EXAMPLES OF REAL CONIC BUNDLES WITH QUARTIC DISCRIMINANT CURVE

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Abstract. We construct examples of real conic bundles over $\mathbb{P}^2$ whose discriminant curve is a smooth plane quartic curve Δ. In each isotopy class of smooth plane quartics, we construct an example where the total space of the conic bundle is rational, and for several isotopy classes we construct examples that are $\mathbb{C}$-rational but that have topological obstructions to rationality over $\mathbb{R}$. In particular, we show that for five of the six isotopy classes of the discriminant double cover $\tilde{\Delta} \to \Delta$ with $\tilde{\Delta}(\mathbb{R}) = \emptyset$, there are both rational and irrational conic bundles. Our examples are double covers of $\mathbb{P}^1 \times \mathbb{P}^2$ branched over a bidegree $(2, 2)$ divisor; these models were previously studied by Frei, the first author, Sankar, Viray, and Vogt.

1. Introduction

A fundamental question in algebraic geometry is the birational classification of algebraic varieties. The simplest varieties are those that are rational, i.e. birational to projective space. We will be interested in rationality over the field $\mathbb{R}$ of real numbers, and when we write (stable/uni-)rationality without reference to the ground field, we will mean over $\mathbb{R}$.

A smooth projective curve is $\mathbb{C}$-rational if and only if it has genus 0. Over the real numbers, however, there are irrational genus 0 curves—pointless conics—and a genus 0 curve C is $\mathbb{R}$-rational if and only if $C(\mathbb{R}) \neq \emptyset$. Rationality is also understood in dimension two, but for threefolds the rationality problem becomes much more complicated. We focus on the case of conic bundles $X \to \mathbb{P}^2$, which are morphisms whose generic fibers are conics. Over $\mathbb{C}$ the birational isomorphism class of a conic bundle is determined by its discriminant double cover; over $\mathbb{R}$, this data together with a constant Brauer class determines the birational isomorphism class. Our case of interest is when the discriminant curve is a smooth plane quartic.

A classical result of Zeuthen [Zeu74] classifies real smooth plane quartic curves into 6 isotopy classes: empty, one oval, two non-nested ovals, two nested ovals, three ovals, and four ovals. Klein [Kle76] showed that the topological type of a plane quartic determines the connected component of the moduli space that it lies in. We show that for five of the six connected components of the moduli space of real smooth plane quartics, there exist both rational and irrational conic bundles with discriminant curve of this given topological type:

Theorem 1.1. *Let $\mathcal{CB}_{**/*}$ denote the set of geometrically standard conic bundles $X \to \mathbb{P}^2$ over $\mathbb{R}$ with smooth quartic discriminant curve Δ of topological type $*$ and discriminant cover $\tilde{\Delta}$ of topological type $**$.*

(1) If $\tilde{\Delta}(\mathbb{R}) \neq \emptyset$, then every member of $\mathcal{CB}_{**/*}$ is rational;
(2) $\mathcal{CB}_{\emptyset/\emptyset}$ contains both rational members and non-unirational members;
(3) $\mathcal{CB}_{\emptyset/1\ oval}$ contains both rational members and members that are unirational but irrational;
(4) $\mathcal{CB}_{\emptyset/2\ non\text{-}nested\ ovals}$, $\mathcal{CB}_{\emptyset/2\ nested\ ovals}$, and $\mathcal{CB}_{\emptyset/3\ ovals}$ each contain both rational members and members that are unirational but not stably rational; and
(5) $\mathcal{CB}_{\emptyset/4\ ovals}$ contains rational members.

The conic bundles in Theorem 1.1 are all $\mathbb{C}$-rational by work of Iskovskikh [Isk87, Theorem 1]. The irrational conic bundles we exhibit in part (2) have no real points, and those in part (4) have disconnected real loci. Irrational examples with $\Delta(\mathbb{R})$ one oval and disconnected examples with $\Delta(\mathbb{R})$ two non-nested ovals were previously constructed in [FJS$^+$, Theorem 1.3]. Part (1) is [FJS$^+$, Section 6.1] and can be shown by a modification of the argument for $\mathbb{C}$-rationality. We note that any étale double cover $\tilde{\Delta} \to \Delta$ of a smooth plane quartic can be realized over $\mathbb{R}$ as the discriminant curve of a geometrically standard conic bundle by [Bru08] and [FJS$^+$, Section 4].

We focus on the case when $\deg \Delta = 4$ because this is the first case where X is geometrically rational but not necessarily rational over $\mathbb{R}$ (see Section 2.1). The conic bundles that we study are double covers of $\mathbb{P}^1 \times \mathbb{P}^2$ branched over a $(2,2)$ divisor, which have the additional structure of a quadric surface bundle via the first projection. These models were introduced in [FJS$^+$] by Frei, the first author, Sankar, Viray, and Vogt to study rationality over non-closed fields of conic bundles with smooth quartic discriminant. They used this model to construct two examples of irrational conic bundles with different obstructions to rationality over $\mathbb{R}$ [FJS$^+$, Theorem 1.3]. In this article, we further analyze these models to construct rational and irrational examples with a focus on the isotopy class of Δ.

The irrational examples of [FJS$^+$] have $\Delta(\mathbb{R})$ one oval or two non-nested ovals. In their one oval example [FJS$^+$, Theorem 1.3(2)], irrationality is witnessed by the intermediate Jacobian torsor (IJT) obstruction. This obstruction to rationality is a refinement over non-closed fields of the intermediate Jacobian obstruction of Clemens–Griffiths, and was recently introduced by Hassett–Tschinkel [HT21b, HT21a] and Benoist–Wittenberg [BW] (see Section 3.3). However, in the two non-nested ovals example of [FJS$^+$, Theorem 1.3(1)], the IJT obstruction vanishes but the total space of the conic bundle has two real connected components, and hence is irrational. On our way to proving Theorem 1.1, the examples that we construct also show that the failure of the IJT obstruction persists for other types of $\Delta(\mathbb{R})$. Namely, we construct an example with $\Delta(\mathbb{R}) = \emptyset$ and an example with $\Delta(\mathbb{R})$ three ovals (Proposition 3.10), where the IJT obstruction to rationality vanishes but the real locus of Y exhibits an obstruction to (stable) rationality.

In addition to constructing isolated examples, we also construct examples of families of conic bundles with irrational and rational members, and whose discriminant curves have different real isotopy classes. More precisely, in Examples 5.1, 5.2, and 5.3 we construct families $\mathcal{Y} \to \mathbb{A}^1_s$ of real threefolds such that every $\mathcal{Y}_s$ has the structure of a conic bundle over $\mathbb{P}^2_{\mathbb{R}}$ with quartic discriminant curve Δ_s such that the real isotopy class of Δ_s, the number of connected components of $\mathcal{Y}_s$, and the vanishing of the IJT obstruction for $\mathcal{Y}_s$ varies in the family.

1.1. **Outline.** In Section 2, we review background and context for conic bundles over $\mathbb{P}^2$, and we recall the key features of the double cover construction of [FJS$^+$]. In Section 3, we relate the connected components of the total space of the conic bundle to those of its discriminant curve (Section 3.1) and construct the irrational examples of Theorem 1.1(2) and (4) (Section 3.2), and review the intermediate Jacobian torsor obstruction for conic bundles (Section 3.3). In Section 4, we construct the rational examples of Theorem 1.1 (2)–(5). In Section 5 we construct examples of families of conic bundles.

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2. Preliminaries

2.1. **Rationality of standard conic bundles over $\mathbb{P}^2$.** We first review some preliminary notions about conic bundle threefolds and rationality. For more details on conic bundle threefolds, see [Pro18, Section 3].

Let k be a field of characteristic $\neq 2$. A conic bundle over $\mathbb{P}^2$ is a proper flat k morphism $\pi\colon X \to \mathbb{P}^2$ whose generic fiber is a smooth conic over $\mathbf{k}(\mathbb{P}^2)$. The *discriminant cover* $\varpi\colon \tilde{\Delta} \to \Delta$ parametrizes the components of the singular fibers of π. A conic bundle is *standard* if π has relative Picard number one.

The models that we consider will have the property that Δ is smooth and $\pi\colon X \to \mathbb{P}^2$ is geometrically standard, i.e. $\rho(X_{\overline{k}}/\mathbb{P}^2_{\overline{k}}) = 1$. Then ϖ is an étale double cover, and the fibers of π are all reduced.

Let W be a smooth projective variety of dimension n over k. Recall that W is said to be *rational over* k (or k*-rational*) if there is a birational map $W \dashrightarrow \mathbb{P}^n$ defined over k, *stably rational over* k if $W \times \mathbb{P}^m$ is k-rational for some m, and *unirational over* k if there is a dominant rational map $\mathbb{P}^n \dashrightarrow W$ defined over k. If $k \subset k'$ is a field extension, then k-rationality implies k'-rationality (and similarly for stable rationality and unirationality), but the converse need not hold, as demonstrated by a pointless real conic. We say that W is *geometrically rational* if the base change $W_{\overline{k}}$ to the algebraic closure of k is $\overline{k}$-rational.

For the majority of this article, we work over $\mathbb{R}$. As mentioned in the introduction, when we say that a variety is rational without specifying the ground field, we mean $\mathbb{R}$-rationality, *not* $\mathbb{C}$-rationality.

In order to show that a variety is not rational, one must show that it has an obstruction to rationality. One obstruction is given by the Lang–Nishimura lemma, which implies that if W is k-rational (or even k-unrational), then it must contain a k-point. Over the real numbers, the locus $W(\mathbb{R})$ of real points also provides an obstruction to rationality: the number of real connected components is a birational invariant of smooth projective real varieties [BCR98, Theorem 3.4.12], so if $W(\mathbb{R})$ is disconnected, then W has an obstruction to stable rationality over $\mathbb{R}$ (see also [CTP90] for an interpretation using unramified cohomology).

Over the complex numbers, rationality of conic bundles over $\mathbb{P}^2$ is well understood. Namely, let $X \to \mathbb{P}^2$ be a geometrically standard conic bundle with smooth discriminant curve Δ. Over $\mathbb{C}$, X is rational if and only if $\deg \Delta \leq 4$, or if $\deg \Delta = 5$ and $\tilde{\Delta} \to \Delta$ is defined by an even theta characteristic. The proof of rationality in the $\deg \Delta \leq 4$ case uses results of Iskovskikh showing that conic bundle surfaces with low degree discriminant are rational, and applies his surface classification to the generic fiber of a pencil of rational curves in $\mathbb{P}^2$ [Isk87, Theorem 1]. In the degree 4 case, one needs to blow down a divisor coming from a singular fiber of π to reduce to the degree 3 conic bundle surface case. The higher degree results are due to the combined work of Tyurin, Masiewicki, Panin ($\deg \Delta = 5$), and Beauville ($\deg \Delta \geq 6$). In addition, if $\deg \Delta \leq 8$, then X is unirational over $\mathbb{C}$. We refer the reader to [Pro18, Theorem 9.1 and Corollary 14.3.4] for an overview of these results.

Over the real numbers, we recall [FJS$^+$, Proposition 6.1], which in particular contains Theorem 1.1(1). If $\deg \Delta \leq 3$, then X is rational if and only if $X(\mathbb{R}) \neq \emptyset$ (for instance this happens if $\Delta(\mathbb{R}) \neq \emptyset$). The Lang–Nishimura lemma shows necessity of a $\mathbb{R}$-point, and if X admits a $\mathbb{R}$-point then a modification of the proof over $\mathbb{C}$ shows that X is rational. In degree 4, the proof of geometric rationality does not always descend, even if $X(\mathbb{R}) \neq \emptyset$, because the singular fibers of π need not be split over $\mathbb{R}$. When $\tilde{\Delta}(\mathbb{R}) \neq \emptyset$, however, the argument over $\mathbb{C}$ goes through if the pencil is chosen through the image of a point of $\tilde{\Delta}(\mathbb{R})$. Similarly, if X has an $\mathbb{R}$-point away from X_Δ, then X is $\mathbb{R}$-unirational by a modification of the argument over $\mathbb{C}$. (See [FJS$^+$, Section 6.1]. More generally, these results hold over any field of characteristic $\neq 2$.)

2.2. **Conic bundle threefolds realized as double covers of $\mathbb{P}^1 \times \mathbb{P}^2$.** We recall the following models of conic bundles, which were studied in [FJS$^+$]. Let k be a field of characteristic $\neq 2$. First, we recall a result of Bruin that allows us express étale double covers of smooth plane quartics in a particular form.

Theorem 2.1 ([Bru08, Section 3])**.** *Let $\varpi\colon \tilde{\Delta} \to \Delta$ be an étale double cover of a smooth plane quartic. Then there exist quadratic forms $Q_1, Q_2, Q_3 \in k[u, v, w]$ such that $\tilde{\Delta} \to \Delta$ is of the form*

$$\begin{gathered}\Delta = (Q_1 Q_3 - Q_2^2 = 0),\\ \tilde{\Delta} = (Q_1 - r^2 = Q_2 - rs = Q_3 - s^2 = 0).\end{gathered} \tag{1}$$

Now let $Q_1, Q_2, Q_3 \in k[u,v,w]$ be quadratic forms as in Theorem 2.1, and define the double cover $\tilde{\pi}\colon Y \to \mathbb{P}^1_{[t_0:t_1]} \times \mathbb{P}^2_{[u:v:w]}$ by

$$z^2 = t_0^2 Q_1 + 2t_0 t_1 Q_2 + t_1^2 Q_3. \tag{2}$$

The second projection $\pi_2\colon Y \to \mathbb{P}^2$ is a conic bundle whose discriminant double cover is defined by (1). The isomorphism class of Y only depends on the double cover $\tilde{\Delta} \to \Delta$, not on the choice of the quadrics Q_i (see [FJS+, Section 4]), and so we will denote the double cover given above by $Y_{\tilde{\Delta}/\Delta}$, or by Y when the context is clear. We review the following properties of Y.

