METRIC DIOPHANTINE APPROXIMATION ON NON-DEGENERATE MANIFOLDS

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Abstract. Metric Diophantine approximation has been a significant area of interest in analytic number theory in the past century. In this report, we will be focusing on the simultaneous Diophantine approximation on manifolds. We will first review some classical results in $\mathbb{R}^n$. Then, we will concentrate on Beresnevich and Yang's [1] paper which resolves a long-standing problem about Khintchine's theorem on manifolds. In our project, we want to generalize some of these results to the context of p-adic numbers.

Contents

1. Introduction

We first recall the fact that $\mathbb{Q}$ is dense in $\mathbb{R}$. That is to say that, for any $x \in \mathbb{R}$ and $\epsilon > 0$, there exists a rational number $\frac{p}{q} \in \mathbb{Q}$ where $p \in \mathbb{Z}$ and $q \in \mathbb{N}$, such that $\left|x - \frac{p}{q}\right| < \epsilon$.
Now, given that we fix $x \in \mathbb{R}$ and $q \in \mathbb{N}$ as the denominator of the rational number we wish to use to approximate x. Just with this information, how small can we make ϵ such that the above statement is still true?
This leads us to an important classical result: Dirichlet's Approximation Theorem.

Theorem 1.1. *For any $x \in \mathbb{R}$ and $N \in \mathbb{N}$, there exist $p, q \in \mathbb{Z}$ with $1 \leq q \leq N$ such that*

$$\left|x - \frac{p}{q}\right| < \frac{1}{qN}. \tag{1.1}$$

Proof. For some $x \in \mathbb{R}$, we let $[x] := \max\{n \in \mathbb{Z} : n \leq x\}$ denote the integer part of x and let $\{x\} := x - [x]$ denote the fractional part of x. Then, we observe that there are $N+1$ numbers

$$\{0x\}, \{x\}, \ldots, \{Nx\} \tag{1.2}$$

in the unit interval $[0, 1)$. If we then divide the unit interval into N equal semi-open subintervals, then by the pigeonhole principle there exists some $\{q_1 x\}, \{q_2 x\}$ in a single semi-open subinterval

$[\frac{u}{N}, \frac{u+1}{N})$ for some $u \in \{1, \ldots, N\}$, where $1 \leq q_1 < q_2 \leq N$ without loss of generality. Then, we have that

$$|\{q_2 x\} - \{q_1 x\}| < \frac{1}{N}. \tag{1.3}$$

We also have that $\{qx\} = p_i x - [qx]$, where $p_i := [q_i x] \in \mathbb{Z}$, and so

$$|\{q_2 x\} - \{q_1 x\}| = |q_2 x - p_2 x - (q_1 x - p_1)| = |(q_2 - q_1)x - (p_2 - p_1)| \tag{1.4}$$

Now, we can define $q = q_2 - q_1$ and $p = p_2 - p_1$, and therefore we get

$$|qx - p| < \frac{1}{qN}, \tag{1.5}$$

as desired. □

Corollary 1.1. *Let $x \in \mathbb{R} \setminus \mathbb{Q}$. Then there exist infinitely many coprime p and q in $\mathbb{Z}$ with $q > 0$ such that*

$$\left| x - \frac{p}{q} \right| < \frac{1}{q^2}. \tag{1.6}$$

Remark 1.1. *This theorem is true for all $x \in \mathbb{R}$ if we remove the condition that p and q must be coprime, or in other words, if we allow approximation by non-reduced fractions.*

Proof. Suppose $x \in \mathbb{R} \setminus \mathbb{Q}$.
The existence of at least one rational $\frac{p}{q}$ fulfilling this condition is an immediate consequence of Theorem 1.1, simply by letting $N = q$. Now, suppose for contradiction that only finitely many such rationals exist:

$$\frac{p_1}{q_1}, \ldots, \frac{p_n}{q_n}, \tag{1.7}$$

where for each $1 \leq i \leq n$, p_i and q_i are coprime, $q_i > 0$ and

$$\left| x - \frac{p_i}{q_i} \right| < \frac{1}{q_i^2}. \tag{1.8}$$

Since x is irrational, we know that $x - \frac{p_i}{q_i} \neq 0$ for all i. Then, we know by the Archimedean principle that there exists some $N \in \mathbb{N}$ such that for all i,

$$\left| x - \frac{p_i}{q_i} \right| > \frac{1}{N}. \tag{1.9}$$

Then, by Theorem 1.1, we know there exist p and q such that

$$\left| x - \frac{p}{q} \right| < \frac{1}{qN} \leq \frac{1}{q^2}, \tag{1.10}$$

since $1 \leq q \leq N$.
Therefore, we have that $\frac{p}{q} \neq \frac{p_i}{q_i}$ for all $1 \leq i \leq n$, but $\left| x - \frac{p}{q} \right| < \frac{1}{q^2}$. Therefore, a contradiction arises, and we conclude that there are infinitely many such rationals. □

2. Review of Results in $\mathbb{R}$

A natural question we can ask is: how small can we make the right hand side of the above inequality? This is where we introduce the concept of approximating functions.

Definition 2.1. *Given a monotonic function $\psi : \mathbb{N} \to \mathbb{R}^{\geq 0}$, $x \in \mathbb{R}$ is said to be ψ-approximable if there are infinitely many $p \in \mathbb{Z}$ and $q \in \mathbb{N}$ such that*

$$\left| x - \frac{p}{q} \right| < \frac{\psi(q)}{q}. \tag{2.1}$$

Definition 2.2. *Given an approximating function ψ, we denote the set of all ψ-approximable elements of $\mathbb{R}^n$ as $S_n(\psi)$. In other words,*

$$\begin{aligned} S_n(\psi) :=\{ \mathbf{x} \in \mathbb{R}^n : \ & \textit{there are infinitely many } (p_1, \ldots, p_n, q) \in \mathbb{Z}^n \times \mathbb{N} \\ & \textit{such that } \left| x_i - \frac{p_i}{q} \right| < \frac{\psi(q)}{q} \textit{ for all } 1 \leq i \leq n \}. \end{aligned} \tag{2.2}$$

In the special case that $\psi(q) = q^{-t}$ for some $t > 0$, we will instead denote $S_n(\psi)$ as $S_n(\psi)$.

Remark 2.1. *We have by Dirichlet's Approximation Theorem that*

$$S_n(1/n) = \mathbb{R}^n. \tag{2.3}$$

Theorem 2.1 (Khintchine, 1924)**.** *Let m denote the usual Lebesgue measure. Let $\psi : \mathbb{N} \to \mathbb{R}^{\geq 0}$ be a monotonic function. Then,*

$$m(S_n(\psi)) = \begin{cases} 0 & \textit{if } \sum_{q=1}^{\infty} \psi(q)^n < \infty \\ FULL & \textit{if } \sum_{q=1}^{\infty} \psi(q)^n = \infty \end{cases}. \tag{2.4}$$

We want to find an analog for this result in the case of a manifold in $\mathbb{R}^n$. We restrict our attention to non-degenerate manifolds, which we define as follows:

Definition 2.3. *Let U be an open subset of R^d. A map $\mathbf{f} : U \to \mathbb{R}^n$ is said to be non-degenerate at $\mathbf{x} \in U$ if there exists some $l \in \mathbb{N}$ such that $\mathbf{f}$ is l times continuously differentiable on some sufficiently small ball centered at $\mathbf{x}$, and the partial derivatives of $\mathbf{f}$ at $\mathbf{x}$ of orders up to l span $\mathbb{R}^n$. In turn, the map $\mathbf{f}$ is non-degenerate if it is non-degenerate at almost every point (in terms of d-dimensional Lebesgue measure) on U.*

For the rest of this paper, we will without loss of generality assume that there is a constant $M \geq 1$ such that

$$\max_{1 \leq k \leq m} \max_{1 \leq i,j \leq d} \max_{\mathbf{x} \in U} \max\{ \left| \frac{\partial f_k(\mathbf{x})}{\partial x_i} \right|, \left| \frac{\partial^2 f_k(\mathbf{x})}{\partial x_i \partial x_j} \right| \} \leq M. \tag{2.5}$$

Definition 2.4. *A manifold M of dimension d embedded in $\mathbb{R}^n$ is said to be non-degenerate if it arises from a non-degenerate map $\mathbf{f} : U \to \mathbb{R}^n$ where U is an open subset of $\mathbb{R}^d$ and $M := \mathbf{f}(U)$.*

Remark 2.2. *We think of the non-degeneracy condition intuitively as the manifold M being "sufficiently" curved, in order to develop a Khintchine type theory on $M \cap S_n(\psi)$.*

In 2021, Beresnevich and Yang developed a theory for the convergence case of Khintchine's theorem on non-degenerate manifolds; see [1]. They prove the following:

Theorem 2.2. *Let $n \geq 2$, a submanifold $M \subseteq \mathbb{R}^n$ be nondegenerate, ψ be monotonic and $\sum_{q=1}^{\infty} \psi(q)^n$ converges. Then, almost all points on M are not ψ-approximable.*

In order to prove Theorem 2.2, we will introduce a new theorem that implies Theorem 2.2. The focus of the rest of the paper will be on proving this theorem.

