Existence of uncountably many singular vectors in $\mathbb{R}^n$ for higher dimensional generalizations of Dani correspondence

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Abstract

The relationship between singular vectors in $\mathbb{R}^n$ and homogeneous dynamics has been well known ever since Dani's original paper on the subject in 1985 [1]. However, all work published on the subject has been focused towards a 1-dimensional homogeneous system. In this paper, we summarize some key results of the 1-dimensional case and examine similar results in two main higher-dimensional generalizations. The paper concludes with an extension of the work to approximation with weights.

1 Diophantine Approximation and Singular Vectors

The field of Diophantine approximation is the study of how well real vectors can be approximated by rational vectors. The results of the field would be indispensable without Dirichlet's Approximation Theorem:

Theorem 1.1 (Dirichlet's Approximation Theorem)**.** *For any real vector* $\mathbf{x} \in \mathbb{R}^n$ *and any* $Q \in \mathbb{N}$*, there exists* $\mathbf{p} \in \mathbb{Z}^n$ *and* $q \in \mathbb{N}$ *such that*

$$\|q\mathbf{x} - \mathbf{p}\| \leq \frac{1}{Q^{1/n}} \qquad\qquad q \leq Q$$

Throughout this paper, $\|\cdot\|$ is taken to be the max-norm of the respective dimension.

Proof. Construct the set

$$S = \left\{(q, p_1, \ldots, p_n) \in \mathbb{R}^{n+1} : -Q - \frac{1}{2} \leq q \leq Q + \frac{1}{2}, |qx_i - p_i| \leq \frac{1}{Q^{1/n}}; i = 1, \ldots, n\right\}$$

The volume of S is $2^{n+1} + \frac{2^n}{Q} > 2^{n+1}$. Thus, by Minkowski's Theorem, there is a non-trivial point in S with integral coordinates. Since S is symmetric, we choose the point such that $q > 0$. Thus, for this point, which we shall label $(q, \mathbf{p})$, we see that $q \leq Q$ and

$$\|q\mathbf{x} - \mathbf{p}\| = \max_{i=1,\ldots,n} |qx_i - p_i| \leq \frac{1}{Q^{1/n}}$$

□

There is an equivalent way of writing Theorem 1.1 using what is called the ***transference principle***:

Theorem 1.2 (Transferred Dirichlet's Approximation Theorem)**.** *For any real vector* $\mathbf{x} \in \mathbb{R}^n$ *and any* $Q \in \mathbb{N}$*, there exists* $p \in \mathbb{Z}$ *and* $\mathbf{q} \in \mathbb{Z}^n$ *such that*

$$|\mathbf{q} \cdot \mathbf{x} + p| \leq \frac{1}{Q^n} \qquad\qquad \|\mathbf{q}\| \leq Q$$

Importantly, Theorems 1.1 and 1.2 give soft upper bounds on the quality of the approximation. We would like to explore vectors for which we can improve this approximation arbitrarily well: singular vectors.

Definition 1.1. *A vector $\mathbf{x} \in \mathbb{R}^n$ is* ***singular***[1] *if for any $\epsilon > 0$, there exists $Q_\epsilon \geq 0$ such that for all $Q \geq Q_\epsilon$, there are infinitely many $q \in \mathbb{N}$ and $\mathbf{p} \in \mathbb{Z}^n$ such that*

$$\|q\mathbf{x} - \mathbf{p}\| \leq \frac{\epsilon}{Q^{1/n}} \qquad\qquad q \leq Q$$

Using the transference principle, Definition 1.1 can be written as

Definition 1.2. *A vector $\mathbf{x} \in \mathbb{R}^n$ is* ***singular*** *if for any $\epsilon > 0$, there exists $Q_\epsilon \geq 0$ such that for all $Q \geq Q_\epsilon$, there are infinitely many $\mathbf{q} \in \mathbb{Z}^n$ and $p \in \mathbb{Z}$ such that*

$$|\mathbf{q} \cdot \mathbf{x} + p| \leq \frac{\epsilon}{Q^n} \qquad\qquad \|\mathbf{q}\| \leq Q$$

The set of singular vectors in $\mathbb{R}^n$, denoted $\mathrm{Sing}\,(n)$, is known to be a set of measure zero [1]. Despite this set being rather small, it is not obvious what all the elements of $\mathrm{Sing}\,(n)$ are for a given n. This issue will be the main focus of this paper.

Another way to measure how well we can improve Theorems 1.1 and 1.2 is by increasing the exponent on the denominator. By increasing the exponent in the denominator, the maximum error bound becomes much smaller, yielding a better approximation.

Definition 1.3. *The* ***uniform exponent*** *of $\mathbf{x} \in \mathbb{R}^n$, denoted $\hat{\omega}\,(\mathbf{x})$, is the supremum of $\gamma > 0$ such that*

$$\|q\mathbf{x} - \mathbf{p}\| \leq \frac{1}{Q^\gamma} \qquad\qquad q \leq Q$$

Similarly, we can define the ***dual uniform exponent*** *of $\mathbf{x} \in \mathbb{R}^n$, denoted $\hat{\omega}^*\,(\mathbf{x})$, is the supremum of $\gamma > 0$ such that*

$$|\mathbf{q} \cdot \mathbf{x} + p| \leq \frac{1}{Q^\gamma} \qquad\qquad \|\mathbf{q}\| \leq Q$$

Using Theorems 1.1 and 1.2, we see that

$$\hat{\omega}\,(\mathbf{x}) \geq \frac{1}{n} \qquad\qquad \hat{\omega}^*\,(\mathbf{x}) \geq n$$

If these are instead strict inequalities, we refer to $\mathbf{x}$ as ***very singular***. Clearly, a very singular number is also singular.

As we shall see in Theorem 3.2, rational numbers behave differently than irrational numbers in the realm of Diophantine Approximation. To generalize this difference to higher dimensions, we need to define what it means for a vector to be "irrational":

Definition 1.4. *A vector $\mathbf{x} \in \mathbb{R}^n$ is* ***totally irrational*** *if $1, x_1, \ldots, x_n$ are all linearly independent over $\mathbb{Q}$.*

It can be shown that if $\mathbf{x}$ is not totally irrational, then

$$\hat{\omega}\,(\mathbf{x}) \geq \frac{1}{n-1} \qquad\qquad \hat{\omega}^*\,(\mathbf{x}) = \infty$$

implying that $\mathbf{x}$ is singular. However, for $n \geq 2$, there exist totally irrational $\mathbf{x} \in \mathrm{Sing}\,(n)$. We will prove this fact later in this paper.

We can already generalize our work thus far to more general error bounds. If $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a non-increasing function, then we can generalize Definition 1.1 (and all of our work thus far) as follows:

Definition 1.5. *A vector $\mathbf{x} \in \mathbb{R}^n$ is φ-**singular** if for any $\epsilon > 0$, there exists $Q_\epsilon \geq 0$ such that for all $Q \geq Q_\epsilon$, there are infinitely many $q \in \mathbb{N}$ and $\mathbf{p} \in \mathbb{Z}^n$ such that*

$$\|q\mathbf{x} - \mathbf{p}\| \leq \epsilon\varphi\,(Q) \qquad\qquad q \leq Q$$

[1]The notion of "singular" extends beyond just vectors. See [1] for the generalization to singular matrices.

2 Trajectories in the Space of Unimdoular Lattices

For $n \geq 2$, we denote by $\mathscr{L}_n$ the space of unimodular lattices in $\mathbb{R}^n$. It is known that we can identify $\mathscr{L}_n$ with the space $\mathrm{SL}_n(\mathbb{R}) / \mathrm{SL}_n(\mathbb{Z})$ since the correspondence $g\,\mathrm{SL}_n(\mathbb{Z}) \leftrightarrow g\mathbb{Z}^n$ for $g \in \mathrm{SL}_n(\mathbb{R})$ is a well-defined bijection. Notably, this space is not compact, which begs the question of what it means for a sequence to diverge in this space.

Proposition 2.1 (Mahler Compactness Criterion)**.** *If $\{g_i\}$ is a sequence in $\mathrm{SL}_n(\mathbb{R})$, then the sequence $\{g_i\,\mathrm{SL}_n(\mathbb{Z})\}$ diverges (or alternatively has no limit points in $\mathrm{SL}_n(\mathbb{R}) / \mathrm{SL}_n(\mathbb{Z})$) if and only if for any neighborhood Ω of 0 in $\mathbb{R}^n$, there exists I such that for all $i \geq I$, $g_i(\mathbb{Z}^n) \cap \Omega \neq \{0\}$.*

One sequence in $\mathrm{SL}_n(\mathbb{R})$ of particular importance is elements of the following form

$$D_t = \begin{pmatrix} e^{-t} & 0_{n-1\times 1} \\ 0_{1\times n-1} & e^{t/(n-1)} I_{n-1\times n-1} \end{pmatrix}$$

where $0_{j\times k}$ is $j \times k$ matrix with all zero entries and $I_{j\times j}$ is the $j \times j$ identity matrix. Note that t is taken to be a non-negative real number.

