The Logic of Choiceless Cardinality Comparison

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Abstract

We work in the setting of Zermelo-Fraenkel set theory without assuming the Axiom of Choice. We consider the Dedekind-finite sets as a Boolean algebra together with the additional structure of comparing cardinality (in the Cantorian sense of injections). What principles does one need to reason not only about intersection, union, and complementation of Dedekind-finite sets, but also about the relative size of Dedekind-finite sets? We give a complete axiomatization: They are exactly the same principles as for reasoning about imprecise probability comparisons, the central one being Generalized Finite Cancellation.

1 Introduction

Assume that Zermelo-Fraenkel axiomatic set theory (ZF) is consistent, so that it has models. Recall that the axioms of ZF are as follows:

Definition 1.1 (Axioms of ZF, refer Section 3.2 of [Jec73])**.** The axioms of ZF are as follows.

A1. Extensionality:

$$\forall u\ (u \in x \leftrightarrow u \in y) \rightarrow x = y$$

A2. Pairing:

$$\forall u\ \forall v\ \exists x\ \forall z\ (z \in x \leftrightarrow z = u \vee z = v)$$

A3. Comprehension: Suppose φ is a formula of set theory. Then

$$\forall \vec{p}\ \forall x\ \exists y\ \forall u\ (u \in y \leftrightarrow u \in x \wedge \varphi(u, \vec{p}))^1$$

A4. Union:

$$\forall a\ \exists b\ \forall x\ (x \in b \leftrightarrow \exists y\ (x \in y \wedge y \in a))^2$$

A5. Power-Set:

$$\forall a\ \exists b\ \forall x\ (x \in b \leftrightarrow x \subseteq a)^3$$

[1] $\vec{p}$ stands for $p_1, \ldots, p_n$

[2] We say that $b = \bigcup a$

[3] $x \subseteq a$ means that $\forall z\ (z \in x \rightarrow z \in a)$

A6. Replacement:

$$(\forall x \in a)\ (\exists! y)\ \varphi(x,y) \rightarrow \exists b\ \forall y\ (y \in b \leftrightarrow (\exists x \in a)\ \varphi(x,y))$$

A7. Infinity:

$$\exists a\ (\varnothing \in a \wedge \forall x\ (x \in a \rightarrow x \cup \{x\} \in a))$$

A8. Regularity:

$$(\forall s \neq \varnothing)\ (\exists x \in s)\ [x \cap s = \varnothing]^4$$

In [DHTH20], Ding, Harrison-Trainor, and Holliday work to answer the following question: can we completely axiomatize reasoning about the relative size of finite sets, of infinite sets, and of arbitrary sets in a formal set-theoretic language? Indeed, they answer this question for a particular language and definition of relative size (see Definition 2.1 of [DHTH20]). Note that Ding, Harrison-Trainor, and Holliday worked in the setting where the Axiom of Choice is true. For example, the axiom (BC3) (see Definition 2.7 of [DHTH20]) says that any two sets are comparable, which is only true in a set theoretic universe with Choice.

The goal of this paper is to answer a modified version of the above question. First, we work in the setting of ZF set theory without assuming the Axiom of Choice; and second, we restrict to the case of Dedekind-finite sets. We answer this question for a particular language (see Definition 2.1) using Cantorian definition of relative size:

Definition 1.2. Given sets x and y, we say $|x| \leqslant |y|$ if and only if there is an injection from x into y, and we say $|x| = |y|$ if and only if there is a bijection between x and y.

Of course the Cantor-Schröder-Bernstein Theorem says that if there are injections in both directions, then there is a bijection; this is true even without the Axiom of Choice.

Theorem 1.3 (The Cantor-Schröder-Bernstein Theorem). *If $|x| \leqslant |y|$ and $|y| \leqslant |x|$, then $|x| = |y|$.*

On the other hand, it is no longer true that if there is an injection from one set to another, then there is a surjection the other way; that is, $|x| \leqslant |y|$ is not the same as saying that there is a surjection from y onto x.

2 Formal Setup

We will be defining a language that can be used to talk about sets and their Boolean combinations, and to compare their cardinalities.

Definition 2.1. Given a set Φ of *set labels*, the *set terms* t and *formulas* φ of the language $\mathcal{L}$ are generated by the following grammar:

$$t ::= a \mid t^c \mid (t \cap t)$$
$$\varphi ::= |t| \geqslant |t| \mid \neg\varphi \mid (\varphi \wedge \varphi)$$

[4]Note that $\varnothing$ is our symbol for the empty set; its existence follows from (A3) and its uniqueness follows from (A1). So, $s \neq \varnothing$ is equivalent to saying that $\exists x\ (x \in s)$.

where $a \in \Phi$. The other sentential connectives $\vee, \rightarrow$, and $\leftrightarrow$ are defined as usual, and we use $\varphi \oplus \psi$ as an abbreviation for $(\varphi \vee \psi) \wedge \neg(\varphi \wedge \psi)$. Standard set-theoretic notion may be defined as follows:

- $\varnothing := t \cap t^c$;
- $t \subseteq s := |\varnothing| \geqslant |t \cap s^c|$;
- $t = s := (t \subseteq s \wedge s \subseteq t)$ and $t \neq s := \neg(t = s)$;
- $t \nsubseteq s := \neg(t \subseteq s)$ and $t \subsetneq s := (t \subseteq s \wedge s \nsubseteq t)$.

We also use $|s| \leqslant |t|$ for $|t| \geqslant |s|$, $|s| > |t|$ for $\neg(|t| \geqslant |s|)$, and $|s| = |t|$ for $|s| \geqslant |t| \wedge |t| \geqslant |s|$.

Definition 2.2. A *field of sets* is a pair $\langle X, \mathcal{F} \rangle$ where X is a nonempty set and $\mathcal{F}$ is a collection of subsets of X closed under intersection and set-theoretic complementation.

We will now be defining our first type of model for the language $\mathcal{L}$: the (pure) model. This model will involve a field of sets $\langle X, \mathcal{F} \rangle$, along with this "naming" function V which assigns to each set label, an actual set in $\mathcal{F}$. More formally:

Definition 2.3. A *(pure) model* is a quadruple $\mathcal{N} = \langle \mathcal{W}, X, \mathcal{F}, V \rangle$, where $\mathcal{W}$ is a model of ZF, X is a non-empty set in $\mathcal{W}$, $\langle X, \mathcal{F} \rangle$ is a field of sets in $\mathcal{W}$, and $V : \Phi \rightarrow \mathcal{F}$.