Theorem 2.3. *Suppose $U \subseteq \mathbb{R}^d$ is open, and $f : U \to \mathbb{R}^n$ is nondegenerate. Then, for any $0 < \epsilon < 1$, and every $t > 0$, there is a subset $\mathfrak{M}(\epsilon, t) \subseteq U$ which we call the* ***minor arcs****, which can be written as a union of balls in U of radius $\frac{\epsilon}{e^{t/2}}$ satisfying the following properties:*

(1) *For every $x_0 \in U$ such that f is l-nondegenerate at x_0 there is a ball B_0 centered at x_0 and $K_0, t_0 > 0$ such that for $t \geq t_0$,*

$$m(\mathfrak{M}(\epsilon, t) \cap B_0) \leq K_0(\epsilon^n e^{3t/2})^{-\frac{1}{d(2l-1)(n+1)}} \tag{2.6}$$

(2) *For every ball $B \subseteq U$ and for all sufficiently large t, we have that the number of rational points $\boldsymbol{p}/q$ of denominator $0 < q < e^t$ lying $\frac{\epsilon}{e^t}$ close to $\mathbf{f}(B \setminus \mathfrak{M}(\epsilon, t) \cap U)$ is less than or equal to $K_1 \epsilon^m e^{(d+1)t} m(B)$, where K_1 depends on n and f only.*

2.1. **Proof of Theorem 2.2 modulo Theorem 2.3.** We will use the following two lemmas to prove Theorem 2.2.

Lemma 2.1. *If $\mathbf{f}(\mathbf{x}) \in W_n(\psi)$, then there are infinitely many $t \in \mathbb{N}$ such that*

$$\left\| \mathbf{f}(\mathbf{x}) - \frac{\mathbf{p}}{q} \right\|_\infty < \frac{\psi(e^{t-1})}{e^{t-1}}, \tag{2.7}$$

for some $(\mathbf{p}, q) \in \mathbb{Z}^n \times \mathbb{N}$ with $e^{t-1} \leq q < e^t$, where $\|\cdot\|_\infty$ denotes the supremum norm.

Proof. Since $\mathbf{x} = (x_1, \ldots, x_n)$ is ψ-approximable, we know that the system

$$\left| x_i - \frac{p_i}{q} \right| < \frac{\psi(q)}{q} \tag{2.8}$$

holds for infinitely many $(p_1, \ldots, p_n, q) \in \mathbb{Z}^n \times \mathbb{N}$. For each q, we can find a corresponding $t \in \mathbb{N}$ such that $e^{t-1} \leq q < e^t$. Since q is unbounded, we can find infinitely many $t \in \mathbb{N}$ where this holds. Finally, ψ is monotonically decreasing, so it follows that

$$\left| x_i - \frac{p_i}{q} \right| < \frac{\psi(q)}{q} < \frac{\psi(e^{t-1})}{e^{t-1}} \tag{2.9}$$

for all i. □

Lemma 2.2.

$$\sum_{q=1}^{\infty} \psi(q)^n < \infty \iff \sum_{q=1}^{\infty} \psi(e^t)^n e^t < \infty. \tag{2.10}$$

Proof. This is an application of the Cauchy condensation test, which says that for a non-increasing sequence of real numbers $f(n)$, the series $\sum_{n=1}^{\infty} f(n)$ converges if and only if the "condensed" series $\sum_{n=0}^{\infty} 2^n f(2^n)$ converges. □

Proof of Theorem 2.2 modulo Theorem 2.3. Without loss of generality, let $M = \mathbf{f}(U)$, where $\mathbf{f} : U \to \mathbb{R}^n$ is a nondegenerate immersion on open $U \subseteq \mathbb{R}^d$. Since $\mathbf{f}$ is nondegenerate, it suffices to show that if $\sum_{q=1}^{\infty} \psi^n(q) < \infty$ and ψ is monotonic, then:

$$m(\{\mathbf{x} \in B_0 : \mathbf{f}(\mathbf{x}) \in S_n(\psi)\}) = 0 \tag{2.11}$$

for a sufficiently small ball B_0 centered at $x_0 \in U$ where $\mathbf{f}$ is l-nondegenerate at x_0 for some $l \in \mathbb{N}$.

Without loss of generality, we assume $\psi(q) \geq q^{-5/4n}$. Otherwise, we can replace $\psi(q)$ with $\max\{\psi(q), q^{-5/4n}\}$.

Given $t > 0$, $0 < \epsilon < 1$ and $\Delta \subseteq \mathbb{R}^d$, we define

$$\mathcal{R}(\Delta; \epsilon, t) := \{(\mathbf{p}, q) \in \mathbb{Z}^{n+1} : 0 < q < e^t \text{ and } \inf_{\mathbf{x} \in \Delta \cap U} \left\| \mathbf{f}(\mathbf{x}) - \frac{\mathbf{p}}{q} \right\|_\infty < \frac{\epsilon}{e^t}\}, \tag{2.12}$$

and

$$N(\Delta;\epsilon,t)=\#\mathcal{R}(\Delta;\epsilon,t). \tag{2.13}$$

So, essentially, $N(\Delta;\epsilon,t)$ counts rational points of bounded denominator ($0<q<e^t$) lying ϵe^{-t} close to $\mathbf{f}(\Delta\cap U)\subseteq M$. We define two sets A_t and B_t, where

$$A_t=\mathfrak{M}(e\psi(e^{t-1}),t)\cap B_0, \tag{2.14}$$

$$B_t=\bigcup_{(\mathbf{p},q)\in\mathcal{R}(B_0\backslash\mathfrak{M}(e\psi(e^{t-1}),t);e\psi(e^{t-1},t)}\{\mathbf{x}\in B_0:\left\|\mathbf{x}-\frac{\mathbf{p}'}{q}\right\|_\infty<\frac{\psi(e^{t-1})}{e^{t-1}}\} \tag{2.15}$$

. We know that $\mathfrak{M}(e\psi(e^{t-1}),t)$ exists thanks to Theorem 2.3.
Then, by Lemma 2.1, we know that for any $T\geq 1$,

$$\{\mathbf{x}\in B_0:\mathbf{f}(\mathbf{x})\in S_n(\psi)\}=\bigcup_{t\geq T}A_t\cup\bigcup_{t\geq T}B_t. \tag{2.16}$$

By our assumption in Theorem 2.3 and that $\psi(q)\geq q^{-5/4n}$, we get that:

$$m(A_t)\ll(e^{(t-1)/4})^{-\frac{1}{d(2l-1)(n+1)}}, \tag{2.17}$$

and

$$m(B_t)\ll\psi(e^{t-1})^m e^{(d+1)(t-1+}\cdot(\frac{\psi(e^{t-1})}{e^{t-1}})^d=\psi(e^{t-1})^n e^{t-1}, \tag{2.18}$$

where m is the d-dimensional Lebesgue measure.

We now apply Lemma 2.2 to get that:

$$m(\{x\in B_0:\text{x is }\psi\text{-approximable}\})\ll\sum_{t\geq T}(e^{(t-1)/4})^{-\frac{1}{d(2l-1)(n+1)}}+\sum_{t\geq T}\psi(e^{t-1})^n e^{t-1}. \tag{2.19}$$

This tends to 0 as $T\to\infty$, since both of the above series are convergent. Therefore, we conclude that the Lebesgue measure of ψ-approximable points in B_0 is 0, and since x_0 and B_0 were arbitrarily chosen, we conclude that almost all points on the manifold M are not ψ-approximable. □

3. Preliminaries from Convex and Discrete Geometry

3.1. **Lattices.** In order to understand Theorem 2.3, we have to understand how we formulate this subset called the minor arcs. To do this, we need an understanding of the discrete algebraic structure known as the lattice.

Definition 3.1. *Let G be a locally compact second countable group, and Λ be a subgroup of G. We call Λ a lattice in G if Λ is discrete and G/Λ has a finite G-invariant Haar measure, which means that there exists a finite measure μ in G/Λ such that $\mu(gA)=\mu(A)$ for any $g\in G$ and any $A\subset G/\Lambda$.*

Definition 3.2. *Given the same setup as in Definition 3.1, there exists a homomorphism $\Delta_G:G\to\mathbb{R}_{>0}$ such that $\mu(Sg)=\Delta_G(g)\cdot\mu(S)$. Δ_G is called the modular function of G. G is said to be unimodular if Δ_G is identically 1. In that case, the measure μ is said to be right-invariant as well.*

Theorem 3.1. *A left invariant Haar measure on G/H is unique up to scalar multiples. The quotient G/H carries a left invariant measure if and only if*

$$\Delta_G(h) = \Delta_H(h) \tag{3.1}$$

for all $h \in H$.

