UNIVERSAL AND HOMOGENEOUS MEDIAN ALGEBRAS

RUOHAN HU

Contents

ABSTRACT. In this paper, we present the basic theory of median algebras. Then we construct two types of universal and homogeneous median algebras explicitly, and prove their universality and homogeneity.

1. Introduction

Median algebras are a kind of algebraic structure closely related to distributive lattices and the boolean algebras. They were studied extensively by Birkhoff and Kiss in [2] and by Isbell in [8]. They also have close relations with graphs and cube complexes, and this aspect is studied in detail by [3].

On the other hand, a significant focus in model theory is the study of universal, homogeneous models.

Definition 1.1 (Homogeneous Model, Universal Model). Suppose $\mathbf{K}$ is a class of finite models, and $\leq$ is a class of embeddings.

A model M is $(\mathbf{K}, \leq)$-*homogeneous*, if for any two isomorphic finite submodels $A, B \subseteq M$ and isomorphism $f : A \to B$ such that $A, B \in \mathbf{K}$ and $A, B \leq M$, there exists an automorphism $g : M \to M$ such that $g|_A = f$.

For a class of finite structures $\mathbf{K}$, a model M is $(\mathbf{K}, \leq)$-*universal* if for every structure $A \in \mathbf{K}$ we have $A \leq M$.

When $\mathbf{K}$ is the set of finite models of a known theory, we would informally simply say M is universal.

Close relatives of median algebra, the boolean algebra and distributive lattice, are known to produce concrete examples of universal, homogeneous structures, as discussed in [5] and Section 6.3 of [9], which motivates the discussion of universal, homogeneous median algebra.

Now, an interesting type of universal and homogeneous structure is the Fraïssé Limit, whose theory is well-studied.

Definition 1.2 (Fraïssé Limit). Given a class of finite structures $\mathbf{K}$, the Fraïssé Limit of $\mathbf{K}$ is a model D such that

(1) $\mathbf{K} = \{$finite submodels of $D\}$ up to isomorphism.
(2) D is homogeneous.
(3) D has cardinality $\aleph_0$.

Theorem 1.3 (Fraïssé's Theorem). *If K satisfies*

(1) closure under substructure (Hereditary Property, HP)
(2) For $A, B \in K$, there is $C \in K$ such that A, B embeds into C (Joint Embedding Property, JEP)
(3) For $A, B, C \in K$ such that A embeds into B and C, there is $D \in K$ such that B, C embeds into D and the embeddings commute with each other (Amalgamation Property, AP)

then there is a unique Fraïssé Limit of $\mathbf{K}$ up to isomorphism.

The theorem and its proof are discussed in [7] as Theorem 7.1.2. Notice that if D is Fraïssé Limit of $\mathbf{K}$, then $\mathbf{K}$ must satisfies HP, JEP and AP, suggesting that a Fraïssé Limit, if exists, must be unique.

In this paper, we give the explicit constructions of two median algebras $\mathbb{M}$ and $\mathcal{M}$ that are homogeneous and universal in different senses.

Theorem 1.4 (Main Theorem 1). $\mathbb{M}$ *is the Fraïssé Limit of all finite median algebras.*

$\mathbb{M}$ can be said as $(\mathbf{K}, \subseteq)$-homogeneous and universal model when $\mathbf{K}$ is the class of all finite median algebras, that is, here we take $\leq$ to be all embeddings. On the other hand, if we take $\leq$ to be the convex embeddings, then we yield a different model $\mathcal{M}$ which is $(\mathbf{K}, \leq)$-homogeneous and universal.

Theorem 1.5 (Main Theorem 2). $\mathcal{M}$ *is the unique median algebra satisfies the following.*

(1) Every finite median algebra embeds into $\mathcal{M}$ as a convex subset.
(2) $\mathcal{M}$ is homogeneous only for convex subsets.
(3) $\mathcal{M}$ has cardinality $\aleph_0$.
(4) For every finite subset $A \subseteq \mathcal{M}$, the convex hull $cl(A)$ of A is finite.

The detail of the definitions will become clear as the paper proceeds.
Our two main tools in proving the above theorems are

(1) a theory of median algebras and their amalgamation, allowing us to do induction on the set of finite median algebras;
(2) duality theory, allowing us to find enough automorphisms of the universal homogeneous median algebra.

2. Median Algebra

2.1. The Algebraic Structure.

Definition 2.1. Consider the language $\mathcal{L}_{ma} = \{\langle -, -, - \rangle\}$ of one single ternary operator. The theory of median algebra T_{ma} is axiomatized by the following axioms.

(1) $\langle a, a, b \rangle = a$.
(2) $\langle a, b, c \rangle = \langle a, c, b \rangle = \langle b, a, c \rangle$.
(3) $\langle \langle a, x, b \rangle, x, c \rangle = \langle a, x, \langle b, x, c \rangle \rangle$.