Proposition 2.2 ([FJS+, Theorem 2.6, Propositions 4.1 and 4.3])**.** *If Y is the threefold defined in* (2)*, then:*

1. *Y is smooth, and the second projection $\pi_2\colon Y \to \mathbb{P}^2$ is a geometrically standard conic bundle with discriminant cover $\varpi\colon \tilde{\Delta} \to \Delta$. In particular, Y is geometrically rational, and if a smooth fiber of π_2 contains a k point then Y is k-unirational.*
2. *The first projection $\pi_1\colon Y \to \mathbb{P}^1$ is a quadric surface bundle. In particular, if π_1 has section defined over k, then Y is k-rational.*
3. *If $\tilde{\Delta}(k) \neq \emptyset$, then π_1 has a section defined over k.*
4. *The Stein factorization of the relative variety of lines is $\mathcal{F}_1(Y/\mathbb{P}^1) \to \Gamma \to \mathbb{P}^1$, where Γ is the genus 2 curve defined by*
$$y^2 = -\det(t^2M_1 + 2tM_2 + M_3),$$
where M_i is the symmetrix 3×3 matrix corresponding to Q_i.
5. [Wit37, Satz 22] *If $k \subseteq \mathbb{R}$ and π_1 is surjective on real points, then π_1 has a section defined over $\mathbb{R}$.*
6. [CTS21, Theorem 3.7.2] *If $X \to \mathbb{P}^2$ is a standard conic bundle with discriminant cover $\tilde{\Delta} \to \Delta$, then* $[(Y_{\tilde{\Delta}/\Delta})_\eta] - [X_\eta] \in \operatorname{Im}(\operatorname{Br} k \to \operatorname{Br} \mathbf{k}(\mathbb{P}^2))$.

In particular, Theorem 2.1 and Proposition 2.2(6) imply that, up to a constant Brauer class, any geometrically conic bundle with smooth quartic discriminant curve is birationally equivalent to one form (2).

Remark. Proposition 2.2(3) shows that a k-point of $\tilde{\Delta}$ gives rise to a section of π_1. However, not every section of π_1 arises in this way: the rational examples constructed in the proof of Theorem 1.1(2)–(5) all admits sections of π_1 over $\mathbb{R}$.

We now specialize to the case $k = \mathbb{R}$. The images of the Weierstrass points of Γ are the zeroes of $-\det(t^2M_1 + 2tM_2 + M_3)$, which give the singular fibers of the quadric surface fibration π_1. Let $\mathfrak{Z}$ denote the set of real points of $-\det(t^2M_1 + 2tM_2 + M_3) = 0$. The signature of the 4×4 matrix

$$\left(\begin{array}{c|c} t^2M_1 + 2tM_2 + M_3 & 0 \\ \hline 0 & -1 \end{array}\right)$$

corresponding to the quadric Y_t is constant on each interval $\mathbb{P}^1(\mathbb{R}) \setminus \mathfrak{Z}$, and at each point of $\mathfrak{Z}$ the number of positive eigenvalues changes by ± 1.

Note that $Y_t(\mathbb{R}) \neq \emptyset$ if and only if the matrix corresponding to Y_t is indefinite. By Witt's Decomposition Theorem [EKM08, Section 8] Y_t contains lines defined over $\mathbb{R}$ if and only if Y_t has signature $(2,2)$.

To each étale double cover $\tilde{\Delta} \to \Delta$, we also associate a twisted double cover of $\mathbb{P}^1 \times \mathbb{P}^2$.

Definition 2.3. Let Q_1, Q_2, Q_3 and Δ be as in Theorem 2.1. The twisted double cover $Y_{\tilde{\Delta}^-/\Delta} \to \mathbb{P}^1 \times \mathbb{P}^2$ associated to $\tilde{\Delta} \to \Delta$ is defined by the equation

$$z^2 = -t_0^2 Q_1 + 2t_0 t_1 Q_2 - t_1^2 Q_3.$$

Since this double cover is of the form in Equation (2) obtained by replacing Q_1 and Q_3 with $-Q_1$ and $-Q_3$, the threefold $Y_{\tilde{\Delta}^-/\Delta}$ satisfies the properties of Proposition 2.2 (with the appropriate substitutions). In particular, $Y_{\tilde{\Delta}^-/\Delta} \to \mathbb{P}^2$ is a conic bundle with discriminant cover $\varpi^- \colon \tilde{\Delta}^- \to \Delta$, where

$$\tilde{\Delta}^- = (Q_1 + r^2 = Q_2 - rs = Q_3 + s^2 = 0)$$

is the quadratic twist of $\tilde{\Delta}$.

3. Conic bundles with topological obstructions to rationality

In this section, we will construct examples of conic bundles $Y \to \mathbb{P}^2$ where irrationality of Y is witnessed by the topology of its the real locus. Throughout, we work over the real numbers. We first make some observations about the real connected components of Y and the real isotopy class of the discriminant curve Δ.

3.1. **Real connected components of Y.** For a morphism $f \colon V \to W$ of quasi-projective algebraic varieties over $\mathbb{R}$, we let $f(\mathbb{R}) \colon V(\mathbb{R}) \to W(\mathbb{R})$ denote the induced map of topological spaces on the sets of real points (with the Euclidean topology).

If $F \in \mathbb{R}[u, v, w]$ is a homogeneous polynomial defining a smooth curve of even degree, then the sign of $F(P)$ for $P \in \mathbb{P}^2(\mathbb{R})$ is well defined. We denote by $(F > 0)_{\mathbb{R}}$ the set of real points for which $F(P) > 0$ (similarly for $\geq, =, \leq$, and $<$). Every connected component of $(F = 0)_{\mathbb{R}}$ is an oval, and the complement of $(F = 0)_{\mathbb{R}}$ in $\mathbb{P}^2(\mathbb{R})$ is a disjoint union of a non-orientable set U_F and a finite number of discs [Man20, Section 2.7]. The non-orientable set U_F is the exterior of the curve defined by F.

In the case where F defines a smooth quartic curve Δ, Zeuthen [Zeu74] proved the following classification result for the real isotopy class of Δ. (Recall that Δ has 28 complex bitangents.) We will sometimes denote the real locus of the plane curve Δ by $(\Delta = 0)_{\mathbb{R}} \coloneqq \Delta(\mathbb{R})$, and we will denote the exterior of Δ by U_Δ.

$\Delta(\mathbb{R})$	$\emptyset$	One oval	Two nested ovals	Two non-nested ovals	Three ovals	Four ovals
Real bitangents	4	4	4	8	16	28

Lemma 3.1. *Let $Y = Y_{\tilde{\Delta}/\Delta}$ be as defined in Section 2.2. The number of connected components of $Y(\mathbb{R})$ is equal to the number of connected components of its image under $\pi_i \colon Y \to \mathbb{P}^i$ for $i = 1, 2$.*

Proof. Since $\pi_i \colon Y \to \mathbb{P}^i$ is the finite morphism $\tilde{\pi}$ composed with the projection $\mathbb{P}^1 \times \mathbb{P}^2 \to \mathbb{P}^i$, it follows from [DK81, Theorem 4.2] and compactness of $\mathbb{P}^n(\mathbb{R})$ that $\pi_i(\mathbb{R})$ is a continuous closed map. The claim then holds since the fibers of π_i are positive-dimensional quadrics and in particular have connected real loci. ☐

Lemma 3.2. *If $Y_{\tilde{\Delta}/\Delta}$ is as defined in Section 2.2, then $Y(\mathbb{R})$ has at most 3 connected components.*

Proof. By Lemma 3.1, it suffices to show that the image of $\pi_1(\mathbb{R})$ has at most 3 components. The signature of the fibers of π_1 can only change at the real branch points of the genus 2 curve Γ defined in Proposition 2.2(4), so the number of connected components of $\pi_1(\mathbb{R})$ is at most half the number of real branch points and so is at most $\frac{1}{2} \cdot 6 = 3$. ☐

Lemma 3.3. *In the setting of Section 2.2, the image of $\pi_2(\mathbb{R})$ is $(Q_1 \geq 0)_{\mathbb{R}} \cup (Q_1Q_3 - Q_2^2 \leq 0)_{\mathbb{R}} \subseteq \mathbb{P}^2(\mathbb{R})$.*

Proof. The fiber of π_2 above $P \in \mathbb{P}^2(\mathbb{R})$ is the conic corresponding to the symmetric matrix

$$\begin{pmatrix} Q_1(P) & Q_2(P) & 0 \\ Q_2(P) & Q_3(P) & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

so the fiber contains an $\mathbb{R}$-point if and only if the top 2×2 submatrix is not negative definite. By Sylvester's criterion, this submatrix is negative definite if and only if $Q_1(P) < 0$ and $(Q_1Q_3 - Q_2^2)(P) > 0$. $\square$

From Lemma 3.3 and Proposition 2.2(1), it immediately follows that:

Corollary 3.4. *If $Y_{\tilde{\Delta}/\Delta}(\mathbb{R}) = \emptyset$, then $\Delta(\mathbb{R}) = \emptyset$. If $\Delta(\mathbb{R}) \neq \emptyset$, then $Y_{\tilde{\Delta}/\Delta}$ is unirational (over $\mathbb{R}$).*

Proposition 3.5. *If $Y_{\tilde{\Delta}/\Delta}(\mathbb{R})$ is disconnected, then $\Delta(\mathbb{R})$ must be two or three ovals. More precisely:*

(1) If $Y_{\tilde{\Delta}/\Delta}(\mathbb{R})$ has three connected components, then $\Delta(\mathbb{R})$ is three ovals; and
(2) If $Y_{\tilde{\Delta}/\Delta}(\mathbb{R})$ has two connected components, then $\Delta(\mathbb{R})$ is two non-nested ovals or two nested ovals.

Proof. If Q_1 is positive definite, then the image of $\pi_2(\mathbb{R})$ is $\mathbb{P}^2(\mathbb{R})$ by Lemma 3.3, so we may assume that Q_1 is negative definite or indefinite. First suppose Q_1 is negative definite. Then the image of $\pi_2(\mathbb{R})$ is $(Q_1Q_3 - Q_2^2 \leq 0)_{\mathbb{R}}$, which can only be disconnected if $\Delta(\mathbb{R})$ is two or more ovals. If $(Q_1Q_3 - Q_2^2 \leq 0)_{\mathbb{R}}$ is disconnected, then it has the same number of connected components as $\Delta(\mathbb{R})$ and, by Lemma 3.1, it also has the same number of connected components as $Y(\mathbb{R})$. So by Lemma 3.2, $\Delta(\mathbb{R})$ is either two or three ovals.

It remains to consider the case when Q_1 is indefinite, so its real locus is one oval. Since $(Q_1 = 0)_{\mathbb{R}} \subset (Q_1Q_3 - Q_2^2 \leq 0)_{\mathbb{R}}$, we have that $(Q_1 \geq 0)_{\mathbb{R}} \cup (Q_1Q_3 - Q_2^2 \leq 0)_{\mathbb{R}}$ is either equal to $(Q_1Q_3 - Q_2^2 \leq 0)_{\mathbb{R}}$ or all of $\mathbb{P}^2(\mathbb{R})$. Thus, again using Lemma 3.2 to rule out the four ovals case when $Y(\mathbb{R})$ is disconnected, we conclude that $Y(\mathbb{R})$ is disconnected if and only if $\Delta(\mathbb{R})$ is either two or three ovals, and that in the disconnected case $Y(\mathbb{R})$ and $\Delta(\mathbb{R})$ have the same number of connected components. $\square$

Remark. All cases in Proposition 3.5 occur; see Section 3.2 and Section 5.

Remark. If $Y_{\tilde{\Delta}/\Delta}(\mathbb{R})$ is disconnected and π_1 has a fiber with signature $(2, 2)$, then the real isotopy class of Δ is two nested ovals. Indeed, Proposition 3.5 and the fact that Γ has at most 6 real Weierstrass points imply that the image of $Y_{\tilde{\Delta}/\Delta}(\mathbb{R})$ in $\mathbb{P}^2(\mathbb{R})$ is $(Q_1Q_3 - Q_2^2 \leq 0)_{\mathbb{R}}$ and that $\Delta(\mathbb{R})$ consists of two ovals. After a coordinate change on $\mathbb{P}^1$ we may assume that Q_1 has signature $(2, 1)$ (see [FJS$^+$, Theorem 2.6]). Recalling that U_{Q_1} denotes the exterior of the plane conic Q_1, the signature assumption on Q_1 implies $U_{Q_1} = (Q_1 > 0)_{\mathbb{R}}$, and since U_{Q_1} is not orientable it cannot be contained in a disc. Then $U_{Q_1} \subset (Q_1Q_3 - Q_2^2 \leq 0)_{\mathbb{R}}$ implies that U_Δ is one of the two connected components of $(Q_1Q_3 - Q_2^2 < 0)_{\mathbb{R}}$, which implies the two ovals of Δ must be nested.

We now relate the real points of $\tilde{\Delta}$ to those of the corresponding curve on the the twisted double cover.