The following lemma is a well-known result:

Lemma 3.1. *Show that $\mathbb{Z}^n$ is a lattice in $\mathbb{R}^n$.*

Proof. First, we show $\mathbb{Z}^n$ is discrete in $\mathbb{R}^n$. It suffices to show that for an arbitrary point in $\mathbb{Z}^n$ we can find a neighborhood around that point with no limit points. It is easy to see that using the product topologies on $\mathbb{Z}^n$ and $\mathbb{R}^n$, and for any point $z = (z_1, ..., z_m) \in \mathbb{Z}^n$, we can simply take the neighborhood $((z_1 - 0.5, z_1 + 0.5) \times \cdots \times (z_m - 0.5, z_m + 0.5)) \subseteq \mathbb{R}^n$ and observe that there is only one point of $\mathbb{Z}^n$ (z) that is in this neighborhood. Therefore, $\mathbb{Z}^n$ is discrete in $\mathbb{R}^n$. By Theorem 3.1, $\mathbb{R}^n/\mathbb{Z}^n$ carries a left invariant measure if and only if their modular functions agree on all $z \in \mathbb{Z}^n$ (both are abelian, so both unimodular). In this case, we can use the Lebesgue measure for both groups, which is both left and right-invariant, and therefore their modular functions are both 1. Therefore, the left invariant measure μ exists. We just have to show μ is finite and $\mathbb{Z}^n$-invariant.
It is well-known that $\mathbb{R}^n/\mathbb{Z}^n$ is isomorphic to $\prod_n S^1$, and so the measure of the entire space would be $(2\pi)^n$, and any subset of the space would have measure less than or equal to $(2\pi)^n$, and we conclude that μ is finite. To show that μ is $\mathbb{Z}^n$-invariant, take a subset $S \subseteq \mathbb{R}^n/\mathbb{Z}^n$, and let us say that S is isomorphic to $(a_1, b_1) \times \cdots \times (a_n, b_n) \subseteq \prod_n S^1$. Then, fix $z = (z_1, \ldots, z_n) \in \mathbb{Z}^n$, and we observe that

$$\begin{aligned} \mu(z + S) &= \mu((z_1 + a_1, z_1 + b_n) \times \cdots \times (z_n + a_n, z_n + b_n)) \\ &= \mu((a_1, b_1) \times \cdots \times (a_n, b_n)) \\ &= \mu(S). \end{aligned} \tag{3.2}$$

Therefore, μ is finite and $\mathbb{Z}^n$-invariant, and we conclude that $\mathbb{Z}^n$ is a lattice in $\mathbb{R}^n$. □

Remark 3.1. *Any lattice in $\mathbb{R}^n$ can be written as $g\mathbb{Z}^n$, where $g \in GL_n(\mathbb{R})$, the set of invertible $n \times n$ matrices. Any unimodular lattice looks like $g\mathbb{Z}^n$, where $g \in SL_n(\mathbb{R})$, the set of $n \times n$ matrices with determinant ± 1.*

3.2. **Notation.** $\|\cdot\|$ and $\|\cdot\|_\infty$ will denote the Euclidean and supremum norms on $\mathbb{R}^n$ respectively. Given $r > 0$ and $\mathbf{x} \in \mathbb{R}^n$, $B(\mathbf{x}, r)$ denotes the Euclidean ball of radius r centered at $\mathbf{x}$, and $\mathcal{B}(\mathbf{x}, r)$ denotes the $\|\cdot\|_\infty$-ball of radius r centered at $\mathbf{x}$, which is a hypercube.
For the rest of the paper, we write $f \ll g$ if there exists a constant C such that $|f| \leq Cg$ pointwise. We write $f \asymp g$ if $f \ll g$ and $g \ll f$. We also denote $G = \mathrm{SL}(n+1, \mathbb{R})$ and $\Gamma = \mathrm{SL}(n+1, \mathbb{Z})$. Then, the homogeneous space $X_{n+1} := G/\Gamma$ can be identified with the set of all unimodular lattices in $\mathbb{R}^{n+1}$, where the coset $g\Gamma$ in X_{n+1} corresponds to the lattice $g\mathbb{Z}^{n+1}$ in $\mathbb{R}^{n+1}$.

3.3. Polar Lattices.

Definition 3.3. *Given a lattice $\Lambda \in X_{n+1}$ and an integer $1 \leq i \leq n+1$, let*

$$\lambda_i(\Lambda) := \inf\{\lambda > 0 : B(\mathbf{0}, \lambda) \cap \Lambda \textit{ contains } i \textit{ linearly independent vectors}\}. \tag{3.3}$$

It is immediately obvious that $\lambda_1(\Lambda) \leq \cdots \leq \lambda_{n+1}(\Lambda)$, which are the successive minima of the closed unit ball $B(\mathbf{0}, 1)$ with respect to Λ.

Definition 3.4. *Given a lattice $\Lambda \in X_{n+1}$, its polar lattice is defined as follows:*

$$\Lambda^* = \{\mathbf{a} \in \mathbb{R}^{n+1} : \mathbf{a} \cdot \mathbf{b} \in \mathbb{Z} \text{ for every } \mathbf{b} \in \Lambda\}. \tag{3.4}$$

Definition 3.5. *Given a convex body $\mathcal{C}$ in $\mathbb{R}^{n+1}$ symmetric about $\mathbf{0}$, one defines the polar body*

$$\mathcal{C}^* = \{\mathbf{y} \in \mathbb{R}^{n+1} : \mathbf{x} \cdot \mathbf{y} \leq 1 \text{ for all } \mathbf{y} \in \mathcal{C}\}. \tag{3.5}$$

Next, we recall [1, Lemma 3.1].

Lemma 3.2. *Let $g \in G$. Then*

$$(g\mathbb{Z}^{n+1})^* = (g^T)^{-1}\mathbb{Z}^{n+1}, \tag{3.6}$$

where $(g^T)^{-1}$ is the inverse of the transpose of g.

We recall [2, Theorem 23.2] by Gruber.

Theorem 3.2. *Let Λ be a lattice in $\mathbb{R}^{n+1}$. Then, for every $1 \leq i \leq n+1$, we have that*

$$1 \leq \lambda_i(\Lambda)\lambda_{n+2-i}(\Lambda^*) \leq (n+1)!^2. \tag{3.7}$$

Given $k \in \mathbb{N}$, we define the following $k \times k$ matrix

$$\sigma_k = \begin{bmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 1 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{bmatrix}. \tag{3.8}$$

We have that for every $g \in G$,

$$\lambda_i(g\mathbb{Z}^{n+1}) = \lambda_i(\sigma_{n+1}^{-1} g \sigma \mathbb{Z}^{n+1}). \tag{3.9}$$

Given $g \in G$, we define the dual of g as

$$g^* = \sigma_{n+1}^{-1}(g^T)^{-1}\sigma. \tag{3.10}$$

3.4. **Defining Major and Minor Arcs.**

Recall that

$$\mathcal{R}(\Delta;\epsilon,t) = \{(\mathbf{p},q) \in \mathbb{Z}^{n+1} : 0 < q < e^t \text{ and } \exists\, x \in \Delta \cap U \text{ with } \mathbf{f}(\mathbf{x}) \in \mathcal{B}(\frac{\mathbf{p}}{q}, \frac{\epsilon}{e^t})\}. \tag{3.11}$$

We want to interpret the condition of $\mathbf{f}(\mathbf{x}) \in \mathcal{B}(\frac{\mathbf{p}}{q}, \frac{\epsilon}{e^t})$ in terms of properties of the action of some $g_{\epsilon,t} \in G$ (see (3.14)) on a certain lattice in $\mathbb{R}^{n+1}$.
Given $\mathbf{y} = (y_1, \dots, y_n) \in \mathbb{R}^n$, we define

$$U(\mathbf{y}) := \begin{bmatrix} \mathbb{I}_n & \sigma_n^{-1}\mathbf{y}^T \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & & & y_n \\ & \ddots & & \vdots \\ & & 1 & y_1 \\ & & & 1 \end{bmatrix} \in G. \tag{3.12}$$

Given an $m \times d$ matrix $\Theta = [\theta_{i,j}]_{1 \le i \le m, 1 \le j \le d} \in \mathbb{R}^{m \times d}$, we define

$$Z(\Theta) := \begin{bmatrix} \mathbb{I}_m & \sigma_m^{-1}\Theta\sigma_d & 0 \\ 0 & \mathbb{I} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & & & \theta_{m,d} & \dots & \theta_{m,1} & 0 \\ & \ddots & & \vdots & \ddots & \vdots & \vdots \\ & & 1 & \theta_{1,d} & \dots & \theta_{1,1} & 0 \\ & & & 1 & \dots & 0 & 0 \\ & & & & \ddots & \vdots & \vdots \\ & & & & & 1 & 0 \\ & & & & & & 1 \end{bmatrix}. \tag{3.13}$$

Finally, for each $g > 0$ and $0 < \epsilon < 1$, we define the following unimodular diagonal matrix

$$g_{\epsilon,t} := \operatorname{diag}\{\phi\epsilon^{-1}, \dots, \phi\epsilon^{-1}, \phi e^{-t}\}, \tag{3.14}$$

where

$$\phi := (\epsilon^n e^t)^{\frac{1}{n+1}}. \tag{3.15}$$

Lemma 3.3. *For any $t > 0$, $\Theta \in \mathbb{R}^{m \times d}$ and $\mathbf{y} \in \mathbb{R}^n$, we have that*

$$g_{\epsilon,t} U(\mathbf{y}) g_{\epsilon,t}^{-1} = U(e^t \epsilon^{-1} \mathbf{y}) \text{ and } g_{\epsilon,t} Z(\Theta) g_{\epsilon,t}^{-1} = Z(\Theta). \tag{3.16}$$