Construction 2.2. Every distributive lattice induces a median algebra by defining

$$\langle a, b, c \rangle = (a \vee b) \wedge (b \vee c) \wedge (c \vee a).$$

This gives a forgetful functor from the category of distributive lattices to the category of median algebras, which would also mean that every homomorphism between distributive lattices is also a homomorphism between median algebras.

Here we specify a type of median algebra which will be useful in the next subsection.

Definition 2.3 (Discrete Median Algebra). A median algebra M is discrete when for every $a, b \in M$, $[a, b]$ is finite.

2.2. **The Graph Structure.** The main objective of this subsection is to introduce the following bijection.

$$\{\text{Discrete Median Algebras}\} \underset{\Pi}{\overset{\Gamma}{\rightleftharpoons}} \{\text{Median Graphs}\}$$

Definition 2.4 (Median Graph). In a graph G, for three vertices a, b, c, their *median* is a vertex $m(a, b, c)$ such that belongs to a shortest path between every pair of a, b, c.

In other words, using the graph metrics d, a median of a, b, c is a vertex m satisfying

(1) $d(a, b) = d(a, m) + d(m, b)$
(2) $d(b, c) = d(b, m) + d(m, c)$
(3) $d(c, a) = d(c, m) + d(m, a)$.

A graph G is a *median graph* when for every three vertices a, b, c, there is a unique median $m(a, b, c)$ of a, b, c.

Proposition 2.5. *If $G = (V, E)$ is a median graph, then the set of vertices V and the graph theoretic median $m : V^3 \to V$ satisfies T_{ma}.*

Define $\Pi(G)$ to be (V, m), the median algebra induced from the median graph G by the above proposition. This provides the map from median graphs to median algebras. (Moreover, we shall see that the image of the map falls entirely within the discrete median algebras.)

We now provide the other map Γ, from discrete median algebras to median graphs.

Definition 2.6 (Interval). Given a median algebra M and $a, b \in M$, an interval $[a, b]$ is defined as a subset $\{x \in M | \langle a, x, b \rangle = x\}$.

Definition 2.7 (Edge). An interval $[a, b]$ is an *edge* if $[a, b] = \{a, b\}$.

Definition 2.8. Given a median algebra M (not necessarily discrete), define the graph structure of M as the graph $\Gamma(M) = (V, E)$, where $V = M$ and $E = \{(a, b) | [a, b]$ is an edge$\}$.

Proposition 2.9. *When*

$$\Gamma : \{Median\ algebras\} \to \{Graphs\}$$

is restricted to discrete median algebras, it becomes a bijection

$$\Gamma : \{Discrete\ Median\ algebras\} \to \{Median\ Graphs\}.$$

Proof. See [1, Theorem 4.3]. □

Proposition 2.10. $\Gamma^{-1}(G)$ *is defined by the median algebra induced by median graph G. In other words $\Gamma^{-1} = \Pi$*

Proof. We argue that $\Gamma(\Pi(G)) = G$. If $E(a, b)$ in G, then for any $x \in [a, b]$ in $\Pi(G)$, $m(a, x, b) = x$ in G. Thus, x is on a shortest path between a and b, so $x \in \{a, b\}$, so $[a, b]$ is an edge. Otherwise, there is at least one shortest path between a, b having more than two points, so $[a, b]$ is not an edge. Thus, $\Gamma(\Pi(G)) = G$.

Then, we have $\Pi(G) = \Gamma^{-1}(G)$. □

Thus, we have proved that Γ and Π give the following isomorphism

$$\{\text{Discrete Median Algebra}\} \rightleftharpoons \{\text{Median Graph}\}.$$

We will be referring to this isomorphism constantly and sometimes implicitly, both for the proofs and intuition.

Moreover, we have the following results on automorphisms of median algebra and median graphs.

Proposition 2.11. *For each median algebra M, every automorphism f of M, by forgetting the median operation, corresponds to an automorphism $\Gamma(f)$ on the vertices of $\Gamma(M)$. We claim that $\Gamma(f)$ is a graph automorphism. This gives a group isomorphism:*

$$\Gamma : Aut(M) \cong Aut(\Gamma(M)).$$

Proof. Let $f \in Aut(M)$. Then f automatically gives a bijection on the vertices of $\Gamma(M)$. If $[a, b]$ is an edge, then because f is an automorphism, $[f(a), f(b)]$ is also an edge. Conversely, if $[f(a), f(b)]$ is an edge, then $[a, b]$ is also an edge. Thus, f gives rise to a graph automorphism of $\Gamma(M)$. Let this automorphism be denoted as $\Gamma(f)$.

It is easy to prove that $\Gamma : Aut(M) \to Aut(\Gamma(G))$ is a graph homomorphism. It is also easy to prove that it is injective, so we now prove it is surjective. Suppose there is a graph automorphism $g \in Aut(\Gamma(M))$. Then g gives a bijection on the underlying set of M. We need only to argue that g would preserve the median operation. Because Γ and Π are inverses, the median operation on M is the same as the graph-theoretic median on $\Gamma(M)$. Then, because g is a graph automorphism, it preserves the metric on the graph, so g preserves the graph-theoretic median. Thus, g is the image of an element of $Aut(M)$. Thus, Γ is bijective. □

This allows us apply results on automorphism group of graphs to median algebras.