Lemma 3.6. *Let $\tilde{\Delta} \to \Delta$ and $\tilde{\Delta}^- \to \Delta$ be as defined in Section 2.2. Then $\varpi(\mathbb{R}) \colon \tilde{\Delta}(\mathbb{R}) \to \Delta(\mathbb{R})$ is surjective if and only if $\tilde{\Delta}^-(\mathbb{R}) = \emptyset$.*

Proof. First, we note that the real points of $\tilde{\Delta}^-$ lie over the locus $(\Delta = 0)_{\mathbb{R}} \cap (Q_1 \leq 0)_{\mathbb{R}}$. Now if $\varpi(\mathbb{R})$ is surjective, then $(\Delta = 0)_{\mathbb{R}}$ is contained in $(Q_1 \geq 0)_{\mathbb{R}}$, and so $(\Delta = 0)_{\mathbb{R}} \cap (Q_1 \leq 0)_{\mathbb{R}} = (\Delta = 0)_{\mathbb{R}} \cap (Q_1 = 0)_{\mathbb{R}}$ is a finite set of points (possibly empty). Since $\tilde{\Delta}^-$ is a smooth projective curve, its real locus is homeomorphic to a (possibly empty) disjoint union of circles [Man20, Section 3.3]. If nonempty, these circles must map finitely onto a finite number of points, which is impossible, so necessarily $\tilde{\Delta}^-(\mathbb{R}) \neq \emptyset$. Conversely, if $\tilde{\Delta}^-(\mathbb{R}) = \emptyset$ then the set $(\Delta = 0)_{\mathbb{R}} \setminus ((\Delta = 0)_{\mathbb{R}} \cap (Q_1 = 0)_{\mathbb{R}}) = (\Delta = 0)_{\mathbb{R}} \cap (Q_1 > 0)_{\mathbb{R}}$ is contained in the image of $\varpi(\mathbb{R})$, and since $\tilde{\Delta}$ is a smooth projective curve, it follows that all of $(\Delta = 0)_{\mathbb{R}}$ is in the image of $\varpi(\mathbb{R})$. $\square$

In particular, if $Y_{\tilde{\Delta}^-/\Delta}$ is irrational, then $\tilde{\Delta}(\mathbb{R}) \neq \emptyset$ and so $\pi_1 \colon Y_{\tilde{\Delta}/\Delta} \to \mathbb{P}^1$ has a section defined over $\mathbb{R}$. If $\varpi(\mathbb{R})$ is not surjective, then $\tilde{\Delta}^-$ has an $\mathbb{R}$-point and so $Y_{\tilde{\Delta}^-/\Delta}$ is rational.

3.2. **Construction of irrational examples.** In this section, we construct examples of conic bundles by giving equations for Q_1, Q_2, Q_3 and taking $Y \coloneqq Y_{\tilde{\Delta}/\Delta}$ and $\tilde{\Delta} \to \Delta$ to be as defined in Section 2.2. Smoothness of Δ and $\tilde{\Delta}$ is verified using the Jacobian criterion, and the topological type of $\Delta(\mathbb{R})$ is verified with the `Sage` code accompanying [PSV11]. The numerical claims about the signatures of the fibers of π_1 can be verified by hand or with the code `Quadric-bundle-verifications.sage`, which is a `Sage` implementation of the `Magma` code accompanying [FJS$^+$].

Example 3.7 (Pointless example with $\Delta(\mathbb{R}) = \emptyset$)**.** Let Y be the double cover of $\mathbb{P}^1 \times \mathbb{P}^2$ constructed in Section 2.2 for the quartics

$$Q_1 \coloneqq -u^2 - v^2 - w^2, \quad Q_2 \coloneqq -u^2 - v^2 + w^2, \quad Q_3 \coloneqq -2u^2 - 9v^2 - 3w^2.$$

Then $\Delta(\mathbb{R}) = \emptyset$, Γ is defined by $y^2 = t^6 + 2t^5 + 10t^4 + 4t^3 + 19t^2 + 30t + 54$, and Γ has no real Weierstrass points. In particular, $\Gamma(\mathbb{R}) \neq \emptyset$ is connected. The fibers of π_1 all have signature $(0, 4)$, so $Y(\mathbb{R}) = \emptyset$.

In the following examples, we construct conic bundles with disconnected real loci. As mentioned in Section 2.1, these conic bundles will not be stably rational over $\mathbb{R}$. Recall from Proposition 3.5 that if Y is constructed as in Section 2.2 and has disconnected real locus, then $\Delta(\mathbb{R})$ must be two or three ovals. The following examples show that these cases all occur.

[FJS$^+$, Theorem 1.3(1)] have previously given an example where $\Delta(\mathbb{R})$ is two non-nested ovals and $Y(\mathbb{R})$ has two connected components. Here we give examples with two nested ovals and three ovals, Later, in Section 5 we will give additional examples where $Y(\mathbb{R})$ is disconnected in families (Examples 5.1, 5.2 and 5.3).

Example 3.8 (Disconnected example with $\Delta(\mathbb{R})$ two nested ovals)**.** Define

$$Q_1 \coloneqq u^2 + v^2 - w^2, \quad Q_2 \coloneqq u^2 + v^2, \quad Q_3 \coloneqq -24u^2 - 15v^2 + w^2,$$

and let Y be the associated double cover of $\mathbb{P}^1 \times \mathbb{P}^2$ constructed in Section 2.2. Then $\Delta(\mathbb{R})$ is two nested ovals, Γ is defined by $y^2 = t^6 + 4t^5 - 36t^4 - 82t^3 + 395t^2 + 78t - 360$ and has real Weierstrass points over $t = -6, -5, -1, 1, 3, 4$. The signatures of the fibers $Y_{[t:1]}$ are:

t	-5.5	-3	0	2	3.5	5
Signature	$(1,3)$	$(0,4)$	$(1,3)$	$(0,4)$	$(1,3)$	$(2,2)$

Thus, $Y(\mathbb{R})$ has two connected components.

Example 3.9 (Disconnected example with $\Delta(\mathbb{R})$ three ovals)**.** Let Y be as in Section 2.2 for

$$Q_1 \coloneqq -u^2 - v^2 - w^2, \quad Q_2 \coloneqq -5u^2 + 5w^2, \quad Q_3 \coloneqq -24u^2 + 4v^2 - 24w^2.$$

Then $\Delta(\mathbb{R})$ is three ovals. The hyperelliptic curve Γ is defined by $y^2 = t^6 - 56t^4 + 784t^2 - 2304$ and has six real Weierstrass points over $t = -6, -4, -2, 2, 4, 6$. The signatures of the $Y_{[t:1]}$ are:

t	-5	-3	0	3	5	7
Signature	$(1,3)$	$(0,4)$	$(1,3)$	$(0,4)$	$(1,3)$	$(0,4)$

Thus, $Y(\mathbb{R})$ has three connected components.

Remark. In all the examples we have found where $Y(\mathbb{R})$ has two components, it has been the case that Γ has six real Weierstrass points when $\Delta(\mathbb{R})$ is two nested ovals, and four real Weierstrass points when $\Delta(\mathbb{R})$ is two non-nested ovals. By Proposition 3.5, if $\Delta(\mathbb{R})$ is three ovals and $Y(\mathbb{R})$ is disconnected, then Γ necessarily has six real Weierstrass points.

Remark. If X is a smooth complete intersection of two quadrics in $\mathbb{P}^5$ that contains a conic C defined over $\mathbb{R}$, then projection from the conic realizes the blow up of X along C as a conic bundle with quartic discriminant curve [HT21b, Remark 13]. Krasnov's topological classification of intersections of quadrics [Kra18, Theorem 5.4] shows that a conic bundle arising in such a way can have at most two real connected components, so in particular the conic bundles of Example 3.9 are not birational over $\mathbb{R}$ to an intersection of two quadrics.

In the following section, we will also see using [FJS$^+$, Corollary 6.3] that Examples 3.7 and 3.7 cannot be obtained from an intersection of two quadrics by projection from a conic.

3.3. **Failure of the intermediate Jacobian torsor obstruction.** The classical intermediate Jacobian obstruction to rationality, introduced by Clemens–Grifiths in their proof of the irrationality of the cubic threefold [CG72], states that the intermediate Jacobian of a rational threefold must be isomorphic to a product of Jacobians of curves. Over non-closed fields, Hassett–Tschinkel [HT21b, Section 11.5] [HT21a, Sections 3 and 4] and Benoist–Wittenberg [BW, Theorem 3.11] have recently introduced a refinement of this obstruction involving the torsors over the intermediate Jacobian. Assuming for simplicity that the intermediate Jacobian is isomorphic to $\mathbf{Pic}^0_{\Gamma/k}$ for a curve Γ of genus ≥ 2, their refinement states that moreover each torsor over the intermediate Jacobian must be isomorphic to some $\mathbf{Pic}^i_{\Gamma/k}$. Following [FJS$^+$], we refer to this as the *intermediate Jacobian torsor (IJT) obstruction*.

Since we work over $\mathbb{R}$, when we mention the IJT obstruction we always mean the obstruction over $\mathbb{R}$.

Benoist–Wittenberg showed that the IJT obstruction is not sufficient to characterize rationality by constructing an example of a (non geometrically standard) real conic bundle $X \to S$ whose intermediate Jacobian is trivial but such that $S(\mathbb{R})$ is disconnected; hence, X has a Brauer obstruction to (stable) rationality over $\mathbb{R}$ [BW20, Theorem 5.7].

In [FJS$^+$], Frei–Ji–Sankar–Viray–Vogt studied the intermediate Jacobian torsors for geometrically standard conic bundles, relating them to certain torsors over the Prym variety of the discriminant cover $\tilde{\Delta} \to \Delta$ [FJS$^+$, Theorem 1.1]. For the double covers described in Section 2.2, they gave an extended description of these torsors [FJS$^+$, Theorem 4.4]. In this situation, they showed that the intermediate Jacobian of Y is $P \coloneqq \mathrm{Prym}_{\tilde{\Delta}/\Delta} \cong \mathbf{Pic}^0_{\Gamma/k}$, where Γ is the genus 2 curve defined in Proposition 2.2(4); that there are four torsors $P, \tilde{P}, P^{(1)}, \tilde{P}^{(1)}$ satisfying $\tilde{P} + P^{(1)} = \tilde{P}^{(1)}$ as P-torsors; and that $P^{(1)} \cong \mathbf{Pic}^1_{\Gamma}$. In particular, since Γ has genus 2, then $\tilde{P}^{(1)}(\mathbb{R}) \neq \emptyset$ implies the vanishing of the IJT obstruction. [FJS$^+$] also gave a geometric interpretation of the $\mathbb{R}$-points of $\tilde{P}^{(1)}$ as Galois-invariant sets of four points $Q_1, Q_2, Q_3, Q_4 \in \tilde{\Delta}(\mathbb{C})$ such that

(1) Q_1, Q_2, Q_3, Q_4 does not span a 2-plane in $\mathbb{P}^4$, and
(2) $\varpi_*(Q_1 + Q_2 + Q_3 + Q_4) = \Delta \cap \ell$ for a line $\ell \in (\mathbb{P}^2)^\vee(\mathbb{R})$. (If $\tilde{\Delta}(\mathbb{R}) = \emptyset$ then ℓ does not meet Δ transversely in any real points [FJS$^+$, Lemma 5.1].)

Remark. By lower semicontinuity of rank, the property that $\tilde{P}^{(1)}$ has an $\mathbb{R}$-point is an open condition.

Applying this criterion, [FJS$^+$] then showed that IJT obstruction also fails to characterize rationality for geometrically standard conic bundles over $\mathbb{P}^2$ by constructing an example of a conic bundle whose real locus is disconnected where the IJT obstruction vanishes. In [FJS$^+$, Theorem 1.3(1)], they constructed $Y_{\tilde{\Delta}/\Delta}$ such that there is a Galois-invariant set of four points of $\tilde{\Delta}$ spanning a 3-plane in $\mathbb{P}^4$ and whose pushforward under ϖ is $\Delta \cap (w = 0)$, thus exhibiting a point on $\tilde{P}^{(1)}$. (In their example $\Gamma(\mathbb{R}) \neq \emptyset$, and so all the intermediate Jacobian torsors are trivial over $\mathbb{R}$.) We use their method to show that the examples we construct in Section 3.2 also have no IJT obstruction.

Proposition 3.10. *All the intermediate Jacobian torsors are trivial over $\mathbb{R}$ in Examples 3.7 and 3.9. In particular, these conic bundles have no IJT obstruction to rationality, and in this case irrationality is exhibited by the real locus of Y.*

We show that $\tilde{P}^{(1)}(\mathbb{R}) \neq \emptyset$ in each case, which implies that $P \cong \tilde{P}^{(1)} \cong \mathbf{Pic}^0_{\Gamma/\mathbb{R}}$ and $\tilde{P} \cong P^{(1)} \cong \mathbf{Pic}^1_{\Gamma/\mathbb{R}}$. Since $\Gamma(\mathbb{R}) \neq \emptyset$, then $\mathbf{Pic}^1_{\Gamma/\mathbb{R}}$ also has an $\mathbb{R}$-point. We use the line $(w = 0)$ in Example 3.7, and the line $(u + v + w = 0)$ in Example 3.9.

Later, in Section 5, we will give examples of families where many of the members have $\mathbb{R}$-points on $\tilde{P}^{(1)}$, including members where $\Delta(\mathbb{R})$ is two non-nested ovals and $Y(\mathbb{R})$ has two components. In these later examples, the line ℓ that we use is often a real bitangent of Δ.

In the following examples and in the examples of Section 5, the numerical claims can be verified using the `Sage` code in [JJ]. The code that computes the real bitangents of Δ is due to Plaumann–Sturmfels–Vinzant and is included in the supplementary material for their paper [PSV11].

