ROUGH ISOMETRIES OF NILPOTENT GROUPS

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Abstract. The familiar notion of an isometry of metric spaces can be generalized to a wider class of maps known as *rough isometries embedding*, which (under a specific equivalence relation) give rise to a group $\mathrm{RI}(X,d)$ for a given metric space (X,d) which can be considered as an expansion of the isometry group. We seek to determine the structure of $\mathrm{RI}(G,d)$ for a finitely generated nilpotent group G and left-invariant d quasi-isometric to a word metric on G, and introduce some constructions which may be useful in solving this problem generally.

1. Introduction

Geometric group theory is broadly concerned with the study of groups in relation to their actions which are "geometric" in some sense, the motivating example being group action on a metric space by isometries. As a special but important case, we look at a group acting on *itself* (via left multiplication) considered as a metric space, and we are most often concerned with metrics for which this action is also by isometries (called *left-invariant metrics*). Our work in particular focuses on two kinds of mapping between metric spaces, called *quasi-isometries embedding* and *rough isometries emdedding*, see **Definition 2.1** and **2.2**, which can both be understood as strictly "weaker" versions of the more familiar notion of an isometry. Both the set of equivalence classes of quasi- and rough isometries embedding from a space (X,d) to itself can be given a group structure, denoted $\mathrm{QI}(X,d)$ and $\mathrm{RI}(X,d)$ respectively. And it is a central question in geometric group theory to determine $\mathrm{QI}(G,d)$ and $\mathrm{RI}(G,d)$ generally for left-invariant d. As an illustration of the more general theory, we examined the case of $G=\mathbb{R}$ and the results regarding $\mathrm{QI}(\mathbb{R},|\cdot|)$ in **Section 3.1**, based on the work of [San05].

Our research has focused on a special case of this larger problem: we wish to understand $\mathrm{RI}(G,d)$ for a finitely generated nilpotent group G, and a left-invariant metric d which is also quasi-isometric to a word metric on G. The finitely generated and nilpotent conditions in particular allow us to utilize properties and constructions involving these groups demonstrated by [Gro81] and [Pan83]. Our work has applied these ideas to break down the problem of characterizing $\mathrm{RI}(G,d)$ into smaller and more approachable parts which may be useful in further research on the topic, and we test the new methods on some simple cases ($G=\mathbb{Z}^n$).

2. Basic Notions

Definition 2.1. Let (X,d_X) and (Y,d_Y) be metric spaces. A map $f:X\to Y$ is called a *quasi-isometry* provided that the following condition hold:

(i) There exists $K\in[1,+\infty)$ such that for all $x_1,x_2\in X$:

$$\tfrac{1}{K}d_X(x_1,x_2)-K\le d_Y(f(x_1),f(x_2))\le K d_X(x_1,x_2)+K$$

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A quasi-isometry is called a *quasi-isometry embedding* if furthermore,

(ii) there exists $C \in \mathbb{R}_{\geq 0}$ such that $f(X)$ is C-dense in Y; that is, for all $y \in Y$, there exists $x \in X$ such that $d_y(f(x), y) \leq C$.

We also consider a similar but slightly stronger condition:

Definition 2.2. Let X and Y be as above; a map $f : X \to Y$ is a *rough isometry embedding* provided that f satisfies (ii) above and there exists $K \in \mathbb{R}_{\geq 0}$ such that for all $x_1, x_2 \in X$:

$$|d_Y(f(x_1), f(x_2)) - d_X(x_1, x_2)| \leq K$$

and similar for rough isometry.

The value of K satisfying the condition for a rough isometry above also automatically satisfies (i) in the previous definition, so a rough isometry is also necessarily a quasi-isometry. A failure of the converse is included below.

Note that quasi- and rough isometries do **not** need to be continuous, as demonstrated by the following basic examples.

Example 2.3. Considering $(\mathbb{R}, d)$ with the standard absolute-value metric, the floor function $\lfloor \cdot \rfloor : (\mathbb{R}, d) \to (\mathbb{R}, d)$ is a rough isometry, as $\mathbb{Z}$ is $\frac{1}{2}$-dense in $\mathbb{R}$ and the error condition is satisfied by $K = 1$.

