5}^{q} z^{q-i} \sum_{j=0}^{3} P_{q-j-i}Q_j.$$

The first four coefficients must all be 0 because the z-degrees are less than q. Since P_{q-4} is nonzero with degree $q/2 - 2^{r-1}$ and each Q_j has degree $s - j \leqslant s \leqslant 2^{r-1}$, we observe that each $P_{q-4}Q_j$ has degree at most $q/2$, hence must be 0 in $\Bbbk[x,y]$. By iterated substitution, $Q_3 = Q_2 = Q_1 = Q_0 = 0$ as desired. □

Corollary 2.7. For $s \leqslant 3$, $N_{3q/2-4-2^{r-1}+s}$ has dimension at least $\binom{s+2}{2}$ and for $4 \leqslant s \leqslant 2^{r-1}$, $N_{3q/2-4-2^{r-1}+s}$ has dimension at least $4s - 2$.

Proof. By Lemmas 2.5 and 2.6, the cardinality of L_s gives lower bounds. For $s \leqslant 3$, every monomial is indivisible by z^4, hence L_s has cardinality $\binom{s+2}{2}$. For $s \geqslant 4$, L_s corresponds to the monomials of degree s in 3 variables, except for those divisible by z^4, of which there are $\binom{s+2}{2} - \binom{s-2}{2} = 4s - 2$. □

Lemma 2.8. In the same situation as Lemma 2.6, L_s is a basis for $N_{3q/2-4-2^{r-1}+s}$.

Proof. We showed linear independence in Lemma 2.6 so it remains to show L_s spans. Fact 2.4(3) gives that $N_i = 0$ for all $i < 3q/2 - 4 - 2^{r-1}$ when $r \geqslant 2$. In case $r = 1$, we use Fact 2.4(1,6,7) to see

$$3q^2 + 4 = 3q^2 - 4 + 2\left(4 + \sum_{i=0}^{3q/2-6} \dim N_i\right) = 3q^2 + 4 + \sum_{i=0}^{3q/2-6} \dim N_i.$$

Thus each $\dim N_i = 0$ for $i < 3q/2 - 4 - 2^{r-1}$ as well. Then Fact 2.4(6,7) yields

$$3q^2 + 4^r = 3q^2 - 4 + 2\sum_{i=0}^{3q/2-4} \dim N_i = 3q^2 - 4 + 2\sum_{s=0}^{2^{r-1}} \dim N_{3q/2-4-2^{r-1}+s}.$$

We use the bounds from Corollary 2.7 with the above equality to see

$$3q^2 + 4^r \geqslant 3q^2 - 4 + 40 + \sum_{s=4}^{2^{r-1}} (s+1)(s+2) - (s-3)(s-2) = 3q^2 + 36 + 4\sum_{s=4}^{2^{r-1}} 2s - 1$$

with equality if and only if each L_s is a basis. As

$$\sum_{s=4}^{2^{r-1}} 2s - 1 = 2\sum_{s=4}^{2^{r-1}} s - \sum_{s=4}^{2^{r-1}} 1 = (2^{r-1}+1)2^{r-1} - 12 - (2^{r-1} - 3) = 4^{r-1} - 9,$$

we get equality. □

Corollary 2.9. For all $i \leqslant 3q/2 - 4$,

$$\dim N_i = \begin{cases} 0 & \text{if } i < 3q/2 - 4 - 2^{r-1} \\ \frac{(s+2)(s+1)}{2} & \text{if } i = 3q/2 - 4 - 2^{r-1} + s \text{ with } 0 \leqslant s \leqslant 3 \\ 4s - 2 & \text{if } i = 3q/2 - 4 - 2^{r-1} + s \text{ with } 4 \leqslant s \leqslant 2^{r-1}. \end{cases}$$

Proof. By Corollary 2.7 and Lemma 2.8. □

Lemma 2.10. $\mathcal{O}_i$ has dimension $\binom{i+2}{2}$ if $i < q$ and dimension $\binom{i+2}{2} - 3\binom{i-q+2}{2}$ if $q \leqslant i \leqslant 3q/2$.