Proof for Example 3.7 (no ovals, $Y(\mathbb{R}) = \emptyset$). The quartic curve Δ is defined by $u^4 + 9u^2v^2 + 7u^2w^2 + 8v^4 + 14v^2w^2 + 2w^4 = 0$, and the intersection $\Delta \cap (w = 0)$ consists of the four complex points

$$[-i : 1 : 0], \quad [i : 1 : 0], \quad [-2i\sqrt{2} : 1 : 0], \quad [2i\sqrt{2} : 1 : 0].$$

One verifies that the set

$$[i : 1 : 0 : 0 : i\sqrt{7}], \quad [-i : 1 : 0 : 0 : -i\sqrt{7}],$$
$$[2i\sqrt{2} : 1 : 0 : \sqrt{7} : \sqrt{7}], \quad [-2i\sqrt{2} : 1 : 0 : \sqrt{7} : \sqrt{7}].$$

of four points of $\tilde{\Delta}$ is $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-invariant and maps to $\Delta \cap (w = 0)$. Since

$$\det \begin{pmatrix} i & 1 & 0 & i\sqrt{7} \\ -i & 1 & 0 & -i\sqrt{7} \\ 2i\sqrt{2} & 1 & \sqrt{7} & \sqrt{7} \\ -2i\sqrt{2} & 1 & \sqrt{7} & \sqrt{7} \end{pmatrix} = -56\sqrt{2} \neq 0,$$

the four points above span a 3-plane in $\mathbb{P}^4$, so $\tilde{P}^{(1)}(\mathbb{R}) \neq \emptyset$. □

Proof for Example 3.9 (three ovals, $Y(\mathbb{R})$ three connected components). Using `Ptilde1.sage` in the accompanying code [JJ], one verifies that

$$[-0.09772 + 0.360004i : -0.90228 - 0.360004i : 1 : 0.22782 - 1.27135i : 0.49668 + 4.31595i]$$
$$[-0.09772 - 0.360004i : -0.90228 + 0.360004i : 1 : 0.22782 + 1.27135i : 0.49668 - 4.31595i]$$
$$[-0.70228 + 2.58711i : -0.29772 - 2.58711i : 1 : 3.44913 + 0.303445i : 10.81703 + 4.31595i]$$
$$[-0.70228 - 2.58711i : -0.29772 + 2.58711i : 1 : 3.44913 - 0.303445i : 10.81703 - 4.31595i]$$

is a set of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-invariant points in $\tilde{\Delta}$ that maps to $\Delta \cap (u + v + w = 0)$, and that the determinant of

$$\begin{pmatrix} -0.09772 + 0.360004i & -0.90228 - 0.360004i & 0.22782 - 1.27135i & 0.49668 + 4.31595i \\ -0.09772 - 0.360004i & -0.90228 + 0.360004i & 0.22782 + 1.27135i & 0.49668 - 4.31595i \\ -0.70228 + 2.58711i & -0.29772 - 2.58711i & 3.44913 + 0.303445i & 10.81703 + 4.31595i \\ -0.70228 - 2.58711i & -0.29772 + 2.58711i & 3.44913 - 0.303445i & 10.81703 - 4.31595i \end{pmatrix}$$

is $\approx 280.57996 \neq 0$. Therefore, the four points above span a 3-plane in $\mathbb{P}^4$, and so $\tilde{P}^{(1)}(\mathbb{R}) \neq \emptyset$. □

Remark. One can check that in Example 3.8, any Galois-invariant set of four points of $\tilde{\Delta}$ mapping to $\Delta \cap \ell$ spans a 2-plane in $\mathbb{P}^4$ for $\ell = (w = 0)$ or if ℓ is a real bitangent of Δ. These exhibit points of $P^{(1)}(\mathbb{R})$ (see [Bru08, Lemma 4.1], [FJS$^+$, Section 2.3]). We have not been able to construct an example where $\Delta(\mathbb{R})$ is two nested ovals, $Y(\mathbb{R})$ is disconnected, and $\tilde{P}^{(1)}$ has a point.

4. Construction of rational examples

In this section, we construct examples of double covers and twisted double covers as in Section 2.2 where the quadric surface bundle π_1 has a section, which by Proposition 2.2(2) implies rationality of Y. The section will either be exhibited by a real point on $\tilde{\Delta}$ (Proposition 2.2(3)) or by a signature computation to show that π_1 is surjective on real points (Proposition 2.2(5)). In the following examples, smoothness of Δ and $\tilde{\Delta}$ is verified using the Jacobian criterion, and the topological type of $\Delta(\mathbb{R})$ is verified with the `Sage` code accompanying [PSV11]. The numerical claims about the signatures of the fibers of π_1 can be verified by hand or with the code `Quadric-bundle-verifications.sage` in [JJ], which is a `Sage` implementation of the `Magma` code accompanying [FJS+].

Example 4.1 (Rational examples with $\Delta(\mathbb{R}) = \emptyset$)**.** Let Q_1, Q_2, Q_3 be as in Example 3.7, and let $Y = Y_{\tilde{\Delta}^-/\Delta}$ be the twisted double cover defined in Definition 2.3. Then $\Delta(\mathbb{R}) = \emptyset$, and Γ is defined by $y^2 = -t^6 + 2t^5 - 10t^4 + 4t^3 - 19t^2 + 30t - 54$. We note that $\Gamma(\mathbb{R}) = \emptyset$, so in particular Γ has no real Weierstrass points. Every fiber of π_1 has signature $(3, 1)$, so π_1 has a section and Y is $\mathbb{R}$-rational by Proposition 2.2(5).

One can check that in this example, $P \cong \tilde{P}^{(1)} \cong \mathbf{Pic}^0_{\Gamma/\mathbb{R}}$ (see Section 3.3). The torsors $\tilde{P} \cong P^{(1)} \cong \mathbf{Pic}^1_{\Gamma/\mathbb{R}}$ are non-trivial because $\Gamma(\mathbb{R}) = \emptyset$; by [FJS+, Proposition 6.4] this also shows that Y is not obtained from an intersection of two quadrics by projection from a conic. (One can also check that the quadrics $Q'_1 := 2u^2 + 3v^2 + 5w^2, Q'_2 := u^2 + 2v^2 + 3w^2$, and $Q'_3 := 2u^2 + 4v^2 + 2w^2$ give a similar example.)

Example 4.2 (Rational example with $\Delta(\mathbb{R})$ one oval)**.**

(1) ($\tilde{\Delta}(\mathbb{R})$ is empty.) Let $\tilde{\Delta} \to \Delta$ and $Y_{\tilde{\Delta}/\Delta}$ be as defined in Section 2.2 for the quadrics

$$Q_1 := -3u^2 - 10uv + 3v^2 - 8uw + 8vw - w^2, \quad Q_2 := u^2 - 2uv - 3v^2 + 2vw + 3w^2,$$
$$Q_3 := -2u^2 + 6uv + v^2 - 6uw - 6vw - 3w^2.$$

Then $\Delta(\mathbb{R})$ is one oval, and we claim that $\tilde{\Delta}(\mathbb{R})$ is empty. For this, since $\Delta(\mathbb{R})$ is connected, and the zero locus $(Q_1 = 0)_{\mathbb{R}}$ is contained in $(Q_1Q_3 - Q_2^2 \leq 0)_{\mathbb{R}}$, so it suffices to check that Δ has an $\mathbb{R}$-point P such that $Q_1(P) < 0$. Indeed, one verifies that this is satisfied by P in the support of $\Delta \cap (v + w = 0)$.

Every fiber of π_1 has signature $(1, 3)$, and so $\pi_1(\mathbb{R})$ is surjective. Thus, Y is rational.

One can check that Γ has no real Weierstrass points and that $\Gamma(\mathbb{R}) = \emptyset$.

(2) (Image of $\tilde{\Delta}(\mathbb{R})$ is one oval.) Let $\tilde{\Delta} \to \Delta$ and $Y_{\tilde{\Delta}/\Delta}$ be as defined in Section 2.2 for the quadrics

$$Q_1 := 12u^2 + 96uv + 36v^2 - 7w^2, \quad Q_2 := 36u^2 + 180uv + 36v^2 - w^2,$$
$$Q_3 := 90u^2 + 96uv + 36v^2 + 7w^2.$$

Then $[0 : 1 : 0 : 6 : 6]$ exhibits a $\mathbb{Q}$-point of $\tilde{\Delta}$, so $Y_{\tilde{\Delta}/\Delta}$ is $\mathbb{Q}$-rational by Proposition 2.2(3). (We also note that in this example, $Y_{\tilde{\Delta}^-/\Delta}(\mathbb{R})$ is connected, but $Y_{\tilde{\Delta}^-/\Delta}(\mathbb{R}) \to \mathbb{P}^1(\mathbb{R})$ is not surjective.)

Example 4.3 (Rational example with $\Delta(\mathbb{R})$ two non-nested ovals)**.**

(1) ($\tilde{\Delta}(\mathbb{R})$ is empty.) Let $\tilde{\Delta} \to \Delta$ and $Y_{\tilde{\Delta}/\Delta}$ be as defined in Section 2.2 for the quadrics

$$Q_1 := -5u^2 + 10uv + 5v^2 + 10uw + 8vw, \quad Q_2 := u^2 - 10uv - 2v^2 + 10vw + 4w^2,$$
$$Q_3 := 2u^2 - 4uv - 4v^2 - 8uw + 8vw + 2w^2.$$

Then $\Delta(\mathbb{R})$ is two non-nested ovals, and we claim that $\tilde{\Delta}(\mathbb{R}) = \emptyset$. For this, we work on the chart $(w \neq 0)$. One can verify that

- The lines $\ell_1 := (v = 0)$ and $\ell_2 := (v = 1)$ are disjoint from $(\Delta = 0)_{\mathbb{R}}$; and

- The set $(\Delta = 0)_{\mathbb{R}} \cap (Q_1 < 0)_{\mathbb{R}}$ contains one $\mathbb{R}$-point (u, v) with $v = \frac{1}{2}$ that lies above ℓ_1 and below ℓ_2 and another $\mathbb{R}$-point (u, v) with $v = -1$ lying below ℓ_1.

In particular, both connected components of $(\Delta = 0)_{\mathbb{R}}$ contain points where Q_1 is negative. Since $(Q_1 \leq 0)_{\mathbb{R}}$ is connected and $(Q_1 = 0)_{\mathbb{R}} \subset (Q_1Q_3 - Q_2^2 \leq 0)_{\mathbb{R}}$, this shows that Q_1 is not positive for all $p \in (\Delta = 0)_{\mathbb{R}}$. See Figure 1 for a visual depiction.

One can check that Γ has two real Weierstrass points over $[t : 1]$ with $t \approx 0.39460, 1.22782$, and the fibers $Y_{[1:1]}$ and $Y_{[2:1]}$ have signatures $(2, 2)$ and $(1, 3)$, respectively. Therefore $\pi_1(\mathbb{R})$ is surjective.

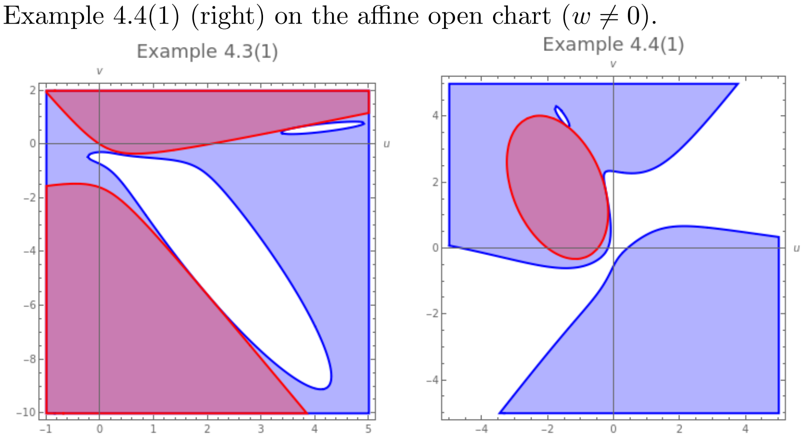
(2) (Image of $\tilde{\Delta}(\mathbb{R})$ is one oval.) Define $\tilde{\Delta} \to \Delta$ and Y as in Section 2.2 for the quadrics

$$Q_1 := 2u^2 - 9v^2 + 12uw - 68vw + 70w^2, \quad Q_2 := u^2 - 6v^2 + 15uw - 19vw - 10w^2,$$
$$Q_3 := u^2 - 8v^2 - 50uw + 51vw - 16w^2.$$

Then $[3 : 1 : 0 : 3 : 1]$ exhibits a $\mathbb{Q}$-point of $\tilde{\Delta}$, so Y is $\mathbb{Q}$-rational by Proposition 2.2(3). The curve $\tilde{\Delta}^-$ also contains $\mathbb{Q}$-points, as shown by $[2 : 1 : 0 : 1 : -2]$. So by Lemma 3.6, the map $\varpi(\mathbb{R})$ is not surjective and so its image is one oval.

(3) (Image of $\tilde{\Delta}(\mathbb{R})$ is two ovals) Let Q_1, Q_2, Q_3, be as in part (1), and define $Y_{\tilde{\Delta}^-/\Delta}$ as in Definition 2.3. By Lemma 3.6, the map $\varpi^-(\mathbb{R})$ is surjective, so in particular $Y_{\tilde{\Delta}^-/\Delta}$ is $\mathbb{R}$-rational.

FIGURE 1. The regions $(Q_1Q_3 - Q_2^2 \leq 0)_{\mathbb{R}}$ (in blue) and $(Q_1 \geq 0)_{\mathbb{R}}$ (in red) of Example 4.3(1) (left) and Example 4.4(1) (right) on the affine open chart $(w \neq 0)$.



Example 4.4 (Rational example with $\Delta(\mathbb{R})$ two nested ovals)**.**

(1) ($\tilde{\Delta}(\mathbb{R})$ is empty.) Let $\tilde{\Delta} \to \Delta$ and $Y_{\tilde{\Delta}/\Delta}$ be as defined in Section 2.2 for the quadrics

$$Q_1 := -4u^2 - 2uv - 2v^2 - 10uw + 4vw - 4w^2, \quad Q_2 := u^2 - 4uv - 3v^2 - 6uw + 2vw + 2w^2,$$
$$Q_3 := -u^2 - 6uv + 8uw - 6vw - 3w^2.$$

Then $\Delta(\mathbb{R})$ is two nested ovals, and we claim that $\tilde{\Delta}(\mathbb{R})$ is empty (see Figure 1).