Proof. Using (3.14) and (3.12), we have the following:

$$\begin{aligned} g_{\epsilon,t} U(\mathbf{y}) g_{\epsilon,t}^{-1} &= \begin{bmatrix} \phi\epsilon^{-1} & & 0 & 0 \\ & \ddots & \vdots & \vdots \\ & & \phi\epsilon^{-1} & 0 \\ 0 & \dots & 0 & \phi e^{-t} \end{bmatrix} \begin{bmatrix} 1 & & & y_n \\ & \ddots & & \vdots \\ & & 1 & y_1 \\ 0 & \dots & 0 & 1 \end{bmatrix} \begin{bmatrix} \phi^{-1}\epsilon & & 0 & 0 \\ & \ddots & \vdots & \vdots \\ & & \phi^{-1}\epsilon & 0 \\ 0 & \dots & 0 & \phi^{-1}e^t \end{bmatrix} \\ &= \begin{bmatrix} \phi\epsilon^{-1} & & & \phi\epsilon^{-1}y_n \\ & \ddots & & \vdots \\ & & \phi\epsilon^{-1} & \phi\epsilon^{-1}y_1 \\ 0 & \dots & 0 & \phi e^{-t} \end{bmatrix} \begin{bmatrix} \phi^{-1}\epsilon & & 0 & 0 \\ & \ddots & \vdots & \vdots \\ & & \phi^{-1}\epsilon & 0 \\ 0 & \dots & 0 & \phi^{-1}e^t \end{bmatrix} \\ &= \begin{bmatrix} 1 & & & e^t\epsilon^{-1}y_n \\ & \ddots & & \vdots \\ & & 1 & e^t\epsilon^{-1}y_1 \\ 0 & \dots & 0 & 1 \end{bmatrix} \\ &= U(e^t\epsilon^{-1}\mathbf{y}). \end{aligned}$$

Using (3.14) and (3.13), we have the following:

$$
\begin{aligned}
g_{\epsilon,t}Z(\Theta)g_{\epsilon,t}^{-1} &= \begin{bmatrix} \phi\epsilon^{-1} & & 0 & 0 \\ & \ddots & \vdots & \vdots \\ & & \phi\epsilon^{-1} & 0 \\ 0 & \dots & 0 & \phi e^{-t} \end{bmatrix} \begin{bmatrix} 1 & & & \theta_{m,d} & \dots & \theta_{m,1} & 0 \\ & \ddots & & \vdots & & \vdots & \vdots \\ & & 1 & \theta_{1,d} & \dots & \theta_{1,1} & 0 \\ & & & 1 & & & 0 \\ & & & & \ddots & & \vdots \\ & & & & & 1 & 0 \\ 0 & & \dots & & 0 & & 1 \end{bmatrix} \begin{bmatrix} \phi^{-1}\epsilon & & 0 & 0 \\ & \ddots & \vdots & \vdots \\ & & \phi^{-1}\epsilon & 0 \\ 0 & \dots & 0 & \phi^{-1}e^{t} \end{bmatrix} \\
&= \begin{bmatrix} \phi\epsilon^{-1} & & & \phi\epsilon^{-1}\theta_{m,d} & \dots & \phi\epsilon^{-1}\theta_{m,1} & 0 \\ & \ddots & & \vdots & & \vdots & \vdots \\ & & \phi\epsilon^{-1} & \phi\epsilon^{-1}\theta_{1,d} & \dots & \phi\epsilon^{-1}\theta_{1,1} & 0 \\ & & & \phi\epsilon^{-1} & & & 0 \\ & & & & \ddots & & \vdots \\ & & & & & \phi\epsilon^{-1} & 0 \\ 0 & & \dots & & 0 & & \phi e^{-t} \end{bmatrix} \begin{bmatrix} \phi^{-1}\epsilon & & 0 & 0 \\ & \ddots & \vdots & \vdots \\ & & \phi^{-1}\epsilon & 0 \\ 0 & \dots & 0 & \phi^{-1}e^{t} \end{bmatrix} \\
&= \begin{bmatrix} 1 & & & \theta_{m,d} & \dots & \theta_{m,1} & 0 \\ & \ddots & & \vdots & & \vdots & \vdots \\ & & 1 & \theta_{1,d} & \dots & \theta_{1,1} & 0 \\ & & & 1 & & & 0 \\ & & & & \ddots & & \vdots \\ & & & & & 1 & 0 \\ 0 & & \dots & & 0 & & 1 \end{bmatrix} \\
&= Z(\Theta).
\end{aligned}
$$

□

Lemma 3.4. *Let $\mathbf{y} \in \mathbb{R}^n$. Then, for any $t > 0$, any $\Theta \in \mathbb{R}^{m\times d}$, if $\mathbf{y} \in \mathcal{B}(\frac{\mathbf{p}}{q}, \frac{\epsilon}{e^t})$ for some $(\mathbf{p}, q) \in \mathbb{Z}^{n+1}$ with $0 < q < e^t$, then*

$$\left\| g_{\epsilon,t}Z(\Theta)U(\mathbf{y})(-\mathbf{p}\sigma_n, q)^T \right\| \leq c_0\phi, \tag{3.17}$$

where

$$c_0 = \sqrt{n+1}\max_{1\leq i\leq m}(1 + |\theta_{i,1}| + \cdots + |\theta_{i,d}|). \tag{3.18}$$

.

Proof. Note that since $\mathbf{y} \in B(\frac{\mathbf{p}}{q}, \frac{\epsilon}{e^t})$, trivially we have that $||g_{\epsilon,t}U(\mathbf{y})(-\mathbf{p}\sigma_n, q)^T||_\infty < \phi$.

$$g_{\epsilon,t}U(\mathbf{y})(-\mathbf{p}\sigma_n, q)^T||_\infty < \phi = \left\| \begin{bmatrix} \phi\epsilon^{-1}(qy_n - p_n) \\ \vdots \\ \phi\epsilon^{-1}(qy_1 - p_1) \\ \phi e^{-t}q \end{bmatrix} \right\| < \left\| \begin{bmatrix} \phi\epsilon^{-1}\frac{\epsilon|q|}{e^t} \\ \vdots \\ \phi\epsilon^{-1}\frac{\epsilon|q|}{e^t} \\ \phi e^{-t}q \end{bmatrix} \right\|. \tag{3.19}$$

Then, using Lemma 3.3, we get:

$$
\begin{aligned}
||g_{\epsilon,t}Z(\Theta)U(\mathbf{y})(-\mathbf{p}\sigma_n, q)^T||_\infty &= ||Z(\Theta)g_{\epsilon,t}U(\mathbf{y})(-\mathbf{p}\sigma_n, q)^T|| \\
&\leq ||Z(\Theta)||_\infty ||g_{\epsilon,t}U(\mathbf{y})(-\mathbf{p}\sigma_n, q)^T|| \\
&\leq ||Z(\Theta)||_\infty \cdot \phi.
\end{aligned} \tag{3.20}
$$

$||Z(\Theta)||_\infty$ is the operator norm of $Z(\Theta)$ as a linear transformation from $\mathbb{R}^{n+1}$ to itself with the supremum norm. So, note that $||Z(\Theta)||_\infty = max_{1\leq i\leq m}(1 + |\theta_{i,1} + \cdots + |\theta_{i,d}|)$. Therefore, we have that $||Z(\Theta)||_\infty \leq c_0$, and so we are done.

□

Now, let $\mathbf{y} = \mathbf{f}(\mathbf{x}) := (\mathbf{x}, f(\mathbf{x})) = (x_1, \ldots, x_d, f_1(\mathbf{x}), \ldots, f_m(\mathbf{x}))$, where $\mathbf{f}$ is a non-degenerate map. For $\mathbf{x} = (x_1, \ldots, x_d) \in U$, we define

$$u(\mathbf{x}) := U(\mathbf{f}(\mathbf{x})), \tag{3.21}$$

and let

$$J(\mathbf{x}) := [\frac{\partial f_i}{\partial x_j}(\mathbf{x})]_{1\leq i\leq m, 1\leq j\leq d} \in \mathbb{R}^{m\times d}. \tag{3.22}$$

Next, for $\mathbf{x} \in U$, we define

$$z(\mathbf{x}) := Z(-J(\mathbf{x})), \tag{3.23}$$

and we let

$$u_1(\mathbf{x}) = z(\mathbf{x})u(\mathbf{x}). \tag{3.24}$$

Lemma 3.5. *Let $\mathbf{x} \in U$. If $\mathbf{f}(\mathbf{x}) \in \mathcal{B}(\frac{\mathbf{p}}{q}, \frac{\epsilon}{e^t})$ for some $(\mathbf{p}, q) \in \mathbb{Z}^{n+1}$ with $0 < q < e^t$, then*

$$\lambda_1(g_{\epsilon,t}u_1(\mathbf{x})\mathbb{Z}^{n+1}) \leq c_1\phi, \tag{3.25}$$

where

$$c_1 = \sqrt{n+1}(d+1)M. \tag{3.26}$$

Proof. Firstly, if we set $\Theta = -J(\mathbf{x})$ and $\mathbf{y} = \mathbf{f}(\mathbf{x})$, then we get that

$$Z(\Theta)U(\mathbf{y}) = z(\mathbf{x})u(\mathbf{x}) = u_1(\mathbf{x}). \tag{3.27}$$