We say pure model to distinguish these models from the urelement models we introduce later, where $\mathcal{W}$ is allowed to be a model with urelements. We say a pure model is a Dedekind-finite model if all of the sets in its field of sets are Dedekind-finite.

Definition 2.4. Given a pure model $\mathcal{N} = \langle \mathcal{W}, X, \mathcal{F}, V \rangle$, we define a function $\hat{V}$, which assigns to each set term a set in $\mathcal{F}$, by:

- $\hat{V}(a) = V(a)$ for $a \in \Phi$
- $\hat{V}(t^c) = X - \hat{V}(t)$
- $\hat{V}(t \cap s) = \hat{V}(t) \cap \hat{V}(s)$

We then define a satisfaction relation $\models$ as follows:

- $\mathcal{N} \models |t| \geqslant |s|$ if and only if $\mathcal{W} \models |\hat{V}(t)| \geqslant |\hat{V}(s)|$;
- $\mathcal{N} \models \neg\varphi$ if and only if $\mathcal{N} \not\models \varphi$;
- $\mathcal{N} \models \varphi \wedge \psi$ if and only if $\mathcal{N} \models \varphi$ and $\mathcal{N} \models \psi$.

Given a class K of urelement models, φ is *valid* over K if and only if $\mathcal{N} \models \varphi$ for all $\mathcal{N} \in K$.

3 Dedekind-finite Sets and Soundness

Recall that we say a set is finite if it bijects onto some $n = \{0, 1, \ldots, n-1\} \in \mathbb{N}$. Moreover, we say a set is infinite, if it is not finite; i.e., if it does not biject onto any such natural number n. There are alternative notions of finite and infinite, which in the presence of Choice, are equivalent to the above.

We say that a set is Dedekind-infinite if it is in bijective correspondence with some proper subset of itself. So, for example, $\mathbb{Z}$ is Dedekind-infinite, because it bijects onto the set of even integers $2\mathbb{Z}$. So, we can think of Dedekind-finite sets as sets that are strictly larger than their proper subsets, or equivalently, as sets that get strictly larger when a single element is added to them. More formally:

Definition 3.1 (Dedekind-finite Sets)**.** A set x is said to be *Dedekind-finite* if

$$\forall y \subsetneq x,\ |y| < |x|$$

In the presence of Choice, a set is Dedekind-finite if and only if it is finite. Although, even in ZF, every finite set is Dedekind-finite, the converse is not necessarily true. That is, there may exist infinite but Dedekind-finite sets. So, we must be careful about which universe we are working in while using these notions.

Now that we have defined what it means for sets to be Dedekind-finite, we can shed some light on amorphous sets. Intuitively, amorphous sets are infinite, Dedekind-finite sets that cannot be split into two disjoint infinite sets. More formally:

Definition 3.2 (Amorphous Sets)**.** An infinite set x is said to be *amorphous* if there do not exist infinite disjoint sets $y, z \subseteq x$ such that $x = y \cup z$.

It is not difficult to prove that every amorphous set is Dedekind-finite, so we leave the proof as an exercise for the reader.

Now that we have some background on Dedekind-finite sets, we look at some important results in ZF involving them. We will first look at Subtraction and Division by m; then we will prove Generalized Finite Cancellation (Theorem 3.6), which relies on these results, and is, in fact, a generalization of their combination.

Theorem 3.3 (Subtraction, Doyle and Conway [DC94])**.** *For any Dedekind-finite set x, if $|y \sqcup x| = |z \sqcup x|$, then $|y| = |x|$.*

We will now look at a version of the Division by m theorem, which is required in our proof of Generalized Finite Cancellation. This theorem has two common versions: the one with bijections, which corresponds to dividing an equality by m, and the one with injections, which corresponds to dividing an inequality by m. The first version was first proved by Lindenbaum, but the proof was unpublished and lost. A proof for the $m = 2$ case was given by Bernstrein [Ber05] and Sierpinski [Sie22]. Later, Sierpinski also gave a proof for dividing an inequality by two [Sie47]. A proof for the $m = 3$ case was given by Tarksi [Tar49], which can be generalized for any $m \neq 0$. Finally, an easy-to-read proof for dividing an inequality by three can be found in Doyle and Conway's paper [DC94]. The general case of dividing an inequality will be most useful to us, and so we have the Division by m theorem:

Theorem 3.4 (Division by m, Lindenbaum (unpublished), Tarski [Tar49])**.** *If* $|m \times A| \leqslant |m \times B|$*, then* $|A| \leqslant |B|$*.*

Before we finally prove Generalized Finite Cancellation, we require a formal notion of balanced sequences of sets, which we state as follows:

Definition 3.5. We say that two sequences of sets $\langle E_1, \dots, E_k\rangle$ and $\langle F_1, \dots, F_k\rangle$ are *balanced*, and write $\langle E_1, \dots, E_k\rangle =_0 \langle F_1, \dots, F_k\rangle$, if and only if for all s, the cardinality of $\{i \mid s \in E_i\}$ is equal to the cardinality of $\{i \mid s \in F_i\}$; that is, if every s appears the same number of times on the left side as on the right side.

GFC (Generalized Finite Cancellation) generalizes the balancing of unions of sets in the natural way. More specifically, it generalizes the fact that if A and B taken together are of the same size as C and D taken together, and A is at least as big as C, then it must be the case that D is at least as big as B.

The GFC principal first appeared in work of Rios Insua [Ins92] and Alon and Lehrer [AL14] on characterisations of relations of imprecise probability. We prove the same principal, though in a completely different context: We have Dedekind-finite sets instead of events in a probability space, and we compare using cardinality instead of relative likelihood.