An important feature of elements $g \in \mathrm{SL}_n(\mathbb{R})$ is that they can be decomposed as below:

$$g = \begin{pmatrix} A & B \\ 0_{1\times n-1} & C \end{pmatrix} \begin{pmatrix} 1 & 0_{n-1\times 1} \\ \mathbf{x} & I_{n-1\times n-1} \end{pmatrix} \sigma$$

where $A \in \mathbb{R}$, B is a $n-1 \times 1$ real matrix, C is an $n-1 \times n-1$ real matrix, $\mathbf{x} \in \mathbb{R}^{n-1}$, and σ is a permuttion matrix except for signs. This decomposition allows us to simplify our trajectories in $\mathscr{L}_n$:

Theorem 2.1. *For $g \in \mathrm{SL}_n(\mathbb{R})$, the trajectory $\{D_t g\,\mathrm{SL}_n(\mathbb{Z})\}$, or when written in a decomposed form $\left\{D_t \begin{pmatrix} A & B \\ 0_{1\times n-1} & C \end{pmatrix} \begin{pmatrix} 1 & 0_{n-1\times 1} \\ \mathbf{x} & I_{n-1\times n-1} \end{pmatrix} \sigma\,\mathrm{SL}_n(\mathbb{Z})\right\}$, diverges if and only if $\left\{D_t \begin{pmatrix} 1 & 0_{n-1\times 1} \\ \mathbf{x} & I_{n-1\times n-1} \end{pmatrix} \mathrm{SL}_n(\mathbb{Z})\right\}$ diverges.*

Proof. First, $\sigma \in \mathrm{SL}_n(\mathbb{Z})$, so there is no harm in absorbing it into the $\mathrm{SL}_n(\mathbb{Z})$ term. Next, let us examine the trajectory of $\left\{D_t \begin{pmatrix} A & B \\ 0_{1\times n-1} & C \end{pmatrix} D_{-t}\right\}$. We see that

$$D_t \begin{pmatrix} A & B \\ 0_{1\times n-1} & C \end{pmatrix} D_{-t} = \begin{pmatrix} A & e^{-t\frac{n+1}{n}} B \\ 0_{1\times n-1} & C \end{pmatrix}$$

Taking the limit $t \to \infty$, we see that $\left\{D_t \begin{pmatrix} A & B \\ 0_{1\times n-1} & C \end{pmatrix} D_{-t}\right\}$ is bounded. We can then rewrite $D_t \begin{pmatrix} 1 & 0_{n-1\times 1} \\ \mathbf{x} & I_{n-1\times n-1} \end{pmatrix} \mathrm{SL}_n(\mathbb{Z})$ as $\left(D_t \begin{pmatrix} A & B \\ 0_{1\times n-1} & C \end{pmatrix} D_{-t}\right) \left(D_t \begin{pmatrix} 1 & 0_{n-1\times 1} \\ \mathbf{x} & I_{n-1\times n-1} \end{pmatrix} \mathrm{SL}_n(\mathbb{Z})\right)$ for all $t \geq 0$, from which the claim is immediate. □

A particularly interesting class of divergent trajectories are degenerate divergent trajectories.

Definition 2.1. *Let $\{g_t\}$ be a 1-parameter subgroup of $\mathrm{SL}_n(\mathbb{R})$ and $\Lambda \in \mathscr{L}_n$ such that $\{g_t(\Lambda)\}$ is divergent. If there exists a non-zero subgroup Σ of Λ such that the volume of the fundamental domain of $g_t(\Sigma)$ vanishes as $t \to \infty$, we say that $\{g_t(\Lambda)\}$ is* ***degenerate****. Divergent trajectories without this property we refer to as* ***non-degenerate****.*

Proposition 2.2. *Let $\{g_t\}$ be a 1-parameter subgroup in $\mathrm{SL}_n(\mathbb{R})$ and let $\Lambda \in \mathrm{SL}_n(\mathbb{R}) / \mathrm{SL}_n(\mathbb{Z})$. Then $\{g_t(\Lambda)\}$ is a degenerate divergent trajectory if and only if there exists $g \in \mathrm{SL}_n(\mathbb{R})$ and $1 \leq p \leq n-1$ such that $g\mathbb{Z}^n = \Lambda$ and $\bigwedge^p (g_t g)(e_1 \wedge \cdots \wedge e_p) \to 0$ as $t \to \infty$, where $e_1, \ldots, e_n$ are the standard basis vectors of $\mathbb{R}^n$.*

Proof. Let Σ_0 be the subgroup generated by $e_1, \dots e_p$ and $\Sigma = g(\Sigma_0)$. Then,

$$\left\| \bigwedge^p (g_t g)(e_1 \wedge \dots \wedge e_p) \right\| = |\det g_t g| \to 0$$

for a suitable norm on the space of exteriors.

Now suppose that there exists $\Sigma \subset \Lambda$ such that the volume of the fundamental domain of $g_t(\Sigma)$ goes to zero as $t \to \infty$. Let S be the largest subgroup of $\bigwedge$ generating the same subspace. We assume there is a basis $v_1, \dots v_n$ of Λ such that $v_1, \dots, v_p$ form a basis of Σ. Up to the sign of the basis vectors, we may assume there exists $g \in \mathrm{SL}_n(\mathbb{R})$ such that $g(e_i) = v_i$ for $i = 1, \dots n$. Now g has the required properties. □

3 Using Trajectories to Show Singularity

Using Theorem 2.1, we can focus our efforts on trajectories of the form $\left\{ D_t \begin{pmatrix} 1 & 0_{n-1\times 1} \\ \mathbf{x} & I_{n-1\times n-1} \end{pmatrix} \mathrm{SL}_n(\mathbb{Z}) \right\}$ for $\mathbf{x} \in \mathbb{R}^{n-1}$. A consequence of this is a strong theorem.

Theorem 3.1 (Dani Correspondence)**.** *The trajectory* $\left\{ D_t \begin{pmatrix} 1 & 0_{n-1\times 1} \\ \mathbf{x} & I_{n-1\times n-1} \end{pmatrix} \mathrm{SL}_n(\mathbb{Z}) \right\}$ *diverges if any only if* $\mathbf{x}$ *is singular.*

Proof. Assume that $\mathbf{x} \in \mathbb{R}^{n-1}$ is singular. Then for any $\delta > 0$, let B_δ be the ball of radius δ centered at the origin in $\mathbb{R}^n$ with respect to $\|\cdot\|$. By Proposition 2.1, it is enough to show that for any $\delta > 0$, there exists $T \geq 0$ such that for all $t \geq T$, $B_\delta \cap \left\{ D_t \begin{pmatrix} 1 & 0_{n-1\times 1} \\ \mathbf{x} & I_{n-1\times n-1} \end{pmatrix} \mathrm{SL}_n(\mathbb{Z}) \right\} \neq \{0\}$.

Now fix $\delta \in (0,1)$ and choose $\epsilon < \delta^n$. Since $\mathbf{x}$ is singular, we know there exists Q_ϵ such that for all $Q \geq Q_\epsilon$, there exists infinitely many $q \in \mathbb{N}$ and $\mathbf{p} \in \mathbb{Z}^{n-1}$ such that

$$\|q\mathbf{x} - \mathbf{p}\| \leq \frac{\epsilon}{Q^{1/(n-1)}} \qquad\qquad q \leq Q$$

Let $q, \mathbf{p}$ satisfy the above conditions. We see that

$$\begin{aligned} \left\| D_t \begin{pmatrix} 1 & 0_{n-1\times 1} \\ \mathbf{x} & I_{n-1\times n-1} \end{pmatrix} \mathrm{SL}_n(\mathbb{Z}) \right\| &= \left\| \begin{pmatrix} e^{-t} & 0_{n-1\times 1} \\ 0_{1\times n-1} & e^{t/(n-1)} I_{n-1\times n-1} \end{pmatrix} \begin{pmatrix} 1 & 0_{n-1\times 1} \\ \mathbf{x} & I_{n-1\times n-1} \end{pmatrix} \begin{pmatrix} q \\ -\mathbf{p} \end{pmatrix} \right\| \\ &= \left\| \begin{pmatrix} e^{-t} q \\ e^{t/(n-1)} (q\mathbf{x} - \mathbf{p}) \end{pmatrix} \right\| \\ &= \max\left(\left| e^{-t} q \right|, \left\| e^{t/(n-1)} (q\mathbf{x} - \mathbf{p}) \right\| \right) \\ &\leq \max\left(e^{-t} Q, e^{t/(n-1)} \frac{\epsilon}{Q^{1/(n-1)}} \right) \end{aligned}$$

If we want to bound the last inequality above by δ, we see that $t \in \left(\ln \frac{Q}{\delta}, (n-1) \ln \frac{\delta Q^{1/(n-1)}}{\epsilon} \right) := I_Q$. For Q large, we see that since $\epsilon < \delta^n$, I_Q and I_{Q+1} overlap. Thus, the interval $\bigcup I_Q$ has a subinterval $[T, \infty)$ such that for $t \in [T, \infty)$, $B_\delta \cap \left\{ D_t \begin{pmatrix} 1 & 0_{n-1\times 1} \\ \mathbf{x} & I_{n-1\times n-1} \end{pmatrix} \mathrm{SL}_n(\mathbb{Z}) \right\} \neq \{0\}$.

If alternatively $\mathbf{x}$ is not singular, then there exists $\epsilon > 0$ and a sequence Q_i such that for infinitely many $q \in \mathbb{N}$ and $\mathbf{p} \in \mathbb{Z}^{n-1}$

$$\|q\mathbf{x} - \mathbf{p}\| > \frac{\epsilon}{Q_i^{1/(n-1)}} \qquad\qquad q \leq Q_i$$

Construct a sequence t_i such that $Q_i < e^{t_i} < \frac{Q_i}{\epsilon}$. Then we see that for $q \leq Q_i$,

$$e^{t_i/(n-1)} \|q\mathbf{x} - \mathbf{p}\| > e^{t_i/(n-1)} \frac{\epsilon}{Q_i^{1/(n-1)}} > \epsilon$$

and for $q > Q_i$,

$$e^{-t_i} q > e^{-t_i} Q_i > \epsilon$$

Thus, $B_\epsilon \cap \left\{ D_t \begin{pmatrix} 1 & 0_{n-1\times 1} \\ \mathbf{x} & I_{n-1\times n-1} \end{pmatrix} \mathrm{SL}_n(\mathbb{Z}) \right\} = \{0\}$, implying that the trajectory is not divergent.