Secondly, the quantity $\max_{1\leq i\leq m}(1+|\theta_{i,1}|+\cdots+|\theta_{i,d}|$ is bounded by $(d+1)M$ from Definition 2.3. This means that $c_0 \leq c_1$, and hence this follows from Lemma 3.3. □

Now, we can finally set up major and minor arcs. For each $t \in \mathbb{N}$, we define

$$b_t := \begin{bmatrix} e^{\frac{dt}{2(n+1)}} I_m & & \\ & e^{-\frac{(m+1)t}{2(n+1)}} I_d & \\ & & e^{\frac{dt}{2(n+1)}} \end{bmatrix} \in G. \tag{3.28}$$

Now, we define the 'raw' set of minor arcs:

$$\mathfrak{M}_0(\epsilon, t) := \{\mathbf{x} \in U : \lambda_{n+1}(b_t g_{\epsilon,t} u_1(\mathbf{x})\mathbb{Z}^{n+1}) > \phi e^{\frac{dt}{2(n+1)}}\}. \tag{3.29}$$

Now, we define the set of minor arcs as a "thickening" of $\mathfrak{M}_0(\epsilon, t)$:

$$\mathfrak{M}(\epsilon, t) := \bigcup_{\mathbf{x}\in\mathfrak{M}_0(\epsilon,t)} B(\mathbf{x}, \epsilon e^{-t/2}) \cup U. \tag{3.30}$$

The set of major arcs is simply the complement of the minor arcs:

$$\mathfrak{M}'(\epsilon, t) := U \setminus \mathfrak{M}(\epsilon, t). \tag{3.31}$$

Lemma 3.6. *For any $t > 0$, $\Theta \in \mathbb{R}^{m\times d}$ and $\mathbf{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d$ we have that*

$$b_t U(\mathbf{x}) b_{-t} = U(e^{-t/2}\mathbf{x}), \tag{3.32}$$

$$b_t Z(\Theta) b_{-t} = Z(e^{t/2}\Theta). \tag{3.33}$$

Proof. The proof involves long matrix multiplications, see the proof of Lemma 3.3 for a similar proof. ☐

Lemma 3.7. *For any* $\mathbf{x} \in U$ *and* $\mathbf{x}' = (x'_1, \ldots, x'_d) \in \mathbb{R}^d$ *such that the line segment joining* $\mathbf{x}$ *and* $\mathbf{x} + \mathbf{x}'$ *is contained in* U *we have that*

$$u_1(\mathbf{x} + \mathbf{x}') = Z(O(\|\mathbf{x}'\|)U(O(\|\mathbf{x}'\|^2))U(\mathbf{x}')u_1(\mathbf{x}). \tag{3.34}$$

Proof. The proof is obtained using the Taylor's expansion of $f(\mathbf{x}')$ and the definition of nondegeneracy. ☐

4. Proving Theorem 1.3

4.1. Minor Arcs.

Proposition 4.1. *Suppose* $U \subseteq \mathbb{R}^d$ *is open,* $\mathbf{x}_0 \in U$, $\mathbf{f} : U \to \mathbb{R}^n$ *be* l*-nondegenerate at* $\mathbf{x}_0$*. Then, there is a ball* $B_0 \subseteq U$ *centered at* $\mathbf{x}_0$ *and constants* K_0, $t_0 > 0$ *depending on* $\mathbf{f}$ *and* B_0 *only such that for any* $0 < \epsilon \leq 1$ *and every* $t \geq t_0$*, we have that*

$$m(\mathfrak{M}(\epsilon, t) \cap B_0) \leq K_0(\epsilon^n e^{\frac{3t}{2}})^{-\frac{1}{d(2l-1)(n+1)}}. \tag{4.1}$$

Furthermore, $\mathfrak{M}(\epsilon, t)$ *can be written as a union of balls in* U *of radius* $\epsilon e^{-t/2}$ *of intersection multiplicity* $\leq N_d$*.*

Proof. By definition of minor arcs, for any $\mathbf{x} \in M_0(\epsilon, t)$, we have that:

$$\lambda_{n+1}(b_t g_{\epsilon,t} u_1(\mathbf{x})\mathbb{Z}^{n+1}) > \phi e^{\frac{dt}{2(n+1)}}. \tag{4.2}$$

Additionally, by Theorem 3.2, we have the following:

$$1 \leq \lambda_1(b_t^* g_{\epsilon,t}^* u_1^*(\mathbf{x})\mathbb{Z}^{n+1})\lambda_{n+1}(b_t g_{\epsilon,t} u_1(\mathbf{x})\mathbb{Z}^{n+1}) \leq (n+1)!^2.$$

So, we conclude that

$$\lambda_1(b_t^* g_t^* u_1^*(\mathbf{x})\mathbb{Z}^{n+1}) \leq (n+1)!^2 \phi^{-1} e^{-\frac{dt}{2(n+1)}}. \tag{4.3}$$

Therefore, by the above, we have that for any $\mathbf{x} \in \mathfrak{M}_0(\epsilon, t)$, there exists $(a_0, \mathbf{a}) \in \mathbb{Z} \times \mathbb{Z}^n \setminus \{\mathbf{0}\}$ such that:

$$\begin{aligned}
&|a_0 + f(\mathbf{x})\mathbf{a}^T| < c_2 e^{-t},\\
&||\nabla f(\mathbf{x})\mathbf{a}^T||_\infty < c_2 \epsilon^{-1} e^{-t/2},\\
&\max\{|a_{d+1}, \ldots, |a_n|\} < c_2 \epsilon^{-1}.
\end{aligned}$$

Let us first observe that:

$$\begin{aligned}
b_t^* g_t^* &= \begin{bmatrix} e^{-\frac{dt}{2(n+1)}} & & \\ & e^{\frac{(m+1)t}{2(n+1)}} I_d & \\ & & e^{-\frac{dt}{2(n+1)}} I_m \end{bmatrix} \phi^{-1} \begin{bmatrix} e^t & & & \\ & \epsilon & & \\ & & \ddots & \\ & & & \epsilon \end{bmatrix} \\
&= \phi^{-1} \begin{bmatrix} e^{t-\frac{dt}{2(n+1)}} & & \\ & \epsilon e^{\frac{(m+1)t}{2(n+1)}} I_d & \\ & & \epsilon e^{-\frac{dt}{2(n+1)}} I_m \end{bmatrix}.
\end{aligned} \tag{4.4}$$

Keeping in mind that $\mathbf{f}(\mathbf{x}) := (x_1, \dots, x_d, f_1(\mathbf{x}), \dots, f_m(\mathbf{x}))$, we also know that:

$$
\begin{aligned}
u_1^*(\mathbf{x}) \begin{bmatrix} a_0 \\ \mathbf{a}^T \end{bmatrix} &= \begin{bmatrix} 1 & -\mathbf{x} & -(f_1(\mathbf{x}), \dots, f_m(\mathbf{x})) \\ 0 & I_d & J(\mathbf{x}) \\ 0 & 0 & I_m \end{bmatrix} \begin{bmatrix} a_0 \\ \mathbf{a}^T \end{bmatrix} \\
&= \begin{bmatrix} a_0 - x_1 a_1 - \cdots - x_d a_d -_1 (\mathbf{x}) a_{d+1} - \cdots - f_m(\mathbf{x}) a_n \\ a_1 + \frac{\partial f_1}{\partial x_1}(\mathbf{x}) a_{d+1} + \cdots + \frac{\partial f_m}{\partial x_1}(\mathbf{x}) a_n \\ \vdots \\ a_d + \frac{\partial f_1}{\partial x_d}(\mathbf{x}) a_{d+1} + \cdots + \frac{\partial f_m}{\partial x_d}(\mathbf{x}) a_n \\ a_{d+1} \\ \vdots \\ a_n \end{bmatrix} \\
&= \begin{bmatrix} a_0 - \mathbf{f}(\mathbf{x})\mathbf{a}^T \\ \nabla \mathbf{f}(\mathbf{x})\mathbf{a}^T \\ a_{d+1} \\ \vdots \\ a_n \end{bmatrix}.
\end{aligned} \tag{4.5}
$$