With $(\mathbb{R}, d)$ the same as above, we can fix any $c > 1$ and consider the map $g : (\mathbb{R}, d) \to (\mathbb{R}, d)$ defined by $g(x) = \lfloor cx \rfloor$. This is a quasi-isometry (with constants $C = \frac{c+1}{2}$ and $K = c$) that fails to be a rough isometry, as we note that $d(g(0), g(x)) - d(0, x) \geq (c-1)x - 1$ is unbounded for positive x.

Neither type of map need be bijective, so in order for the self-maps of either type to have group structure similar to isometries, we must introduce an equivalence relation.

Definition 2.4. Let (X, d) be a metric space. We define two sets:

$$\mathrm{QI}(X, d) := \{f : X \to X \mid f \text{ quasi-isometry } \} \,/\, \sim$$
$$\mathrm{RI}(X, d) := \{f : X \to X \mid f \text{ rough isometry } \} \,/\, \sim$$

The equivalence relation $\sim$ is the same for both sets; we say $f \sim g$ if there exists C such that $d(f(x), g(x)) \leq C$ for all $x \in X$.

The following proposition is central to our research problem, and its verification is very straightforward from the definitions above.

Proposition 2.5. *Composition of maps in QI(X, d) and RI(X, d) is well defined, and it endows both sets with a group structure.*

Giving $\mathrm{Isom}(X, d)$ the same equivalence relation, we have subgroup containments $\mathrm{RI}(X, d) \subseteq \mathrm{QI}(X, d)$.

In light of these definitions, the broadest problem one could ask in the context of geometric group theory is to determine $\mathrm{QI}(G, d)$ and $\mathrm{RI}(G, d)$ for a group G with a left invariant metric d. However, with the significant relaxed restriction on maps in $\mathrm{QI}(G, d)$, questions about its structure for even familiar groups and metrics are much more difficult to answer, as we explore in the next section.

3. Starting Examples

3.1. **Quasi-Isometries of $\mathbb{R}$.** We turn to an example of the quasi-isometry group that demonstrates the considerably complexity introduced by this wider class of maps in comparison to the group of isometries. We again consider $\mathbb{R}$ with the standard absolute-value metric (which, as an aside, we note is invariant under the group structure of $\mathbb{R}$ given by addition). We examine a few interesting properties of the group $\mathrm{QI}(\mathbb{R}, d)$, which were first outlined and proven in [San05].

Definition 3.1. Let $f : \mathbb{R} \to \mathbb{R}$ be a piecewise linear homeomorphism (i.e. a continuous, piecewise linear map which is either strictly increasing or strictly decreasing). We denote by $\Lambda(f)$ the set of *slopes of* f; that is, the set containing $f'(t)$ for all $t \in \mathbb{R}$ where the derivative is defined. The map f is said to have *bounded slopes* if there is some $M \in \mathbb{R}_{>0}$ for which $|\Lambda(f)| \subseteq (1/M, M)$, and the set of such piecewise linear homeomorphisms with bounded slope is denoted $\mathrm{PL}_\delta(\mathbb{R})$.

The set $\mathrm{PL}_\delta(\mathbb{R})$ forms a subgroup of the group of piecewise linear homeomorphisms of $\mathbb{R}$.

Lemma 3.2. *For all $f \in PL_\delta(\mathbb{R})$, f is a quasi-isometry. Furthermore, there is a group homomorphism $\varphi : PL_\delta(\mathbb{R}) \to QI(\mathbb{R}, |\cdot|)$ defined by $f \mapsto [f]$.*

Proof. We first note that for any differentiable function of bounded slope, the quasi-isometry condition is immediately true by the Mean Value Theorem; if $M \in \mathbb{R}_{>0}$ is the value for f as in the definition above (and without loss of generality f strictly increasing), then for any distinct $a < b$ let $c \in (a, b)$ be the value such that $f'(c)(b - a) = f(b) - f(a)$. Since $f(c) \in (1/M, M)$, we find:

$$\tfrac{1}{M}(b-a) < \underbrace{f'(c)(a-b)}_{=f(b)-f(a)} < M(b-a)$$

For the general piecewise linear case, we need only adjust for the points between a and b which are not differentiable and apply the mean value theorem to each interval between these "break points", which we denote $a_1 < a_2 < \ldots < a_{n-1}$ (and set $a_0 = a$, $a_n = b$). Let $c_i \in (a_i, a_{i+1})$ be such that $f'(c_i)(a_{i+1} - a_i) = f(a_{i+1}) - f(a_i)$. Again since $f'(c_i) \in (1/M, M)$ for all i, we find:

$$\frac{1}{M}(a-b) = \frac{1}{M}\left(\sum_{i=0}^{n-1} a_{i+1} - a_i\right) < \underbrace{\sum_{i=0}^{n-1} f'(c_i)(a_{i+1} - a_i)}_{=f(b)-f(a)} < M\left(\sum_{i=0}^{n-1} a_{i+1} - a_i\right) = M(b-a)$$

Thus f is a quasi-isometry. The homomorphism $\varphi : \mathrm{PL}_\delta(\mathbb{R}) \to \mathrm{QI}(\mathbb{R}, |\cdot|)$ immediately follows. □

Thus the set $\mathrm{PL}_\delta(\mathbb{R})$ is a large collection of quasi-isometries. The following theorem states that all quasi-isometries have an equivalent representative in $\mathrm{PL}_\delta(\mathbb{R})$.

Theorem 3.3 (Sankaran)**.** *The homomorphism $\varphi : PL_\delta(\mathbb{R}) \to QI(\mathbb{R}, |\cdot|)$ is surjective.*

Proof. Again without loss of generality we consider a map $f \in \mathrm{QI}(\mathbb{R}, |\cdot|)$ with constant K as in (i) such that $\lim_{x \to +\infty} f(x) = +\infty$ and likewise for $-\infty$ (since these form the index 2 subgroup of *endpoint-preserving maps* in $\mathrm{QI}(\mathbb{R}, |\cdot|)$, and both $\mathrm{QI}(\mathbb{R}, |\cdot|)$ and $\mathrm{PL}_\delta(\mathbb{R})$ contain orientation/endpoint reversing maps). The general idea of the proof is to produce an increasing collection of points $(y_k)_k \in \mathbb{Z}$ (i.e. $k < k' \implies y_k < y'_k$) such that

1. $(y_k)_{k \in \mathbb{Z}}$ is dense in $\mathbb{R}$ in the quasi-isometric sense,
2. $(f(y_k))_{k \in \mathbb{Z}}$ is also strictly increasing,

(3) the piecewise linear homeomorphism g defined by setting $g(y_k) = f(y_k)$ and making g linear with slope $[f(y_{k+1}) - f(y_k)]/(y_{k+1} - y_k)$ on the interval (y_k, y_{k+1}) for all $k \in \mathbb{Z}$ is a quasi-isometry.

To produce a sequence satisfying (1) and (2), we opt to cite a lemma from [San05]:

Lemma 3.4. *Given f an endpoint preserving quasi-isometry as above and $x \in \mathbb{R}$, there exist numbers $v \in [x - 4K^2, x)$ and $z \in (x, x + 4K^2]$ such that $z - x$ and $x - v$ are both integers, and in addition $f(v) < f(x) < f(z)$.*

(In essence, the lemma uses the quasi-isometry condition to show that the limit behavior of f carries over to smaller-scale behavior.) With this lemma, we can perform induction in both directions:

- Let $x_0 = 0$
- For $k \in \mathbb{Z}_{\geq 0}$, define x_{k+1} to be the value of z given by setting $x = x_k$ in the lemma above.
- For $k \in \mathbb{Z}_{\leq 0}$, define x_{k-1} to be the value of v given by setting $x = x_k$ in the lemma above.

Note that $(x_k)_{k\in\mathbb{Z}}$ is a sequence of integers which by the lemma we know satisfies (1) (it is $4K^2$-dense in $\mathbb{R}$) and (2). We need only make a slight adjustment to produce a sequence that in addition satisfies (3); define $(y_k)_{k\in\mathbb{Z}}$ by setting $y_k := x_{kK^3}$. The properties (1)-(2) readily carry over from (x_k) (now with density $4K^5$), so we consider the piecewise linear homeomorphism g as defined above. It suffices to show that this map has bounded slope (i.e. $g \in \mathrm{PL}_\delta(\mathbb{R})$), so we need only find a bounding constant on $\Lambda(g) = \{[f(y_{k+1}) - f(y_k)]/[y_{k+1} - y_k] \mid k \in \mathbb{Z}\}$. We can find such a bound using the inequality (i) for f and the fact that as defined we have $y_{k+1} - y_k \geq K^3$ for all k:

$$\frac{f(y_{k+1}) - f(y_k)}{y_{k+1} - y_k} > K^{-1} - \frac{K}{y_{k+1} - y_k} \geq K^{-1} - K^{-2}$$
$$\frac{f(y_{k+1}) - f(y_k)}{y_{k+1} - y_k} < K + \frac{K}{y_{k+1} - y_k} \leq K + K^{-2}$$

Thus $\Lambda(g) \subset [K^{-1} - K^{-2}, K + K^{-2}]$ and $g \in \mathrm{PL}_\delta(\mathbb{R})$.

Since f and g are quasi-isometries that have the same value on the $4K^5$-dense subset $(y_k)_{k\in\mathbb{Z}}$, it follows that the are equivalent in $\mathrm{QI}(\mathbb{R}, |\cdot|)$, so $[f] = \varphi(g)$, as desired. □

Note that this result still does not fully characterize $\mathrm{QI}(\mathbb{R}, |\cdot|)$, as the homomorphism φ is not injective (the kernel contains the elements of $\mathrm{PL}_\delta(\mathbb{R})$ with slope $\neq 1$ only in a compact interval), and a number of additional properties of the group can be found in [San05].

3.2. **Rough Isometries of $\mathbb{Z}$.** We now turn to a different example, which illustrates the challenges of working over a familiar space with a broader class of metrics. We outline a possible argument to show that the rough isometry group is the same for all metrics on $\mathbb{Z}$ of the kind we describe below.

Let d_1 be any metric on $\mathbb{Z}$ which is (1) left-invariant and (2) such that there exists a quasi-isometry $f : (\mathbb{Z}, d_1) \to (\mathbb{Z}, d)$, where d is the subspace metric for the standard absolute value metric on $\mathbb{R}$. (This is a special case of the problem that we consider in the next section.)

Our broad idea for these metrics is that for the special case of the word metric, an element $[\psi] \in \mathrm{RI}(\mathbb{Z}, d)$ is characterized by its limit behavior (i.e. $\lim_{n\to\infty} \psi(n)$), which must be either $+\infty$ or $-\infty$ because of the rough-isometry condition. These then correspond to ψ being equivalent to the identity or the map $x \mapsto -x$ in $\mathrm{RI}(\mathbb{Z}, d)$ respectively. The thought then is that the metric d_1 which is quasi-isometry to d necessarily has enough structural similarity for this characterization to carry over to $(\mathbb{Z}, d_1)$. The following sketch attempts to formalize these notions with constructions that quantify the "discrepancy" between d_1 and d, as well as ψ and id.

To begin, consider the sequence $c_n := d_1(0, n)$; because d_1 is left-invariant, we know this sequence is subadditive (i.e. for all $m, n \in \mathbb{N}$, we have $c_{m+n} \leq c_n + c_m$). We can then cite a useful previously established result:

Lemma 3.5 (Fekete). *For any non-negative subadditive sequence $(c_n)_{n\in\mathbb{N}}$, the limit $\lim_{n\to\infty} \frac{c_n}{n}$ exists and is equal to $\inf \frac{c_n}{n}$.*

Thus we have some $C > 0$ for which $\lim_{n\to\infty} \frac{c_n}{n} = C$, and without loss of generality we may alter d_1 by a scaling such that $C = 1$. If we then define another sequence $C_n := d_1(0, n) - n$, from this constraint we can again deduce that (C_n) is non-negative and subadditive.