□

Corollary 3.1. *Given $\Lambda \in \mathscr{L}_n$ such that $\{D_t\Lambda\}$ is divergent, that same trajectory is degenerate as well if and only if $\Lambda \cap \{\mathbf{v} \in \mathbb{R}^n : D_t(\mathbf{v}) \to 0 \text{ as } t \to \infty\} \neq \{0\}$. If every nonzero element of Λ is of the form $\sum p_i e_i$ where e_i are the standard basis vectors in $\mathbb{R}^n$ and $p_n \neq 0$, then for any $x^1, \ldots, x^{n-1} \in \Lambda$, $x^1 \wedge \cdots \wedge x^{n-1}$ is not in $\left\{ \mathbf{v}^1 \wedge \cdots \wedge \mathbf{v}^{n-1} \in \bigwedge^{n-1} \mathbb{R}^n : \bigwedge^{n-1} D_t \left(\mathbf{v}^1 \wedge \cdots \wedge \mathbf{v}^{n-1}\right) \to 0 \text{ as } t \to \infty \right\}$ unless it is zero. In light of Theorem 3.1, we see that this only happens if $\mathbf{x}$ is not totally irrational.*

Theorem 3.2. *The only singular real numbers are the rationals.*

Proof. By Corollary 3.1 we know that $x \in \mathbb{Q}$ implies $x \in \mathrm{Sing}(1)$. So, we need only check irrational x. Assume there exists $x \in \mathbb{R} \setminus \mathbb{Q}$ singular and let $\frac{r_n}{s_n}$ for $r_n \in \mathbb{Z}$ and $s_n \in \mathbb{N}$ be a sequence of best approximations of x. That means that for a given large n there is no $s \in \mathbb{N}$ and $r \in \mathbb{Z}$ such that $1 \leq s < s_n$ and $|sx - r| < |s_n x - r_n|$. We also know that $|s_n x - r_n| > \frac{1}{2s_{n+1}}$. If we set $Q = s_{n+1}$, then since x is singular, $|s_n x - r_n| \leq \frac{\epsilon}{Q}$ for some $\epsilon > 0$. This implies that $\frac{\epsilon}{Q} > \frac{1}{2Q}$ or equivalently $\epsilon > \frac{1}{2}$. But if this holds for all n large, then x cannot be singular. □

The results of Theorem 3.2 do not extend to $\mathrm{Sing}(n)$ for $n \geq 2$, as we shall see. But first, some quick notation.

If $\mathbf{m} = (m_0, \ldots, m_n) \in \mathbb{Z}^{n+1}$ is a primitive vector, then we let $A_\mathbf{m}$ denote the hyperplane

$$A_\mathbf{m} := \left\{ x \in \mathbb{R}^n : \sum_{i=1}^{n} m_i x_i = m_0 \right\}$$

We also define

$$|A_\mathbf{m}| := \|m_1, \ldots, m_n\|$$

Let $\Phi : \mathbb{Z}^n \setminus \{0\} \to \mathbb{R}_+$ be a proper function, i.e.

$$\{\mathbf{q} \in \mathbb{Z}^n \setminus \{0\} : \Phi(q) \leq C\} \text{ is finite for any } C > 0$$

We also use $\langle x \rangle$ to be the distance from $x \in \mathbb{R}$ to the closest integer. This is equivalent to $|qx - p|$ without specifying particular q, p. If instead we were in the transferred case, $\langle \mathbf{q} \cdot \mathbf{x} \rangle$ for $\mathbf{q} \in \mathbb{Z}^n$ and $\mathbf{x} \in \mathbb{R}^n$ is equivalent to $|\mathbf{q} \cdot \mathbf{x} - p|$ without specifying p.

Given Φ, we define the ***irrationality measure***

$$\psi_{\Phi,\mathbf{x}}(Q) = \min_{\mathbf{q} \in \mathbb{Z}^n \setminus \{0\}, \Phi(\mathbf{q}) \leq Q} \langle \mathbf{q} \cdot \mathbf{x} \rangle$$

Note that in the case $\Phi = \|\cdot\|$,

$$\hat{\omega}^*(\mathbf{x}) = \sup \left\{ \gamma : \limsup_{Q \to \infty} Q^\gamma \psi_{\|\cdot\|,\mathbf{x}}(Q) < \infty \right\}$$

With these tools, we are now ready to prove the foundational theorem of this paper.

Theorem 3.3. *Let $S \subset \mathbb{R}^n$ be a nonempty locally closed subset, let $\mathcal{L} = \{L_1, L_2, \ldots\}$ and $\mathcal{L}' = \{L'_1, L'_2, \ldots\}$ be disjoint collections of distinct closed subsets of S, each of which is contained in a rational affine hyperplane in $\mathbb{R}^n$, and for each i let A_i be a rational affine hyperplane containing L_i. Assume the following hold:*

a) $\bigcup_i L_i \cup \bigcup_j L'_j = \{\mathbf{x} \in S : \mathbf{x}$ *is contained in a rational affine hyperplane*$\}$

b) For each i, $T > 0$, $L_i = \overline{\bigcup_{|A_j|>T} L_i \cap L_j}$

c) For each i *and for any finite subsets of indices* F, F' *with* $i \notin F$, $L_i = \overline{L_i \setminus \left(\bigcup_{k\in F} L_k \cup \bigcup_{k'\in F'} L'_{k'}\right)}$

d) $\bigcup_i L_i$ *is dense in* S

Then for an arbitrary proper function $\Phi : \mathbb{Z}^n / \{0\} \to \mathbb{R}_+$ and any non-increasing function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$, there exists uncountably many totally irrational $\mathbf{x} \in S$ such that $\psi_{\Phi,\mathbf{x}}(Q) \leq \varphi(Q)$ for all large enough Q.

Proof. Let

$$\mathcal{B} := \{\mathbf{x} \in S : \exists Q_0 \text{ s.t. } \forall Q \geq Q_0, \psi_{\Phi,\mathbf{x}}(Q) \leq \varphi(Q) \text{ and } \mathbf{x} \text{ is totally irrational}\}$$

Suppose for the sake of contradiction that $\mathcal{B}$ is at most countably infinite. We can then write $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots\}$ (in the case that $\mathcal{B}$ is finite, then this is just a finite list). Let $\mathcal{W}$ be an open subset of $\mathbb{R}^n$ such that $S = \overline{S} \cap \mathcal{W}$. Put $\mathcal{U}_0 = \mathcal{W}$, $\mathbf{q}_0 = 0$, $p_0 = 0$, $i_0 = 0$, and $\Phi(0) = 0$. We will see that for each $\nu \in \mathbb{N}$ there is a bounded open set $\mathcal{U}_\nu \subseteq \mathcal{W}$ and an index $i_\nu \in \mathbb{N}$ such that with the notation $(p_\nu, \mathbf{q}_\nu) = \mathbf{m}_{i_\nu}$ the following conditions are satisfied:

1) $\emptyset \neq \overline{S \cap \mathcal{U}_\nu} \subset \mathcal{U}_{\nu-1}$
2) $i_\nu > i_{\nu-1}$, $\Phi(\mathbf{q}_\nu) > \Phi(\mathbf{q}_{\nu-1})$ for all $\nu \in \mathbb{N}$
3) For all $k < \nu$, $\mathcal{U}_\nu$ is disjoint from $L_k \cup L'_k \cup \{\mathbf{b}_k\}$
4) For all $\nu \in \mathbb{N}$ and $\mathbf{x} \in \mathcal{U}_\nu$ we have
$$|\mathbf{q}_{\nu-1} \cdot \mathbf{x} - p_{\nu-1}| < \varphi(\Phi(\mathbf{q}_\nu))$$
5) For all $\nu \in \mathbb{N}$, $\mathcal{U}_\nu \cap L_{i_\nu} \neq \emptyset$

To see this suffices, take a point

$$\mathbf{x} \in S \cap \bigcap_\nu \mathcal{U}_\nu = \bigcap_\nu \overline{S \cap \mathcal{U}_\nu}$$

This intersection is nonempty since the right-hand side is by condition 1) an intersection of nonempty nested compact sets, and the equality follows that for $\nu \geq 2$, the sets $\overline{\mathcal{U}_\nu}$ are contained in $\mathcal{W}$. We will reach a contradiction by showing that both $\mathbf{x} \in \mathcal{B}$ and $\mathbf{x} \notin \mathcal{B}$. By condition 3), $\mathbf{x}$ is not equal to any of the $\mathbf{b}_i$, and hence $\mathbf{x} \notin \mathcal{B}$. Also by condition 3), $\mathbf{x}$ is not contained in any of the sets in the collections $\mathcal{L}, \mathcal{L}'$ and thus by hypothesis a) $\mathbf{x}$ is totally irrational. The function $\varphi(Q)$ is non-increasing by assumption, and so is the irrationality measure function $Q \mapsto \psi_{\Phi,\mathbf{x}}(Q)$, as follows from its definition. The properness condition guarantees that $\Phi(\mathbf{q}_\nu) \to \infty$ as $\nu \to \infty$. By condition 2), for any $Q \geq Q_0 := \Phi(\mathbf{q}_1)$, there is ν with $Q \in [\Phi(\mathbf{q}_\nu), \Phi(\mathbf{q}_{\nu+1})]$ and by condition 4) we have

$$\psi_{\Phi,\mathbf{x}}(Q) \leq \psi_{\Phi,\mathbf{x}}(\Phi(\mathbf{q}_\nu)) \leq \langle \mathbf{q}_\nu \cdot \mathbf{x} \rangle \leq |\mathbf{q}_\nu \cdot \mathbf{x} - p_\nu| < \varphi(\Phi(\mathbf{q}_{\nu+1})) \leq \varphi(Q)$$

which shows that $\mathbf{x} \in \mathcal{B}$.

Now we shall construct our sequences such that conditions 1) through 5) hold. Let $\nu = 1$. Choose $i_1 := \min\{i \in \mathbb{N} : L_i \neq \emptyset\}$, which must exist by hypothesis d). Define $\mathcal{U}_1$ to be some open set containing a point in L_{i_1} such that $\overline{\mathcal{U}_1} \subset \mathcal{W}$. We see that conditions 1), 3), and 5) follow immediately from this choice. If $p_1, \mathbf{q}_1$ are the elements of the primitive vector corresponding to A_{i_1}, then conditions 2) and 4) hold as well.

Now suppose we have constructed $\mathcal{U}_k, i_k$ for $k = 1, \ldots, \nu$. Let $i = i_\nu$. By condition 5), for $k = \nu$ we have $\mathcal{U}_k \cap L_i \neq \emptyset$. By hypothesis b), there is an infinite subsequence of indices j such that $|A_j| \to \infty$ as $j \to \infty$ and $\mathcal{U}_k \cap L_i \cap L_j \neq \emptyset$. For each such j, let $A_j = A_{\mathbf{m}_j}$ with $\mathbf{m}_j = \left(p'_j, \mathbf{q}'_j\right)$. Then by the definition of $|A_j|$, along this subsequence $\left\|\mathbf{q}'_j\right\| \to \infty$. Thus, by the properness of Φ, we can choose $j > i$ such that $\Phi\left(\mathbf{q}'_j\right) > \Phi\left(\mathbf{q}_\nu\right)$. If we set $i_{\nu+1} := j$, this ensures that condition 2) holds for $\nu + 1$.