To show $\tilde{\Delta}(\mathbb{R}) = \emptyset$, we work on the chart $(w \neq 0)$ and define the box $B := \{(u, v) \mid -2 \leq u \leq -1,\ 3.5 \leq v \leq 4.5\}$. One can verify that the boundary of B is disjoint from $(\Delta = 0)_{\mathbb{R}}$, that the set $(\Delta = 0)_{\mathbb{R}} \cap (Q_1 < 0)_{\mathbb{R}} \cap B$ contains an $\mathbb{R}$-point (u, v) with $v = 4$ and $-2 < u < -1$, and the set $(\Delta = 0)_{\mathbb{R}} \cap (Q_1 < 0)_{\mathbb{R}}$ contains an $\mathbb{R}$-point in the complement of B whose v-coordinate is 1. In particular, there are points on both connected components of $(\Delta = 0)_{\mathbb{R}}$ where Q_1 is negative. Since $(Q_1 \leq 0)_{\mathbb{R}}$ is connected and $(Q_1 = 0)_{\mathbb{R}} \subset (Q_1Q_3 - Q_2^2 \leq 0)_{\mathbb{R}}$, we have that Q_1 is not positive for all $P \in (\Delta = 0)_{\mathbb{R}}$.

Γ has no real Weierstrass points, and all fibers of π_1 have signature $(1,3)$, so $\pi_1(\mathbb{R})$ is surjective. Thus Y is rational.

(2) (Image of $\tilde{\Delta}(\mathbb{R})$ is one oval.) Let $\tilde{\Delta} \to \Delta$ and $Y_{\tilde{\Delta}/\Delta}$ be as defined in Section 2.2 for the quadrics

$$Q_1 \coloneqq 2u^2 - 9v^2 + 13w^2, \quad Q_2 \coloneqq u^2 - 6v^2 - 3w^2, \quad Q_3 \coloneqq u^2 - 8v^2 + 6w^2$$

Then Δ has real isotopy class two nested ovals, and $[3:1:0:3:1]$ exhibits a $\mathbb{Q}$-point of $\tilde{\Delta}$, so Y is $\mathbb{Q}$-rational by Proposition 2.2(3).

Since $[2:1:0:1:-2]$ exhibits a $\mathbb{Q}$-point of $\tilde{\Delta}^-$, Lemma 3.6 implies that $\varpi(\mathbb{R})$ is not surjective. Thus, the image of $\varpi(\mathbb{R})$ is one oval.

(3) (Image of $\tilde{\Delta}(\mathbb{R})$ is two ovals.) Let Q_1, Q_2, Q_3 be as in Example 3.8, and let $Y_{\tilde{\Delta}^-/\Delta}$ be the associated twisted double cover (Definition 2.3). Since the threefold $Y_{\tilde{\Delta}/\Delta}$ of Example 3.8 is irrational over $\mathbb{R}$, then $\varpi^-(\mathbb{R})$ is surjective and in particular $Y_{\tilde{\Delta}^-/\Delta}$ is $\mathbb{R}$-rational. (Note that $\tilde{\Delta}^-$ does not contain any $\mathbb{Q}$-points, as it has no $\mathbb{Q}_2$-points.)

Example 4.5 (Rational example with $\Delta(\mathbb{R})$ three ovals)**.**

(1) ($\tilde{\Delta}(\mathbb{R})$ is empty.) Let $\tilde{\Delta} \to \Delta$ and $Y_{\tilde{\Delta}/\Delta}$ be as defined in Section 2.2 for the quadrics

$$Q_1 \coloneqq -3u^2 - 4uv + v^2 + 10uw + 4vw - 2w^2, \quad Q_2 \coloneqq 5u^2 + 4uv - 2v^2 + 8uw - 6vw + 5w^2,$$
$$Q_3 \coloneqq -2u^2 + 2uv - 3v^2 - 8uw + 2vw - 2w^2.$$

Then $\Delta(\mathbb{R})$ is three ovals, and we claim that $\tilde{\Delta}(\mathbb{R}) = \emptyset$.

For this, we work on the chart $(w \neq 0)$. Define the boxes $B_1 \coloneqq \{(u,v) \mid -0.5 \leq u \leq 0,\ 0 \leq v \leq 1\}$ and $B_2 \coloneqq \{(u,v) \mid -30 \leq u \leq -2,\ 0 \leq v \leq 17\}$. One can verify that the boundary of each B_i is disjoint from $(\Delta = 0)_{\mathbb{R}}$, and that

- The set $(\Delta = 0)_{\mathbb{R}} \cap (Q_1 < 0)_{\mathbb{R}} \cap B_1$ contains an $\mathbb{R}$-point $(-0.125, v)$ with $0 < v < 1$;
- The set $(\Delta = 0)_{\mathbb{R}} \cap (Q_1 < 0)_{\mathbb{R}} \cap B_2$ contains an $\mathbb{R}$-point $(-4, v)$ with $0 < v < 17$; and
- The set $(\Delta = 0)_{\mathbb{R}} \cap (Q_1 < 0)_{\mathbb{R}}$ contains an $\mathbb{R}$-point $(0.125, v)$ disjoint from both B_1 and B_2.

In particular, there exists a point on each of the three connected components of $(\Delta = 0)_{\mathbb{R}}$ where Q_1 is negative. Since $(Q_1 \leq 0)_{\mathbb{R}}$ is connected and $(Q_1 = 0)_{\mathbb{R}} \subset (Q_1Q_3 - Q_2^2 \leq 0)_{\mathbb{R}}$, this shows that Q_1 is not positive for all $p \in (\Delta = 0)_{\mathbb{R}}$.

The associated genus 2 curve Γ has equation $y^2 = 39t^6 + 102t^5 - 1335t^4 + 1114t^3 + 47t^2 + 20t - 32$ and has four real Weierstrass points over $[t:1]$ where $t \approx -7.60663, 0.31045, 0.95547, 4.06172$. One can check that the fibers of $Y_{[t:1]}$ have signatures as in the table below, so $\pi_1(\mathbb{R})$ is surjective.

t	-3	0.5	2	6
Signature	$(1,3)$	$(2,2)$	$(1,3)$	$(2,2)$

(2) (Image of $\tilde{\Delta}(\mathbb{R})$ is one oval.) Let $\tilde{\Delta} \to \Delta$ and $Y_{\tilde{\Delta}/\Delta}$ be as defined in Section 2.2 for the quadrics

$$Q_1 \coloneqq -4u^2 + 2v^2 + 2w^2, \quad Q_2 \coloneqq 3u^2 - v^2 - 3w^2, \quad Q_3 \coloneqq -2u^2 + 2v^2 + 2w^2.$$

Then $\Delta(\mathbb{R})$ is three ovals and $[0:1:1:2:-2]$ exhibits a $\mathbb{Q}$-point of $\tilde{\Delta}$, so Y is $\mathbb{Q}$-rational by Proposition 2.2(3). We will show that the image of $\varpi(\mathbb{R})$ is one oval in part (3).

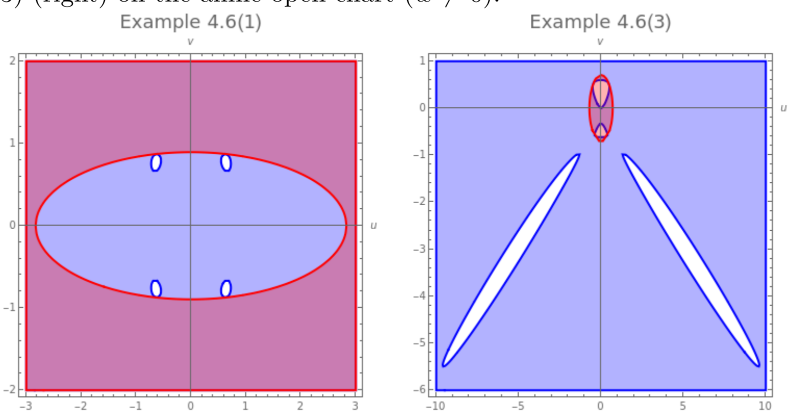
(3) (Image of $\tilde{\Delta}(\mathbb{R})$ is two ovals.) Let Q_1, Q_2, Q_3 be as in part (2), and let $Y_{\tilde{\Delta}^-/\Delta}$ and $\tilde{\Delta}^- \to \Delta$ be as defined in Definition 2.3. We claim that the image of $\varpi^-(\mathbb{R})$ is two of the connected components of $\Delta(\mathbb{R})$, and in particular $Y_{\tilde{\Delta}^-/\Delta}$ is $\mathbb{R}$-rational. To show this, we work on the chart $(w \neq 0)$ of $\mathbb{P}^2$. Define the boxes $B_1 \coloneqq \{(u,v) \mid -2.5 \leq u \leq -0.5, -0.6 \leq v \leq 0.6\}$ and $B_2 \coloneqq \{(u,v) \mid 0.5 \leq u \leq 2.5, -0.6 \leq 0.6\}$. One can verify that the boundary of each B_i is disjoint from $(\Delta = 0)_{\mathbb{R}}$ and that

- The set $(\Delta = 0)_{\mathbb{R}} \cap (-Q_1 \geq 0)_{\mathbb{R}} \cap B_1$ contains the $\mathbb{R}$-points $(-1, 0)$, and
- The set $(\Delta = 0)_{\mathbb{R}} \cap (-Q_1 \geq 0) \cap B_2$ contains the $\mathbb{R}$-points $(1, 0)$.

It follows that the set $(\Delta = 0)_{\mathbb{R}} \cap (-Q_1 \geq 0)_{\mathbb{R}}$ in $\mathbb{P}^2(\mathbb{R})$ has at least two connected components. Since $\Delta(\mathbb{R})$ is three ovals and $\tilde{\Delta}(\mathbb{R}) \neq \emptyset$, as shown in part (2) by the $\mathbb{Q}$-point $[0 : 1 : 1 : 2 : -2]$, we have that the image of $\varpi^-(\mathbb{R})$ cannot be all three ovals. From this, we conclude that the image of $\varpi^-(\mathbb{R})$ is two ovals. This also shows that the image of $\varpi(\mathbb{R})$ in part (2) is one oval.

(4) (Image of $\tilde{\Delta}(\mathbb{R})$ is three ovals.) Let Q_1, Q_2, Q_3 be as defined in Example 3.9 and $Y_{\tilde{\Delta}^-/\Delta}$ as defined in Definition 2.3. Since $-Q_1$ is positive definite, $\tilde{\Delta}(\mathbb{R})$ surjects onto $\Delta(\mathbb{R})$ and so $Y_{\tilde{\Delta}^-/\Delta}$ is rational.

FIGURE 2. The regions $(Q_1Q_3 - Q_2^2 \leq 0)_{\mathbb{R}}$ (in blue) and $(Q_1 \geq 0)_{\mathbb{R}}$ (in red) of Example 4.6(1) (left) and (3) (right) on the affine open chart $(w \neq 0)$.



Example 4.6 (Rational examples with $\Delta(\mathbb{R})$ four ovals)**.**

(1) ($\tilde{\Delta}(\mathbb{R})$ is empty.) Let $\tilde{\Delta} \to \Delta$ and $Y_{\tilde{\Delta}/\Delta}$ be as defined in Section 2.2 for the quadrics

$$Q_1 := u^2 + 10v^2 - 8w^2, \quad Q_2 := 10u^2 - 4w^2, \quad Q_3 := -2u^2 - 5v^2 + 3w^2.$$

Then $\Delta(\mathbb{R})$ is four ovals, and we claim that $\tilde{\Delta}(\mathbb{R}) = \emptyset$. For this, we work on the chart $(w \neq 0)$. Define disjoint boxes $B_1 := \{(u, v) \mid 0.5 \leq u \leq 1,\ 0.5 \leq v \leq 1\}$, $B_2 := \{(u, v) \mid -1 \leq u \leq -0.5,\ 0.5 \leq v \leq 1\}$, $B_3 := \{(u, v) \mid -1 \leq u \leq -0.5,\ -1 \leq v \leq -0.5\}$, and $B_4 := \{(u, v) \mid 0.5 \leq u \leq 1,\ -1 \leq v \leq -0.5\}$. Then, one can verify that the boundary of each B_i is disjoint from $(\Delta = 0)_{\mathbb{R}}$, and that

- The set $(\Delta = 0)_{\mathbb{R}} \cap (Q_1 < 0)_{\mathbb{R}} \cap B_1$ contains an $\mathbb{R}$-point $(0.6, v)$ with $0.5 < v < 1$;
- The set $(\Delta = 0)_{\mathbb{R}} \cap (Q_1 < 0)_{\mathbb{R}} \cap B_2$ contains an $\mathbb{R}$-point $(-0.6, v)$ with $0.5 < v < 1$;
- The set $(\Delta = 0)_{\mathbb{R}} \cap (Q_1 < 0)_{\mathbb{R}} \cap B_3$ contains an $\mathbb{R}$-point $(-0.6, v)$ with $-1 < v < -0.5$; and
- The set $(\Delta = 0)_{\mathbb{R}} \cap (Q_1 < 0)_{\mathbb{R}} \cap B_4$ contains an $\mathbb{R}$-point $(0.5, v)$ with $-1 < v < -0.5$.

In particular, this shows that there are points on all four connected components of $(\Delta = 0)_{\mathbb{R}}$ where Q_1 is negative. Since $(Q_1 \leq 0)_{\mathbb{R}}$ is connected and $(Q_1 = 0)_{\mathbb{R}} \subset (Q_1Q_3 - Q_2^2 \leq 0)_{\mathbb{R}}$, this shows that Q_1 is not positive for all $p \in (\Delta = 0)_{\mathbb{R}}$. See Figure 2 for a visual depiction.

The genus 2 curve $\Gamma_{\tilde{\Delta}/\Delta}$ is defined by $y^2 = 80t^6 + 1680t^5 + 1370t^4 - 1600t^3 - 645t^2 + 380t - 30$ and has 6 real Weierstrass points over the points $[\alpha_i \colon 1] \in \mathbb{P}^1(\mathbb{R})$ where $-21 < \alpha_1 < -2 < \alpha_2 < -1 < \alpha_3 < 0 < \alpha_4 < 0.1 < \alpha_5 < 0.5 < \alpha_6$. The fibers $Y_{[t:1]}$ have signatures

t	-21	-2	-1	0	0.1	0.5
Signature	$(2, 2)$	$(1, 3)$	$(2, 2)$	$(1, 3)$	$(2, 2)$	$(1, 3)$

so $\pi_1(\mathbb{R})$ is surjective, and in particular $Y_{\tilde{\Delta}/\Delta}$ is $\mathbb{R}$-rational.