Proof. A basis for $\mathcal{O}_i$ is given by monomials of degree i indivisible by x^q, y^q, and z^q. Because $3q/2 \leqslant 2q-1$, it follows that a monomial of degree i is divisible by at most one of x^q, y^q, or z^q. If $i < q$, $\mathcal{O}_i$ has basis given by the monomials of degree i, of which there are $\binom{i+2}{2}$. If $i \geqslant q$, there are exactly $\binom{i-q+2}{2}$ monomials divisible by x^q, and the same is true of y^q and z^q. Thus $\mathcal{O}_i$ has dimension $\binom{i+2}{2} - 3\binom{i-q+2}{2}$. □

Lemma 2.11. For all i, C_i has dimension $\dim C_i = \dim N_i - \dim \mathcal{O}_i + \dim \mathcal{O}_{i+4}$. In addition, $\dim C_i = \dim N_{3q-7-i}$.

Proof. The first claim follows from additivity of dimension on exact sequences applied to

$$0 \to N_i \to \mathcal{O}_i \to \mathcal{O}_{i+4} \to C_i \to 0.$$

The second claim follows by applying the duality functor $\operatorname{Hom}_{\mathcal{O}}(-, \mathcal{O}(3q-3))$ to the above exact sequence. □

Lemma 2.12. For $i \geqslant -4$ and $r \geqslant 3$, we compute the dimensions of C_i below.

Proof. We divide into cases for i and will use Corollary 2.9 and Lemmas 2.10 and 2.11 repeatedly.

Case 1: $-4 \leqslant i < 0$. Here $N_i = 0 = \mathcal{O}_i$, so $\dim \mathcal{O}_{i+4} = \dim C_i$, thus

$$\dim C_i = \binom{i+6}{2} = \frac{(i+6)(i+5)}{2}.$$

Case 2: $0 \leqslant i < q-4$. Since $i < 3q/2 - 4 - 2^{r-1}$, we get $\dim N_i = 0$, $\dim \mathcal{O}_i = \binom{i+2}{2}$ and $\dim \mathcal{O}_{i+4} = \binom{i+6}{2}$. Thus

$$\dim C_i = \frac{(i+6)(i+5)}{2} - \frac{(i+2)(i+1)}{2} = 4i + 14.$$

Case 3: $i = q-4+k$ with $k = 0,1,2,3$. Because $i < 3q/2 - 4 - 2^{r-1}$, we have $\dim N_i = 0$, $\dim \mathcal{O}_i = \binom{i+2}{2}$ and $\dim \mathcal{O}_{i+4} = \binom{i+6}{2} - 3\binom{k+2}{2}$. Thus

$$\dim C_i = 4i + 14 - \frac{3}{2}(k+2)(k+1) = 4i + 14 - \frac{3}{2}(i-q+6)(i-q+5).$$

Case 4: $i = q+k$, $0 \leqslant k < q/2 - 2^{r-1} - 4$. We check that $i < 3q/2 - 4 - 2^{r-1}$. In this case, $\dim N_i = 0$, $\dim \mathcal{O}_i = \binom{i+2}{2} - 3\binom{k+2}{2}$ and $\dim \mathcal{O}_{i+4} = \binom{i+6}{2} - 3\binom{k+6}{2}$ so

$$\dim C_i = 4i - 12k - 28 = 12q - 8i - 28.$$

Case 5: $i = q+k$, $k = q/2 - 2^{r-1} - 4 + s$ with $s = 0,1,2,3$. Here we have $i = 3q/2 - 4 - 2^{r-1} + s$, so $\dim N_i = \binom{s+2}{2}$, $\dim \mathcal{O}_i = \binom{i+2}{2} - 3\binom{k+2}{2}$ and $\dim \mathcal{O}_{i+4} = \binom{i+6}{2} - 3\binom{k+6}{2}$, hence

$$\dim C_i = \frac{(s+2)(s+1)}{2} + 4i - 12k - 28 = \frac{(s+2)(s+1)}{2} + 12q - 8i - 28.$$