Next, let $\mathbf{x}' \in \mathcal{U}_\nu \cap L_i \cap L_j$. Since $\mathbf{x}' \in L_i$, we see that $\mathbf{q}_\nu \cdot \mathbf{x}' = p_\nu$. By continuity, we choose a small neighborhood $\mathcal{V} \subset \mathcal{U}_\nu$ of $\mathbf{x}'$ such that for all $\mathbf{x} \in \mathcal{V}$

$$|\mathbf{q}_\nu \cdot \mathbf{x} - p_\nu| < \psi\left(\Phi\left(\mathbf{q}_{\nu+1}\right)\right)$$

Thus, condition 4) holds for $\nu + 1$.

Since $\mathbf{x}' \in L_j$, we must have $\mathcal{V} \cap L_{i_{\nu+1}} \neq \emptyset$. Thus by hypothesis c), there exists $\mathbf{x}''$ such that

$$\mathbf{x}'' \in L_j \cap \mathcal{V} \setminus \bigcup_{k<\nu+1} \left(L_k \cup L'_{k'} \cup \{\mathbf{b}_k\}\right)$$

Further, we can take a neighborhood $\mathcal{U}_{\nu+1}$ of $\mathbf{x}''$ such that $\overline{\mathcal{U}_{\nu+1}} \subset \mathcal{U}_\nu$ and

$$\mathcal{U}_{\nu+1} \cap \bigcup_{k<\nu+1} \left(L_k \cup L'_{k'} \cup \{\mathbf{b}_k\}\right) = \emptyset$$

Consequently, conditions 1), 3), and 5) now hold. Thus concludes the induction and this proof.

□

Corollary 3.2. *For $S \subset \mathbb{R}^n$ $(n \geq 2)$ and $\mathcal{L}, \mathcal{L}'$ satisfying Theorem 3.3, if we choose $\Phi = \|\cdot\|$ and $\varphi(Q) = \frac{1}{Q^n}$, there exists an uncountable number of totally-irrational vectors $\mathbf{x} \in \mathrm{Sing}(n)$.*

4 Expanding the Trajectories

4.1 Maximal Diagonal Trajectories

We are now ready to explore generalizations of this work to a n-parameter driving matrix:

$$D_{\mathbf{t}} = \mathrm{diag}\left(e^{\sum_{i=1}^n t_i}, e^{-t_1}, \ldots, e^{-t_n}\right)$$

where the t_i are taken to be independent of each other. Based on our previous work, we should expect (and will soon prove) that there is a relationship between this driving matrix and singular vectors, for some new definition of singular. But to redefine singular, we must first have a new version of Theorem 1.1:

Theorem 4.1. *For any real vector $\mathbf{x} \in \mathbb{R}^n$ and any $Q_1, \ldots, Q_n \in \mathbb{N}$, there exists $p \in \mathbb{Z}$ and $q_1, \ldots, q_n \in \mathbb{Z}$ such that*

$$\left|-p + \sum_{i=1}^n q_i x_i\right| \leq \prod_{i=1}^n \frac{1}{Q_i} \qquad |q_1| \leq Q_1, \ldots, |q_n| \leq Q_n \tag{1}$$

Proof. Suppose $Q_1 = \ldots = Q_n = 1$. Then we can apply Theorem 1.2 and we are done.

Now assume some $Q_i \neq 1$. Construct the set

$$S = \left\{(q_1, \ldots, q_n, p) \in \mathbb{R}^{n+1} : -Q_i - \frac{1}{2} \leq q_i \leq Q_i + \frac{1}{2} \text{ for } i = 1, \ldots, n, \left|-p + \sum_{i=1}^n q_i x_i\right| \leq \prod_{i=1}^n \frac{1}{Q_i}\right\}$$

The volume of S is greater than 2^{n+1}. Thus, by Minkowski's Theorem, there is a non-trivial point in S with integral coordinates. Since S is symmetric, we choose the point such that one of $q_1, \ldots, q_n > 0$. Thus, for this point, which we shall label $(q_1, \ldots, q_n, p)$, we see that $q_i \leq Q_i$, and

$$\left|-p + \sum_{i=1}^n q_i x_i\right| \leq \prod_{i=1}^n \frac{1}{Q_i}$$

□

Remark 4.1. *The proof of Theorem 4.1 shows that if $Q_j > 1$ for some $j = 1, \ldots, n$ then one of the q_is is strictly positive.*

As highlighted in the proof of Theorem 4.1, this setup looks very similar Theorem 1.2: If we set $\mathbf{q} = (q_1, \ldots, q_n) \in \mathbb{Z}^n$, then Theorem 4.1 becomes

Theorem 4.2. *For any real vector $\mathbf{x} \in \mathbb{R}^2$ and any $Q_1, \ldots, Q_n \in \mathbb{N}$, there exists $p \in \mathbb{Z}$ and $\mathbf{q} \in \mathbb{Z}^n$ such that*

$$|\mathbf{q} \cdot \mathbf{x} + p| \leq \prod_{i=1}^{n} \frac{1}{Q_i} \qquad |q_1| \leq Q_1, \ldots, |q_n| \leq Q_n$$

Definition 4.1. *The collection $(x_1, \ldots, x_n)$ is **singular** if for any $\epsilon > 0$, there exists $(Q_1)_\epsilon, \ldots, (Q_n)_\epsilon \geq 0$ such that for all $Q_i \geq (Q_i)_\epsilon$ for $i = 1, \ldots, n$, there are infinitely many $q_1, \ldots, q_n \in \mathbb{N}$ and $p \in \mathbb{Z}$ such that*

$$\left| -p + \sum_{i=1}^{n} q_i x_i \right| \leq \frac{\epsilon}{\prod_{i=1}^{n} Q_i} \qquad q_1 \leq Q_1, \ldots, q_n \leq Q_n$$

Now having properly redefined what it means for a vector to be singular, we can now show the relationship between this new definition and our new driving matrix:

Theorem 4.3. *Let $\mathbf{x} = (x_1, \ldots, x_n)$. The trajectory $\left\{ D_{\mathbf{t}} \begin{pmatrix} 1 & \mathbf{x} \\ 0_{1\times n} & I_{n\times n} \end{pmatrix} \mathrm{SL}_{n+1}(\mathbb{Z}) \right\}$ diverges if any only if the collection $(x_1, \ldots, x_n)$ is singular.*

Note that the middle matrix is transposed to the way it was presented in Section 2. This is merely a change in convention and still allows us to apply the results from Sections 2 and 3 to this new trajectory.

Proof. Assume that the collection $(x_1, \ldots, x_n)$ is singular. Then for any $\delta > 0$, let B_δ be the ball of radius δ centered at the origin in $\mathbb{R}^{n+1}$ with respect to $\|\cdot\|$. By Proposition 2.1, it is enough to show that for any $\delta > 0$, there exists $T_1 \ldots, T_n \geq 0$ such that for all $t_i \geq T_i$ for $i = 1, \ldots, n$, $B_\delta \cap \left\{ D_{\mathbf{t}} \begin{pmatrix} 1 & \mathbf{x} \\ 0_{1\times n} & I_{n\times n} \end{pmatrix} \mathrm{SL}_{n+1}(\mathbb{Z}) \right\} \neq \{0\}$.

Now fix $\delta \in (0, 1)$ and choose $\epsilon < \delta^{n+1}$. Since the collection $(x_1, \ldots, x_n)$ is singular, we know there exists $(Q_1)_\epsilon, \ldots, (Q_n)_\epsilon$ such that for all $Q_i \geq (Q_i)_\epsilon$ for $i = 1, \ldots, n$, there exists infinitely many $q_1, \ldots, q_n \in \mathbb{N}$ and $p \in \mathbb{Z}^n$ such that

$$\left| -p + \sum_{i=1}^{n} q_i x_i \right| \leq \frac{\epsilon}{\prod_{i=1}^{n} Q_i} \qquad q_1 \leq Q_1, \ldots, q_n \leq Q_n$$

Let $q_1, \ldots, q_n, p$ satisfy the above conditions. We see that

$$\begin{aligned}
\left\| D_{\mathbf{t}} \begin{pmatrix} 1 & \mathbf{x} \\ 0_{1\times n} & I_{n\times n} \end{pmatrix} \mathrm{SL}_{n+1}(\mathbb{Z}) \right\| &= \left\| \mathrm{diag}\left(e^{\sum_{i=1}^n t_i}, e^{-t_1}, \ldots, e^{-t_n} \right) \begin{pmatrix} 1 & \mathbf{x} \\ 0_{1\times n} & I_{n\times n} \end{pmatrix} \begin{pmatrix} -p \\ q_1 \\ \vdots \\ q_n \end{pmatrix} \right\| \\
&= \left\| \begin{pmatrix} e^{\sum_{i=1}^n t_i} \left(-p + \sum_{i=1}^n q_i x_i \right) \\ e^{-t_1} q_1 \\ \vdots \\ e^{-t_n} q_n \end{pmatrix} \right\| \\
&= \max\left(\left| e^{\sum_{i=1}^n t_i} \left(-p + \sum_{i=1}^{n} q_i x_i \right) \right|, \left| e^{-t_1} q_1 \right|, \ldots, \left| e^{-t_n} q_n \right| \right) \\
&\leq \max\left(\frac{\epsilon e^{\sum_{i=1}^n t_i}}{\prod_{i=1}^n Q_i}, e^{-t_1} Q_1, \ldots, e^{-t_n} Q_n \right)
\end{aligned}$$

If we want to bound the last inequality above by δ, we see that $(t_1, \ldots, t_n)$ must lie within the simplex with vertices $\left(\ln \frac{Q_1}{\delta}, \ldots, \ln \frac{Q_n}{\delta}\right)$ and $\left(\ln \frac{Q_1}{\delta}, \ldots, \ln \frac{Q_{i-1}}{\delta}, \ln \frac{\delta^n Q_i}{\epsilon}, \ln \frac{Q_{i+1}}{\delta}, \ldots, \ln \frac{Q_n}{\delta}\right)$ for $i = 1, \ldots, n$, which we shall denote as $\Delta_{Q_1,\ldots,Q_n}$. For $Q_1, \ldots Q_n$ large, we see that since $\epsilon < \delta^{n+1}$, $\Delta_{Q_1,\ldots,Q_n}$ overlaps with $\Delta_{Q_1,\ldots,Q_{i-1},Q_i+1,Q_{i+1},\ldots,Q_n}$ for all $i = 1, \ldots n$. Thus, the subset of $\mathbb{R}^n$ $\bigcup \Delta_{Q_1,\ldots,Q_n}$ has diverging trajectory with initial point $(T_1, \ldots, T_n)$ such that for $(t_1, \ldots, t_n)$ along this trajectory, $B_\delta \cap \left\{ D_{\mathbf{t}} \begin{pmatrix} 1 & \mathbf{x} \\ 0_{1\times n} & I_{n\times n} \end{pmatrix} \mathrm{SL}_{n+1}(\mathbb{Z}) \right\}$ has a non-zero element.