2.3. Convex sets.

Definition 2.12. Given a median algebra M, a subset A is *convex* if for every $a, b \in A$, we have $[a, b] \subseteq A$.

Notice that because $\{\langle a, b, c \rangle\} = [a, b] \cap [b, c] \cap [c, a]$, a convex subset is naturally a median subalgebra. However, not every median subalgebra is a convex subset.

Notice that convexity of a subset is relative to the median algebra the subset is in. If A is a convex subset of B, we say $A \leq B$.

We now give some lemmas concerning convexity.

Lemma 2.13. *For median algebras A, B, C,*

(1) If $A \subseteq B \subseteq C$, and if $A \leq B$ and $B \leq C$, then $A \leq C$.
(2) If $A \subseteq B \subseteq C$, and if $A \leq C$, then $A \leq B$;
(3) If $A, C \subseteq B$ and $A \leq B$, then $A \cap C \leq C$.

Proof. (1): Obvious.

(2): For $a, b \in A$ and $x \in B$ if $\langle a, x, b \rangle = x$ in B, then because $B \subseteq C$, the same equation holds in C, so because $A \leq C$, we have $x \in A$.

(3): Assume there is $x, y \in A \cap C$ and $z \in C$ such that $\langle x, z, y \rangle = z$. Then by definition of convexity $z \in A$, so $z \in A \cap C$. Thus, $A \cap C \leq C$. $\qquad\square$

Proposition 2.14. *Given a median algebra M, for any finite convex subset I, and an element $p \in M$ of distance 1 to I, there exists a convex subset $I' \subseteq I$, and a finite convex set J disjoint from I such that*

(1) $p \in J$;
(2) $J \cong I'$ as median subalgebras;
(3) $J \cup I' = I' \times 2$ as median subalgebras;
(4) $I \cup J = I \sqcup_{I'} (I' \times 2)$ is also a convex set in M.

Proof. See [8, Lemma 6.8] $\qquad\square$

The above propositions gives us a controlled way of expanding a convex subset in a median algebra. If M is finite, then starting with any finite convex subset M_0 (say, a subset of a single point), we can expand M_0 step-by-step according to the above process, namely by "**duplicating a convex subset**." Because M is finite, we can have a finite chain of inclusion

$$M_0 \subset M_1 \subset ... \subset M_n = M.$$

Notice that because every M_i is convex in M, by the lemma, each M_i is convex in M_{i+1}.

Moreover, because every convex subset is a median subalgebra, the above chain of subsets can also be viewed an inclusion chain of median algebras. This can be expressed with the following proposition.

Proposition 2.15. *For every finite median algebra M such that $|M| \geq 2$, there is a finite convex subset N such that $|N| < |M|$, and M is constructed by duplicating a convex subset of N according to the proposition.*

This proposition allows us to make argument on the class of all finite median algebras by inducting on the number of elements. In other words, if we want to prove every finite median algebra has property P, we need only to prove that

(1) The trivial median algebra of one element has property P;

(2) if M has property P, then every M' which expands M by duplicating a convex subset of M also has property P.

The following lemma is an example of this technique.

Lemma 2.16. *Every finite median algebra can be embedded inside a hypercube median algebra 2^n for some n(which is the median algebra induced by the boolean algebra 2^n by the forgetful functor).*

Proof. The trivial median algebra of one element first obviously can be embedded in any 2^n. Suppose M can be embedded in 2^n by a map f, i.e. for every element $a \in M$, we assign it $f(a)$ a 01-string of length n. Suppose M is expanded to N by duplicating an ideal I. Denote the "new point" as $a_1, ..., a_m$ and the corresponding old points as $b_1, ..., b_m$. Then, we define $g : N \to 2^{n+1}$ as follows: for $x \in M$,

$$g(x) = f(x) \frown 0$$

and for a_i where $i = 1, ..., m$, we have

$$g(a_i) = f(b_i) \frown 1.$$

Thus, we have constructed an embedding of N into 2^{n+1}. Thus, by induction, every finite median algebra can be embedded into 2^n for some n. $\square$

3. Duality Theory

This section is a brief digresssion to the duality theory of distributive lattices. Duality theory is able to realize abstract lattices as concrete, topological constructions, which can in some cases greatly simplify the discussion of lattices.

Even for median algebras induced from distributive lattices, not every median algebra automorphism is a lattice homomorphism. Nevertheless, duality theory for distributive lattice is still useful, because for the universal homogeneous median algebra and its automorphisms, enough many of them are lattice automorphisms, meaning that duality theory still simplify a large enough part of the discussion.

We now dive into the topological counterparts of the distributive lattice.