Case 6: $i = q+k$, $k = q/2 - 2^{r-1} - 4 + s$ with $4 \leqslant s \leqslant 2^{r-1}$. We have $\dim N_i = 4s - 2$, $\dim \mathcal{O}_i = \binom{i+2}{2} - 3\binom{k+2}{2}$, and $\dim \mathcal{O}_{i+4} = \binom{i+6}{2} - 3\binom{k+6}{2}$. Then

$$\dim C_i = 4s + 4i - 12k - 30.$$

Expanding, and letting $s = i - 3q/2 + 2^{r-1} + 4$,

$$\dim C_i = 12q - 8i - 28 + 4s - 2 = (6 + \frac{1}{2^{m-1}})q - 4i - 14.$$

Case 7: $3q/2-3 \leqslant i < 3q/2-6+2^{r-1}$. Let $s = 3q/2-3+2^{r-1}-i$, so $4 \leqslant s \leqslant 2^{r-1}$. We observe $3q/2-2^{r-1} \leqslant 3q-7-i \leqslant 3q/2-4$, so we let $j = 3q-7-i$ and $j = 3q/2-4-2^{r-1}+s$. By Lemma 2.11 and Corollary 2.9,

$$\dim C_i = \dim N_j = 4s-2 = 6q-14+2^{r+1}-4i = (6+\frac{1}{2^{m-1}})q-4i-14.$$

Case 8: $3q/2-6+2^{r-1} \leqslant i < 3q/2-2+2^{r-1}$. We have $3q/2-2^{r-1}-4 \leqslant 3q-7-i < 3q/2-2^{r-1}$, so we let $j = 3q-7-i = 3q/2-4-2^{r-1}+s$ with $0 \leqslant s \leqslant 3$. Then

$$\dim C_i = \dim N_j = \frac{(s+2)(s+1)}{2} = \frac{(3q/2-1+2^{r-1}-i)(3q/2-2+2^{r-1}-i)}{2}.$$

Case 9: $3q/2-2+2^{r-1} \leqslant i$. Then $3q-7-i < 3q/2-4-2^{r-1}$, so

$$\dim C_i = \dim N_{3q-7-i} = 0.$$

□

Corollary 2.13. In the notation of Theorem 2.1, $f_n(x)$ is a piecewise function supported on $[-\frac{4}{q}, \frac{3}{2}-\frac{2}{q}+2^{-m-1}]$ given below for $r \geqslant 2$ where $i = \lfloor wq \rfloor$ and $q = 2^n = 2^{m+r}$:

$$q f_n(w) = \begin{cases} \frac{(i+6)(i+5)}{2} & \text{if } -\frac{4}{q} \leqslant w < 0 \\ 4i+14 & \text{if } 0 \leqslant w < 1-\frac{4}{q} \\ 4i+14-\frac{3}{2}(i-q+6)(i-q+5) & \text{if } 1-\frac{4}{q} \leqslant w < 1 \\ 4(3q-2i-7) & \text{if } 1 \leqslant w < \frac{3}{2}-\frac{1}{2^{m+1}}-\frac{4}{q} \\ \frac{(i-3q/2+2^{r-1}+6)(i-3q/2+2^{r-1}+5)}{2}+4(3q-2i-7) & \text{if } \frac{3}{2}-\frac{1}{2^{m+1}}-\frac{4}{q} \leqslant w < \frac{3}{2}-\frac{1}{2^{m+1}} \\ (6+\frac{1}{2^{m-1}})q-4i-14 & \text{if } \frac{3}{2}-\frac{1}{2^{m+1}} \leqslant w < \frac{3}{2}-\frac{6}{q}+\frac{1}{2^{m+1}} \\ \frac{(3q/2-1+2^{r-1}-i)(3q/2-2+2^{r-1}-i)}{2} & \text{if } \frac{3}{2}-\frac{6}{q}+\frac{1}{2^{m+1}} \leqslant w < \frac{3}{2}-\frac{2}{q}+\frac{1}{2^{m+1}}. \end{cases}$$

Proof. The claim follows from Fact 2.4(5) and the fact that $a \leqslant \lfloor wq \rfloor < b$ iff $a/q \leqslant w < b/q$ for any $a, b \in \mathbb{Z}$.