Now we consider some $\psi \in \mathrm{RI}(\mathbb{Z}, d_1)$ and suppose $\lim_{n\to\infty} \psi(n) = +\infty$. We want to show that $[\psi] = [\mathrm{id}]$, so suppose for contradiction that it does not, and we have some sequence $L_n \in \mathbb{Z}$ such that $d_1(\psi(L_n), L_n)$ is unbounded. Then we define another sequence $D_n := \psi(L_n) - L_n$; our aim is the express the defining inequality of a rough isometry in terms of C_n and D_n and derive some contradictory limit. We know that there exists some $M \in \mathbb{R}_{\geq 0}$ such that $|d_1(\psi(k), \psi(l)) - d_1(k, l)| \leq M$ for all $k, l \in \mathbb{Z}$, so in particular we have for $k, k' \in \mathbb{Z}$:

$$\begin{aligned} |d_1(\psi(L_k), \psi(L_{k'})) - d_1(L_k, L_{k'})| &= |d_1(L_k + D_k, L_{k'} + D_{k'}) - d_1(L_k, L_{k'})| \\ = \big|[C_{L_{k'}+D_{k'}-L_k-D_k} + (L_{k'} + D_{k'} - L_k - D_k)] &- [C_{L_{k'}-L_k} + (L_{k'} - L_k)]\big| \\ &= \big|C_{L_{k'}+D_{k'}-L_k-D_k} - C_{L_{k'}-L_k} + D_{k'} - D_k\big| \leq M \end{aligned}$$

(Implicitly, we restrict to the values of k and k' such that that the sequence elements of (C_n) are defined.) If we then rearrange and take the limit as k' tends to $+\infty$, we have:

$$\lim_{k'\to\infty} \frac{C_{L_{k'}+D_{k'}-L_k-D_k} - C_{L_{k'}-L_k}}{D_{k'} - D_k} = -1$$

The step remaining is to derive the desired contradiction; some mild constraints on the sequences above (e.g. suppose (C_n) is an increasing sequence, implying that the limit above must be positive) suggest that this may fail in the general case. Hence we have the following:

Proposition 3.6. *Let $(\mathbb{Z}, d_1)$ be a metric space with d_1 left-invariant and quasi-isometric to the word metric. If the qequence C_n defined as above is non-decreasing, $RI(\mathbb{Z}, d_1) \cong \mathbb{Z}/2$.*

4. The Nilpotent Case

We now narrow our focus to a more specific and accessible sub-problem of the general one stated above, and which most of our work was intended to address. Firstly, we are interested specifically in the group $\mathrm{RI}(G, d)$, which is generally easier to work with than the larger group $\mathrm{QI}(G, d)$. Next, we impose the restriction that G is both finitely generated and nilpotent (which in particular includes all finitely generated abelian groups); the relevance of the nilpotent condition is addressed in both this and the next section. To give our restriction on the metric d, we first introduce a familiar notion of geometric group theory:

Definition 4.1. Let G be a finitely generated group with a fixed generating set $\{g_1, g_2, \ldots, g_m\}$. We can define a metric d on G, called the *word metric*, as follows:

- Given $g \in G$, define the *norm* $|g| \in \mathbb{Z}_{\geq 0}$ to be the smallest n such that $g = g_{i_1} g_{i_2} \dots g_{i_n}$ for some $i_j \in \{1, \dots, m\}$. (By convention, $|e| = 0$.)
- Given $g, h \in G$, define $d(g, h) := |g^{-1}h|$

Example 4.2. $G = \mathbb{Z}^2$ with generators $\{(1,0),(0,1)\}$. Then d is the "taxicab metric" on $\mathbb{Z}^2 \subset \mathbb{R}^2$, with $d((n_1, m_1), (n_2, m_2)) = |n_1 - n_2| + |m_1 - m_2|$

Remark 4.3. The word metric depends on the choice of generating set, but all word metrics are quasi-isometric.

We can now state the refined question in full:

Problem: *Let G be a finitely generated nilpotent group, and let d be a left invariant metric on G which is quasi-isometric to a word metric on G. What is the group RI(G, d)?*

5. Limits of Metric Spaces

This section gives an overview of the theory that informs our approach for the specific problem of finitely generated nilpotent groups. Central to our methods is the notion of metric space convergence first outlined in [Gro81], which along with [Pan83] provides useful results specific to nilpotent groups. We again begin with definitions for general metric spaces and then restrict our attention to groups.

Definition 5.1. Let (X, d) be a metric space, and $A \subseteq X$ a closed subset. For $r \geq 0$ we define the *r-neighborhood of A*:

$$A^{(r)} := \bigcup_{x \in A} B_a(r)$$

Where $B_a(r)$ denotes the metric ball of radius r around a.