If alternatively the collection $(x_1, \ldots, x_n)$ is not singular, then there exists $\epsilon > 0$ and sequences $(Q_1)_j, \ldots, (Q_n)_j$ such that for infinitely many $q_1, \ldots, q_n \in \mathbb{N}$ and $p \in \mathbb{Z}$

$$\left| -p + \sum_{i=1}^{n} q_i x_i \right| > \frac{\epsilon}{\prod_{i=1}^{n} (Q_i)_j} \qquad\qquad q_1 \leq (Q_1)_j, \ldots, q_n \leq (Q_n)_j$$

Construct sequence $(t_1)_j, \ldots, (t_n)_j$ such that $(Q_i)_j < e^{(t_i)_j} < \frac{(Q_i)_j}{\epsilon}$ for all $i = 1, \ldots, n$. Then we see that for a given $i \in \{1, \ldots, n\}$, $q_i \leq (Q_i)_j$,

$$e^{\sum_{i=1}^{n} (t_i)_j} \left| -p + \sum_{i=1}^{n} q_i x_i \right| > e^{\sum_{i=1}^{n} (t_i)_j} \frac{\epsilon}{\prod_{i=1}^{n} (Q_i)_j} > \epsilon$$

and for $q_i > (Q_i)_j$,

$$e^{-(t_i)_j} q > e^{-(t_i)_j} (Q_i)_j > \epsilon$$

Thus, $B_\epsilon \cap \left\{ D_{\mathbf{t}} \begin{pmatrix} 1 & \mathbf{x} \\ 0_{1\times n} & I_{n\times n} \end{pmatrix} \mathrm{SL}_{n+1}(\mathbb{Z}) \right\} = \{0\}$, implying that the trajectory is not divergent.

□

We can further generalize our work from the previous section to show that there exists an uncountable number of totally irrational singular vectors.

Theorem 4.4. *Let $S \subset \mathbb{R}^2$ be a nonempty locally closed subset, let $\mathcal{L} = \{L_1, L_2, \ldots\}$ and $\mathcal{L}' = \{L'_1, L'_2, \ldots\}$ be disjoint collections of distinct closed subsets of S, each of which is contained in a rational affine hyperplane in $\mathbb{R}^2$, and for each i let A_i be a rational affine hyperplane containing L_i. Assume the following hold:*

a) $\bigcup_i L_i \cup \bigcup_j L'_j = \{\mathbf{x} \in S : \mathbf{x}$ is contained in a rational affine hyperplane$\}$

b) For each i, $T > 0$, $L_i = \overline{\bigcup_{|A_j|>T} L_i \cap L_j}$

c) For each i and for any finite subsets of indices F, F' with $i \notin F$, $L_i = \overline{L_i \setminus \left(\bigcup_{k\in F} L_k \cup \bigcup_{k'\in F'} L'_{k'}\right)}$

d) $\bigcup_i L_i$ is dense in S

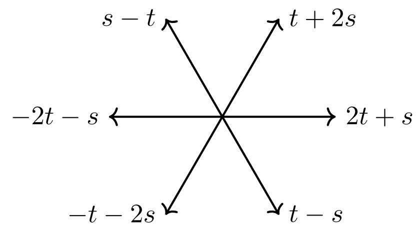
Then for arbitrary proper functions $\Phi_1, \Phi_2 : \mathbb{Z}^n/\{0\} \to \mathbb{R}_+$ and any non-increasing function $\varphi : \mathbb{R}_+ \times R_+ \to \mathbb{R}_+$, there exists uncountably many totally irrational pairs $(x, y) \in S$ such that

$$\psi_{\Phi_1,\Phi_2,x,y}(Q, R) := \min_{q,r\in\mathbb{Z}^2\setminus\{0\},\Phi_1(q)\leq Q,\Phi_2(r)\leq R} \langle qx + ry \rangle \leq \varphi(Q, R)$$

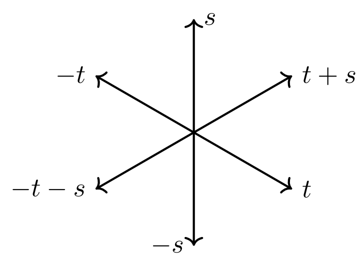
for all large Q, R.

Remark 4.2. *We know by Theorem 4.3 that singularity is related to $D_{\mathbf{t}}$. Importantly, the dimension of $\{D_{\mathbf{t}} : \mathbf{t} \in \mathbb{R}^n\}$ is the same as the real rank of $SL_{n+1}(\mathbb{R})$, which is also equal to the rational rank of $SL_{n+1}(\mathbb{R})$. Due to results in [4], we should expect there to be regions of the root system for which there are no divergent trajectories. Thus, some work must be done to show that our choice of S and $\mathcal{L}, \mathcal{L}'$ guarantees that we are not in those corresponding regions.*

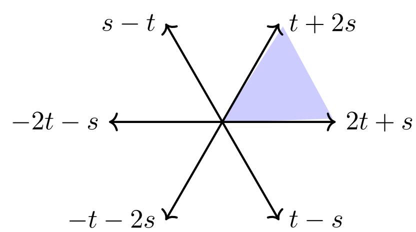
Proof. Let $t := t_1$ and $s := t_2$ (for ease of readability). We find the root system corresponding to $D_{\mathbf{t}}$ to be



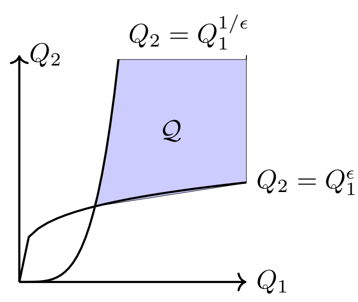
Thus the orthogonal root system is



For reasons that will be clear soon, let us work in the region of the root system such that $s > \epsilon t$ and $t > \epsilon s$ for some $\epsilon \in (0, 1)$:



We also know from Theorem 4.3 that there is a relationship between t, s and Q_1, Q_2. For given large $Q_1, Q_2 \in \mathbb{N}$ and $\delta \in (0, 1)$, we know that the trajectory of t, s can be taken to pass through the point $\left(\ln \frac{Q_1}{\delta}, \ln \frac{Q_2}{\delta}\right)$. This implies the relation $e^{-t} Q_1 = e^{-s} Q_2$. But our region in the root system implies that $Q_1^{\epsilon} < Q_2 < Q_1^{1/\epsilon}$, as shown in the region below:



Importantly, this shaded region is infinite and has a boundary well-suited for this proof. We shall denote this region as $\mathcal{Q}$.

Let

$$\mathcal{B} := \{(x, y) \in S : \exists Q_0, R_0 \text{ s.t. } \forall Q \geq Q_0, R \geq R_0, \psi_{\Phi_1, \Phi_2, x, y}(Q, R) \leq \varphi(Q, R) \text{ and } (x, y) \text{ is totally irrational}\}$$

Suppose for the sake of contradiction that $\mathcal{B}$ is at most countably infinite. We can then write $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots\}$ (in the case that $\mathcal{B}$ is finite, then this is just a finite list). Let $\mathcal{W}$ be an open subset of $\mathbb{R}^2$ such that $S = \overline{S} \cap \mathcal{W}$. Put $\mathcal{U}_0 = \mathcal{W}$, $q_0 = 0$, $r_0 = 0$, $p_0 = 0$, $i_0 = 0$, $\Phi_1(0) = 0$, and $\Phi_2(0) = 0$. We will see that for each $\nu \in \mathbb{N}$ there is a bounded open set $\mathcal{U}_\nu \subseteq \mathcal{W}$ and an index $i_\nu \in \mathbb{N}$ such that with the notation $(p_\nu, q_\nu, r_\nu) = \mathbf{m}_{i_\nu}$ the following conditions are satisfied:

1) $\emptyset \neq \overline{S \cap \mathcal{U}_\nu} \subset \mathcal{U}_{\nu-1}$

2) $i_\nu > i_{\nu-1}$, $\Phi_1(q_\nu) > \Phi_1(q_{\nu-1})$, $\Phi_2(r_\nu) > \Phi_2(r_{\nu-1})$ for all $\nu \in \mathbb{N}$

3) For all $k < \nu$, $\mathcal{U}_\nu$ is disjoint from $L_k \cup L'_k \cup \{\mathbf{b}_k\}$

4) For all $\nu \in \mathbb{N}$ and $(x, y) \in \mathcal{U}_\nu$ we have

$$|xq_{\nu-1} + yr_{\nu-1} - p_{n-1}| < \varphi(\Phi_1(q_\nu), \Phi_2(r_\nu))$$

5) For all $\nu \in \mathbb{N}$, $\mathcal{U}_\nu \cap L_{i_\nu} \neq \emptyset$

To see this suffices, take a point

$$(x, y) \in S \cap \bigcap_\nu \mathcal{U}_\nu = \bigcap_\nu \overline{S \cap \mathcal{U}_\nu}$$