Theorem 3.6 (Generalized Finite Cancellation)**.** *Suppose that*

$$\langle A_1, \dots, A_n, \underbrace{E, \dots, E}_{m}\rangle =_0 \langle B_1, \dots, B_n, \underbrace{F, \dots, F}_{m}\rangle,$$

that $B_1, \dots, B_n$ *are Dedekind-finite, and that* $|A_i| \geqslant |B_i|$ *for each* i*. Then* $|E| \leqslant |F|$*.*

Proof. Among the elements of $\langle A_1, \dots, A_n, E, \dots, E\rangle$ there may be repeated elements. We may replace the ith appearance of each element a in the sequence by the ordered pair (a, i). In doing so, we obtain a new sequence of pairwise disjoint sets $\langle A'_1, \dots, A'_n, E_1, \dots, E_m\rangle$, and each set in this sequence is in bijection with the corresponding set of the original sequence. Similarly, we can replace the second sequence by $\langle B'_1, \dots, B'_n, F_1, \dots, F_m\rangle$. We still have that $|A'_i| \geqslant |B'_i|$ for each i, and that the sequences are balanced:

$$\langle A'_1, \dots, A'_n, E_1, \dots, E_m\rangle =_0 \langle B'_1, \dots, B'_n, F_1, \dots, F_m\rangle.$$

Indeed we can now replace $A'_1, \dots, A'_n$ and $B'_1, \dots, B'_n$ by their disjoint unions $A = A'_1 \cup \dots \cup A'_n$ and $B = B'_1 \cup \dots \cup B'_n$. Then $|A| \geqslant |B|$, and we have balanced sequences

$$\langle A, E_1, \dots, E_m\rangle =_0 \langle B, F_1, \dots, F_m\rangle.$$

Moreover, B is Dedekind-finite. We will show that $|m \times E| = |E_1 \cup \dots \cup E_m| \leqslant |F_1 \cup \dots \cup F_m| = |m \times F|$, from which it follows by dividing by m (Theorem 3.4) that $|E| \leqslant |F|$.

Fix an injection $f : B \to A$ witnessing that $|A| \geqslant |B|$. We will define an injection $g : E_1 \cup \ldots \cup E_m \to F_1 \cup \ldots \cup F_m$. Given $x \in E_1 \cup \ldots \cup E_m$, we must define $g(x)$. By the balancing assumption, $x \in B \cup F_1 \cup \ldots \cup F_m$. If $x \in F_1 \cup \ldots \cup F_m$, let $g(x) = x$. Otherwise, if $x \in B$, $f(x) \in A$. By the balancing assumption, $f(x) \in B \cup F_1 \cup \ldots \cup F_m$; if $f(x) \in F_1 \cup \ldots \cup F_m$, set $g(x) = f(x)$. Otherwise, $f(x) \in B$ and so $f(f(x)) \in A$. Continue until, for some k, $f^k(x) \in F_1 \cup \ldots \cup F_m$ and we define $g(x) = f^k(x)$. If we never find that $f^k(x) \in F_1 \cup \ldots \cup F_m$, then it must be that $x, f(x), f(f(x)), \ldots$ are all in B. Since f is injective, this sequence has no repetition, and so B contains an ω-sequence. This contradicts the fact that B is Dedekind finite. So the construction eventually terminates and defines $g(x)$.

We must argue that g is injective. For distinct x and y in $E_1 \cup \cdots \cup E_m$, consider the two sequences $x, f(x), f(f(x)), \ldots, f^k(x) = g(x)$ and $y, f(y), f(f(y)), \ldots, f^\ell(y) = g(y)$. Each element of these two sequences, except the first elements x and y, are in A. Thus x does not appear anywhere in the sequence for y, and y does not appear anywhere in the sequence for f. Since f is injective, it must be that $g(x) \neq g(y)$. Hence g is injective.

Thus we have shown that $|E_1 \cup \ldots \cup E_m| \leqslant |F_1 \cup \ldots \cup F_m|$, i.e., that $|m \times E| \leqslant |m \times F|$. Recall that by applying division by m (Theorem 3.4), it follows that $|E| \leqslant |F|$. ☐

Now that we have formalized some results pertaining to Dedekind-finite sets, we will define the logic for $\mathcal{L}$ (Definition 2.1).

Definition 3.7 (DedFinLogic)**.** The logic for Dedekind-finite sets DedFinLogic is the logic for $\mathcal{L}$ with the following axiom schemas

(D1) All substitution instances of classical propositional tautologies;

(D2) $\neg|\varnothing| \geqslant |\varnothing^c|$ (Non-triviality);

(D3) $|s| \geqslant |\varnothing|$ (Positivity);

(D4) $|s| \geqslant |s|$ (Reflexivity);

(D5) $|s| \geqslant |t| \wedge |t| \geqslant |u| \to |s| \geqslant |u|$ (Transitivity);

(D6) $\mathrm{DedFin}(t_1, \ldots, t_n) \to \mathrm{GFC}_{n,m}(s_1, \ldots, s_n, e; t_1, \ldots, t_n, f)$;

and the following rules

(R1) $[\varphi \wedge (\varphi \to \psi)] \to \psi$ (Modus Ponens);

(R2) If $t = 0$ is provable in the equational theory of Boolean algebras, then $|\varnothing| \geqslant |t|$ is a theorem.

$\mathrm{GFC}_{n,m}(s_1, \ldots, s_n, e; t_1, \ldots, t_n, f)$ in the axiom schema (D6) is the formal expression of the GFC principal and will be defined in an analogous manner to FC_n in Definition 2.8 of [DHTH20].

First, for each k such that $1 \leqslant k \leqslant n$, define the term $\mathcal{S}_k$ as the union of the terms of the form $s_1^{c_{s,1}} \cap \ldots \cap s_n^{c_{s,n}} \cap e^{c_{e,1}} \cap \ldots \cap e^{c_{e,m}}$, where exactly k many c_i's are c and the rest are empty. Similarly, define $\mathcal{T}_k$ by replacing the s's with t's and the e's with f's. So, intuitively $\mathcal{S}_k$ denotes the set of elements which are in exactly k many sets among $s_1, \ldots, s_n, \underbrace{e, \ldots, e}_{m}$.