□

Lemma 2.14. Continuing the notation from Corollary 2.13, the Hilbert–Kunz density function $f = \lim_{n\to\infty} f_n$ of $\Bbbk[x,y,z]/(g_\alpha)$ is supported on $[0, \frac{3}{2}+2^{-m-1}]$ and is given below:

$$f(w) = \begin{cases} 4w & \text{if } 0 \leqslant w < 1 \\ 12-8w & \text{if } 1 \leqslant w < \frac{3}{2}-\frac{1}{2^{m+1}} \\ 6+2^{1-m}-4w & \text{if } \frac{3}{2}-\frac{1}{2^{m+1}} \leqslant w < \frac{3}{2}+\frac{1}{2^{m+1}}. \end{cases}$$

Proof. We notice $\lim_{n\to\infty} \frac{i}{q} = w$ since

$$\lfloor wq \rfloor \leqslant wq < \lfloor wq \rfloor + 1.$$

The result is then easily derived from Corollary 2.13. □

We now let $\alpha \in \Bbbk^*$ be transcendental over $\mathbb{F}_2$.

Lemma 2.15. For each i, the dimension of C_i is given below:

$$\dim C_i = \begin{cases} \binom{i+6}{2} & \text{if } -4 \leqslant i < 0 \\ 4i + 14 & \text{if } 0 \leqslant i < q-4 \\ 4i + 14 - 3\binom{i-q+6}{2} & \text{if } q-4 \leqslant i < q \\ 12q - 8i - 28 & \text{if } q \leqslant i < \frac{3q}{2} - 3 \\ 0 & \text{if } i \geqslant 3q/2 - 3. \end{cases}$$

Proof. Since α is transcendental, $m = \infty$. Thus $n < m$ for all n, hence Fact 2.4(7) tells us that $e_n(g) = 3q^2 - 4$. By comparing with Fact 2.4(6), we get that N_i is trivial for $i \leqslant \frac{3q}{2} - 4$. The claim then follows from Lemmas 2.10 and 2.11. □

Lemma 2.16. With the notation of Theorem 2.1, $f_n(w)$ is a piecewise function supported on $[-\frac{4}{q}, \frac{3}{2} - \frac{3}{q}]$ and is given below, where $i = \lfloor wq \rfloor$ and $q = 2^n$:

$$q f_n(w) = \begin{cases} \binom{i+6}{2} & \text{if } -4/q \leqslant w < 0 \\ 4i + 14 & \text{if } 0 \leqslant w < 1 - \frac{4}{q} \\ 4i + 14 - 3\binom{i-q+6}{2} & \text{if } 1 - \frac{4}{q} \leqslant w < 1 \\ 12q - 8i - 28 & \text{if } 1 \leqslant w < \frac{3}{2} - \frac{3}{q} \\ 0 & \text{if } w \geqslant \frac{3}{2} - \frac{3}{q}. \end{cases}$$

Proof. This follows from Lemma 2.15 and $a \leqslant \lfloor wq \rfloor < b$ iff $a/q \leqslant w < b/q$ for integers a, b. □

3. Density Function: Line-S_4 Quartics

In this section we compute the Hilbert–Kunz density function of Monsky's line-S_4 quartic polynomial. We assume the following setup.

Setting 3.1. For $\alpha \in \Bbbk^*$, let $\varphi_\alpha : \Bbbk \cup \{\infty\} \to \Bbbk \cup \{\infty\}$ given by $t \mapsto t^4 + \alpha t^{-4}$. We define $\varphi_\alpha^{(r)}$ to be the r-fold composition of φ_α with itself, and say the *escape time* $l(\alpha)$ of α is the positive integer r such that $\varphi_\alpha^{(r)}(1) = 0$ (if there is no such integer, we set $l(\alpha) = \infty$). Note that there is at most one integer r such that $\varphi_\alpha^{(r)}(1) = 0$ because if $\varphi_\alpha^{(r)}(1) = 0$, then $\varphi_\alpha^{(s)}(1) = \infty$ for all $s > r$. Let $h = h_\alpha = \alpha z^4 + (x^2 + yz)(y^2 + xz)$, $l = l(\alpha)$ and $q = 2^{2l+r}$ for $r \geqslant 1$ and finite l. In addition, $\varphi_i : \mathcal{O}_i \to \mathcal{O}_{i+4}$ denotes multiplication by h, with kernel and cokernel N_i and C_i, respectively. Lastly, we consider elements of $\mathcal{O}$ to be polynomials in z with coefficients being polynomials in x and y. We then say the *lowest term* of a polynomial is its term with minimal z-degree.