(2) (Image of $\tilde{\Delta}(\mathbb{R})$ is one oval.) Let $\tilde{\Delta} \to \Delta$ and $Y_{\tilde{\Delta}/\Delta}$ be as defined in Section 2.2 for the quadrics

$$Q_1 \coloneqq -8uv - v^2 + 10uw - 10vw - 2w^2, \quad Q_2 \coloneqq 5u^2 + 8uv - 3v^2 - 8uw - 2w^2,$$
$$Q_3 \coloneqq -5u^2 - 4uv + 3v^2 + 4vw.$$

Then $\Delta(\mathbb{R})$ is four ovals, and we claim that the image of $\varpi(\mathbb{R})$ is one oval, which implies that $\tilde{\Delta}(\mathbb{R}) \neq \emptyset$ so $Y_{\tilde{\Delta}/\Delta}$ is $\mathbb{R}$-rational. (In this example $\tilde{\Delta}(\mathbb{Q}_2) = \emptyset$, so $\tilde{\Delta}(\mathbb{Q}) = \emptyset$.)

To show that the image of $\varpi(\mathbb{R})$ is one oval, we note that $(\Delta = 0)_{\mathbb{R}}$ does not meet the line $(w = 0)$, and we work on the chart $(w \neq 0)$. Define disjoint boxes $B_1 \coloneqq \{(u, v) \mid -4 \leq u \leq -3, 2 \leq v \leq 3\}, B_2 \coloneqq \{(u, v) \mid -1 \leq u \leq 0, -1 \leq v \leq 1\}, B_3 \coloneqq \{(u, v) \mid -1 \leq u \leq 0, -3 \leq v \leq 1.5\}$, and $B_4 \coloneqq \{(u, v) \mid 0.5 \leq u \leq 3, 0 \leq v \leq 3\}$. Then, one can verify that the boundary of each B_i is disjoint from $(\Delta = 0)_{\mathbb{R}}$ and that

- The set $(\Delta = 0)_{\mathbb{R}} \cap (Q_1 \geq 0)_{\mathbb{R}} \cap B_1$ contains an $\mathbb{R}$-point $(-3.5, v)$ with $2 < v < 3$,
- The set $(\Delta = 0)_{\mathbb{R}} \cap (Q_1 < 0)_{\mathbb{R}} \cap B_2$ contains an $\mathbb{R}$-point $(u, -0.5)$ with $-1 < u < 0$,
- The set $(\Delta = 0)_{\mathbb{R}} \cap (Q_1 < 0)_{\mathbb{R}} \cap B_3$ contains an $\mathbb{R}$-point $(u, -2)$ with $-1 < u < 0$, and
- The set $(\Delta = 0)_{\mathbb{R}} \cap (Q_1 < 0)_{\mathbb{R}} \cap B_4$ contains an $\mathbb{R}$-point $(2, v)$ with $0 < v < 3$.

This shows that there are points on three connected components of $(\Delta = 0)_{\mathbb{R}}$ where Q_1 is negative, and there are points on the last connected components of $(\Delta = 0)_{\mathbb{R}}$ where Q_1 is positive. Since $(Q_1 = 0)_{\mathbb{R}} \subset (Q_1Q_3 - Q_2^2 \leq 0)_{\mathbb{R}}$ is connected, this means that exactly one connected component of $(\Delta = 0)_{\mathbb{R}}$ lies in $(Q_1 \geq 0)_{\mathbb{R}}$. It follows that the image of $\varpi(\mathbb{R})$ is one oval.

(3) (Image of $\tilde{\Delta}(\mathbb{R})$ is two ovals.) Define $\tilde{\Delta} \to \Delta$ and Y as in Section 2.2 for the quadrics

$$Q_1 \coloneqq -2u^2 - 2v^2 + w^2, \quad Q_2 \coloneqq 4u^2 - 9v^2 - vw + 2w^2, \quad Q_3 \coloneqq -3u^2 + 9v^2 + 8vw + 4w^2$$

Then $[0 : 0 : 1 : 1 : 2]$ exhibits a $\mathbb{Q}$-point of $\tilde{\Delta}$, so Y is $\mathbb{Q}$-rational by Proposition 2.2(3).

To show that the image of $\varpi(\mathbb{R})$ is two ovals, it suffices for us to show that only two connected components of $(\Delta = 0)_{\mathbb{R}}$ is contained in $(Q_1 \geq 0)_{\mathbb{R}}$. (See Figure 2.)

For this, we work on the chart $(w \neq 0)$. Define disjoint boxes $B_1 \coloneqq \{(u, v) \mid -1 \leq u \leq 1, 0.1 \leq v \leq 1\}, B_2 \coloneqq \{(u, v) \mid -1 \leq u \leq 1, -0.8 \leq v \leq -0.2\}, B_3 \coloneqq \{(u, v) \mid -10 \leq u \leq -1.1, -6 \leq v \leq -0.9\}$, and $B_4 \coloneqq \{(u, v) \mid 1.1 \leq u \leq 10, -6 \leq v \leq -0.9\}$. Then, one can verify that the boundary of each B_i is disjoint from $(\Delta = 0)_{\mathbb{R}}$ and that

- The set $(\Delta = 0)_{\mathbb{R}} \cap (Q_1 \geq 0)_{\mathbb{R}} \cap B_1$ contains the $\mathbb{R}$ point $(0, 3/5)$,
- $(\Delta = 0)_{\mathbb{R}} \cap (Q_1 \geq 0)_{\mathbb{R}} \cap B_2$ contains an $\mathbb{R}$ point $(0, v)$ with $-0.8 < v < -0.2$,
- $(\Delta = 0)_{\mathbb{R}} \cap (Q_1 < 0)_{\mathbb{R}} \cap B_3$ contains an $\mathbb{R}$-point $(-2, v)$ with $-6 < v < -0.9$, and
- $(\Delta = 0)_{\mathbb{R}} \cap (Q_1 < 0)_{\mathbb{R}} \cap B_3$ contains an $\mathbb{R}$-point $(2, v)$ with $-6 < v < -0.9$.

In particular, this shows that there are points on two connected components of $(\Delta = 0)_{\mathbb{R}}$ where Q_1 is positive, and there are points on the two other connected components of $(\Delta = 0)_{\mathbb{R}}$ where Q_1 is negative. Since $(Q_1 = 0)_{\mathbb{R}} \subset (Q_1Q_3 - Q_2^2 \leq 0)_{\mathbb{R}}$ is connected, this means that exactly two connected components of $(\Delta = 0)_{\mathbb{R}}$ lies in $(Q_1 \geq 0)_{\mathbb{R}}$, so the image of $\varpi(\mathbb{R})$ is two ovals.

(4) (Image of $\tilde{\Delta}(\mathbb{R})$ is three ovals.) Let $Y_{\tilde{\Delta}/\Delta}$ be as defined in Section 2.2 for the quadrics

$$Q_1 \coloneqq -4u^2 - uv + 9v^2 - 19uw + 6vw - 20w^2, \quad Q_2 \coloneqq 4u^2 - 14uv + 6v^2 + 21uw + 38vw + 38w^2$$

$$Q_3 \coloneqq u^2 - 6uv + 4v^2 - 21uw + 28vw + 49w^2.$$

Then $\Delta(\mathbb{R})$ is four ovals and $[0 : 1 : 0 : 3 : 2]$ exhibits a $\mathbb{Q}$-point of $\tilde{\Delta}$, so Y is $\mathbb{Q}$-rational by Proposition 2.2(3). Define $\tilde{\Delta}^-$ as in Definition 2.3. One can verify that the image of $\tilde{\Delta}^-(\mathbb{R})$ is one oval, similar to what's done in Example 4.6(2), and one concludes that the image of $\varpi^-(\mathbb{R})$ is three ovals. Note that $\tilde{\Delta}^-(\mathbb{R})$ is also $\mathbb{Q}$-rational as $[2 : 1 : 0 : 3 : -2]$ exhibits a $\mathbb{Q}$-point of $\tilde{\Delta}^-(\mathbb{R})$.

(5) (Image of $\tilde{\Delta}(\mathbb{R})$ is four ovals.) Let Q_1, Q_2, Q_3 be as in Example 4.6(1), and let $\tilde{\Delta}^- \to \Delta$ and $Y_{\tilde{\Delta}^-/\Delta}$ be as defined in Definition 2.3. Moreover, the map $\varpi^-(\mathbb{R})\colon \tilde{\Delta}^-(\mathbb{R}) \to \Delta(\mathbb{R})$ is surjective by Lemma 3.6, so $Y_{\tilde{\Delta}^-/\Delta}$ is $\mathbb{R}$-rational. (In this example $\tilde{\Delta}^-(\mathbb{Q}_5) = \emptyset$, and so $\tilde{\Delta}^-(\mathbb{Q}) = \emptyset$.)

5. Families of conic bundles with rational and non-rational members

In Sections 3 and 4, we used the double covers of $\mathbb{P}^1 \times \mathbb{P}^2$ defined in Section 2.2 to construct examples of irrational and rational conic bundles. We next give two examples of these double covers in one-parameter families, where the rationality of Y (and number of real connected components) and the real topological type of the discriminant curve vary in the family. We also exhibit interesting behavior of the intermediate Jacobian torsor (IJT) obstruction to rationality. Namely, in Example 5.1 we expect that the IJT obstruction vanishes for every smooth member; Example 5.2 has both rational and irrational members for which the IJT obstruction vanishes, but also contains irrational members for which we cannot show that the IJT obstruction vanishes; and Example 5.3 contains rational members, but we are not able to show that the IJT obstruction vanishes for any irrational members.

Recall that the IJT obstruction and $\tilde{P}^{(1)}$ were defined in Section 3.3. In the examples below, the numerical claims about $\tilde{P}^{(1)}$ can be verified using the code in [JJ]. The code that computes the real bitangents of Δ is due to Plaumann–Sturmfels–Vinzant and is included in the supplementary material for their paper [PSV11]. Recall also that since rank is a lower semicontinuous function for matrices, the existence of an $\mathbb{R}$-point on $\tilde{P}^{(1)}$ is an open condition.

In the following proofs, the numerical claims about $-\det M_{t,s}$, its discriminant, and the resultant of the partial derivatives $\partial_u\Delta_s, \partial_v\Delta_s, \partial_w\Delta_s$ can be verified by hand or using the `Macaulay2` code `Section-5-singular-members.m2` in [JJ]. Smoothness of Δ and $\tilde{\Delta}$ and the claims about the signatures of the fibers of π_1 can be verified by hand or using the code `Quadric-bundle-verifications.sage`, which is a `Sage` implementation of the `Magma` code accompanying [FJS+].

We first give an example of a family whose fibers include rational members with $\Delta(\mathbb{R})$ two nested ovals, rational members $\Delta(\mathbb{R})$ two non-nested ovals, and irrational members with $\Delta(\mathbb{R})$ two non-nested ovals.

Example 5.1. Let $\mathcal{Y} \to \mathbb{A}^1_s$ be the family of real conic bundle threefolds defined by the equation

$$z^2 = t_0^2(su^2-5v^2+4uw-5vw-2w^2)+2t_0t_1(u^2+v^2+10uw+2vw+3w^2)+t_1^2(-u^2-v^2-10uw-2vw-4w^2).$$

Then a general member $\mathcal{Y}_s$ has the structure of a geometrically standard conic bundle over $\mathbb{P}^2$ with smooth quartic discriminant curve Δ_s (and hence is $\mathbb{C}$-rational), and

(1) $\mathcal{Y}_s(\mathbb{R})$ is disconnected and therefore $\mathcal{Y}_s$ is not stably rational for $s \in (-\infty, \beta_1)$;
(2) $\mathcal{Y}_s$ is rational for $s \in (\beta_2, \infty)$;
(3) $\mathcal{Y}_s(\mathbb{R})$ is connected but the map $\pi_{1,s}(\mathbb{R})$ is not surjective for $s \in (\beta_1, \beta_2)$. In particular there is no known rationality construction for $\mathcal{Y}_s$ over $\mathbb{R}$; and
(4) Every interval on which Δ_s is smooth contains values of s such that $\mathcal{Y}_s$ has no IJT obstruction. Moreover, the IJT obstruction vanishes for every for $s \in (-\infty, -1] \cup (\beta_2, \infty)$ with $\Delta_s, \mathcal{Y}_s$ smooth.

The quartic curve Δ_s is singular for $s = -5, \beta_1, \beta_2, \beta_3$. For values of s with Δ_s and $\mathcal{Y}_s$ smooth, the number of real connected components of $\mathcal{Y}_s$ and the real isotopy class of Δ_s are:

s	$(-\infty,-5) \cup (-5,\beta_1)$	(β_1,β_2)	(β_2,β_3)	(β_3,∞)
$\mathcal{Y}_s(\mathbb{R})$	Two components	One component	One component (rational)	One component (rational)
$\Delta_s(\mathbb{R})$	Two non-nested ovals	One oval	Two nested ovals	Two non-nested ovals

Here $\frac{2}{5} < \beta_1 < \beta_2 < \frac{1}{2} < \beta_3$ and the β_i are the real roots of the discriminant of $-\det(t^2 M_{1,s} + 2tM_2 + M_3)$, where $M_{1,s}, M_2, M_3$ are the 3×3 symmetric matrices corresponding to $su^2 - 5v^2 + 4uw - 5vw - 2w^2, u^2 + v^2 + 10uw + 2vw + 3w^2, -u^2 - v^2 - 10uw - 2vw - 4w^2$, respectively.

We expect that in fact the IJT obstruction vanishes for $\mathcal{Y}_s$ for every $s \in (-\infty, \infty)$ such that Δ_s is smooth.