Then, $\mathrm{GFC}_{n,m}(s_1, \ldots, s_n, e; t_1, \ldots, t_n, f)$ is defined by:

$$\Big(\bigwedge_{i=1}^{n} \mathcal{S}_i = \mathcal{T}_i\Big) \to \Big(\Big(\bigwedge_{i=1}^{n} |s_i| \geqslant |t_i|\Big) \to |e| \leqslant |f|\Big)$$

Theorem 3.8 (Soundness)**.** *DedFinLogic (3.7) is sound with respect to Dedekind-finite pure models.*

Proof. Note that most of the axiom schemas for DedFinLogic, like non-triviality (D2), reflexivity (D4), and transitivity (D5), are clearly valid according to the pure model semantics. So, in proving the soundness of DedFinLogic, our main task is to show that axiom schema (D6), i.e. Generalized Finite Cancellation, is valid. This, however, follows from Theorem 3.6. And so, DedFinLogic is Sound with respect to Dedekind-finite pure models. □

4 Urelements and Permutation Models

We introduce set theory with atoms, which we will be using as a tool to make some of our proofs easier to deal with. For example, our completeness proof will pass through set theory with atoms. Eventually, we will see that the models of set theory with atoms of our interest and those models without atoms do have a similar structure (see Theorem 4.7).

The *set theory with atoms*, ZFA, is a modified version of set theory (ZF), and is characterized by the fact that it admits objects other than sets, *atoms*. Atoms are objects which do not have any elements.

Definition 4.1 (The Language of ZFA, refer Section 4.1 of [Jec73])**.** The language of ZFA consists of $=$ and $\in$ and two constant symbols $\varnothing$ and A (the empty set and the set of all atoms). The axioms of ZFA are like the axioms of ZF except for the following changes:

$\varnothing$. Null set:

$$\neg\exists x\ (x \in \varnothing)$$

A. Atoms:

$$\forall z\ (z \in A \leftrightarrow z \neq \varnothing \wedge \neg\exists x\ (x \in z))$$

(*Atoms* are the elements of A and *sets* are all objects which are not atoms.)

A1. Extensionality:

$$(\forall \text{ set } x)(\forall \text{ set } y)\ [\forall u\ (u \in x \leftrightarrow u \in y) \rightarrow x = y]$$

A8. Regularity:

$$(\forall \text{ nonempty } s)\ (\exists x \in s)\ [x \cap s = \varnothing]$$

Note that 's is nonempty' is not necessarily the same as '$s \neq \varnothing$'; it is the same only if s is a set. Furthermore, some operations like $\bigcup x$ or $\mathcal{P}(x)$ make sense only for sets and some like $\{x, y\}$ also for atoms. Finally, if we add to ZFA the axiom $A = \varnothing$, then we get ZF.

Definition 4.2 (ZFA+AC)**.** The language of ZFA+AC is similar to that of ZFA; however, ZFA+AC has the following additional axiom:

A9. Choice:

$$\forall F\ [\varnothing \notin F \wedge (\forall x, y \in F)\ (x \neq y \rightarrow x \cap y = \varnothing)] \rightarrow \exists C\ [(\forall x \in F)(\exists! z \in x)\ (z \in C)]$$

Note that axiom [A9] has numerous equivalent forms, many of which can be found in Part I and II of [HR98]

Definition 4.3. Let $\mathcal{X}$ and $\mathcal{Y}$ be models of ZF. We say $\phi : \mathcal{X} \to \mathcal{Y}$ is an $\in$-*isomorphism* if and only if ϕ is a bijection and for all $x, y \in \mathcal{X}$, $x \in y$ if and only if $\phi(x) \in \phi(y)$

Definition 4.4. For any set S we define the cumulative hierarchy above S, $\mathcal{P}^\alpha(S)$, inductively as follows

$$\begin{aligned} \mathcal{P}^0(S) &= S, \\ \mathcal{P}^{\alpha+1}(S) &= \mathcal{P}^\alpha(S) \cup \mathcal{P}(\mathcal{P}^\alpha(S)), \\ \mathcal{P}^\lambda(S) &= \bigcup_{\beta<\lambda} \mathcal{P}^\beta(S) \qquad (\lambda \text{ limit}) \end{aligned}$$

We will now look at permutation models. Note that the axioms of ZFA do not distinguish between the atoms, and this is the underlying idea of such permutation models. Furthermore, they are also used to construct models in which the set of atoms, A, is not well-orderable.

Definition 4.5 (Permutation Models, refer section 4.2 of [Jec73])**.** Consider set theory with atoms and let A be the set of atoms. Let π be a permutation of the set A. Using the hierarchy of $\mathcal{P}^\alpha(A)$'s, we can define πx for every x as follows:

$$\pi(\varnothing) = \varnothing,\ \pi(x) = \pi[x] = \{\pi(y) : y \in x\}$$

(either by $\in$-recursion or by recursion on the rank of x.) Under this definition π becomes an $\in$-*automorphism* of the universe and one can easily verify the following facts about π:

(a) $x \in y \iff \pi x \in \pi y$;

(b) $\phi(x_1, \dots, x_n) \iff \phi(\pi x_1, \dots, \pi x_n)$;

(c) $\text{rank}(x) = \text{rank}(\pi x)$;

(d) $\pi\{x, y\} = \{\pi x, \pi y\}$ and $\pi(x, y) = (\pi x, \pi y)$;

(e) If R is a relation then πR is a relation and $(x, y) \in R \iff (\pi x, \pi y) \in \pi R$;

(f) If f is a function on X then πf is a function on πX and $(\pi f)(\pi x) = \pi(f(x))$;

(g) $\pi x = x$ for every x in the kernel;

(h) $(\pi \cdot \rho)x = \pi(\rho(x))$.

Let $\mathcal{G}$ be a group of permutations of A. A set $\mathcal{F}$ of subgroups of $\mathcal{G}$ is a *normal filter* on $\mathcal{G}$ is for all subgroups H, K of $\mathcal{G}$:

(i) $\mathcal{G} \in \mathcal{F}$;

(ii) if $H \in \mathcal{F}$ and $H \subseteq K$, then $K \in \mathcal{F}$;

(iii) if $H \in \mathcal{F}$ and $K \in \mathcal{F}$, then $H \cap K \in \mathcal{F}$;

(iv) if $\pi \in \mathcal{G}$ and $H \in \mathcal{F}$, then $\pi H \pi^{-1} \in \mathcal{F}$;

(v) for each $a \in A$, $\{\pi \in \mathcal{G} : \pi a = a\} \in \mathcal{F}$.