In analogy with Fact 2.4, we list some facts about h_α established in [Mon98b].

Fact 3.2. Let $n \geq 2$.

(1) [Mon98b, Theorem 4.8] When $n = 2l + 1$, $N_{3q/2-5} \neq (0)$.

(2) [Mon98a, Lemma 4.1] $\dim_\Bbbk(\mathcal{O}_n / g_\alpha \mathcal{O}_n) = 3 \cdot 4^n - 4 + 2 \sum_{i=0}^{3 \cdot 2^{n-1} - 4} \dim_\Bbbk N_i$.

(3) [Mon98b, Corollary 3.15 & Theorem 5.8] $\dim_\Bbbk(\mathcal{O}_n / g_\alpha \mathcal{O}_n) = 3 \cdot 4^n - 4$ for $1 \leqslant n \leqslant 2l$, and $\dim_\Bbbk(\mathcal{O}_n / g_\alpha \mathcal{O}_n) = 3 \cdot 4^n + 4^{n-2l}$ otherwise.

Lemma 3.3. Let $q = 2^{2l+1}$ and $u \in N_{3q/2-5}$ have lowest term Pz^c. Then $c \leqslant q/2 - 3$.

Proof. We notice that the lowest term of hu is $x^2y^2Pz^c$. Since h annihilates u and $c < q$ (else $u = 0$), x^2y^2 annihilates P. Thus every term in P is divisible by either x^{q-2} or y^{q-2}. This implies that P has degree at least $q-2$. On the other hand, P has degree $3q/2-5-c$. Putting the two together gives the result. □

Lemma 3.4. In the setting of Lemma 3.3, P is a $\Bbbk$-linear combination of $x^{q-2}y^{q/2-3-c}$ and $x^{q/2-3-c}y^{q-2}$.

Proof. Lemma 3.8 in [Mon98a] tells us P is a polynomial in x^2 and y^2. This means no term of P is divisible by x^{q-1} or y^{q-1} (since $q-1$ is odd and $x^q = y^q = 0$). On the other hand, every term of P is divisible by x^{q-2} or y^{q-2}, which gives the result. □

Lemma 3.5. In the setting of Lemma 3.3, $P = ax^{q-2}y^{q/2-3-c}+bx^{q/2-3-c}y^{q-2}$ for some $a,b \in \Bbbk^*$.

Proof. By Lemma 3.4, $P = ax^{q-2}y^{q/2-3-c} + bx^{q/2-3-c}y^{q-2}$ for some $a,b \in \Bbbk$. Suppose for a contradiction that $b = 0$. We may then take $P = x^{q-2}y^{q/2-3-c}$ by rescaling, and will write $u = \sum_{i=c}^{q-1} P_i z^i$ (so $P_c = P$). Because $h = \alpha z^4 + xyz^2 + (x^3+y^3)z + x^2y^2$ annihilates u, we get the following expression:

$$\begin{aligned}
0 &= hu \\
&= x^2y^2P_cz^c + \left((x^3+y^3)P_c + x^2y^2P_{c+1}\right)z^{c+1} + \left(xyP_c + (x^3+y^3)P_{c+1} + x^2y^2P_{c+2}\right)z^{c+2} \\
&+ \left(xyP_{c+1} + (x^3+y^3)P_{c+2} + x^2y^2P_{c+3}\right)z^{c+3} \\
&+ \sum_{k=0}^{q-c-5}\left(\alpha P_{c+k} + xyP_{c+k+2} + (x^3+y^3)P_{c+k+3} + x^2y^2P_{c+k+4}\right)z^{c+k+4}.
\end{aligned}$$