This intersection is nonempty since the right-hand side is by condition 1) an intersection of nonempty nested compact sets, and the equality follows that for $\nu \geq 2$, the sets $\overline{\mathcal{U}_\nu}$ are contained in $\mathcal{W}$. We will reach a contradiction by showing that both $(x, y) \in \mathcal{B}$ and $(x, y) \notin \mathcal{B}$. By condition 3), (x, y) is not equal to any of the $\mathbf{b}_i$, and hence $(x, y) \notin \mathcal{B}$. Also by condition 3), (x, y) is not contained in any of the sets in the collection $\{L_1, \ldots\}, \{L'_1, \ldots\}$ and thus by hypothesis a) (x, y) is totally irrational. The function $\varphi(Q, R)$ is non-increasing in both parameters by assumption, and so is the irrationality measure function $(Q, R) \mapsto \psi_{\Phi_1, \Phi_2, x, y}(Q, R)$, as follows from its definition. The properness condition guarantees that $\Phi_1(q_\nu) \to \infty, \Phi_2(r_\nu) \to \infty$ as $\nu \to \infty$. By condition 2), for any $Q \geq Q_0 := \Phi_1(q_1)$, there is ν with $Q \in [\Phi_1(q_\nu), \Phi_1(q_{\nu+1})]$ and there exists $R \in [\Phi_2(r_\nu), \Phi_2(r_{\nu+1})]$. By condition 4) for those Q, R we have

$$\begin{aligned}
\psi_{\Phi_1, \Phi_2, x, y}(Q, R) &\leq \psi_{\Phi_1, \Phi_2, x, y}(\Phi_1(q_\nu), \Phi_2(r_\nu)) \\
&\leq \langle q_\nu x + r_\nu y \rangle \\
&\leq |q_\nu x + r_\nu y - p_\nu| \\
&< \varphi(\Phi_1(q_{\nu+1}), \Phi_2(r_{\nu+1})) \\
&\leq \varphi(Q, R)
\end{aligned}$$

which shows that $(x, y) \in \mathcal{B}$.

Now we shall construct our sequences such that conditions 1) through 5) hold. Let $\nu = 1$. Choose $i_1 := \min\{i \in \mathbb{N} : L_i \neq \emptyset, (q_i, r_i) \in \mathcal{Q}\}$, which must exist by hypothesis d). Define $\mathcal{U}_1$ to be some open set containing a point in L_{i_1} such that $\overline{\mathcal{U}_1} \subset \mathcal{W}$. We see that conditions 1) through 5) follow immediately from this choice.

Now suppose we have constructed $\mathcal{U}_k, i_k$ for $k = 1, \ldots, \nu$. Let $i = i_\nu$. By condition 5), for $k = \nu$ we have $\mathcal{U}_k \cap L_i \neq \emptyset$. By hypothesis b) and d), there is an infinite subsequence of indices j such that $|A_j| \to \infty$ as $j \to \infty$, $\mathcal{U}_k \cap L_i \cap L_j \neq \emptyset$, and $(q_j, r_j) \in \mathcal{Q}$. For each such j, let $A_j = A_{\mathbf{m}_j}$ with $\mathbf{m}_j = \left(p'_j, q'_j, r'_j\right)$. Then by the definition of $|A_j|$, along this subsequence $q'_j \to \infty$ or $r'_j \to \infty$. Thus, by the properness of Φ_1, Φ_2 and the geometry of $\mathcal{Q}$, we can choose $j > i$ such that $\Phi_1\left(q'_j\right) > \Phi_1(q_\nu)$ and $\Phi_2\left(r'_j\right) > \Phi_2(r_\nu)$. If we set $i_{\nu+1} := j$, this ensures that condition 2) holds for $\nu + 1$.

Next, let $(x', y') \in \mathcal{U}_\nu \cap L_i \cap L_j$. Since $(x', y') \in L_i$, we see that $q_\nu x' + r_\nu y' = p_\nu$. By continuity, we choose a small neighborhood $\mathcal{V} \subset \mathcal{U}_\nu$ of (x', y') such that for all $(x, y) \in \mathcal{V}$

$$|q_\nu x + r_\nu y - p_\nu| < \psi(\Phi_1(q_{\nu+1}), \Phi_2(r_{\nu+1}))$$

Thus, condition 4) holds for $\nu + 1$.

Since $(x', y') \in L_j$, we must have $\mathcal{V} \cap L_{i_{\nu+1}} \neq \emptyset$. Thus by hypothesis c), there exists (x'', y'') such that

$$(x'', y'') \in L_j \cap \mathcal{V} \setminus \bigcup_{k<\nu+1} (L_k \cup L'_{k'} \cup \{\mathbf{b}_k\})$$

Further, we can take a neighborhood $\mathcal{U}_{\nu+1}$ of (x'', y'') such that $\overline{\mathcal{U}_{\nu+1}} \subset \mathcal{U}_\nu$ and

$$\mathcal{U}_{\nu+1} \cap \bigcup_{k<\nu+1} (L_k \cup L'_{k'} \cup \{\mathbf{b}_k\}) = \emptyset$$

Consequently, conditions 1), 3), and 5) now hold. Thus concludes the induction and this proof.

□

Corollary 4.1. *For S, $\mathcal{L}$, $\mathcal{L}'$, and A_i satisfying Theorem 4.4, if we choose $\Phi_1 = \Phi_2 = \|\cdot\|$ and $\varphi(Q,R) = \frac{1}{QR}$, there exists an uncountable number of singular pairs (x, y) that are totally irrational.*

Remark 4.3. *If we took $\mathcal{Q}$ to be any larger, we would have a problem. In the limit $\delta \to 0$, we could have trajectories of (q, r) in $\mathcal{Q}$ such that no matter how much one parameter increased, the other could remain the same. This would prohibit us from evolving the q_ν, r_ν properly. Any region in our root system containing points outside of the Weyl Chamber bounded between $2t + s$ and $t + 2s$, we either find the same issues, find that one of t, s is negative, or we find that $\mathcal{Q}$ is finite if it exists at all. Of course, we could work in subsets of $\mathcal{Q}$, but $\mathcal{Q}$ is the maximal subset of the root system for which we can guarantee the existence of unaccountably many non-obvious singular vectors.*

Remark 4.4. *The ideas of Theorem 4.4 generalize to higher dimensions. However, the root system of these higher-dimensional driving matrices is much more complicated and this complication would detract from the main idea of the proof.*

4.2 Sub-maximal Diagonal Trajectories

In subsection 4.1 we assumed that the t_i were independent of each other. Now let us suppose that some t_i are identical. Let $a_1, \ldots, a_k \in \mathbb{N}$ such that $\sum_{i=1}^k a_i = n$ for $k \le n$. Then we construct the weighted driving matrix

$$D_{a_1,\ldots,a_k} := \operatorname{diag}\left(e^{\sum_{i=\ell}^k a_\ell t_\ell}, \underbrace{e^{-t_1}, \ldots, e^{-t_1}}_{a_1 \text{ times}}, \ldots, \underbrace{e^{-t_k}, \ldots, e^{-t_k}}_{a_k \text{ times}} \right)$$

Since our driving matrix is "submaximal", then we should also expect our assortment of Q_i to also be "submaximal". If we think about the case in Section 2, that was the case that $a_1 = n$ and no other a_1 exist. Similarly, $Q_1 = Q$ and no other Q_i exist. Thus, we must change what we mean for a vector to be singular.

Theorem 4.5. *For any real vector $\mathbf{x} \in \mathbb{R}^n$ and any $Q_1, \ldots, Q_k \in \mathbb{N}$, with $k \le n$ there exists $p \in \mathbb{Z}$ and $q_1, \ldots, q_n \in \mathbb{Z}$ such that*

$$\left| -p + \sum_{i=1}^n q_i x_i \right| \le \prod_{\ell=1}^k \frac{1}{Q_\ell^{a_\ell}} \qquad |q_1|, \ldots, |q_{a_1}| \le Q_1, \ldots, |q_{n+1-a_k}|, \ldots, |q_n| \le Q_k$$

The proof of Theorem 4.5 is so similar to its related proofs that we shall omit it. Theorem 4.5 naturally leads to the new definition of singular:

Definition 4.2. *The collection $(x_1, \ldots, x_n)$ is **singular** if for any $\epsilon > 0$, there exists $(Q_1)_\epsilon, \ldots, (Q_k)_\epsilon \ge 0$ for $k \le n$ such that for all $Q_\ell \ge (Q_\ell)_\epsilon$ for $\ell = 1, \ldots, k$, there are infinitely many $q_1, \ldots, q_n \in \mathbb{N}$ and $p \in \mathbb{Z}$ such that*

$$\left| -p + \sum_{i=1}^n q_i x_i \right| \le \frac{\epsilon}{\prod_{\ell=1}^k Q_\ell^{a_\ell}} \qquad q_1, \ldots, q_{a_1} \le Q_1, \ldots, q_{n+1-a_k}, \ldots, q_n \le Q_k$$

Now all that needs to be constructed is an anologue of Theorem 3.1.

Theorem 4.6. *Let* $\mathbf{x} = (x_1, \ldots, x_n)$*. The trajectory* $\left\{D_{a_1,\ldots,a_k}\begin{pmatrix}1 & \mathbf{x}\\ 0_{1\times n} & I_{n\times n}\end{pmatrix}\mathrm{SL}_{n+1}(\mathbb{Z})\right\}$ *diverges if any only if the collection* $(x_1, \ldots, x_n)$ *is singular.*

Proof. Assume that the collection $(x_1, \ldots, x_n)$ is singular. Then for any $\delta > 0$, let B_δ be the ball of radius δ centered at the origin in $\mathbb{R}^{n+1}$ with respect to $\|\cdot\|$. By Proposition 2.1, it is enough to show that for any $\delta > 0$, there exists $T_1 \ldots, T_k \geq 0$ such that for all $t_\ell \geq T_\ell$ for $\ell = 1, \ldots, k$, $B_\delta \cap \left\{D_{a_1,\ldots,a_k}\begin{pmatrix}1 & \mathbf{x}\\ 0_{1\times n} & I_{n\times n}\end{pmatrix}\mathrm{SL}_{n+1}(\mathbb{Z})\right\} \neq \{0\}$.