So, multiplying together the left hand sides of (4.4) and (4.5), we get:

$$
\begin{aligned}
b_t^* g_t^* u_1^*(\mathbf{x}) \begin{bmatrix} a_0 \\ \mathbf{a}^T \end{bmatrix} &= \phi^{-1} \begin{bmatrix} e^{t - \frac{dt}{2(n+1)}} & & \\ & \epsilon e^{\frac{(m+1)t}{2(n+1)}} I_d & \\ & & \epsilon e^{-\frac{dt}{2(n+1)}} I_m \end{bmatrix} \begin{bmatrix} a_0 - \mathbf{f}(\mathbf{x})\mathbf{a}^T \\ \nabla \mathbf{f}(\mathbf{x})\mathbf{a}^T \\ a_{d+1} \\ \vdots \\ a_n \end{bmatrix} \\
&= \begin{bmatrix} e^t e^{-\frac{dt}{2(n+1}} (a_0 - \mathbf{f}(\mathbf{x})\mathbf{a}^T) \\ \epsilon e^{\frac{(m+1)t}{2(n+1)}} \nabla \mathbf{f}(\mathbf{x})\mathbf{a}^T \\ \epsilon e^{-\frac{dt}{2(n+1)}} \begin{bmatrix} a_{d+1} & \dots & a_n \end{bmatrix}^T \end{bmatrix}.
\end{aligned} \tag{4.6}
$$

Now, from Theorem 3.2, we have that $\lambda_1(b_t^* g_t^* u_1^*(\mathbf{x})\mathbb{Z}^{n+1}) \le c_2 \phi^{-1} e^{-\frac{dt}{2(n+1)}}$, where $c_2 = (n+1)!^2$. Therefore, we get that:

$$
\left\| b_t^* g_t^* u_1^*(\mathbf{x})\mathbb{Z}^{n+1} \right\|_\infty \le c_2 \phi^{-1} e^{-\frac{dt}{2(n+1)}}.
$$

So, we get the inequalities:

$$
\begin{aligned}
& e^t e^{-\frac{dt}{2(n+1)}} (a_0 - \mathbf{f}(\mathbf{x})) \le c_2 \phi^{-1} e^{-\frac{dt}{2(n+1)}} \\
& \Rightarrow (a_0 - \mathbf{f}(\mathbf{x})) \le c_2 e^{-t}
\end{aligned}
$$

$$
\begin{aligned}
& \epsilon e^{\frac{(m+1)t}{2(n+1)}} \nabla \mathbf{f}(\mathbf{x})\mathbf{a}^T \le c_2 \phi^{-1} e^{-\frac{dt}{2(n+1)}} \\
& \nabla \mathbf{f}(\mathbf{x})\mathbf{a}^T \le c_2 \epsilon^{-1} e^{-\frac{t(d+m+1)}{2(n+1)}} \\
& \Rightarrow \nabla \mathbf{f}(\mathbf{x})\mathbf{a}^T \le c_2 \epsilon^{-1} e^{-\frac{t}{2}}
\end{aligned}
$$

$$
\begin{aligned}
& \epsilon e^{-\frac{dt}{2(n+1)}} \max\{|a_{d+1}, \dots, |a_n|\} \le c_2 \phi^{-1} e^{-\frac{dt}{2(n+1)}} \\
& \Rightarrow \max\{|a_{d+1}, \dots, |a_n|\} \le c_2 \epsilon^{-1}
\end{aligned}
$$

Now, using the three inequalities above, as well as our upper bound M on all the partial derivatives of f, for every $\mathbf{x}' \in M(\epsilon, t)$, we want to find a strict upper bound on $|a_0 + \mathbf{f}(\mathbf{x}')\mathbf{a}^T|$. Using the Taylor expansion of the function $a_0 + \mathbf{f}(\mathbf{x}')\mathbf{a}^T$, one has that for every $\mathbf{x}' \in \mathfrak{M}(\epsilon, t)$,

$$\left|a_0 + \mathbf{f}(\mathbf{x}')\mathbf{a}^T\right| < c_2 e^{-t} + c_2 d e^{-t} + \frac{1}{2} d^2 m M c_2 \epsilon e^{-t} \le c_3 e^{-t}, \tag{4.7}$$

where c_3 is some constant that depends on n and $\mathbf{f}$ only. Similarly, using the Taylor's expansion of the gradient $\nabla \mathbf{f}(\mathbf{x})\mathbf{a}^T$, one has that for every $\mathbf{x}' \in \mathfrak{M}(\epsilon, t)$,

$$\left\|\nabla \mathbf{f}(\mathbf{x}')\mathbf{a}^T\right\|_\infty \le c_2 \epsilon^{-1} e^{-t/2} + d c_2 e^{-t/2} \le c_3 \epsilon^{-1} e^{-t/2}. \tag{4.8}$$

Using this information, we refer to the quantitative non-divergence estimate on the space of lattices due to Bernik, Kleinbock and Margulis [3], and we get there exists a ball $B_0 \subseteq U$ centered at some $\mathbf{x}_0 \in \mathfrak{M}(\epsilon, t)$ and a constant $E \ge 1$ such that for any choice of

$$0 < \delta \le 1, T \ge 1 \text{ and } K > 0 \text{ satisfying } \delta^n < K T^{n-1}, \tag{4.9}$$

the Lebesgue measure of $\mathfrak{M}(\epsilon, t)$ satisfies the inequality

$$m(\mathfrak{M}(\epsilon, t) \cap B_0) \le E(c_3^{n+1} \epsilon^{-n} e^{-3t/2})^{-\frac{1}{d(2l-1)(n+1)}} m(B_0). \tag{4.10}$$

□

4.2. **Major Arcs.**

Proposition 4.2. *Suppose $U \subseteq \mathbb{R}^d$ is open, $\mathbf{f} : U \to \mathbb{R}^n$ be non-degenerate. Then, for any $0 < \epsilon \le 1$, any ball $B \subseteq U$ and all sufficiently large t, we have that*

$$N(B \setminus \mathfrak{M}(\epsilon, t); \epsilon, t) \le K_1 \epsilon^m e^{(d+1)T} m(B), \tag{4.11}$$

where K_1 depends on n and $\mathbf{f}$ only.

We need some lemmas in order to prove this proposition.

Lemma 4.1. $N(\Delta_1 \cup \Delta_2; \epsilon, t) \le N(\Delta_1; \epsilon, t) + N(\Delta_2; \epsilon, t)$.

Proof. The proof of this is trivial. □

Lemma 4.2. *For all sufficiently large $t > 0$ we have that*

$$N(B \setminus \mathfrak{M}(\epsilon, t); \epsilon, t) \le 2(\epsilon e^{-t})^{-\frac{d}{2}} m(B) \max_{x_0 \in \mathfrak{M}'(\epsilon, t) \cap B} N(\Delta_t(x_0) \cap B; \epsilon, t). \tag{4.12}$$

Proof. We denote

$$\Delta_t(x_0) := B_\infty(x_0, (\epsilon e^{-t})^{1/2}, \tag{4.13}$$

and

$$M = \bigcap_{x \in \mathfrak{M}_0} B_2(x, \epsilon e^{-t/2}) \cap U. \tag{4.14}$$

Fix N such that for sufficiently large t, B can be covered by N hypercubes of sidelength $(\epsilon e^{-t})^{1/2}$. The volume of each such hypercube would be $(\epsilon e^{-t})^{d/2}$.
Now, we also know that

$$N(\epsilon e^{-t})^{d/2} \asymp m(B) \le 2m(B). \tag{4.15}$$

So, we have that

$$N \le 2(\epsilon e^{-t})^{-d/2} m(B). \tag{4.16}$$

We observe that any of these hypercubes which intersect $\mathfrak{M}'(\epsilon, t) \cap B$ can be covered by a hypercube $\Delta_t(x_0)$ with $x_0 \in \mathfrak{M}' \cap B \cap \Delta$. Thus, the collection of the sets $\Delta_t(x_0) \cap B$ is a cover for $\mathfrak{M}'(\epsilon, t) \cap B$. Let's say the number of such sets that would form the cover is $N(\Delta; \epsilon, t)$.

Then, we can observe that

$$N(B \setminus \epsilon, \mathbf{t}; \epsilon, t) \leq 2(\epsilon e^{-t})^{-d/2} m(B) N(\Delta; \epsilon, t). \tag{4.17}$$

Additionally,

$$N(\Delta; \epsilon, t) \leq \max_{x_0 \in \mathfrak{M}' \cap B} N(\Delta_t(x_0) \cap B; \epsilon, t). \tag{4.18}$$

Combining the two inequalities above gets the lemma as desired. □

Lemma 4.3. *Let a ball $B \subseteq U$ be given. Then, $\forall t > 0$ and $\mathbf{x}_0 \in \mathfrak{M}'(\epsilon, t) \cap B$, we have:*

$$N(\Delta_t(\mathbf{x}_0) \cap B; \epsilon, t) \ll \epsilon^n e^t (\epsilon e^{-t})^{-d/2}. \tag{4.19}$$

Proof. Let us assume that $N(\Delta_t(\mathbf{x}_0) \cap B; \epsilon, t) \neq 0$. We take any $(\mathbf{p}, q) \in R(\Delta_t(\mathbf{x}_0) \cap B; \epsilon, t)$. Then by definition, there exists $\mathbf{x} \in \Delta_t(\mathbf{x}_0) \cap B$ such that $\left\| f(x) - \frac{\mathbf{q}}{q} \right\|_\infty < \frac{\epsilon}{e^t}$
Then, by Lemma 4.2, we have that:

$$\| g_{\epsilon,t} u_1(\mathbf{x})(-\mathbf{p}\sigma_n, q) \| \leq c_1 \phi. \tag{4.20}$$