Now considering two closed subsets $A, B \subseteq X$, we can define the *Hausdorff distance* between them:

$$H(A, B) := \inf\{\varepsilon \geq 0 \mid A \subseteq B^{(\varepsilon)} \text{ and } B \subseteq A^{(\varepsilon)}\}$$

If X is a compact metric space, then H gives a metric on the set of *closed* subsets of X (to see why this fails on the full power set, note that the Hausdorff distance between an open and closed ball of the same radius and center is 0). If X is not compact, the Hausdorff distance can be infinite but it otherwise maintains the properties of a metric. The Hausdorff distance also allows us to compare metric spaces more generally:

Definition 5.2. Let (X, x) and (Y, y) be metric spaces with a chosen base point. Let $\mathcal{E}$ be the set of triples (Z, φ, ψ), where Z is a metric space, and $\varphi : X \to Z$, $\psi : Y \to Z$ are isometric embeddings such that $\varphi(x) = \psi(y)$. The *(modified) Gromov-Hausdorff distance* $\tilde{H}(X, Y)$ is defined as:

$$\tilde{H}((X, x), (Y, y)) := \inf_{(Z, \varphi, \psi) \in \mathcal{E}} \{H(\varphi(X), \psi(Y))\}$$

Where $H(\varphi(X), \psi(Y))$ is the Hausdorff distance between closed subsets in Z.

The addition of a base point in this definition might seem to be an unnecessary choice, but for a group with a word metric the identity $e \in G$ is a natural choice.

Example 5.3. Let $X = \mathbb{Z}^2$ with the word metric as in §4, and let $Y = (\mathbb{R}^2, d_1)$, where d_1 is the "taxicab" metric on $\mathbb{R}^2$. (We set $x = y = (0,0)$ for both spaces.) Then we can readily see that $\tilde{H}((X,x),(Y,y)) \leq \frac{1}{2}$, as we can let $Z = Y$, $\varphi : \mathbb{Z}^2 \hookrightarrow \mathbb{R}^2$ be the (isometric) inclusion and $\psi = \mathrm{id}_{\mathbb{R}^2}$.

It is again true that the Gromov-Hausdorff distance gives a metric only on *compact* metric spaces. But for the wider class of *proper* metric spaces (i.e. where every closed ball is compact) there is still a useful notion of convergence:

Definition 5.4. Let (X_j, x_j) be a sequence of based metric spaces. For a given metric space (Y, y), we say that (X_j, x_j) *converges to* (Y, y) if $\lim_{n\to\infty} \tilde{H}((X_j, x_j), (Y, y)) = 0$.

Example 5.5. Let X and Y be as in the previous example, and define $X_j = (\mathbb{Z}^2, \frac{1}{j}d)$ (i.e. a scaling of the original word metric) with $x_j = x$. Then we again have an isometric inclusion $\varphi_j : X_j \hookrightarrow \mathbb{R}^2$ which shows that $(X_j, \tilde{x_j}), (Y, y) \leq \frac{1}{j}$ (since the integer lattice in $\mathbb{R}^2$ is just scaled down by j). Thus $\lim_{j\to\infty}((\mathbb{Z}^2, \frac{1}{j}d), (0,0)) = (\mathbb{R}^2, (0,0))$

Remark 5.6. The distance $\tilde{H}$ gives a metric on *compact* metric spaces, but for general metric spaces it only satisfies weaker properties. For reasons which are outlined further in [Gro81], the notion of convergence in particular benefits from the constraint that the metric spaces involved are *proper* (i.e. all closed balls are compact). To see why this constraint is important, consider the example above but replace $\mathbb{R}^2$ with $\mathbb{Q}^2$ given the induced metric.