Because the z-degrees are sufficiently small, we get the following equations for all $0 \leqslant k < q-c-4$:

$$\begin{aligned}
x^2y^2P_{c+1} &= (x^3+y^3)P_c \\
x^2y^2P_{c+2} &= xyP_c + (x^3+y^3)P_{c+1} \\
x^2y^2P_{c+3} &= xyP_{c+1} + (x^3+y^3)P_{c+2} \\
x^2y^2P_{c+k+4} &= \alpha P_{c+k} + xyP_{c+k+2} + (x^3+y^3)P_{c+k+3}.
\end{aligned}$$

By our assumption that $P_c = x^{q-2}y^{q/2-3-c}$, we get the following equalities:

$$\begin{aligned}
P_{c+1} &= x^{q-4}y^{q/2-2-c} \\
P_{c+2} &= x^{q-6}y^{q/2-1-c} \\
P_{c+3} &= x^{q-8}y^{q/2-c}.
\end{aligned}$$

We now claim that the minimal x-degree of P_{c+k} is $q-2(k+1)$ for each k. The cases $k = 0,1,2,3$ are handled above. For $k \geqslant 4$, we have

$$x^2y^2P_{c+k} = \alpha P_{c+k-4} + xyP_{c+k-2} + (x^3+y^3)P_{c+k-1}.$$

By induction, P_{c+k-4} has minimal x-degree $q-2(k-3)$, P_{c+k-2} has minimal x-degree $q-2(k-1)$ and P_{c+k-1} has minimal x-degree $q-2k$. On the right side of the equality, we have a term

$y^3 P_{c+k-1}$ with minimal x-degree $q-2k$, and every other term has minimal x-degree strictly larger than $q-2k$. Thus the minimal x-degree of the right hand side is $q-2k$. Our claim follows.

Now letting $k = q/2$, we notice that the minimal x-degree of P_{c+k-1} is 0 by the above. In addition, by Lemma 3.3, $c+k \leqslant q-3 < q$, hence

$$x^2 y^2 P_{c+k} = \alpha P_{c+k-4} + xyP_{c+k-2} + (x^3+y^3)P_{c+k-1}.$$

Then the right hand side has minimal x-degree 0, which tells us there are no polynomials P_{c+k} satisfying the equation, contradicting our assumptions. The argument in showing $a \neq 0$ is completely analogous, so we conclude $a \neq 0$ and $b \neq 0$ as desired. □

Lemma 3.6. For each $r \geqslant 1$, there is a nontrivial element $u \in N_{3q/2-4-2^{r-1}}$ with lowest term of the form $(ax^{q-2}y^{\lambda} + bx^{\lambda}y^{q-2})z^c$ with $a, b \in \Bbbk^*$. In addition, $c \leqslant q/2 - 3 \cdot 2^{r-1}$.

Proof. Argue by induction on r, with $r = 1$ given by Lemmas 3.3, 3.5, and 4.8 in [Mon98a]. Let $u \in N_{3q/2-4-2^{r-1}}$ be an element given to us inductively. It is clear that $h^2u^2 \in \mathfrak{m}^{[2q]}$, so it suffices to show hu^2 has the other desired properties. Let Pz^c be the lowest term of u. The lowest term of hu^2 is $x^2y^2P^2z^{2c}$ (as long as $x^2y^2P^2 \notin \mathfrak{m}^{[2q]}$). We will now show $x^2y^2P^2 \notin \mathfrak{m}^{[2q]}$ by writing P as $ax^{q-2}y^{\lambda} + bx^{\lambda}y^{q-2}$ where $\lambda = q/2 - 2 - 2^{r-1} - c$ by induction. Thus $P^2 = a^2x^{2q-4}y^{2\lambda} + b^2x^{2\lambda}y^{2q-4}$, so x^2y^2 does not annihilate P^2. By induction it's also clear that the z-degree requirement is satisfied. □

Lemma 3.7. Let u be as in Lemma 3.6. For each $k \leqslant 2^{r-1}$, the set

$$T_k := \{x^{d_1}y^{d_2}z^{d_3}u \mid d_1 + d_2 + d_3 = k, d_1 < 2 \text{ or } d_2 < 2\}$$

is linearly independent in $N_{3q/2-4-2^{r-1}+k}$.