Now fix $\delta \in (0,1)$ and choose $\epsilon < \delta^{n+1}$. Since the collection $(x_1, \ldots, x_n)$ is singular, we know there exists $(Q_1)_\epsilon, \ldots, (Q_k)_\epsilon$ for $k \leq n$ such that for all $Q_\ell \geq (Q_\ell)_\epsilon$ for $\ell = 1, \ldots, k$, there exists infinitely many $q_1, \ldots, q_n \in \mathbb{N}$ and $p \in \mathbb{Z}^n$ such that

$$\left|-p + \sum_{i=1}^n q_i x_i\right| \leq \frac{\epsilon}{\prod_{\ell=1}^k Q_\ell^{a_\ell}} \qquad q_1, \ldots, q_{a_1} \leq Q_1, \ldots, q_{n+1-a_k}, \ldots, q_n \leq Q_k$$

Let $q_1, \ldots, q_n, p$ satisfy the above conditions. We see that

$$\begin{aligned}
\left\|D_{a_1,\ldots,a_k}\begin{pmatrix}1 & \mathbf{x}\\ 0_{1\times n} & I_{n\times n}\end{pmatrix}\mathrm{SL}_{n+1}(\mathbb{Z})\right\| &= \left\|\operatorname{diag}\left(e^{\sum_{\ell=1}^k a_\ell t_\ell}, \underbrace{t_1, \ldots, t_1}_{a_1 \text{ times}}, \ldots, \underbrace{t_k, \ldots, t_k}_{a_k \text{ times}}\right)\begin{pmatrix}1 & \mathbf{x}\\ 0_{1\times n} & I_{n\times n}\end{pmatrix}\begin{pmatrix}-p\\ q_1\\ \vdots\\ q_n\end{pmatrix}\right\| \\
&= \left\|\begin{pmatrix}e^{\sum_{\ell=1}^k a_\ell t_\ell}\left(-p + \sum_{i=1}^n q_i x_i\right)\\ e^{-t_1} q_1\\ \vdots\\ e^{-t_k} q_n\end{pmatrix}\right\| \\
&= \max\left(\left|e^{\sum_{\ell=1}^k a_\ell t_\ell}\left(-p + \sum_{i=1}^n q_i x_i\right)\right|, \left|e^{-t_1} q_1\right|, \ldots, \left|e^{-t_k} q_n\right|\right) \\
&\leq \max\left(\frac{\epsilon e^{\sum_{\ell=1}^k a_\ell t_\ell}}{\prod_{\ell=1}^k Q_\ell^{a_\ell}}, e^{-t_1} Q_1, \ldots, e^{-t_k} Q_k\right)
\end{aligned}$$

If we want to bound the last inequality above by δ, we see that $(t_1, \ldots, t_k)$ must lie within the simplex in $\mathbb{R}^k$ with vertices $\left(\ln\frac{Q_1}{\delta}, \ldots, \ln\frac{Q_k}{\delta}\right)$ and $\left(\ln\frac{Q_1}{\delta}, \ldots, \ln\frac{\delta^{(n+1-a_\ell)/a_\ell} Q_\ell}{\epsilon^{1/a_\ell}}, \ldots, \ln\frac{Q_k}{\delta}\right)$ for $\ell = 1, \ldots, k$, which we shall denote as $\Delta_{Q_1,\ldots,Q_k}$. For $Q_1, \ldots Q_k$ large, we see that since $\epsilon < \delta^{n+1}$, $\Delta_{Q_1,\ldots,Q_k}$ overlaps with $\Delta_{Q_1,\ldots,Q_{\ell-1},Q_\ell+1,Q_{\ell+1},\ldots,Q_k}$ for all $\ell = 1, \ldots, k$. Thus, the subset of $\mathbb{R}^k$ $\bigcup \Delta_{Q_1,\ldots,Q_k}$ has diverging trajectory with initial point $(T_1, \ldots, T_k)$ such that for $(t_1, \ldots, t_k)$ along this trajectory, $B_\delta \cap \left\{D_{a_1,\ldots,a_k}\begin{pmatrix}1 & \mathbf{x}\\ 0_{1\times n} & I_{n\times n}\end{pmatrix}\mathrm{SL}_{n+1}(\mathbb{Z})\right\} \neq \{0\}$.

If alternatively the collection $(x_1, \ldots, x_n)$ is not singular, then there exists $\epsilon > 0$ and sequences $(Q_1)_j, \ldots, (Q_k)_j$ such that for infinitely many $q_1, \ldots, q_n \in \mathbb{N}$ and $p \in \mathbb{Z}$

$$\left|-p + \sum_{i=1}^n q_i x_i\right| > \frac{\epsilon}{\prod_{\ell=1}^k (Q_\ell)_j^{a_\ell}} \qquad q_1, \ldots, q_{a_1} \leq (Q_1)_j, \ldots, q_{n+1-a_k}, \ldots, q_n \leq (Q_k)_j$$

Construct sequence $(t_1)_j, \ldots, (t_k)_j$ such that $(Q_\ell)_j < e^{(t_\ell)_j} < \frac{(Q_\ell)_j}{\epsilon}$ for all $\ell = 1, \ldots, k$. Then we see that for a given $i \in \{1, \ldots, n\}$, and corresponding $m \in (1, \ldots, k)$, if $q_i \leq (Q_m)_j$,

$$e^{\sum_{\ell=1}^{k} a_\ell (t_\ell)_j} \left| -p + \sum_{i=1}^{n} q_i x_i \right| > e^{\sum_{\ell=1}^{k} a_\ell (t_\ell)_j} \frac{\epsilon}{\prod_{\ell=1}^{k} (Q_\ell)_j^{a_\ell}} > \epsilon$$

and for $q_i > (Q_m)_j$,

$$e^{-(t_m)_j} q_i > e^{-(t_m)_j} (Q_m)_j > \epsilon$$

Thus, $B_\epsilon \cap \left\{ D_{a_1,\ldots,a_k} \begin{pmatrix} 1 & \mathbf{x} \\ 0_{1\times n} & I_{n\times n} \end{pmatrix} \mathrm{SL}_{n+1}(\mathbb{Z}) \right\} = \{0\}$, implying that the trajectory is not divergent. $\square$

We conclude this section with the generalization of our proof that uncountably many totally irrational singular vectors exist.

Theorem 4.7. *Let $S \subset \mathbb{R}^n$ be a nonempty locally closed subset, let $\mathcal{L} = \{L_1, L_2, \ldots\}$ and $\mathcal{L}' = \{L_1', L_2', \ldots\}$ be disjoint collections of distinct closed subsets of S, each of which is contained in a rational affine hyperplane in $\mathbb{R}^n$, and for each i let A_i be a rational affine hyperplane containing L_i. Assume the following hold:*

a) $\bigcup_i L_i \cup \bigcup_j L_j' = \{\mathbf{x} \in S : \mathbf{x}$ is contained in a rational affine hyperplane$\}$

b) For each i, $T > 0$, $L_i = \overline{\bigcup_{|A_j|>T} L_i \cap L_j}$

c) For each i and for any finite subsets of indices F, F' with $i \notin F$, $L_i = \overline{L_i \setminus \left(\bigcup_{m\in F} L_m \cup \bigcup_{m'\in F'} L_{m'}'\right)}$

d) $\bigcup_i L_i$ is dense in S

Then for arbitrary proper functions $\Phi_1, : \mathbb{Z}^{a_1}/\{0\} \to \mathbb{R}_+, \ldots, \Phi_k : \mathbb{Z}^{a_k}/\{0\} \to \mathbb{R}_+$, and any non-increasing function $\varphi : \mathbb{R}_+^k \to \mathbb{R}_+$, there exists uncountably many totally irrational $\mathbf{x} \in S$ such that for all large enough $Q_1, \ldots, Q_k$,

$$\psi_{\Phi_1,\ldots,\Phi_k,\mathbf{x}}(Q_1, \ldots, Q_k) := \min_{\substack{\mathbf{q}_1 \in \mathbb{Z}^{a_1}\setminus\{0\}, \ldots, \mathbf{q}_k \in \mathbb{Z}^{a_k}\setminus\{0\} \\ \Phi_1(\mathbf{q}_1) \leq Q_1, \ldots, \Phi_k(\mathbf{q}_k) \leq R_k}} \langle \mathbf{q} \cdot \mathbf{x} \rangle \leq \varphi(Q_1, \ldots, Q_k)$$

where $\mathbf{q} := (\mathbf{q}_1, \ldots, \mathbf{q}_k)$.

Proof. Let

$$\mathcal{B} := \left\{ \mathbf{x} \in S : \begin{array}{c} \exists (Q_1)_0, \ldots, (Q_k)_0 \text{ s.t. } \forall Q_\ell \geq (Q_\ell)_0 \text{ for } \ell = 1, \ldots, k, \\ \psi_{\Phi_1,\ldots,\Phi_k,\mathbf{x}}(Q_1, \ldots, Q_k) \leq \varphi(Q_1, \ldots, Q_k) \text{ and } \mathbf{x} \text{ is totally irrational} \end{array} \right\}$$

Suppose for the sake of contradiction that $\mathcal{B}$ is at most countably infinite. We can then write $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots\}$ (in the case that $\mathcal{B}$ is finite, then this is just a finite list). Let $\mathcal{W}$ be an open subset of $\mathbb{R}^n$ such that $S = \overline{S} \cap \mathcal{W}$. Put $\mathcal{U}_0 = \mathcal{W}$, $(\mathbf{q}_1)_0 = 0, \ldots, (\mathbf{q}_k)_0 = 0$, $p_0 = 0$, $i_0 = 0$, and $\Phi_\ell(0) = 0$ for $\ell = 1, \ldots, k$. We will see that for each $\nu \in \mathbb{N}$ there is a bounded open set $\mathcal{U}_\nu \subseteq \mathcal{W}$ and an index $i_\nu \in \mathbb{N}$ such that with the notation $(p_\nu, (\mathbf{q_1})_\nu, \ldots, (\mathbf{q_k})_\nu) = (p_\nu, \mathbf{q}_\nu) = \mathbf{m}_{i_\nu}$ the following conditions are satisfied:

1) $\emptyset \neq \overline{S \cap \mathcal{U}_\nu} \subset \mathcal{U}_{\nu-1}$

2) $i_\nu > i_{\nu-1}$, $\Phi_\ell((\mathbf{q}_\ell)_\nu) > \Phi_\ell\left((\mathbf{q}_\ell)_{\nu-1}\right)$ for all $\nu \in \mathbb{N}$ and $\ell = 1, \ldots k$