For each x, let $\mathrm{sym}_{\mathcal{G}}(x) = \{\pi \in \mathcal{G} : \pi x = x\}$; note that $\mathrm{sym}_{\mathcal{G}}(x)$ is a subgroup of $\mathcal{G}$. Let $\mathcal{G}$ and $\mathcal{F}$ be fixed. We say that x is *symmetric* if $\mathrm{sym}_{\mathcal{G}}(x) \in \mathcal{F}$. The class

$$\mathcal{U} = \{x : x \text{ is symmetric and } x \subseteq \mathcal{U}\}$$

consists of all *hereditarily symmetric* objects. So far $\mathcal{U}$ is just a class; however, we can prove that it is in fact a model of ZF (see Theorem 4.1 of [Jec73]). We call $\mathcal{U}$ a *permutation model.*

For our purposes, it suffices to consider the following simple type of permutation models: Let $\mathcal{G}$ be a group of permutations of A. A family I of subsets of A is a *normal ideal* if for all subsets E, F of A:

(i) $\varnothing \in I$;

(ii) if $E \in I$ and $F \subset E$, then $F \in I$;

(iii) if $E \in I$ and $F \in I$, then $E \cup F \in I$;

(iv) if $\pi \in \mathcal{G}$ and $E \in I$, then $\pi[E] \in I$;

(v) for each $a \in A$, $\{a\} \in I$.

For each x, let $\mathrm{fix}_{\mathcal{G}}(x) = \{\pi \in \mathcal{G} : \pi y = y \text{ for all } y \in x\}$; note that $\mathrm{fix}_{\mathcal{G}}(x)$ is a subgroup of $\mathcal{G}$.

Let $\mathcal{F}$ be the filter on $\mathcal{G}$ generated by the subgroups of $\mathrm{fix}(E), E \in I$. $\mathcal{F}$ is a normal filer, and so it defines a permutation model $\mathcal{U}$. Note that x is symmetric if and only if there exists $S \in I$ such that

$$\mathrm{fix}(S) \subseteq \mathrm{sym}(x).$$

We say that S is the *support* of x.

We now introduce urelement models, which function as the ZFA analogue of our pure models. Note that we use the term "urelements" instead of "atoms" to distinguish these objects from the atoms of a Boolean algebra.

Definition 4.6. An *urelement model* is a quadruple $\mathcal{M} = \langle \mathcal{U}, X, \mathcal{F}, V \rangle$, where $\mathcal{U}$ is a model of ZFA, X is a non-empty set in $\mathcal{U}$, $\langle X, \mathcal{F} \rangle$ is a field of sets in $\mathcal{U}$, and $V : \Phi \to \mathcal{F}$. We say that an urelement model is a *permutation urelement model* if the underlying model $\mathcal{U}$ of ZFA is a permutation model. [5]

[5] We could ask for X to be a non-empty set of atoms, but this adds technical complications to some of our proofs.

Theorem 4.7 (The First Embedding Theorem, Theorem 6.1 of [Jec73])**.** *Let $\mathcal{Y}$ be a model of ZFA+AC, let A be the set of all atoms of $\mathcal{Y}$, let $\mathcal{K}$ be the kernel of $\mathcal{Y}$ and let α be an ordinal in $\mathcal{Y}$. For every permutation model $\mathcal{U} \subseteq \mathcal{Y}$ (a model of ZFA), there exists a symmetric extension $\mathcal{U}^* \supseteq \mathcal{K}$ (a model of ZF) and a set $\tilde{A} \in \mathcal{U}^*$ such that*

$$(\mathcal{P}^\alpha(A))^{\mathcal{U}} \textit{ is } \in\textit{-isomorphic to } (\mathcal{P}^\alpha(\tilde{A}))^{\mathcal{U}^*}$$

Theorem 4.8. *For each permutation urelement model $\mathcal{M}$, there is a symmetric pure model $\mathcal{N}$ such that $\mathcal{M} \equiv \mathcal{N}$.*

Proof. Let $\mathcal{M} = (\mathcal{U}, X, \mathcal{F}, V)$ be a permutation urelement model. Thus $\mathcal{U}$ is a permutation model of ZFA. We will construct a model $\mathcal{U}^*$ of ZF without atoms, a set $X^* \in \mathcal{U}^*$, and a field of sets $\mathcal{F}^*$ on X^* such that there is a bijection $x \mapsto x^*$ from $X \to X^*$, inducing an isomorphism of Boolean algebras $Y \mapsto Y^*$ from $\mathcal{F} \to \mathcal{F}^*$. Moreover, this map will respect cardinality: we will have that $|E| \geqslant |F|$ in $\mathcal{U}$ if and only if $|E^*| \geqslant |F^*|$ in $\mathcal{U}^*$. Then, defining $V^*(p) = V(p)^*$, we will get that $\mathcal{N} = (\mathcal{U}^*, X^*, \mathcal{F}^*, V^*)$ satisfies $\mathcal{M} \equiv \mathcal{N}$.

Let A be the set of all atoms of $\mathcal{U}$ and let $\mathcal{K}$ be the kernel of $\mathcal{U}$ Let β be an ordinal in $\mathcal{U}$ which is sufficiently large that $X, \mathcal{F} \in (\mathcal{P}^\beta(A))^{\mathcal{U}}$. Let $\alpha = \beta + 3$; note that for any $E, F \subseteq X$, and any injection $f : E \to F$, viewing f as a set of ordered pairs we have that $f \in (\mathcal{P}^\alpha(A))^{\mathcal{U}}$. Then by The First Embedding Theorem (Theorem 4.7), there is a model of ZF, $\mathcal{U}^* \supseteq \mathcal{K}$, and a set $\tilde{A}$ in $\mathcal{U}^*$ such that

$$(\mathcal{P}^\alpha(A))^{\mathcal{U}} \text{ is } \in\text{-isomorphic to } (\mathcal{P}^\alpha(\tilde{A}))^{\mathcal{U}^*}$$