Definition 3.1 (Bounded Priestley Space)**.** A *bounded Priestley space* is a 4-tuple $(X, \leq, 0, 1)$ composed of a Stone space, a partial order and the lowest and highest points of the partial order satisfying the following condition.

- If $x \not\leq y$, then there is a clopen up-set U such that $x \in U$ and $y \notin U$.

A morphism between Priestley space is defined as a continuous map preserving the order, 0 and 1. Denote the category of Bounded Priestley spaces as **BPSpace**.

Construction 3.2. Define a functor $\mathbf{F} : \mathbf{BPSpace} \to \mathbf{DLat}^{op}$ as follows. (Notice that **DLat** is the category of distributive lattice in general, i.e. not necessarily bounded.)

For each bounded priestley space X, define

$$\mathbf{F}(X) := \{\text{clopen low-sets of } X\}$$

with join and meet defined as union and intersection.

$\mathbf{F}(f)$, where f is a morphism between Priestley spaces, is defined by f^{-1} on the clopen sets.

Define $\mathbf{G} : \mathbf{DLat}^{op} \to \mathbf{BPSpace}$ as follows. For each distributive lattice D, define

$$\mathbf{G}(D) := \{f : D \to 2 \text{ is a homomorphism}\} \subseteq 2^D$$

with the order defined by $f \leq g \Leftrightarrow f^{-1}(0) \subseteq g^{-1}(0)$, and topology defined as the subspace of 2^D (with product topology and 2 given the discrete topology).

$\mathbf{G}(f)$, when f is a morphism between distributive lattices, is defined by taking $- \circ f$ on the set of homomorphisms to 2.

Theorem 3.3 (Priestley Duality). $\mathbf{F}$ *and* $\mathbf{G}$ *form an equivalence of category between* $\mathbf{BPSpace}$ *and* $\mathbf{DLat}^{op}$.

Proof. See [4, Theorem 2.5] □

4. First Kind of Universal Homogeneous Structure

Consider the Cantor set $\mathcal{C}$, which is a subspace of the real interval $[0,1]$. Let $\leq$ be the weakest partial order on $\mathcal{C}$ such that 0 is the unique $\leq$-lowest element and 1 is the unique $\leq$-highest element. In other words, $0 \leq x$ and $x \leq 1$ for every $x \in \mathcal{C}$, and for any $x, y \in \mathcal{C} \setminus \{0, 1\}$, x, y are incomparable.

Construction 4.1. Define $\mathbb{M}$ as the median algebra induced by the distributive lattice which the dual of the Priestley space $(\mathcal{C}, 0, 1, \leq)$.

In other words,

$$\mathbb{M} = \{A \subseteq \mathcal{C} | A \text{ is clopen and } 0 \in A, 1 \notin A\}$$

and the median operation is defined as

$$\langle a, b, c \rangle := (a \cap b) \cup (b \cap c) \cup (c \cap a).$$

Now we state the theorem.

Theorem 4.2 (Main Theorem 1). $\mathbb{M}$ *satisfies the follows:*

(1) $\mathbb{M}$ *is universal, i.e. every finite median algebra embeds into* $\mathbb{M}$;
(2) $\mathbb{M}$ *is homogeneous;*
(3) $|\mathbb{M}| = \aleph_0$.

Thus by Fraïssé's Theorem $\mathbb{M}$ *is the unique countable, universal, homogeneous median algebra.*

The proof for $|\mathbb{M}| = \aleph_0$ can be done easily by a point-set topological argument.

We notice that every finite partition of $\mathcal{C}$ into clopen sets induces a subalgebra of $\mathbb{M}$. For a partition P, we call the resulting subalgebra $\mathbb{M}_P$. Explicitly, when $P = \{c_0, ..., c_n\}$ is a partition of $\mathcal{C}$ into clopen sets, where $0 \in c_0$ and $1 \in c_n$, we have

$$\mathbb{M}_P = \left\{ A \subset \mathcal{C} \middle| A = c_0 \cup \bigcup_{i \in S} c_i, S \subseteq \{1, ..., n-1\} \right\} \cong 2^{n-1}.$$

We thus can prove universality.

Proof of Universality. By Lemma 2.16, every finite median algebra embeds into a median algebra 2^{n-1} for some n. Because $\mathbb{M}_P \cong 2^{n-1}$ is a subalgebra of $\mathbb{M}$ for P a partition of $\mathcal{C}$ into n clopen subsets, we conclude that every finite median algebra embeds into a median algebra. □

We now begin the proof of homogeneity. We again apply the construction of finite partition.

Lemma 4.3. *Given a finite partition P of $\mathfrak{C}$, every automorphism of $\mathbb{M}_P$ extends to an automorphism of* $\mathbb{M}$.

In order to prove this, we need the two following lemmas.

Lemma 4.4.   *(1) Every automorphism of $\mathcal{C}$ which preserves 0 and 1 induces an automorphism of $\mathbb{M}$.*
   (2) For a clopen subset $A \subseteq \mathfrak{C} \setminus \{0,1\}$, $X \mapsto X \oplus A$ gives an automorphism of $\mathbb{M}$.

