CHRISTOFFEL TRANSFORMATIONS OF LEGENDRE POLYNOMIALS

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1. Introduction

In this paper we are interested in class of orthogonal polynomials (OPs). OPs with respect to the measure $\mathrm{d}\mu(x)$ are sequences $\{P_n\}_{n=0}^{\infty}$ of polynomials so that

$$\int_{-1}^{1} x^k P_n \mathrm{d}\mu(x) = 0$$

for $k = 0, 1, \cdots, n-1$.

Here we are interested only in the case where our measure $\mathrm{d}\mu(x)$ can be expressed in terms of some weight function i.e. $\mathrm{d}\mu(x) = w(x)\mathrm{d}x$. In particular in the case when $w(x)$ is a polynomial. In the case when $w(x) > 0$ the OPs will always exist and they will satisfy $\deg P_n = n$. For a reference on classical OPs see [2]. In this paper we will be concerned with weight functions of the form

$$w(x) = \prod_{i=1}^{m} (x - z_i).$$

In particular we are interested in how the OPs P_n^q with weight $w(x)$ change when we very our z_is as parameters. Previous studies of orthogonal polynomials with polynomial weight functions have used that we can think of OPs with respect to the measure $w(x)\mathrm{d}x$ as the Chistoffel transform (for more information see the next section) of the Lengendre polynomials (orthogonal polynomials with respect to the weight function $w(x) \equiv 1$). In the first section some classsical results and ways of thinking about OPs are introduced. In the second section we introduce the Chistoffel transform. In the third section introduce the idea of recurrence relations and give a table of some recurrence coefficients for $w(x) = x - z$. In section four we use apply a method that was first used by Magnus in [4] to derive a set of equations on the recurrence coefficients of for semi-classical OPs. Magnus found that the equations that recurrence coeffients the semi-classical OPs with weight $w(x) = \mathrm{e}^{-x^4/4 - tx^2}$ on $\mathbb{R}$ satisfy the Painlevé IV equation. However, unlike in Magnus's the paper our string equations also involve coefficients of our OPs. In sections six to nine we introduced the Riemann-Hilbert problem (RHP) for OPs with weight function $w(x) = x - z$ and proved uniquess of the solutions for our RHP. In the final sections we used the RHP for OPs with weight $w(x) = x - z$ and we used a method similiar to the method used by Celsus et al. in a paper on kissing polynomial to find x and z differential equations that our OPs satisfy [7].

2. Some Classical Properties of Orthogonal Polynomials

There are other ways to think about this definition of OPs. Firstly we can we can define a bi-linear form

$$\langle f, g \rangle_w := \int_{-1}^{1} f(x)g(x)w(x)\mathrm{d}x. \tag{1}$$

Another way to think about OPs is using Gram-Smchidt algorithm. We can construct a set of OPs by starting from the basis $\{1, x, \cdots, x^n\}$ and constructing a set of OPs using the Gram-Smchidt procedure

$$\begin{aligned}
P_0^q(x) &:= 1, \\
P_j^q(x) &:= x^j - \sum_{j=0}^{j-1} \frac{\langle P_i^q(x), x^j \rangle_w}{\langle P_i^q(x), P_i^q(x) \rangle_w} P_i^q(x).
\end{aligned}$$

However, because $\langle \cdot, \cdot \rangle_w$ is a bi-linear form and not a an inner product it can happen that $\langle f, f \rangle_w = 0$. When this happens Gram-Schmidt can fail.

A natural question that arises when thinking about OPs is for which values of n do OPs exist and are they unique. A second way that we can think about this problem is as a linear system of equations on the coefficients c_j ($P_n^q(x) = \sum_{i=0}^n c_i x^i$) of our polynomials. If we define

$$\mu_k = \int_{-1}^{1} x^k q(x) \mathrm{d}x$$

then we have the equations

$$\begin{aligned}
\mu_0 c_0 + \mu_1 c_1 + \cdots + \mu_n c_n &= 0 \\
\mu_1 c_0 + \mu_2 c_1 + \cdots + \mu_{n+1} c_n &= 0 \\
&\vdots \\
\mu_{n-1} c_0 + \mu_n c_1 + \cdots + \mu_{2n-1} c_n &= 0.
\end{aligned}$$

If we assume that we are looking for a monic polynomial we can set $c_n = 1$ and therefore we can express our system of equations as

$$\begin{aligned}
\mu_0 c_0 + \mu_1 c_1 + \cdots + \mu_{n-1} c_{n-1} &= -\mu_n \\
\mu_1 c_0 + \mu_2 c_1 + \cdots + \mu_n c_{n-1} &= -\mu_{n+1} \\
&\vdots \\
\mu_{n-1} c_0 + \mu_n c_1 + \cdots + \mu_{2n-2} c_{n-1} &= -\mu_{2n-1}.
\end{aligned}$$

From linear algebra this system will have a solution when

$$\det(D_{n-1}) = \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_{n-1} \\ \mu_1 & \mu_2 & \cdots & \mu_n \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{n-1} & \mu_n & \cdots & \mu_{2n-2} \end{vmatrix} \neq 0.$$

One example of a set OPs is the Legendre polynomials with weight function $w(x) \equiv 1$. We will denote the degree n monic Legendre polynomial by $P_n(x)$.