Proof of properties in Example 5.1. Each fiber $\mathcal{Y}_s$ is the double cover of $\mathbb{P}^1 \times \mathbb{P}^2$ associated to the quadrics

$$Q_{1,s} \coloneqq su^2 - 5v^2 + 4uw - 5vw - 2w^2, \quad Q_2 \coloneqq u^2 + v^2 + 10uw + 2vw + 3w^2, \quad Q_3 \coloneqq -u^2 - v^2 - 10uw - 2vw - 4w^2$$

as constructed in Section 2.2. For each s, let Γ_s be the associated genus two curve as defined in Proposition 2.2(4). Define the matrix $M_{t,s} \coloneqq t^2 M_{1,s} + 2tM_2 + M_3$. Then

$$-\det M_{t,s} = \left(-\frac{15s}{4} - 20\right) t^6 + \left(24s - \frac{399}{2}\right) t^5 - \left(25s + \frac{1089}{4}\right) t^4 + (10s + 546)t^3 - (3s + 360)t^2 + 134t - 22.$$

The discriminant of the polynomial $-\det M_{t,s}$ is a degree 9 polynomial in s with three real roots $\beta_1, \beta_2, \beta_3$, whose values are approximately

$$\beta_1 \approx 0.417608, \quad \beta_2 \approx 0.469848, \quad \beta_3 \approx 45.0611.$$

One then verifies that the genus two curve Γ_s has four real Weierstrass points for $s < \beta_1$, two real Weierstrass points for $\beta_1 < s < \beta_2$, no real Weierstrass points for $\beta_2 < s < \beta_3$, and two real Weierstrass points for $\beta_3 < s$. We now compute the signatures of the fibers of $\pi_{1,s} \colon \mathcal{Y}_s \to \mathbb{P}^1$ for s in each of these intervals.

(1) For $s < \beta_1$: Γ_s has four real Weierstrass points (note that when $s = -\frac{16}{3}$ one of these is over $[1:0]$). The signatures of the fibers of $\pi_{1,s}$ have the sequence $(0,4), (1,3), (0,4), (1,3)$, and therefore $\mathcal{Y}_s(\mathbb{R})$ has connected components.
(2) For $\beta_1 < s < \beta_2$: Γ_s has two real Weierstrass points, and the signature sequence is $(1,3), (0,4)$. Hence $\mathcal{Y}_s(\mathbb{R})$ is connected; however, π_1 is not surjective on real points and hence does not admit a section over $\mathbb{R}$.
(3) For $\beta_2 < s < \beta_3$: Γ_s has no real Weierstrass points, and every fiber has signature $(1,3)$. Hence $\pi_{1,s}$ has a section defined over $\mathbb{R}$, and in particular $\mathcal{Y}_s$ is rational over $\mathbb{R}$. One can also check that $\tilde{\Delta}_s$ has an $\mathbb{R}$-point for s in this interval.
(4) $\beta_3 < s$: Γ_s has two real Weierstrass points, and the signature sequence is $(1,3), (2,2)$. Therefore $\pi_{1,s}$ has a section defined over $\mathbb{R}$, and in particular $\mathcal{Y}_s$ is rational over $\mathbb{R}$. One can also check that $\tilde{\Delta}_s$ has an $\mathbb{R}$-point for values of s in this interval.

We now compute the locus where Δ_s is singular. The resultant of the partial derivatives of Δ_s is a degree 17 polynomial in s with four real roots $s = -5, \beta_1, \beta_2, \beta_3$. The real isotopy class of Δ_s is constant in each interval of $\mathbb{A}^1(\mathbb{R}) \setminus \{-5, \beta_1, \beta_2, \beta_3\}$, so it suffices to check at a single point on each interval. Using the `Sage` code accompanying [PSV11], one verifies that Δ_s has real rigid isotopy class

(1) Two non-nested ovals for $s < -5$ and $-5 < s < \beta_1$,
(2) One oval for $\beta_1 < s < \beta_2$,
(3) Two nested ovals for $\beta_2 < s < \beta_3$, and
(4) Two non-nested ovals for $\beta_3 < s$.

We also note that $\mathcal{Y}_{-5}(\mathbb{R})$ has two connected components, and that $\mathcal{Y}_s(\mathbb{R})$ is connected for $s \in \{\beta_1, \beta_2, \beta_3\}$.

It remains to show that for $s \in (-\infty, -5) \cup (-5, -1)$, the IJT obstruction vanishes for $\mathcal{Y}_s$. We will exhibit a point on $\tilde{P}^{(1)}$ for these values of s. The intersection $\Delta_s \cap (w = 0)$ is given by the equation $(su^2 - 5v^2)(-u^2 - v^2) - (u^2 + v^2)^2 = 0$ and consists of the four complex points

$$[1 : -i : 0], \quad [1 : i : 0], \quad [2 : \sqrt{s+1} : 0], \quad [-2 : \sqrt{s+1} : 0].$$

First suppose $-5 < s < -1$. Then $\sqrt{s+5} \in \mathbb{R}$ and $\sqrt{s+1}, \sqrt{-s-5} \notin \mathbb{R}$, so

$$[1 : i : 0 : \sqrt{s+5} : 0], \quad [1 : -i : 0 : \sqrt{s+5} : 0],$$
$$[2 : \sqrt{s+1} : 0 : \sqrt{-s-5} : -\sqrt{-s-5}], \quad [2 : -\sqrt{s+1} : 0 : -\sqrt{-s-5} : \sqrt{-s-5}]$$

is a set of four $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-invariant points of $\tilde{\Delta}$ mapping to $\Delta_s \cap (w = 0)$. These span a 3-plane in $\mathbb{P}^4$ since

$$\det \begin{pmatrix} 1 & i & \sqrt{s+5} & 0 \\ 1 & -i & \sqrt{s+5} & 0 \\ 2 & \sqrt{s+1} & \sqrt{-s-5} & -\sqrt{-s-5} \\ 2 & -\sqrt{s+1} & -\sqrt{-s-5} & \sqrt{-s-5} \end{pmatrix} = 8i\sqrt{s+5}\sqrt{-s-5} \in \mathbb{R} \setminus \{0\}$$

and so the above set of points on $\tilde{\Delta}$ exhibits an $\mathbb{R}$-point on $\tilde{P}^{(1)}$.

If $s < -5$, then $\sqrt{-s-5} \in \mathbb{R}$ and $\sqrt{s+1}, \sqrt{s+5} \notin \mathbb{R}$, so

$$[1 : i : 0 : \sqrt{s+5} : 0], \quad [1 : -i : 0 : -\sqrt{s+5} : 0],$$
$$[2 : \sqrt{s+1} : 0 : \sqrt{-s-5} : -\sqrt{-s-5}], \quad [2 : -\sqrt{s+1} : 0 : \sqrt{-s-5} : -\sqrt{-s-5}]$$

is a set of four $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-invariant points mapping to $\Delta_s \cap (w = 0)$ and gives an $\mathbb{R}$-point on $\tilde{P}^{(1)}$ since

$$\det \begin{pmatrix} 1 & i & \sqrt{s+5} & 0 \\ 1 & -i & -\sqrt{s+5} & 0 \\ 2 & \sqrt{s+1} & \sqrt{-s-5} & -\sqrt{-s-5} \\ 2 & -\sqrt{s+1} & \sqrt{-s-5} & -\sqrt{-s-5} \end{pmatrix} = -4\sqrt{s+5}\sqrt{s+1}\sqrt{-s-5} \in \mathbb{R} \setminus \{0\}.$$

When $s = -1$, one can check using the `Sage` code `Ptilde1.sage` in [JJ] that $\tilde{P}^{(1)}$ contains an $\mathbb{R}$-point over $\Delta_{-1} \cap (u + 10w = 0)$. For $s \in (\beta_2, \infty)$, the IJT obstruction vanishes because $\mathcal{Y}_s$ is rational. (Moreover, on $\tilde{P}^{(1)}$ has an $\mathbb{R}$-point on (β_3, ∞) because $\Gamma_s(\mathbb{R}) \neq \emptyset$ and so all the intermediate Jacobian torsors are trivial, see Section 3.3. One can also check that $\tilde{P}^{(1)}$ has a point for many values of $s \in (\beta_2, \beta_3)$, e.g. $\frac{1}{2}, 1$.) It remains to exhibit values of s in the intervals $(-1, \beta_1)$ and (β_1, β_2) for which $\tilde{P}^{(1)}$ has a point. Using the code `Ptilde1-bitangents.sage` in [JJ], one verifies this for $-\frac{1}{2}, 0 \in (-1, \beta_1)$. For the interval (β_1, β_2), one can check using `Ptilde1.sage` in [JJ] that $s = \frac{45}{100}$ has a point on $\tilde{P}^{(1)}$ mapping to $\Delta \cap (u + \frac{1}{10}v + \frac{1}{5}w = 0)$. $\square$

The next example contains members where $Y(\mathbb{R})$ has one, two, or three connected components.

Example 5.2. Let $\mathcal{Y} \to \mathbb{A}^1_s$ be the family of real conic bundle threefolds defined by the equation

$$z^2 = t_0^2(-u^2 - v^2 + suv - w^2) + 2t_0t_1(-5u^2 + 5w^2) + t_1^2(-24u^2 + 4v^2 - 24w^2).$$

Then a general fiber $\mathcal{Y}_s$ has the structure of a geometrically standard conic bundle over $\mathbb{P}^2$ with smooth quartic discriminant curve Δ_s (and hence is $\mathbb{C}$-rational). The quartic curve Δ_s is singular over ten points $s = \pm\beta_1, \pm\beta_2, \pm\beta_3, \pm\beta_4, \pm 7/\sqrt{3}$, and

(1) $\mathcal{Y}_s$ is rational for $s \in (-\infty, -\beta_4) \cup (\beta_4, \infty)$;
(2) $\mathcal{Y}_s(\mathbb{R})$ is connected but the map $\pi_{1,s}(\mathbb{R})$ is not surjective for $s \in (-\beta_4, -\beta_2) \cup (\beta_2, \beta_4)$. Hence there is no known rationality construction for $\mathcal{Y}_s$ over $\mathbb{R}$; and
(3) $\mathcal{Y}_s(\mathbb{R})$ is disconnected and hence $\mathcal{Y}_s$ is not stably rational for $s \in (-\beta_2, -\beta_2)$.

For smooth Δ_s and $\mathcal{Y}_s$, the number of connected components of $\mathcal{Y}_s(\mathbb{R})$ and the isotopy class of $\Delta_s(\mathbb{R})$ are:

s	$(-\infty, -\beta_4) \cup (\beta_4, \infty)$	$(-\beta_4, -\beta_2) \cup (\beta_2, \beta_4)$	$(-\beta_2, -\beta_1) \cup (\beta_1, \beta_2)$	$(-\beta_1, \beta_1)$
$\mathcal{Y}_s(\mathbb{R})$	One component (rational)	One component	Two components	Three components
$\Delta_s(\mathbb{R})$	Two nested ovals	One oval	Two non-nested ovals	Three ovals

Here $1 < \beta_1 < \beta_2 < \beta_3 < \beta_4 < 4$ and the β_i are the positive real roots of the discriminant of $-\det(t^2 M_{1,s} + 2tM_2 + M_3)$, where $M_{1,s}, M_2, M_3$ are the 3×3 symmetric matrices corresponding to $-u^2 - v^2 + suv - w^2, -5u^2 + 5w^2, -24u^2 + 4v^2 - 24w^2$, respectively.

In this example, every interval on which Δ_s is smooth *except* the intervals $(-\beta_4, -\beta_2)$ and (β_2, β_4) contains points s such that the IJT obstruction vanishes for $\mathcal{Y}_s$. We expect that the real IJT obstruction vanishes for $\mathcal{Y}_s$ for every $s \notin (-\beta_4, -\beta_2) \cup (\beta_2, \beta_4)$ such that Δ_s is smooth. For $s \in (-\beta_4, -\beta_2) \cup (\beta_2, \beta_4)$, we are not able to exhibit an $\mathbb{R}$-point on $\tilde{P}^{(1)}$—it is likely that $\mathcal{Y}_s$ has an IJT obstruction to $\mathbb{R}$-rationality in these intervals.

Proof of properties in Example 5.2. $\mathcal{Y}_s$ is the double cover of $\mathbb{P}^1 \times \mathbb{P}^2$ associated to the quadrics

$$Q_{1,s} := -u^2 - v^2 + suv - w^2, \quad Q_2 := -5u^2 + 5w^2, \quad Q_3 := -24u^2 + 4v^2 - 24w^2$$

as constructed in Section 2.2. Let Γ_s be the associated genus two curve as defined in Proposition 2.2(4). Define the matrix $M_{t,s} = t^2 M_{1,s} + 2tM_2 + M_3$. Then

$$-\det M_{t,s} = \left(1 - \frac{s^2}{4}\right) t^6 + \frac{5s^2}{2} t^5 - (6s^2 + 56)t^4 + 784t^2 - 2304.$$

The discriminant of $-\det M_{t,s}$ is a degree 14 polynomial with eight real roots $s = \pm\beta_1, \pm\beta_2, \pm\beta_3, \pm\beta_4$ with

$$\beta_1 \approx 1.1067, \quad \beta_2 = \tfrac{8}{3}\sqrt{5/3} \approx 3.44265, \quad \beta_3 = \sqrt{15} \approx 3.87298, \quad \beta_4 \approx 3.9724.$$

We now compute the signatures of the fibers of $\pi_{1,s}$.