The proof of lemma is straightforward. (1) is a direct application of Priestley duality, and (2) is clear by some boolean-algebraic calculations.

Lemma 4.5. *The automorphism group of a hypercube is generated by*

   (1) flipping along a coordinate;
   (2) Permutation of coordinates.

Proof. See [6]. $\qquad\qquad\square$

Proof of Lemma 4.2. Let $P = \{c_0, ..., c_n\}$ be a partition. By Proposition 2.11, we have $Aut(\mathbb{M}_P) = Aut(2^{n-1})$ isomorphic to the automorphism group of the hypercube graph. Then, by Lemma 4.5, $Aut(2^{n-1})$ is generated by flipping a coordinate or permutation of coordinates. Thus, $Aut(\mathbb{M}_P)$ is generated by

   (1) the dual of an automorphism of $\mathcal{C}$ which permutes $c_1, ..., c_{n-1}$ (this automorphism of $\mathcal{C}$ exists because every clopen subset of $\mathcal{C}$ is homeomorphic to $\mathcal{C}$);
   (2) $X \mapsto X \oplus c_i$ for $i = 1, ..., n-1$.

Both type of automorphisms can be extended to an automorphism of $\mathbb{M}$ by Lemma 4.4. Thus, every automorphism of $\mathbb{M}_P$ extends to an automorphism of $\mathbb{M}$. $\qquad\square$

Now we extends Lemma 4.2 to isomorphism between two subalgebras $\mathbb{M}_A, \mathbb{M}_B$ given by two different partitions A, B of $\mathcal{C}$ into the same finite number of clopen subsets.

Lemma 4.6. *Given two finite partitions A, B of $\mathcal{C}$ into the same number of clopen sets, if we have an isomorphism $g : \mathbb{M}_A \to \mathbb{M}_B$, then f extends to an automorphism f of $\mathbb{M}$ such that $f|_{\mathbb{M}_A} = g$.*

Proof. Because every clopen subset of $\mathcal{C}$ is isomorphic to $\mathcal{C}$, there is an automorphism on $\mathcal{C}$ taking partition A to partition B. Let the dual of this automorphism be f_1. Then, f_1 restricts to an isomorphism $f_1|_{\mathbb{M}_B} : \mathbb{M}_B \to \mathbb{M}_A$.
   Then, by Lemma 4.2, we have can extend $f_1|_{\mathbb{M}_B} \circ g$ (which is an automorphism of $\mathbb{M}_A$) to an automorphism of $\mathbb{M}$. Let this automorphism be f_2. Then, we have $f_1|_{\mathbb{M}_B} \circ g = f_2|_{\mathbb{M}_A}$ and thus $g = (f_1^{-1} \circ f_2)|_{\mathbb{M}_A}$. Thus, g extends to $f_1^{-1} \circ f_2$ which is an automorphism of $\mathbb{M}$. $\qquad\square$

Now, we generalize the result to isomorphism between two finite subalgebras in general, which is one of the most technical parts of the paper.

Proof of Homogeneity. For $f : A \to B$ isomorphism of finite subalgebras of $\mathbb{M}$. We shall choose two subalgebras C_A and C_B that come from finite partitions of $\mathcal{C}$ which contain A and B respectively, so that there is an isomorphism $g : C_A \to C_B$ that extends f. Then we can apply Lemma 4.6.
   We try to construct an invariant on ordered, median subalgebras of $\mathbb{M}$ of n elements.

Let A have a canonical order $a_1, ..., a_n$. Notice that each a_i is a clopen subset of the Cantor set containing 0 and excluding 1. Consider the **coarsest** partition of the Cantor set into clopen subsets $c_0, ..., c_m$, such that each a_i us a disjoint union of $(c_i)_{i=0}^m$. Then, define the matrix form of A with respect to the partition as

$$m(A)_{i,j} = \begin{cases} 1 & \text{if } c_j \subseteq a_i \\ 0 & \text{if } c_j \nsubseteq a_i \end{cases}$$

Then, define character of A, $ch(A) \subseteq 2^n/\neg$ (i.e. 2^n modulo negation operation) as

$$ch(A) = \{\text{columns of } m(A) \text{ modulo } \neg\} \subseteq 2^n/\neg.$$

Notice that the character is not dependent upon the partition, since all partitions (such that every a_i is a disjoint union of segments) are finer than $c_1, ..., c_m$, and all finer partitions would yield the same character.

First, if $ch(A) = ch(B)$, then the coarsest partition of Cantor set would (modulo some splitting of segments) give C_A and C_B and an isomorphism between C_A to C_B would bring A to B.

Thus, we need only to prove that if $A \cong B$ (suppose an order on both is preserved by the isomorphism), then $ch(A) = ch(B)$. We shall prove this by strong induction on number of elements.

We will index the elements of A and B by natural numbers, and naturally an order is induced on both.