Since $\mathbf{x} \in \Delta_t(\mathbf{x}_0)$, we have that $\mathbf{x}_0 = \mathbf{x} + (\epsilon e^{-t})^{1/2} \mathbf{x}'$, where $\|\mathbf{x}'\| \leq 1$. We know that since $\mathbf{x}, \mathbf{x}' \in U$, the line segment joining them is contained in U as well. So, by Lemma 3.7, we have that:

$$u_1(\mathbf{x}_0) = Z(d_1(\epsilon e^{-t})^{1/2}) U(d_2 \epsilon e^{-t}) U((\epsilon e^{-t 1/2} \mathbf{x}') u_1(\mathbf{x}). \tag{4.21}$$

Then, we have that:

$$\begin{aligned} g_{\epsilon,t} u_1(\mathbf{x}_0) &= g_{\epsilon,t} Z(d_1(\epsilon e^{-t})^{1/2}) U(d_2 \epsilon e^{-t}) U((\epsilon e^{-t 1/2} \mathbf{x}') u_1(\mathbf{x}) \\ &= g_{\epsilon,t} Z(d_1(\epsilon e^{-t})^{1/2}) g_{\epsilon,t}^{-1} g_{\epsilon,t} U(d_2 \epsilon e^{-t}) g_{\epsilon,t}^{-1} g_{\epsilon,t} U((\epsilon e^{-t 1/2} \mathbf{x}') g_{\epsilon,t}^{-1} g_{\epsilon,t} u_1(\mathbf{x}) \\ &= Z(d_1(\epsilon e^{-t})^{1/2}) U(d_2) U((\epsilon e^{-t-1/2} \mathbf{x}') g_{\epsilon,t} u_1(\mathbf{x}). \end{aligned} \tag{4.22}$$

Therefore,

$$g_{\epsilon,t} u_1(\mathbf{x}_0)(-\mathbf{p}\sigma_n, q) = Z(d_1(\epsilon e^{-t})^{1/2}) U(d_2) U((\epsilon e^{-t-1/2} \mathbf{x}') g_{\epsilon,t} u_1(\mathbf{x})(-\mathbf{p}\sigma_n, q). \tag{4.23}$$

Now, let us denote:
$g_{\epsilon,t} u_1(\mathbf{x})(-\mathbf{p}\sigma_n, q) =: \mathbf{v} = (v_n, \ldots, v_1, v_0)$.
Then, by the above, we have:

$$\begin{aligned} g_{\epsilon,t} u_1(\mathbf{x}_0)(-\mathbf{p}\sigma_n, q) &= Z(d_1(\epsilon e^{-t})^{1/2}) U(d_2) U((\epsilon e^{-t-1/2} \mathbf{x}') \mathbf{v} \\ &= Z(d_1(\epsilon e^{-t})^{1/2}) U(d_2) \mathbf{v}', \end{aligned} \tag{4.24}$$

where $\mathbf{v}' = U((\epsilon e^{-t-1/2} \mathbf{x}') \mathbf{v} = (v_n, \ldots, v_{d+1}, v_d + (\epsilon e^{-t-1/2} x'_d v_0, \ldots, v_1 + (\epsilon e^{-t})^{-1/2} x'_1 v_0, v_0)$.

We know from the above that $\|\mathbf{v}\| \leq c_1 \phi$. Furthermore, since $0 < q < e^t$, we get $|v_0| = \phi e^{-t} q \leq \phi$. Using the fact that $\|b x'\| \leq 1$, we know that:
When $d < i \leq m$, we have $|v_i| \leq c_1 \phi$, from the fact that $\|\mathbf{v}\| \leq c_1 \phi$.
When $1 \leq i \leq d$, and given that $(\epsilon e^{-t})^{-1/2}) \geq 1$,

$$\begin{aligned} |v_i + (\epsilon e^{-t})^{-1/2} x'_i v_0| &\leq |v_i| + (\epsilon e^{-t})^{-1/2} \cdot |x'_i| \cdot |v_0| \\ &\leq c_1 \phi + (\epsilon e^{-t})^{-1/2}) \cdot 1 \cdot \phi \\ &\leq c_1 \phi (\epsilon e^{-t})^{-1/2} + (\epsilon e^{-t})^{-1/2}) \cdot \phi \\ &= (c_1 + 1) \phi (\epsilon e^{-t})^{-1/2}). \end{aligned} \tag{4.25}$$

Therefore, denoting $[a]$ as the closed interval $[-a,a]$, in conclusion we get that:

$$\mathbf{v}' \in [c_1\phi]^m \times (c_1+1)\phi(\epsilon e^{-t})^{-1/2}]^d \times [\phi]. \tag{4.26}$$

Then, we get from the definition of $\mathbf{v}'$ that:

$$\begin{aligned} g_{\epsilon,t}u_1(\mathbf{x}_0)(-\mathbf{p}\sigma_n, q) &= Z(d_1(\epsilon e^{-t})^{1/2})U(d_2)\mathbf{v}' \\ &\in [c_4\phi]^m \times [c_4\phi(\epsilon e^{-t-1/2}]^d \times [c_4\phi], \end{aligned} \tag{4.27}$$

for some constant $c_4 > 0$ depending on n and f only. Then, we get that:

$$b_t g_{\epsilon,t}u_1(\mathbf{x}_0)(-\mathbf{p}\sigma_n, q) \in [c_4\phi e^h]^m \times [c_4\phi\epsilon^{-1/2}e^h]^d \times [c_4\phi e^h], \tag{4.28}$$

where $h = dt/2(n+1)$.
Denote $\Omega = [c_4\phi e^h]^m \times [c_4\phi\epsilon^{-1/2}e^h]^d \times [c_4\phi e^h]$
Then, we get that:

$$b_t g_{\epsilon,t}u_1(\mathbf{x}_0)(-\mathbf{p}\sigma_n, q) \in \Omega \cap b_t g_{\epsilon,t}u_1(\mathbf{x}_0)\mathbb{Z}^{n+1} \subseteq (c_6\Omega) \cap b_t g_{\epsilon,t}u_1(\mathbf{x}_0)\mathbb{Z}^{n+1}. \tag{4.29}$$

For any $c_6 > 1$. On the other hand, since $\mathbf{x}_0 \in \mathfrak{M}'(\epsilon, t)$, we have that:

$$\lambda_{n+1}(b_t g_{\epsilon,t}u_1(\mathbf{x}_0)\mathbb{Z}^{n+1}) \le \phi e^h. \tag{4.30}$$

This implies that there exists a constant $c_6 > 1$ such that $c_6\Omega$ contains a full fundamental domain of $b_t g_{\epsilon,t}u_1(\mathbf{x}_0)\mathbb{Z}^{n+1}$. Therefore,

$$\#((c_6\Omega) \cap b_t g_{\epsilon,t}u_1(\mathbf{x}_0)\mathbb{Z}^{n+1}) \ll L_{n+1}(c_6\Omega) \asymp \phi^{n+1}\epsilon^{-d/2}e^{(n+1)h}. \tag{4.31}$$

□

5. Understanding $\mathbb{Q}_S$

Given a prime p, any nonzero rational number x can be written as $x = \frac{p^\alpha r}{s}$, where r and s are integers not divisible by p, and α is a unique integer. Then, we define the p-adic norm of x as $|x|_p = p^{-\alpha}$. $\mathbb{Q}_p$ is the completion of $\mathbb{Q}$ with respect to the p-adic norm.

Remark 5.1. *Any $x \in \mathbb{Q}_p$ can be written uniquely as $x = \sum_{n=n_0}^{\infty} a_n p^n$, where $a_n \in \{0,\dots,p-1\}$ and $n_0 \in \mathbb{Z}$ such that $|x|_p = p^{-n_0}$ and $a_{n_0} \neq 0$.*

We use the following notation (where p is some fixed prime):

$$\mathbb{Z}_S := \mathbb{Z}[1/p] = \{\frac{a}{p^k} \mid a \in \mathbb{Z},\, k \in \mathbb{N} \cup \{0\}\}. \tag{5.1}$$

Let $x = (x_p, x_\infty) \in \mathbb{Q}_S := \mathbb{Q}_p \times \mathbb{R}$, then we define $|x|_S = \max\{|x_1|_p, |x_2|_\infty\}$. Let $\Lambda \subseteq \mathbb{Q}_S^n$ is a lattice. Then, we define $\Lambda^* = \{y \in \mathbb{Q}_S^n \mid x \cdot y \in \mathbb{Z}_S \forall x \in \Lambda\}$.
The following lemmas are well-known results, but we are re-proving them for the sake of gaining a better understanding of $\mathbb{Q}_S$.

Lemma 5.1. *Show that $\mathbb{Z}[1/p]$ is dense in $\mathbb{Q}_p$.*

Proof. Say $x \in \mathbb{Q}_p$. Then, we can write $x = a_{-m}p^{-m} + \cdots + a_0 + a_1 p + \dots$, where $a_{-m} \neq 0$.
Now, fix $\epsilon > 0$. We know there exists $n \in \mathbb{N}$ such that $\frac{1}{p^n} < \epsilon$. Fix this n.
Our goal is to find some x_0 such that $|x - x_0| < \frac{1}{p^n}$.