Proof. Suppose we have a nontrivial linear combination of monomials of degree k, each indivisible by x^2y^2, annihilating u. Let $Q(x,y)z^d$ be the lowest term of this linear combination. Letting Pz^c be the lowest term of u, it follows that $PQz^{c+d} = 0$. However, the inequalities $d \leqslant k \leqslant 2^{r-1}$ and $c \leqslant q/2 - 3 \cdot 2^{r-1}$ (by Lemma 3.6) imply that $c + d \leqslant q/2 - 2 \cdot 2^{r-1} < q$, hence $PQ = 0$. Now let $Q = \sum_{i=0}^{k} a_i x^i y^{k-i}$. By Lemma 3.6 once more,

$$0 = PQ = a\left(\sum_{j=0}^{k} a_j x^{j+q-2} y^{k+\lambda-j}\right) + b\left(\sum_{i=0}^{k} a_i x^{i+\lambda} y^{k+q-2-i}\right)$$

where $\lambda = q/2 - 2 - 2^{r-1} - c$ and $a, b \in \Bbbk^*$. Then $\lambda \leqslant q/2 - 2 - 2^{r-1}$. As a consequence, none of the powers of x overlap in either sum, for

$$i + \lambda \leqslant 2^{r-1} + q/2 - 2 - 2^{r-1} < q - 2 \leqslant j + q - 2.$$

For the same reason, none of the powers of y overlap, so each monomial appearing in our sum is distinct. Then for each j with $a_j \neq 0$, it must be that

$$j + q - 2 \geqslant q \text{ or } k + \lambda - j \geqslant q.$$

The second possibility is never true by our bounds, hence we get $a_j \neq 0$ implies $j \geqslant 2$. In addition, for each i with $a_i \neq 0$, we get

$$i + \lambda \geqslant q \text{ or } k + q - 2 - i \geqslant q.$$

The first possibility is never realized by our bounds, so it must be that $i \leqslant k-2$. In other words, we have shown

$$Q = \sum_{i=2}^{k-2} a_i x^i y^{k-i} = x^2 y^2 \sum_{i=0}^{k-4} a_{i+2} x^i y^{k-i-4}.$$

Then x^2y^2 divides Q, contrary to assumption. □

Corollary 3.8. For each $0 \leqslant k \leqslant 2^{r-1}$, $N_{3q/2-4-2^{r-1}+k}$ has dimension at least $\binom{k+2}{2}$ for $k \leqslant 3$ and dimension at least $4k-2$ if $k \geqslant 4$.

Proof. For $k \leqslant 3$, no monomial is divisible by x^2y^2, hence Lemma 3.7 provides to us a linearly independent set of size $\binom{k+2}{2}$. If $k \geqslant 4$, there are precisely $\binom{k-2}{2}$ monomials divisible by x^2y^2. □

Lemma 3.9. For each $0 \leqslant k \leqslant 2^{r-1}$, the linearly independent set T_k is a basis for $N_{3q/2-4-2^{r-1}+k}$. Moreover, for $N_i = (0)$ for all $i < 3q/2 - 4 - 2^{r-1} + k$.

Proof. The argument follows *mutatis mutandis* as Lemma 2.8 using Lemma 3.7, Corollary 3.8, and Fact 3.2(2,3). □

Using Lemma 3.9, the calculation of the density function for S_α is identical to that of R_α in Section 2 so we omit the details.

Lemma 3.10. For $\alpha \in \Bbbk^*$ with escape time l, the Hilbert–Kunz density function of $\Bbbk[x,y,z]/(h_\alpha)$ is given below:

$$f(w) = \begin{cases} 4w & \text{if } 0 \leqslant w < 1 \\ 12 - 8w & \text{if } 1 \leqslant w < \frac{3}{2} - 2^{-2l-1} \\ 6 + 2^{1-2l} - 4w & \text{if } \frac{3}{2} - 2^{-2l-1} \leqslant w < \frac{3}{2} + 2^{-2l-1}. \end{cases}$$

4. Hilbert–Kunz Multiplicities for Segre Products

We are now prepared to prove the main result of this article by combining the results of Sections 2 and 3 with [Tri18].