Let ϕ be this map. A vital property of ϕ which we will use implicitly throughout the rest of the proof is that, for any set S in $(\mathcal{P}^\alpha(A))^{\mathcal{U}}$,

$$\phi(S) = \{\phi(x) : x \in S\}.$$

That is, $\phi(S)$ cannot get any new elements which are not in the range of ϕ. This relies not just on the fact that ϕ is an $\in$-isomorphism, but on the particular range of ϕ. (Note that this fact does not apply to atoms a of $\mathcal{U}$; in this case, $\phi(a)$ is a non-empty set.) Another property of ϕ is that it acts level-by-level: for each $\gamma \leqslant \alpha$, ϕ maps

$$(\mathcal{P}^\gamma(A))^{\mathcal{U}} \longrightarrow (\mathcal{P}^\gamma(\tilde{A}))^{\mathcal{U}^*}.$$

For each $x \in X$, define x^* in $\mathcal{U}^*$ by $x^* = \phi(x)$. We let

$$X^* = \phi(X) = \{x^* : x \in X\}$$

and for any $Y \subseteq X$, we have $Y^* \subseteq X^*$ defined by

$$Y^* = \phi(Y) = \{x^* : x \in Y\}.$$

We also define

$$\mathcal{F}^* = \phi(\mathcal{F}) = \{Y^* : Y \in \mathcal{F}\}.$$

Note that $Y \mapsto Y^*$ is an isomorphism of Boolean algebras between the fields of sets $\mathcal{F}$ and $\mathcal{F}^*$.

Now, suppose that $|E| \geqslant |F|$ in $\mathcal{U}$; that is, suppose that there is an injection $f : F \to E$. Then $\phi(f)$ will be an injection from $F^* = \phi(F)$ to $E^* = \phi(E)$ as

$$f(a) = b \Longleftrightarrow \phi(f)(\phi(a)) = \phi(b).$$

(Note that f is in the domain of ϕ by choice of α.) Thus $|E^*| \geqslant |F^*|$.

Conversely, suppose that $|E^*| \geqslant |F^*|$; that is, suppose that there is an injection $g^* : F^* \to E^*$. First note that g^* is in the range of ϕ because E^* and F^* are in $(\mathcal{P}^{\beta}(\tilde{A}))^{\mathcal{U}^*}$ (we use here the fact that ϕ maps level-by-level), and so g^* is in $(\mathcal{P}^{\alpha}(\tilde{A}))^{\mathcal{U}^*}$. Then let g be the preimage of g^* under ϕ, $g^* = \phi(g)$. Then, as ϕ is an $\in$-isomorphism, g will be an injection from E to F and so $|E| \geqslant |F|$. $\square$

Note that each pure model is already an urelement model, and so we get:

Corollary 4.9. *The logic of urelement models is the same as the logic of pure models.*

5 Models for Imprecise Probability and Completeness

In this section, we introduce imprecise probability measures models, or simply, probability measures models. We notice that the axiom schemas for our language are precisely those of the imprecise probability measures, and so we use the completeness result for probability measures models as an intermediate step in proving completeness for our models.

Definition 5.1. A *probability measures model* is a triple $\langle W, P, V\rangle$ such that W is a set of states, P is a set of probability measures on $\mathcal{P}(W)$, and $V : \Phi \to \mathcal{P}(W)$. We define, for any $E, F \in \mathcal{P}(W)$,

$$E \precsim F \iff \forall \mu \in P,\ \mu(E) \leqslant \mu(F).$$

Definition 5.2. Given a probability measures model $\mathcal{N} = \langle W, P, V\rangle$, we define a function $\hat{V}$, which assigns to each set term a set in $\mathcal{P}(W)$, by:

- $\hat{V}(a) = V(a)$ for $a \in \Phi$
- $\hat{V}(u^c) = W - \hat{V}(u)$
- $\hat{V}(u \cap v) = \hat{V}(u) \cap \hat{V}(v)$

We then define a satisfaction relation $\models$ with the usual clauses for propositional variables and Boolean connectives, plus: $\mathcal{N} \models |v| \leqslant |u|$ if and only if $(\hat{V}(v)) \precsim (\hat{V}(u))$.

Notice that on the surface it may look like the languages are different; for example, in imprecise probabilities, we tend to use the $x \precsim y$ notation rather than our standard $|x| \leqslant |y|$ notation. However, there is clearly a direct correspondence between these.

Theorem 5.3 (Theorem 4 of [HTHI17])**.** *DedFinLogic (3.7) is sound and complete with respect to probability measures models.*

Theorem 5.4. *For each finite probability measures model $\mathcal{M}$, there is a Dedekind-finite urelement model $\mathcal{N}$ such that $\mathcal{M} \equiv \mathcal{N}$.*

Proof. Let $\mathcal{M} = \langle W, P, V \rangle$ be a finite probability measures model. We will construct a Dedekind-finite urelement model $\mathcal{N} = (\mathcal{U}, X, \mathcal{F}, V^*)$ such that $\mathcal{M} \equiv \mathcal{N}$.

In fact, we will define $\mathcal{U}$, X, and $\mathcal{F}$ and show that there is a map $E \mapsto E^*$ from $\mathcal{P}(W) \to \mathcal{F}$ which is an isomorphism of Boolean algebras such that for $E, F \subseteq W$,

$$E \precsim F \longleftrightarrow \mathcal{U} \models |E^*| \leqslant |F^*|.$$

Given $s \in \Phi$, and $V(s) = E \subseteq W$, define $V^*(s) = E^* \in \mathcal{F}$. It follows from the following two claims that $\mathcal{M} \equiv \mathcal{N}$.

Claim 1. *If t is a term and $\hat{V}(t) = E$, then $\hat{V}^*(t) = E^*$.*

Proof. Using 2.4, we will prove this claim by induction on set terms as follows:

$$\hat{V}(t) = V(t) \overset{*}{\mapsto} V^*(t) = \hat{V}^*(t)$$

$$\hat{V}(t^c) = W - \hat{V}(t) \overset{*}{\mapsto} X - \hat{V}^*(t) = \hat{V}^*(t^c)$$

$$\hat{V}(t \cap u) = \hat{V}(t) \cap \hat{V}(u) = E \cap F \overset{*}{\mapsto} (E \cap F)^* = E^* \cap F^* = \hat{V}^*(t) \cap \hat{V}^*(u) = \hat{V}^*(t \cap u) \quad \square$$

Claim 2. $\mathcal{M} \models \varphi \Longleftrightarrow \mathcal{N} \models \varphi$.