Let $x_0 = a_{-m}p^{-m} + \cdots + a_n p^n \in \mathbb{Z}[1/p]$. Then, we have that:

$$\begin{aligned} |x - x_0|_p &= \left|a_{n+1}p^{n+1} + \dots\right|_p \\ &\leq p^{-(n+1)} \\ &< p^{-n} < \epsilon. \end{aligned} \tag{5.2}$$

So, we've shown that for any ϵ-neighborhood around an arbitrary $x \in \mathbb{Q}_p$, there is another $x \neq x_0 \in \mathbb{Z}[1/p]$ in that ϵ-neighborhood. Therefore, $\mathbb{Z}[1/p]$ is dense in $\mathbb{Q}_p$. □

Lemma 5.2. *$\mathbb{Z}$ is dense in $B(0,1) \subseteq \mathbb{Q}_p$*

Proof. Firstly, we know that $\mathbb{Z} \subseteq \mathbb{Q}_p$, because for any $a \in \mathbb{Z}$, we can write $a = p^x m$ where m does not divide p and $x \geq 0$. Then, we observe that $|a|_p = p^{-x} \leq 1$. So, $\mathbb{Z} \subseteq B(0,1)$. Then, we can follow a similar proof method to show that $\mathbb{Z}$ is dense in $B(0,1)$. □

Lemma 5.3. *$\mathbb{Q}_p$ does not have a lattice.*

Proof. Recall: Let Γ be a subgroup of G. We call Γ a lattice in G if Γ is discrete and G/Γ has a finite G-invariant volume. It means there exists a finite measure μ in G/Γ such that $\mu(gA) = \mu(A)$ for any $g \in G$ and any $A \subset G/\Gamma$.
Suppose by contradiction that there exists some discrete nontrivial lattice $\Lambda \subseteq \mathbb{Q}_p$. That means that there exists some nontrivial $x \in \Lambda$.
Then, we know that $\subseteq \Lambda$ as well. This is because Λ is a subgroup and closed with respect to addition.
However, we already know that $\mathbb{Z}$ is dense in $B(0,1)$. This means that for any $0 < \epsilon < 1$ and any $x \in \Lambda$, if we consider the ϵ-neighborhood around x, we can find another $y \in \Lambda$ that is in that neighborhood. This contradicts the premise that Λ is discrete. So, we conclude that $\mathbb{Q}_p$ does not have a lattice. □

Lemma 5.4. *$\mathbb{Z}[1/p] \subseteq \mathbb{Q}_p \times \mathbb{R}$.*

Proof. We want to show that $\mathbb{Z}_p$ has no limit points. Assume for contradiction that there exists a sequence (a_n,a_n) that converges to $(x, y) \in \mathbb{Q}_p \times \mathbb{R}$. This means that the sequences $|a_{n+1} - a_n|_p \to 0$ and $|a_{n+1} - a_n|_\infty \to 0$.

If we think of the p-adic valuation as describing how "divisible by p" the number is, then we can think that the more "divisible by p" it is, the smaller its p-adic norm gets.

So, let us first assume that $|a_{n+1} - a_n|_p \to 0$. Then, this would actually mean that with the infinity norm, $|a_{n+1} - a_n|_\infty \to \infty$.

Similarly, assume that $|a_{n+1} - a_n|_\infty \to 0$. This means that $|a_{n+1} - a_n|_\infty$ is getting less "divisible by p", and so we actually get that $|a_{n+1} - a_n|_p$ is a monotonically increasing sequence which is always positive, and therefore cannot converge to 0.

Therefore, we conclude that such a sequence $(a_n, a_n) \to (x, y) \in \mathbb{Q}_p \times \mathbb{R}$ cannot exist. Therefore, conclude that $\mathbb{Z}_p$ is discrete in $\mathbb{Q}_p \times \mathbb{R}$. □

6. Generalizing Results to $\mathbb{Q}_S$

We have not been able to get full results in the case of $\mathbb{Q}_S$. However, there are some lemmas which we expect to fully work in this space and not just in $\mathbb{R}^d$.

Theorem 6.1. *Let C be a proper convex body that is o-symmetric. Then, $\frac{4^d}{d!} \leq V(C)V(C^*)$.*

Remark 6.1. *This theorem would work for all convex bodies in any arbitrary space.*

Proof. Since a non-singular linear transformation does not change $V(C)V(C^*)$, we can assume that $O = \{x : |x_1| + \cdots + |x_d| \leq 1\}$ is inscribed in C and has maximum volume among all such cross-polytopes.
Since O has maximum volume, C is contained in the cube $K = \{x : |x_i| \leq 1\}$. Thus, $O \subseteq C \subseteq K$, and by polarity, $K^* \subseteq C^* \subseteq O^*$. Note that $K^* = O$. Hence,

$$\frac{(2^d)^2}{(d!)^2} = V(O)^2 = V(O)V(K^*) \leq V(C)V(C^*). \tag{6.1}$$

□

Theorem 6.2. *If C is a convex body in $\mathbb{Q}_S^n$, then we define*

$$C^* = \{y : |x \cdot y|_S \leq 1 \text{ for all } x \in C\}. \tag{6.2}$$

If L is a lattice in $\mathbb{Q}_S^n$, then we define

$$L^* = \{y \in \mathbb{Q}_S^n \mid x \cdot y \in \mathbb{Z}_S \text{ for all } x \in L\}. \tag{6.3}$$

Let C be an o-symmetric convex body and L a lattice in $\mathbb{Q}_S^d$. Let $\lambda_i = \lambda_i(C, L)$ and $\lambda_j^* = \lambda_j(C^*, L^*)$. Then, for $k = 1, \ldots, d$,

$$1 \leq \lambda_{d-k+1}\lambda_k^* \leq \frac{4^d}{V(C)V(C^*)} \leq (d!)^2. \tag{6.4}$$

Proof. We know that $V(C)V(C*) \geq \frac{4^d}{(d!)^2}$, and that $d(L)d(L^*) = 1$ in the setting of $\mathbb{Q}_S^d$.
We can choose linearly independent points $l_1, \ldots, l_d \in L$ and $m_1, \ldots, m_d \in L^*$ such that for $i, j = 1, \ldots, d$,

$$l_i \in \lambda_i bdC, m_j \in \lambda_j^* bdC^*, \tag{6.5}$$

where bdC is the boundary of C.
Then, $\pm\frac{1}{\lambda_i} l_i \in bdC$ and $\pm\frac{1}{\lambda_j^*} m_J \in bdC^*$. By the definition of C^*, we have that

$$\pm \frac{l_i}{\lambda_i} \cdot \frac{m_j}{\lambda_j^*} \leq 1 \text{ or } \lambda_i\lambda_j^* \geq \pm l_i \cdot m_j. \tag{6.6}$$

By the definition of $L*$, we then have that for $i, j = 1, \ldots, d$:

$$\lambda_i\lambda_j^* \geq 1 \text{ or } l_i \cdot m_j = 0. \tag{6.7}$$

Let $k \in \{1, \ldots, d\}$. Since $m_1, \ldots, m_k$ are linearly independent, the set $\{x : x \cdot m_1 = \cdots = x \cdot m_k = 0\}$ is a subspace of $\mathbb{Q}_S^d$ of dimension $d - k$. Thus, at least one of the $d - k + 1$ linearly independent points $l_1, \ldots, l_{d-k+1}$ is not contained in this subspace.
Hence, we can choose a suitable i and j such that $l_i \cdot m_j \neq 0$, where $1 \leq i \leq d - k + 1$ and $1 \leq j \leq k$. Then, since $0 < \lambda_1 \leq \cdots \leq \lambda_d < \infty$, we get that

$$\lambda_{d-k+1}\lambda_k^* \leq \lambda_i\lambda_j^* \leq 1. \tag{6.8}$$

We know Minkowski's second fundamental theorem holds for our setup as well. So,

$$\frac{2^d}{d!} d(L) \leq \lambda_1 \ldots \lambda_d V(C) \leq 2^d d(L). \tag{6.9}$$

So, we know that $\lambda_1 \dots \lambda_d V(C) \leq 2^d d(L)$ and $\lambda_1^* \dots \lambda_d^* V(C^*) \leq 2^d d(L^*)$.
Thus,

$$(\lambda_1\lambda_d^*)\dots(\lambda_{d-k+1}\lambda_k^* \dots (\lambda_d\lambda_1^*)V(C)V(C^*) \leq 4^d d(L)d(L^*). \tag{6.10}$$

Combining this with the fact that $d(L)d(L^*) = 1$ and that $1 \leq \lambda_1\lambda_d^*, \dots, \lambda_{d-k+1}\lambda_k^*, \dots, \lambda_d\lambda_1^*$, we get the inequality

$$\lambda_{d-k+1}\lambda_k^* \leq \frac{4^d}{V(C)V(Cj)} \leq (d!)^2. \tag{6.11}$$

And we are done.

□

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