3) For all $m < \nu$, $\mathcal{U}_\nu$ is disjoint from $L_m \cup L_m' \cup \{\mathbf{b}_m\}$

4) For all $\nu \in \mathbb{N}$ and $\mathbf{x} \in \mathcal{U}_\nu$ we have

$$|\mathbf{q}_{\nu-1} \cdot \mathbf{x} - p_{\nu-1}| < \varphi(\Phi_1((\mathbf{q}_1)_\nu), \ldots, \Phi_k((\mathbf{q}_k)_\nu))$$

5) For all $\nu \in \mathbb{N}, \mathcal{U}_\nu \cap L_{i_\nu} \neq \emptyset$

To see this suffices, take a point

$$\mathbf{x} \in S \cap \bigcap_\nu \mathcal{U}_\nu = \bigcap_\nu \overline{S \cap \mathcal{U}_\nu}$$

This intersection is nonempty since the right-hand side is by condition 1) an intersection of nonempty nested compact sets, and the equality follows that for $\nu \geq 2$, the sets $\overline{\mathcal{U}_\nu}$ are contained in $\mathcal{W}$. We will reach a contradiction by showing that both $\mathbf{x} \in \mathcal{B}$ and $\mathbf{x} \notin \mathcal{B}$. By condition 3), $\mathbf{x}$ is not equal to any of the $\mathbf{b}_i$, and hence $\mathbf{x} \notin \mathcal{B}$. Also by condition 3), $\mathbf{x}$ is not contained in any of the sets in the collections $\mathcal{L}, \mathcal{L}'$ and thus by hypothesis a) $\mathbf{x}$ is totally irrational. The function $\varphi(Q_1, \ldots, Q_k)$ is non-increasing by assumption, and so is the irrationality measure function $Q_1, \ldots, Q_k \mapsto \psi_{\Phi_1,\ldots,\Phi_k,\mathbf{x}}(Q_1, \ldots, Q_k)$, as follows from its definition. The properness condition guarantees that $\Phi_\ell((\mathbf{q}_\ell)_\nu) \to \infty$ as $\nu \to \infty$ for each $\ell = 1, \ldots, k$. By condition 2), for any $Q_1 \geq (Q_1)_0 := \Phi_1((\mathbf{q}_1)_1)^2$, there is ν with $Q_1 \in \left[\Phi_1((\mathbf{q}_1)_\nu), \Phi_1\left((\mathbf{q}_1)_{\nu+1}\right)\right]$ and after choosing $Q_\ell \in \left[\Phi_\ell((\mathbf{q}_\ell)_\nu), \Phi_\ell\left((\mathbf{q}_\ell)_{\nu+1}\right)\right]$ for $\ell = 2, \ldots k$ by condition 4) we have

$$\begin{aligned}
\psi_{\Phi_1,\ldots,\Phi_k,\mathbf{x}}(Q_1, \ldots, Q_k) &\leq \psi_{\Phi_1,\ldots,\Phi_k,\mathbf{x}}(\Phi_1((\mathbf{q}_1)_\nu), \ldots, \Phi_k((\mathbf{q}_k)_\nu)) \\
&\leq \langle \mathbf{q}_\nu \cdot \mathbf{x} \rangle \\
&\leq |\mathbf{q}_\nu \cdot \mathbf{x} - p_\nu| \\
&< \varphi\left(\Phi_1\left((\mathbf{q}_1)_{\nu+1}\right), \ldots, \Phi_k\left((\mathbf{q}_k)_{\nu+1}\right)\right) \\
&\leq \varphi(Q_1, \ldots, Q_k)
\end{aligned}$$

which shows that $\mathbf{x} \in \mathcal{B}$.

Now we shall construct our sequences such that conditions 1) through 5) hold. Let $\nu = 1$. Choose $i_1 := \min\{i \in \mathbb{N} : L_i \neq \emptyset\}$, which must exist by hypothesis d). Define $\mathcal{U}_1$ to be some open set containing a point in L_{i_1} such that $\overline{\mathcal{U}_1} \subset \mathcal{W}$. We see that conditions 1), 3), and 5) follow immediately from this choice. If $p_1, (\mathbf{q}_1)_1, \ldots, (\mathbf{q}_k)_1$ are the elements of the primitive vector corresponding to A_{i_1}, then conditions 2) and 4) hold as well.

Now suppose we have constructed $\mathcal{U}_m, i_m$ for $m = 1, \ldots, \nu$. Let $i = i_\nu$. By condition 5), for $m = \nu$ we have $\mathcal{U}_m \cap L_i \neq \emptyset$. By hypothesis b), there is an infinite subsequence of indices j such that $|A_j| \to \infty$ as $j \to \infty$ and $\mathcal{U}_m \cap L_i \cap L_j \neq \emptyset$. For each such j, let $A_j = A_{\mathbf{m}_j}$ with $\mathbf{m}_j = \left(p'_j, (\mathbf{q}'_1)_j, \ldots, (\mathbf{q}'_k)_j\right)$. Then by the definition of $|A_j|$, along this subsequence $\left\|\left((\mathbf{q}'_1)_j, \ldots, (\mathbf{q}'_k)_j\right)\right\| \to \infty$. If we employ a trick similar to that of Theorem 4.4, (i.e. restrict our root system as to construct an appropriately concave subset $\mathcal{Q}$ of $\mathbb{R}^k$), we can ensure that restricting $\left((\mathbf{q}'_1)_j, \ldots, (\mathbf{q}'_k)_j\right)$ to $\mathcal{Q}$ forces $(\mathbf{q}'_1)_j, \ldots, (\mathbf{q}'_k)_j \to \infty$. Thus, by the properness of $\Phi_1, \ldots, \Phi_k$, we can choose $j > i$ such that $\Phi_1\left((\mathbf{q}'_1)_j\right) > \Phi_1((\mathbf{q}_1)_\nu), \ldots, \Phi_k\left((\mathbf{q}'_k)_j\right) > \Phi_k((\mathbf{q}_k)_\nu)$. If we set $i_{\nu+1} := j$, this ensures that condition 2) holds for $\nu + 1$.

Next, let $\mathbf{x}' \in \mathcal{U}_\nu \cap L_i \cap L_j$. Since $\mathbf{x}' \in L_i$, we see that $\mathbf{q}_\nu \cdot \mathbf{x}' = p_\nu$. By continuity, we choose a small neighborhood $\mathcal{V} \subset \mathcal{U}_\nu$ of $\mathbf{x}'$ such that for all $\mathbf{x} \in \mathcal{V}$

$$|\mathbf{q}_\nu \cdot \mathbf{x} - p_\nu| < \psi\left(\Phi_1\left((\mathbf{q}_1)_{\nu+1}\right), \ldots, \Phi_k\left((\mathbf{q}_k)_{\nu+1}\right)\right)$$

Thus, condition 4) holds for $\nu + 1$.

Since $\mathbf{x}' \in L_j$, we must have $\mathcal{V} \cap L_{i_{\nu+1}} \neq \emptyset$. Thus by hypothesis c), there exists $\mathbf{x}''$ such that

$$\mathbf{x}'' \in L_j \cap \mathcal{V} \setminus \bigcup_{m<\nu+1} (L_m \cup L'_{m'} \cup \{\mathbf{b}_m\})$$

Further, we can take a neighborhood $\mathcal{U}_{\nu+1}$ of $\mathbf{x}''$ such that $\overline{\mathcal{U}_{\nu+1}} \subset \mathcal{U}_\nu$ and

$$\mathcal{U}_{\nu+1} \cap \bigcup_{m<\nu+1} (L_m \cup L'_{m'} \cup \{\mathbf{b}_m\}) = \emptyset$$

[2]Note that this is referring to the *vector* $(\mathbf{q}_1)_\nu$ for $\nu = 1$, not a component of a vector.

Consequently, conditions 1), 3), and 5) now hold. Thus concludes the induction and this proof.

□

Corollary 4.2. *For S, $\mathcal{L}$, $\mathcal{L}'$, and A_i satisfying Theorem 4.7, if we choose $\Phi_1 = \cdots = \Phi_k = \|\cdot\|$ and $\varphi(Q_1, \ldots, Q_k) = \prod_{\ell=1}^{k} \frac{1}{Q_\ell}$, there exists an uncountable number of singular pairs $(x_1, \ldots, x_n)$ that are totally irrational.*

4.3 Weighted Approximation

The results of Theorems 3.3, 4.4, and 4.7 are rather general, leaving our desired results as corollaries. In fact, the generality of the Φ_i allow us to expand our results to quasinorms. Let us first set

$$\mathbf{s} = (s_1, \ldots, s_n) \in (0,1)^n \qquad \sum_{i=1}^{n} s_i = 1$$

and define

$$\rho := \max_{1\le i\le n} s_i \qquad \delta := \min_{1\le i\le n} s_i$$

We can then define the **s-*quasinorm*** $\|\cdot\|_{\mathbf{s}}$ on $\mathbb{R}^n$ as

$$\|\mathbf{x}\|_{\mathbf{s}} = \max_{1\le i\le n} |x_i|^{1/s_i}$$

We see that the max-norm that we have been working with thus far is the n-th root of the $\left(\frac{1}{n}, \ldots, \frac{1}{n}\right)$-quasinorm. This also lends to new versions of the weighted uniform exponent. The ***weighted uniform exponent for weights*** **s** is the suprememum of all γ such that

$$\|q\mathbf{x}\|_{\mathbf{s}} \le Q^{-n\gamma} \qquad 0 \le q \le Q$$

has a solution $q \in \mathbb{N}$ for all large Q. We can also naturally extend our work using the transference principle to the dual uniform exponent. We see that for $\mathbf{x}$ totally irrational,

$$\hat{\omega}^*_{\mathbf{s}} \ge n \qquad \frac{1}{n} \le \hat{\omega}_{\mathbf{s}} \le \frac{1}{\rho n}$$

We could replace any of the Φ_i in Theorems 3.3, 4.4, and 4.7 with these quasinorms to prove that under the correct conditions, there is an uncountable number of totally irrational, singular vectors in $\mathbb{R}^n$ with respect to any quasinorm.

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