Proof. If s and t are terms, then

$$\mathcal{M} \models s \succsim t \Longleftrightarrow \hat{V}(s) \succsim \hat{V}(t) \Longleftrightarrow \mathcal{U} \models |\hat{V}^*(s)| \geqslant |\hat{V}^*(t)| \Longleftrightarrow \mathcal{N} \models |s| \geqslant |t|.$$

Furthermore, if φ and ψ are formulas, then

$$\mathcal{M} \models \neg\varphi \Longleftrightarrow \mathcal{M} \not\models \varphi \Longleftrightarrow \mathcal{N} \not\models \varphi \Longleftrightarrow \mathcal{N} \models \neg\varphi;$$

$$\mathcal{M} \models \varphi \wedge \psi \Longleftrightarrow \mathcal{M} \models \varphi \text{ and } \mathcal{M} \models \psi \Longleftrightarrow \mathcal{N} \models \varphi \text{ and } \mathcal{N} \models \psi \Longleftrightarrow \mathcal{N} \models \varphi \wedge \psi. \quad \square$$

We will now define $\mathcal{U}$, X, and $\mathcal{F}$ as above, beginning with $\mathcal{U}$. Let $\mathcal{Y}$ be a model of ZFA+AC and let A be the set of urelements in $\mathcal{Y}$. The model $\mathcal{U}$ will be a permutation model (a model of ZFA) defined from $\mathcal{Y}$. Much of our construction will follow the general argument of Theorem 11.1 of [Jec73], where given a partial order $(P, \preceq)$ in the kernel of $\mathcal{Y}$, Jech constructs a permutation model $\mathcal{U}$ containing sets $(S_p)_{p \in P}$ such that $p \preceq q$ if and only if $|S_p| \leqslant |S_q|$ in $\mathcal{U}$.

Consider the poset $(P, \preceq)$, where P is the set of probability measures, with $\preceq$ being the ordering that no two of them are comparable. Since $(P, \preceq)$ is finite, a copy of it is contained in the kernel of $\mathcal{Y}$.

We then follow the construction of Theorem 11.1 of [Jec73] using $(P,\preceq)$ as our partial order. Assume that the set A of urelements has cardinality $|A| = |P| \cdot \aleph_0$ and let $\{a_{\mu,n} : \mu \in P,\ n \in \mathbb{N}\}$ be an enumeration of A. For each $\mu \in P$, let

$$A_\mu = \{a_{\mu,n} : n \in \mathbb{N}\}.$$

Let $\mathcal{G}$ be the group of all permutations π of A such that $\pi(A_\mu) = A_\mu$ for each $\mu \in P$; that is, if $\pi a_{\mu,n} = \pi a_{\rho,m}$ then $\mu = \rho$. Let $\mathcal{F}$ be the filter on $\mathcal{G}$ given by the ideal of finite subsets of A. Since $\mathcal{F}$ is a normal filter on $\mathcal{G}$, it defines a permutation model $\mathcal{U}$ consisting of all hereditarily symmetric elements of $\mathcal{Y}$. Furthermore, each $x \in \mathcal{U}$ has a finite support $S \subseteq A$ such that $\mathrm{sym}(x) \supseteq \mathrm{fix}(S)$.

Claim 3. *A_μ is amorphous, hence Dedekind-finite.*

Proof. Suppose A_μ is not amorphous. Then it can be split up into two disjoint infinite sets B_μ and C_μ. If B_μ is preserved in $\mathcal{U}$, then it must have some finite support S. Furthermore, we know that any permutation $\pi \in \mathrm{fix}(S)$ would fix B_μ. We choose $b \in B_\mu$ and $c \in C_\mu$ such that $b, c \notin S$. We then let π be the permutation that swaps b and c and fixes everything else; that is, $\pi(b) = c$, $\pi(c) = b$, and $\pi(a) = a$ for all $a \neq b, c$. Note that π fixes each element of S, and so π fixes B_μ as a set. However, $\pi(b) = c \notin B_\mu$, which is a contradiction. An analogous argument can be made for C_μ as well. So, it must be the case that A_μ cannot be split into two disjoint infinite sets; that is A_μ must be amorphous. □

As a result of this claim, if f is a map whose domain (or range) is a subset of A_μ, then its domain (or range) must be finite or cofinite in A_μ.

Claim 4. *There is a partial injection $f\colon D \subseteq A_\mu \to A_\rho$ defined on an infinite subset of A_μ if and only if $\mu = \rho$.*

Proof. For the forward direction, suppose $\mu \neq \rho$ and that there is a partial injection φ : $\mathrm{dom}(\varphi) \subseteq A_\mu \to A_\rho$ in $\mathcal{U}$. Since φ is in $\mathcal{U}$, it must be finitely supported. Let S be a finite support of φ. We choose $a, a' \in \mathrm{dom}(\varphi) \subseteq A_\mu$ such that $a, a' \notin S$. Note that we make this choice in such a way that a, a' are in the domain of φ. Call $\varphi(a) = b$ and $\varphi(a') = b'$. We then let π be the permutation that swaps a and a' and fixes everything else; that is, $\pi(a) = a'$, $\pi(a') = a$ and $\pi(c) = c$ for all $c \neq a, a'$. Since π fixes φ, we have that

$$\pi(b) = \pi(\varphi(a)) = \varphi(\pi a) = \varphi(a') = b'$$

By definition, since $b \neq a, a'$, we have that $\pi b = b$. From the above equation it also follows that $\pi b = b'$. So, we conclude that $b = b'$. However, this would mean that $\varphi(a) = \varphi(a')$, contradicting the injectivity of φ.