Let $A = \{a_1, ..., a_n\}, B = \{b_1, ..., b_n\}$ such that $A \cong B$ (the isomorphism preserves the index) and $ch(A) = ch(B)$. Suppose (without loss of generality) the indices $1, ..., k$ give convex sets in both A and B. Assume $A' = \{a_1, ..., a_{n+k}\}, B' = \{b_1, ..., b_{n+k}\}$ are extended from A, B by duplication of the convex set, where a_{n+i}, b_{n+i} correspond to a_i, b_i, and naturally there is isomorphism $A' \cong B'$.

Assuming $ch(A) = ch(B)$, we shall argue that $ch(A') = ch(B')$: we need only to argue that $ch(A')$ depends only on the median algebra structure of A', without any arbitrary selection.

Assume $\{c_0, ..., c_m\}$ is the coarsest partition of $\mathcal{C}$ corresponding to A. Assume $0 \in c_0$ and $1 \in c_m$ so for every a_i, $c_1 \subseteq a_i$ and $c_m \nsubseteq a_i$. Then we now argue that

$$a_{n+i} \cap c_j = a_i \cap c_j \text{ for } i = 1, ..., k \text{ and } j = 1, ..., m-1.$$

The statement is automatically true for $m = 1$. Assume $m \geq 2$. Then, we would have $n \geq 2$. Select $l \in \{1, ..., n\}$ such that $a_l \cap c_j \neq a_i \cap c_j$. l must exist since $(c_j)_{j=0}^m$ is the coarsest partition and $n \geq 2$. Notice that $a_j \cap c_j, a_i \cap c_j$ both are either c_j or empty.

By property of convex set we know $\langle a_l, a_i, a_{n+i}\rangle = a_i$. Thus, $a_{n+i} \cap c_j = a_i \cap c_j$ for every $i = 1, ..., k$ and $j = 1, ..., m-1$. Thus, $(c_1, ..., c_{m-1})$ is still the coarsest partition of part of the Cantor set $(\mathcal{C} \setminus (c_0 \cup c_m))$ when $a_{n+1}, ..., a_{n+k}$ are added.

Now, we have determined $a_{n+i} \cap c_j$ for $j = 1, ..., m-1$, we shall now determine $a_{n+i} \cap c_1$ and $a_{n+i} \cap c_m$.

Notice that for any $i, j \in 1, ..., k$, we have $\langle a_i, a_{n+i}, a_{n+j}\rangle = a_{n+i}$. Thus, we have

$$(a_{n+i} \cap c_1) \cup (a_{n+j} \cap c_1) = \langle c_1, a_{n+i} \cap c_1, a_{n+j} \cap c_1 \rangle = a_{n+i} \cap c_1$$

meaning that

$$a_{n+j} \cap c_1 \subseteq a_{n+i} \cap c_1.$$

Since the above equation applies to all i, j, we see that $a_{n+j} \cap c_1 = a_{n+i} \cap c_1$ for all $i, j = 1, .., k$ and the same applies to c_m.

Thus, we can partition c_0 into $c_0' = a_{n+i} \cap c_0$ and its complement c_0'' in c_0 and partition c_m into $c_m' = a_{n+i} \cap c_m$ and its complement c_m'' in c_m. Thus, we have reached a partition for $a_1, ..., a_{n+k}$. Then, we conclude

$$ch(A') = \left\{ i \mapsto \begin{cases} 1 & \text{if } i \leq n \\ 0 & \text{if } i > n \end{cases} \right\} \cup \left\{ i \mapsto \begin{cases} a(i) & \text{if } i \leq n \\ a(i-n) & \text{if } i > n \end{cases} \middle| a \in ch(A) \right\}$$

Thus, we have proved that $ch(A')$ depends entirely upon the median algebra structure on A'. □

Thus, by Fraïssé's Theorem, $\mathbb{M}$ is the unique Fraïssé's Limit of all finite median algebras.

5. Second Kind of Universal Homogeneous Structure

We now prove the second main result of this paper, i.e. constructing the unique median algebra that is universal and homogeneous for convex sets.

Construction 5.1. We construct an infinite series of median graphs/median algebras in the following iterated process.

(1) Start with a point.
(2) For every finite convex set $x_1, ..., x_n$ in the current median algebra, duplicate the finite convex set as $y_1, ..., y_n$ so that x_i is connected to y_i by an edge, and $y_1, ..., y_n$ are connected with each other in the same way as $x_1, ..., x_n$.
(3) Repeat the above process.

The initial, trivial median algebra is denoted as $\mathcal{M}_0$. The median algebra corresponding to the median graph reached by the n-th iteration of the above process is called $\mathcal{M}_n$. The union of all $\mathcal{M}_n$ is $\mathcal{M}$.

We now state the second main theorem.