Proof of Theorem 1.1. With the notation of Section 1, Lemma 2.14 and Lemma 3.10 give the Hilbert–Kunz density functions for R_α and S_α for all $\alpha \in \Bbbk^*$. Denote by f_γ the Hilbert–Kunz density function of R_γ. By Theorem 2.2,

$$\mathrm{HKD}(R_\alpha \# R_\beta)(x) = e(R_\alpha) w f_\beta(w) + e(R_\beta) w f_\alpha(w) - f_\alpha(w) f_\beta(w).$$

$\mathrm{HKD}(R_\alpha \# R_\beta)(w)$ is supported on $[0, \frac{3}{2} + 2^{-m(\alpha)-1}]$ and since $e(R_\alpha) = e(R_\beta) = 4$ we have that $\mathrm{HKD}(R_\alpha \# R_\beta)(w)$ equals

$$\begin{cases} 16w^2 & \text{if } 0 \leqslant w < 1 \\ -128w^2 + 288w - 144 & \text{if } 1 \leqslant w < \frac{3}{2} - 2^{-m(\alpha)-1} \\ -80w^2 + 12(14 + 2^{1-m(\alpha)})w - 12(6 + 2^{1-m(\alpha)}) & \text{if } \frac{3}{2} - 2^{-m(\alpha)-1} \leqslant w < \frac{3}{2} - 2^{-m(\beta)-1} \\ -48w^2 + 8(12 + 2^{1-m(\beta)} + 2^{1-m(\alpha)})w - 36 - 6(2^{1-m(\beta)} + 2^{1-m(\alpha)}) - 2^{2-m(\alpha)-m(\beta)} & \text{if } \frac{3}{2} - 2^{-m(\beta)-1} \leqslant w < \frac{3}{2} + 2^{-m(\beta)-1} \\ -16w^2 + 4(6 + 2^{1-m(\alpha)})w & \text{if } \frac{3}{2} + 2^{-m(\beta)-1} \leqslant w < \frac{3}{2} + 2^{-m(\alpha)-1}. \end{cases}$$

By Theorem 2.1, $e_{\mathrm{HK}}(R_\alpha \# R_\beta) = \int_{\mathbb{R}} \mathrm{HKD}(R_\alpha \# R_\beta)(w)\, \mathrm{d}w$, which we compute to be

$$12 - 2 \cdot 2^{-m(\alpha)-2m(\beta)} + 6 \cdot 4^{-m(\alpha)} + 6 \cdot 4^{-m(\beta)} - \frac{2}{3} \cdot 8^{-m(\alpha)}.$$

The calculations for $S_\alpha \# S_\beta$ and $R_\alpha \# S_\beta$ are identical. □

References

[BM10] Holger Brenner and Paul Monsky. "Tight closure does not commute with localization". *Ann. of Math. (2)* 171.1 (2010), pp. 571–588. DOI: 10.4007/annals.2010.171.571.

[Bor+24] Levi Borevitz, Naima Nader, Theodore J. Sandstrom, Amelia Shapiro, Austyn Simpson, and Jenna Zomback. "On localization of tight closure in line-S_4 quartics". *J. Pure Appl. Algebra* 228.9 (2024), Paper No. 107682, 17. DOI: 10.1016/j.jpaa.2024.107682.

[Mon83] Paul Monsky. "The Hilbert-Kunz function". *Math. Ann.* 263.1 (1983), pp. 43–49. DOI: 10.1007/BF01457082.

[Mon98a] Paul Monsky. "Hilbert–Kunz Functions in a Family: Line-S4 Quartics". *Journal of Algebra* 208.1 (1998), pp. 359–371. DOI: 10.1006/jabr.1998.7517.

[Mon98b] Paul Monsky. "Hilbert–Kunz Functions in a Family: Point-S4 Quartics". *Journal of Algebra* 208.1 (1998), pp. 343–358. DOI: 10.1006/jabr.1998.7500.

[Tri18] Vijaylaxmi Trivedi. "Hilbert-Kunz density function and Hilbert-Kunz multiplicity". *Trans. Amer. Math. Soc.* 370.12 (2018), pp. 8403–8428. DOI: 10.1090/tran/7268.

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