6. The Asymptotic Cone

We can now use this framework in the context of finitely generated nilpotent groups, using a construction introduced in [Gro81] and a result pertaining to it from [Pan83]. Let G be a finitely generated nilpotent group with a word metric d. We define a sequence of metric spaces $(G_n, d_n) := (G, \frac{1}{n}d)$, where the underlying set remains the same but the word metric is scaled down by a factor of $\frac{1}{n}$ (and the base point for each space is the identity $e \in G$). (Note that **Example 5.5** is the special case $G = \mathbb{Z}^2$.) Then we have the following result:

Theorem 6.1 (Pansu)**.** *Let (G_n, d_n) be the sequence of metric spaces above for finitely generated nilpotent G. Then the sequence converges to a limit space (G_∞, d_∞), and the group of isometries Isom(G_∞, d_∞) is a Lie group.*

The limit space (G_∞, d_∞) is called *the asymptotic cone of* (G, d). As opposed to reasoning with the points in G_∞ directly, we can use the fact that it is the limit of the spaces G_n to conceptualize them as sequences $(g_m) \in G$ with the constraint that the limit $\lim_{m\to\infty} d_m(e, g_m) = \lim_{m\to\infty} d(e, g_m)/m$ exists and is finite. Again Example 5.5 is helpful to build intuition; in that case, this corresponds to choosing a sequence of points in $\mathbb{Q}^2$ with denominator tending to ∞, and the limit condition ensures that this sequence converges to an element in $\mathbb{R}^2$. As that example demonstrates, however, such sequences do not determine elements in G_∞ uniquely; given sequences (g_m) and (g'_m), we have:

$$d_\infty((g_m), (g'_m)) = \lim_{m\to\infty} \frac{d(g_m, g'_m)}{m}$$

Thus $(g_m) = (g'_m)$ in G_∞ if the limit above is 0.

With the asymptotic cone construction, we can now revisit the group $\mathrm{RI}(G, d)$ and consider some $\psi : G \to G$ in $\mathrm{RI}(G, d)$. We have a sequence of maps $\psi_n : (G_n, d_n) \to (G_n, d_n)$ given by the same map on the underlying set G, and it is immediate that each of these is a rough isometry (with constants C/n as in the definition). While the generalized notion of a convergence of maps on metric spaces is beyond scope of our project, we can in fact define a limit map $\psi_\infty : (G_\infty, d_\infty) \to (G_\infty, d_\infty)$ by setting $\psi_\infty(g_m)_{m\in\mathbb{N}} = (\psi(g_m))_{m\in\mathbb{N}}$. (It is easy to check using the fact that ψ is a rough isometry that the sequence $(\psi(g_m))$ gives a well-defined element of G_∞ satisfying the limit conditions.) In fact, as the rough isometry constants C/n for ψ_n might suggest, the limit map ψ_∞ is an *isometry* of (G_∞, d_∞) and thus an element of the Lie group $\mathrm{Isom}(G_\infty, d_\infty)$. The culmination of this results is given in the following proposition:

Proposition 6.2. *There is a map $\varphi : RI(G,d) \to Isom(G_\infty, d_\infty)$ given by setting $\varphi(\psi) = \psi_\infty$, and this map is a group homomorphism.*

It is immediately clear that this map is well-defined on $\mathrm{RI}(G,d)$ (i.e. for $[\psi] = [\tau]$ we have $(\psi(g_m)) = (\tau(g_m)) \in G_\infty$), and fact that this map is a group homomorphism is also a direct consequence of the definition.

Given this group homomorphism, we can fit the map φ into a short exact sequence of groups:

$$0 \longrightarrow \ker(\varphi) \hookrightarrow \mathrm{RI}(G,d) \xrightarrow{\varphi} \mathrm{im}(\varphi) \longrightarrow 0$$

Thus, our construction breaks the larger problem of determining the structure of $\mathrm{RI}(G,d)$ for finitely generated nilpotent G into two smaller problems:

- **What is** $\mathrm{im}(\varphi)$**?** That is, what isometries of the limit space can be realized as the limit of a rough isometry in this way? This is where the Lie group structure of $\mathrm{Isom}(G_\infty, d_\infty)$ may be useful, as there is more preexisting theory for closed subgroups of Lie groups.
- **What is** $\ker(\varphi)$**?** That is, which rough isometries produce a change in the initial group that becomes "negligible" in the asymptotic cone? This is a question with less background research to draw upon, but the added constraint of $\psi \in \ker(\varphi)$ makes classifying such maps more approachable.

For the second problem, our work with special cases has given an impression what the solution may be:

Conjecture 6.3. *For $G = \mathbb{Z}^n$ with d a* ***word metric****, $\ker(\varphi)$ is trivial.*

References

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