Theorem 5.2 (Main Theorem 2). *$\mathcal{M}$ is the unique median algebra that satisfies the following.*

(1) Every finite median algebra embeds into $\mathcal{M}$ as a convex subset.
(2) $\mathcal{M}$ is ultrahomogeneous only for convex subsets. In other words, for two convex subsets A, B, any isomorphism between A, B extends to an automorphism of $\mathcal{M}$.
(3) $\mathcal{M}$ has cardinality $\aleph_0$.
(4) For every finite subset $A \subseteq \mathcal{M}$, the convex hull $cl(A)$ of A is finite.

Definition 5.3 (Convex Hull). Given a median algebra M and a subset $A \subseteq M$, define the convex hull $cl(A)$ of A to be the smallest convex sets containing A, i.e. intersection of all convex sets containing A.

The cardinality is trivial. We first prove the closure finiteness of $\mathcal{M}$.

Proof of Closure Finiteness. Given a finite set $A \subset \mathcal{M}$, because A is finite, there is n such that $A \subseteq \mathcal{M}_n$. We now prove that $\mathcal{M}_n$ is a convex subset of $\mathcal{M}$.

Suppose for contradiction that $\mathcal{M}_n$ is not convex in $\mathcal{M}$. Then there is $a \in \mathcal{M}$ witnessing the non-convexity of $\mathcal{M}_n$ (i.e. $\langle p, a, q \rangle = a$ for some $p, q \in \mathcal{M}_n$). Then there is $\mathcal{M}_m$ such that $m \geq n$ and $a \in \mathcal{M}_m$. By Proposition 2.13 and Lemma 2.12,

$\mathcal{M}_n$ is convex in $\mathcal{M}_m$, but this contradicts that $\langle p, a, q \rangle = a$ for some $p, q \in \mathcal{M}_n$. Thus, $\mathcal{M}_n$ is convex in $\mathcal{M}$. $\qquad\square$

We then prove the universality.

Proof of Universality. Let A be a finite median algebra. Let $\{*\} = A_0 \subset .... \subset A_n = A$ be a sequence of expansion reached by applying Proposition 2.13.

Then by Proposition 2.13 and the construction of $\mathcal{M}$, we have $A_i \subseteq \mathcal{M}_i$ for $i = 0, ..., n$ and each A_i is a convex subset of $\mathcal{M}_i$. Thus, A is a convex subset of $\mathcal{M}$. $\qquad\square$

We now prove the homogeneity of $\mathcal{M}$. For this purpose, we now switch to an axiomatic treatment of $\mathcal{M}$. Notice that from now on we rely heavily on the graph-theoretic perspective.

Construction 5.4. Let T_{gma} be the theory given by the following axioms.
(1) The axioms of median algebra.
(2) For every convex set $x_1, ..., x_n, x_{n+1}, ..., x_{n+k}$ where $x_1, ..., x_n$ is a smaller convex set, there exist $y_1, ..., y_n$ disjoint from the points $x_{n+1}, ..., x_{n+k}$ such that

$$\left(\bigwedge_i E(x_i, z_i) \right) \wedge \left(\bigwedge_{i,j} (E(x_i, x_j) \leftrightarrow E(z_i, z_j)) \right)$$

and such that $\{x_1, ..., x_{n+k}, z_1, ..., z_n\}$ is also a convex set.

Here $E(x, y)$ denotes there is an edge between x, y in the graph structure. In other words:

$$E(x, y) := \neg \exists a ((\langle x, a, y \rangle = a) \wedge (a \neq x) \wedge (a \neq y)).$$

Theorem 5.5 (Homogeneity). *(1) $\mathcal{M} \models T_{gma}$*
(2) For every countable model $N \models T_{gma}$ and finite convex sets $B \subseteq N$ and $A \subseteq \mathcal{M}$ and an isomorphism $g : A \to B$ of median algebras, there exists an embedding $f : \mathcal{M} \to N$ such that $f|_A = \mathcal{M}$, and $Im(f)$ is a connected component of N (when N is considered as a graph).
(3) $\mathcal{M}$ is homogeneous for convex sets.

Proof. (1) We need only to check the axiom (scheme) 2 of T_{gma}. For convex set $I = \{x_1, ..., x_n, ..., x_{n+k}\}$ and $I' = \{x_1, ..., x_n\}$, there is m such that $I \subseteq \mathcal{M}_m$. Then, I and I' are convex in $\mathcal{M}_m$ by Lemma 2.13. Then, $\mathcal{M}_{m+1}$ contains the duplicated I' as stated in the axiom scheme 2 of T_{gma}. Thus, $\mathcal{M} \models T$.

(2) We first consider the case when $A \cong B$ is one single point x_0. Because for every vertex of N, the edge degree is ω, we argue that every connected component of N can be traversed in a similar way as Construction 5.1.

For each vertex, we give a well-order on vertices connected to x. Then consider the following process which traverses a connected component of N.
(1) Begin with a point x_0 of N. Put it into the queue.
(2) Select the lowest-ordered vertex connected to the front of the queue, that has not yet been traversed. Add it to the back of the queue. Remove the vertex in the front of the queue, and add it to the back of the queue.
(3) Repeat the above process.