For the converse direction, if $\mu = \rho$, then the identity on A_μ is an injection with empty support, i.e., an injection in $\mathcal{U}$. □

For each atomic event E (i.e., E is a singleton $\{w\}$), we can associate a set $f_\mu(E)$ of size $\mu(E)$ such that for any two distinct E and F, the sets $f_\mu(E)$ and $f_\mu(F)$ are disjoint. Since any other event is a union of atomic events, once $f_\mu(E)$ is chosen for each atom E, we can extend it to all sets by setting

$$f_\mu(E) = \bigcup_{E \supseteq F \text{ an atom}} f_\mu(F).$$

Then the map $E \mapsto f_\mu(E)$ is a Boolean algebra isomorphism and $|f_\mu(E)| = \mu(E)$.

We then define E^* to be

$$E^* = \bigcup_{\mu \in P} f_\mu(E) \times A_\mu.$$

We also have that $E \mapsto E^*$ is an isomorphism of Boolean algebras.

Suppose that $E \precsim F$, that is, for any μ in P, $\mu(E) \leqslant \mu(F)$. It follows that $|f_\mu(E)| = \mu(E) \leqslant \mu(F) = |f_\mu(F)|$. So, we have:

$$|E^*| = |\bigcup_{\mu \in P} f_\mu(E) \times A_\mu| \leqslant |\bigcup_{\mu \in P} f_\mu(F) \times A_\mu| = |F^*|.$$

Now we suppose that $\mathcal{U} \models |E^*| \leqslant |F^*|$; that is, suppose there is an injection $g : E^* \to F^*$ in $\mathcal{U}$.

For each $\mu \in P$, we define define an injection $h_\mu : f_\mu(E) \to f_\mu(F)$ witnessing that $\mu(E) \leqslant \mu(F)$. Given $x \in f_\mu(E)$, there is a copy $\{x\} \times A_\mu$ in E^*. We first argue that g maps all but finitely much of $\{x\} \times A_\mu$ to some $\{y\} \times A_\mu$ in F^* [6]; we then argue that there is exactly one such y; and then that two different x's don't map to the same y. Finally, we define $h_\mu(x)$ to be y, and in doing so, we show that h_μ is a well-defined injection.

It is clear that there must be at least one such y such that all but finitely much of $\{x\} \times A_\mu$ maps to $\{y\} \times A_\mu$ in F^*. If this was not the case, and only finitely much of $\{x\} \times A_\mu$ mapped into some $\{y\} \times A_\mu$'s, that would contradict there being only finitely many y's.

We now argue that there is exactly one such y. For a contradiction, suppose that infinitely much of $\{x\} \times A_\mu$ maps to both $\{y\} \times A_\mu$ and $\{y'\} \times A_\mu$ in F^*. This would mean that the pre-images $g^{-1}[\{y\} \times A_\mu]$ and $g^{-1}[\{y'\} \times A_\mu]$ are both disjoint infinite subsets of $\{x\} \times A_\mu$, contradicting that A_μ is amorphous. So, it must be the case that y is unique, i.e. h_μ is well-defined.

We complete this argument by showing that h_μ must, in fact, be an injection. So, for a contradiction, suppose that there are x, x' in $f_\mu(E)$ such that infinitely much of both $\{x\} \times A_\mu$ and $\{x'\} \times A_\mu$ maps into $\{y\} \times A_\mu$ in F^*. However, this would mean that the restricted images $g[\{x\} \times A_\mu]$ and $g[\{x'\} \times A_\mu]$ are both disjoint infinite subsets of $\{y\} \times A_\mu$, contradicting that A_μ is amorphous. This must mean that different x's must map to different y's, and so h_μ is an injection.

Now that we have defined $\mathcal{U}$, we will define X and $\mathcal{F}$

$$X := \bigcup_{E \subseteq W} E^*.$$

Finally, we define $\mathcal{F}$ as the image of $\mathcal{P}(W)$ under $*$. That is, if the mapping $E \mapsto E^*$ is preserved, then $\mathcal{F} = \{E^* \mid E \subseteq W\}$ will perform the function of our field of sets set in the urelement model. That is, $\langle X, \mathcal{F} \rangle$ is precisely a field of sets.

Claim 5. *If P is finite, then each element of $\mathcal{F}$ is Dedekind-finite.*

Proof. Recall that $E^* := \bigcup_{\mu \in P} f_\mu(E) \times A_\mu$. Since each A_μ is Dedekind-finite (Claim 3), we have that $f_\mu(E) \times A_\mu$ is Dedekind-finite. Furthermore, since P is finite, E^* is a finite union of Dedekind-finite sets, and so each E^* is Dedekind-finite; that is, each element of $\mathcal{F}$ is Dedekind-finite. $\square$

The model $\mathcal{N} = (\mathcal{U}, X, \mathcal{F}, V^*)$ as constructed above is a Dedekind-finite urelement model such that $\mathcal{M} \equiv \mathcal{N}$, proving our theorem. $\square$

[6] Note that it would be more precise to begin by saying that infinitely much of $\{x\} \times A_\mu$ maps to some $\{y\} \times A_\mu$. However, since each A_μ is amorphous, as shown in Claim 3, all but finitely much of $\{x\} \times A_\mu$ mapping to some $\{y\} \times A_\mu$ would be equivalent to infinitely much of $\{x\} \times A_\mu$ mapping to some $\{y\} \times A_\mu$.

Since we have already proven soundness (Theorem 3.8), we complete our characterization with the following completeness proof. In this proof, we will be relying on two previously proven results: Theorem 5.4, which proves completeness with respect to urelement models, and Corollary 4.9, which transfers this to completeness with respect to pure models.

Theorem 5.5 (Completeness)**.** *DedFinLogic (3.7) is sound and complete with respect to Dedekind-finite urelement models and also with respect to Dedekind-finite pure models.*

Proof. Recall that Theorem 5.4 says that a Dedekind-finite urelement model can be transformed into a finite probability measures model that satisfies the same formulas according to the semantics of the latter. So, the completeness of DedFinLogic with respect to urelement models follows from the completeness of DedFinLogic with respect to probability measures models (Theorem 5.3).

Furthermore, recall that Corollary 4.9 says that the logic of urelement models is the same as that of pure models. So, the completeness of DedFinLogic with respect to pure models follows from the completeness of DedFinLogic with respect to urelement models, which we have just proven above. Thus, DedFinLogic is complete with respect to Dedekind-finite urelement models and Dedekind-finite pure models. □

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