(1) $s < -\beta_4$: Γ_s has two real Weierstrass points over $t = 4, 6$, and the fibers of $\pi_{1,s}$ have signatures in the sequence $(1,3), (2,2)$. In particular $\pi_{1,s}$ is surjective on real points and has a section over $\mathbb{R}$, so $\mathcal{Y}_s$ is rational over $\mathbb{R}$. One can also check that $\tilde{\Delta}_s(\mathbb{R}) \neq \emptyset$.
(2) $-\beta_4 < s < -\sqrt{15}$: Γ_s has four real Weierstrass points, and the fibers have signature sequence $(1,3), (0,4), (1,3), (2,2)$. In particular, $\mathcal{Y}_s(\mathbb{R})$ is connected but $\pi_{1,s}$ is not surjective on real points and hence does not have a section defined over $\mathbb{R}$.
(3) $-\sqrt{15} < s < -\frac{8}{3}\sqrt{5/3}$: Γ_s has four real Weierstrass points, and the fibers have signature sequence $(1,3), (0,4), (1,3), (2,2)$. In particular, $\mathcal{Y}_s(\mathbb{R})$ is connected but $\pi_{1,s}$ is not surjective on real points and hence does not have a section defined over $\mathbb{R}$.
(4) $-\frac{8}{3}\sqrt{5/3} < s < -\beta_1$: Γ_s has four real Weierstrass points (when $s = -2$ one of the Weierstrass points is over $[1:0]$), and the fibers have signature sequence $(1,3), (0,4), (1,3), (0,4)$. In particular $Y_s(\mathbb{R})$ has two connected components and hence is not stably rational over $\mathbb{R}$.
(5) $-\beta_1 < s < \beta_1$: Γ_s has six real Weierstrass points, and the signatures of the fibers of $\pi_{1,s}$ have the sequence $(0,4), (1,3), (0,4), (1,3), (0,4), (1,3)$. Thus $\mathcal{Y}_s(\mathbb{R})$ has three components.
(6) $\beta_1 < s < \frac{8}{3}\sqrt{5/3}$: Γ_s has four real Weierstrass points (when $s = 2$ one of the Weierstrass points is over $[1:0]$), and the fibers have signature sequence $(1,3), (0,4), (1,3), (0,4)$. Therefore $\mathcal{Y}_s(\mathbb{R})$ has two connected components.
(7) $\frac{8}{3}\sqrt{5/3} < s < \sqrt{15}$: Γ_s has four real Weierstrass points, and the fibers have signature sequence $(1,3), (0,4), (1,3), (2,2)$. In particular, $\mathcal{Y}_s(\mathbb{R})$ is connected but $\pi_{1,s}$ is not surjective on real points and hence does not have a section defined over $\mathbb{R}$.
(8) $\sqrt{15} < s < \beta_4$: Γ_s has four real Weierstrass points, and the fibers have signature sequence $(1,3), (0,4), (1,3), (2,2)$. In particular, $\mathcal{Y}_s(\mathbb{R})$ is connected but $\pi_{1,s}$ is not surjective on real points and hence does not have a section defined over $\mathbb{R}$.
(9) $\beta_4 < s$: Γ_s has two real Weierstrass points over $t = 4, 6$, and the signature sequence of the fibers is $(1,3), (2,2)$. In particular $\pi_{1,s}$ is surjective on real points and has a section over $\mathbb{R}$, so $\mathcal{Y}_s$ is rational over $\mathbb{R}$. One can also check that $\tilde{\Delta}_s(\mathbb{R}) \neq \emptyset$.

We note that $\mathcal{Y}_{\pm\beta_1}$ have disconnected real loci, and that $\mathcal{Y}_{\pm\sqrt{15}}(\mathbb{R})$ are connected but the morphisms $\pi_{1,\pm\sqrt{15}}$ are not surjective on real points.

The quartic curves Δ_s are smooth away from the roots of the resultant $R_3(\partial_u\Delta_s, \partial_v\Delta_s, \partial_w\Delta_s)$, which one can compute is a degree 22 polynomial with ten real roots $s = \pm\beta_1, \pm\beta_2, \pm\beta_3, \pm\beta_4, \pm 7/\sqrt{3}$. We now determine the real isotopy class of the quartic curve Δ_s using the `Sage` code accompanying [PSV11]. It suffices to check one point on each interval of $\mathbb{A}^1(\mathbb{R}) \setminus \{\pm\beta_1, \pm\frac{8}{3}\sqrt{5/3}, \pm\sqrt{15}, \pm\beta_4, \pm 7/\sqrt{3}\}$, and we find that the real rigid isotopy class of Δ_s is:

(1) Two nested ovals for $s < -7/\sqrt{3}$ and $-7/\sqrt{3} < s < -\beta_4$,
(2) One oval for $-\beta_4 < s < -\sqrt{15}$,
(3) One oval for $-\sqrt{15} < s < -\frac{8}{3}\sqrt{5/3}$,
(4) Two non-nested ovals for $-\frac{8}{3}\sqrt{5/3} < s < -\beta_1$,
(5) Three ovals for $-\beta_1 < s < \beta_1$,
(6) Two non-nested ovals for $\beta_1 < s < \frac{8}{3}\sqrt{5/3}$,
(7) One oval for $\frac{8}{3}\sqrt{5/3} < s < \sqrt{15}$,
(8) One oval for $\sqrt{15} < s < \beta_4$, and
(9) Two nested ovals for $\beta_4 < s < 7/\sqrt{3}$ and $7/\sqrt{3} < s$.

It remains to show that each interval of $\mathbb{A}^1(\mathbb{R})\setminus\{\pm\beta_1, \pm\beta_2, \pm\beta_3, \pm\beta_4, \pm 7/\sqrt{3}\}$ except $(-\beta_4, -\beta_3)$, $(-\beta_3, -\beta_2)$, (β_2, β_3), and (β_3, β_4) contains values of s such that the IJT obstruction vanishes for $\mathcal{Y}_s$. First, we note that $\mathcal{Y}_s$ is rational and Γ_s has real points for $s \in (-\infty, -\beta_4) \cup (\beta_4, \infty)$ with Δ_s smooth, so the IJT obstruction vanishes and moreover $\tilde{P}^{(1)}$ has a real point. One the remaining intervals, one can show using `Ptilde1-bitangents.sage` in [JJ] that $\tilde{P}^{(1)}$ has a point for the following values of s: $-3, -2 \in (-\beta_2, -\beta_1)$; $-1, 0 \in (-\beta_1, \beta_1)$; and $2, 3 \in (\beta_1, \beta_2)$. □

Finally, we give an example of a family containing members whose discriminant curves are four ovals, and members with disconnected real loci.

Example 5.3. Let $\mathcal{Y} \to \mathbb{A}^1_s$ be the family of real conic bundle threefolds defined by the equation

$$z^2 = t_0^2(su^2 + v^2 - w^2) + 2t_0t_1\left(-\frac{43}{57}u^2 - \frac{93}{14}v^2 + \frac{85}{39}w^2\right) + t_1^2\left(\frac{8}{57}u^2 - \frac{221}{14}v^2 + \frac{50}{13}w^2\right).$$

We claim that a general fiber $\mathcal{Y}_s$ has the structure of a geometrically standard conic bundle over $\mathbb{P}^2$ with smooth quartic discriminant curve Δ_s. Furthermore, the quartic curve Δ_s is singular over six real points $s = \beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \zeta$, and

(1) $\mathcal{Y}_s(\mathbb{R})$ is disconnected and hence $\mathcal{Y}_s$ is not stably rational for $s \in (-\infty, \beta_1) \cup (\beta_1 < s < \beta_2)$;
(2) $\mathcal{Y}_s(\mathbb{R})$ is connected but the map $\pi_{1,s}(\mathbb{R})$ is not surjective for $s \in (\beta_2, \beta_3) \cup (\beta_3, \beta_4)$. Hence there is no known rationality construction for $\mathcal{Y}_s$ over $\mathbb{R}$; and
(3) $\mathcal{Y}_s$ is rational for $s \in (\beta_4, \zeta) \cup (\zeta, \beta_5) \cup (\beta_5, \infty)$.

Here $\frac{-5}{2} < \beta_1 < \beta_2 < \beta_3 < \beta_4 < \beta_5 < \frac{17}{4}$ are the β_i's that are the real roots of the discriminant of $-\det(t^2M_{1,s} + 2tM_2 + M_3)$ where $M_{1,s}, M_2, M_3$ are the 3×3 symmetric matrices corresponding to $su^2 + v^2 - w^2$, $-\frac{43}{57}u^2 - \frac{93}{14}v^2 + \frac{85}{39}w^2$, and $\frac{8}{57}u^2 - \frac{221}{14}v^2 + \frac{50}{13}w^2$ respectively.

For s such that Δ_s and $\mathcal{Y}_s$ are smooth, the number of real connected components of $\mathcal{Y}_s$ and the real isotopy class of Δ_s are:

s	$(-\infty, \beta_2)$	(β_2, β_4)	(β_4, β_5)	(β_5, ∞)
$\mathcal{Y}_s(\mathbb{R})$	Two components	Connected	Connected (rational)	Connected (rational)
$\Delta_s(\mathbb{R})$	Two nested ovals	Two non-nested ovals	Four ovals	Three ovals

In this example, we are not able to exhibit an $\mathbb{R}$-point of $\tilde{P}^{(1)}$ for $s \in (-\infty, \beta_4)$. For $s \in (\beta_4, \infty)$ with Δ_s and $\mathcal{Y}_s$ smooth, we have that $\mathcal{Y}_s$ is rational and $\Gamma_s(\mathbb{R}) \neq \emptyset$; hence, $\tilde{P}^{(1)}(\mathbb{R}) \neq \emptyset$ for these s.

Proof of properties in Example 5.3. Each fiber of $\mathcal{Y} \to \mathbb{A}^1_s$ is the double cover of $\mathbb{P}^1 \times \mathbb{P}^2$ with the associated quadrics

$$Q_{1,s} := su^2 + v^2 - w^2, \quad Q_2 := -\frac{43}{57}u^2 - \frac{93}{14}v^2 + \frac{85}{39}w^2, \quad Q_3 := \frac{8}{57}u^2 - \frac{221}{14}v^2 + \frac{50}{13}w^2$$

as defined in Section 2.2. Let Γ_s be the associated genus two curve as defined in Proposition 2.2(4). Define the matrix $M_{t,s} = t^2 M_{1,s} + 2tM_2 + M_3$. Then

$$\begin{aligned} -\det M_{t,s} = st^6 &- \left(\frac{4817}{273}s + \frac{86}{57}\right)t^5 - \left(\frac{-6967}{182}s - \frac{416446}{15561}\right)t^4 - \left(\frac{-32735}{273}s + \frac{133897}{2223}\right)t^3 \\ &- \left(\frac{-425}{7}s + \frac{2731606}{15561}\right)t^2 - \frac{1163570}{15561}t + \frac{3400}{399}. \end{aligned}$$

The discriminant of $-\det M_{t,s} \in \mathbb{Q}[s][t]$ is a degree 9 polynomial in s with five real roots $\beta_1, \beta_2, \beta_3, \beta_4, \beta_5$ whose values are approximately

$$\beta_1 \approx -2.253001, \quad \beta_2 \approx -1.491244, \quad \beta_3 \approx 0.10422, \quad \beta_4 \approx 0.289804, \quad \beta_5 \approx 4.05482.$$

We will now compute the signatures of the fibers of $\pi_{1,s}$. It suffices for us to check one point in each interval of $\mathbb{A}(\mathbb{R}) \setminus \{\beta_1, \beta_2, \beta_3, \beta_4, \beta_5\}$. We find that:

(1) $s < \beta_1$ and $\beta_1 < s < \beta_2$: Γ_s has six real Weierstrass points and the fibers of $\pi_{1,s}$ has signatures in sequence $(0,4), (1,3), (2,2), (1,3), (0,4), (1,3)$. In particular, this shows that $\mathcal{Y}_s(\mathbb{R})$ has two connected components. Thus, $\mathcal{Y}_s$ is irrational over $\mathbb{R}$.
(2) $\beta_2 < s < \beta_3$ and $\beta_3 < s < \beta_4$: Γ_s has six real Weierstrass points (when $s = 0$ one of these is over $[1:0]$) and the fibers of $\pi_{1,s}$ has signatures in sequence $(2,2), (1,3), (2,2), (1,3), (0,4), (1,3)$. In particular, $\pi_{1,s}$ is not surjective on real points and hence does not have a section defined over $\mathbb{R}$.
(3) $\beta_4 < s < \beta_5$: Γ_s has six real Weierstrass points and the fibers of $\pi_{1,s}$ has signatures in sequence $(2,2), (1,3), (2,2), (1,3), (2,2), (1,3)$. In particular, $\pi_{1,s}$ is surjective on real points and has a section defined over $\mathbb{R}$, so $\mathcal{Y}_s$ is rational. (One can also check $\tilde{\Delta}_s(\mathbb{R}) \neq \emptyset$.)
(4) $\beta_5 < s$: Γ_s has four real Weierstrass points and the fibers of $\pi_{1,s}$ has signatures in sequence $(2,2), (1,3), (2,2), (1,3)$. In particular, $\pi_{1,s}$ is surjective on real points and has a section defined over $\mathbb{R}$, so $\mathcal{Y}_s$ is rational. (One can also check $\tilde{\Delta}_s(\mathbb{R}) \neq \emptyset$.)

Before determining the real isotopy class of Δ_s, we compute the locus where Δ_s is singular. The resultant of the partial derivatives of Δ is a degree 17 polynomial with six real roots $s = \beta_1, \beta_2, \beta_3, \beta_4, \zeta, \beta_5$, where $\zeta \approx 1.08788$. So the isotopy class of Δ_s, is constant on each interval of $\mathbb{A}(\mathbb{R}) \setminus \{\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \zeta\}$, and using the `Sage` code accompanying [PSV11] we find that the real isotopy class of Δ_s is:

(1) Two nested ovals for $s < \beta_1$ and $\beta_1 < s < \beta_2$,
(2) Two non-nested ovals for $\beta_2 < s < \beta_3$ and $\beta_3 < s < \beta_4$,
(3) Four ovals for $\beta_4 < s < \zeta$ and $\zeta < s < \beta_5$, and
(4) Three ovals for $\beta_5 < s$.

□

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