By induction on the distance to x_0, We can argue that the traversing process covers the entire connected component of x_0.

Let the vertex selected at the i-th step be x_i. We now construct an isomorphism from M_g to N as follows.

Recall that the median graph reached by the i-th step of Construction 5.1 be $\mathcal{M}_i$. Let M_0 be mapped to x_0. Then we shall construct the rest of the isomorphism inductively.

Suppose we have already constructed map from M_{i-1} into N. Then in the i-th step of the traversing process, if x_i is already is already in the image M_{i-1}, then skip this step (we can only skip finitely many steps in a row, so the whole process is still pseudo-finite). If not, then let x_i be of distance 1 to $\mathcal{M}_{i-1}$. Then, by Prop 2.14, we can include x_i by duplicating a finite convex set of $\mathcal{M}_{i-1}$.

Thus, we force x_i to be contained in the duplication of a finite convex set of $\mathcal{M}_{i-1}$ in the construction of $\mathcal{M}_i$ (and the rest of the map of $\mathcal{M}_i$ into N is easy to construct). Thus, we have given an embedding of $\mathcal{M}_i$ into N, and $x_i \in M_i$. Inductively, we have given an embedding of $\mathcal{M}$ into N. Because the traversing process covers one entire connected component of N, $Im(f)$ is a connected component of N.

We now consider the case when $A \cong B$ is a nontrivial finite convex subset. The proof is basically the same, but in the beginning, for a small n such that $A \subseteq \mathcal{M}_n$, we start with a certain function $f|_{\mathcal{M}_n} : \mathcal{M}_n \to N$ which takes A to B according to the isomorphism. We then begin the traversing process with every vertex of $\mathcal{M}_n$ inside the queue. In this way, we would eventually yield an embedding from of $\mathcal{M}$ into N.

(3) Evident by applying part (1) and (2) when we let $N = \mathcal{M}$. $\square$

We now prove the uniqueness.

Theorem 5.6 (Uniqueness).   *(1) Every model satisfying (1) and (2) of Theorem 5.2 models T_{gma}.*
   (2) $\mathcal{M}$ is the unique model of T_{gma} satisfying (3) and (4) of Theorem 5.2.

Proof. (1) Evident by applying homogeneity and universality.

(2) Suppose N is a model of T_{gma} satisfying (3) and (4). Because N satisfies (4), and because for each $a, b \in N$, $[a, b]$ is a convex set, every $[a, b]$ is finite, so N is a discrete median algebra, so by Proposition 2.9 and 2.10, N is induced from a median graph, which is always connected. Thus, N has only one connected component, which is N itself. Thus, by Theorem 5.5, $N \cong \mathcal{M}$. $\square$

Remark 5.7. $\mathcal{M}$ is an example of a kind of construction called Hrushovski's construction, when we take the closed subsets in the construction to be convex subsets. Lemma 2.14 ensures that the axioms of closed subsets are satisfied. In this way, the uniqueness can also be proved by referring to Proposition 2.2 of [10].

6. Acknowledgements

References

[1] Hans-J. Bandelt and Jarmila Hedlíková. "Median algebras". In: *Discrete Mathematics* 45.1 (1983), pp. 1–30. DOI: https://doi.org/10.1016/0012-365X(83)90173-5. URL: https://www.sciencedirect.com/science/article/pii/0012365X83901735.

[2] G. Birkhoff and S. A. Kiss. "A ternary operation in distributive lattices". In: *Bulletin of the American Mathematical Society* (1947).

[3] Brian H. Bowditch. "Median Algebras". 2022. URL: http://homepages.warwick.ac.uk/~masgak/papers/median-algebras.pdf.

[4] David M. Clark and Brian A. Davey. *Natural Duality for the Working Algebraist*. Cambridge University Press, 1998.

[5] Manfred Droste and Dugald Macpherson. "The automorphism group of the universa distributive lattice". In: *algebra universalis* 43.4 (2000), pp. 295–306.

[6] Frank Harary. "The Automorphism Group of a Hypercube". In: *Journal of Universal Computer Science* (2000).

[7] Wilfred Hodges. *Model Theory*. Encyclopedia of Mathematics and its Applications. Cambridge University Press, 1993. ISBN: 9780521304429.

[8] John R. Isbell. "Median Algebra". In: *Transactions of the American Mathematical Society* 260.2 (1980), pp. 319–362. URL: http://www.jstor.org/stable/1998007 (visited on 08/09/2023).

[9] Bruno Poizat. *A course in model theory: an introduction to contemporary mathematical logic*. Springer Science & Business Media, 2012.

[10] Martin Ziegler. "An exposition of Hrushovski's New Strongly Minimal Set". In: *Annals of Pure and Applied Logic* 164.12 (2013). Logic Colloquium 2011, pp. 1507–1519. ISSN: 0168-0072. DOI: https://doi.org/10.1016/j.apal.2013.06.020. URL: https://www.sciencedirect.com/science/article/pii/S0168007213000894.