In the case were $w(x)$ is polynomial function we have a tool call the Chistoffel transform that can be used to study this problem.

Theorem 1 (Chistoffel [1]). *Let $\{P_n\}$ be the set of monic polynomials orthogonal with respect to $w(x)\mathrm{d}x$ and let*

$$q(x) := \prod_{j=1}^{m} (x - z_j),$$

be a positive polynomial on $[-1,1]$*. Then, the polynomials* $P_n^q(x)$ *defined by*

$$C_{n,m}q(x)P_n^q(x) := \begin{vmatrix} P_n(z_1) & P_{n+1}(z_1) & \cdots & P_{n+m}(z_1) \\ P_n(z_2) & P_{n+1}(z_2) & \cdots & P_{n+m}(z_2) \\ \vdots & \vdots & \ddots & \vdots \\ P_n(z_m) & P_{n+1}(z_m) & \cdots & P_{n+m}(z_m) \\ P_n(x) & P_{n+1}(x) & \cdots & P_{n+m}(x) \end{vmatrix}, \tag{2}$$

where

$$C_{n,m} = \begin{vmatrix} P_n(z_1) & P_{n+1}(z_1) & \cdots & P_{n+m-1}(z_1) \\ P_n(z_2) & P_{n+1}(z_2) & \cdots & P_{n+m-1}(z_2) \\ \vdots & \vdots & \ddots & \vdots \\ P_n(z_m) & P_{n+1}(z_m) & \cdots & P_{n+m-1}(z_m) \end{vmatrix},$$

are orthogonal with respect to $q(x)w(x)\mathrm{d}x$*.* P_n^q *has degree* n*. If the zero* x_k *is repeated, replace the corresponding rows with successive derivatives evaluated at* x_k*.*

For a proof of this see [6].

Theorem 2. *The Orthogonal polynomials* $\{P_n^q\}$ *with weight function* $q(x) = x - z$ *always exist and satisfy* $\deg P_n^q = n$ *if and only if* z *is not a root of the* n*th Legendre polynomial.*

Proof. In order to prove this fact we can start by noticing that the moments of $x - z$ are given by

$$\begin{aligned} \mu_k(z) &= \int_{-1}^{1} x^k(x-z)\mathrm{d}x \\ &= \left[\frac{x^{k+2}}{k+2} - \frac{zx^{k+1}}{k+1}\right]_{-1}^{1} \\ &= \frac{1}{k+2}(1-(-1)^k) - \frac{z}{k+1}(1+(-1)^k). \end{aligned}$$

From this formula we can see that each μ_k is polynomial of degree at most 1 in z and therefore it follows that $\det(D_{n-1})$ is a polynomial of degree at most n. Were D_{n-1} is defined by

$$D_{n-1} := \begin{bmatrix} \mu_0 & \mu_1 & \cdots & \mu_{n-1} \\ \mu_1 & \mu_2 & \cdots & \mu_n \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{n-1} & \mu_n & \cdots & \mu_{2n-2} \end{bmatrix}.$$

Now suppose that we consider the polynomial $\check{p}_n$ defined by

$$q(x)\check{p}_n(x) = \begin{vmatrix} P_n(z) & P_{n+1}(z) \\ P_n(x) & P_{n+1}(x) \end{vmatrix}. \tag{3}$$

From the definition of $\check{p}_n(x)$ we can see that

$$\int_{-1}^{1} x^k \check{p}_n(x)q(x)\mathrm{d}x = \int_{-1}^{1} x^k(P_n(z)P_{n+1}(x) - P_{n+1}(z)P_n(x))\mathrm{d}x = 0,$$

for $k = 0, 1, \cdots, n-1$. In addition to this we can notice that the right hand side of (3) as a simply root at $x = z$ and there is divisible by $q(x) = x - z$. That's more if $P_n(z) = 0$ then we can see that the degree of $\check{p}(x)$ will be $n-1$ and therefore we don't have unique OPs of degree n which satisfy n orthogonality conditions. So it must be the case that $\det(D_{n-1}) = 0$.

Because $\det(D_{n-1})$ is a polynomial of degree n and it has root everywhere that $P_n(x)$ has roots. Because Legendre polynomials have simple roots in $(-1, 1)$ only (for a proof of this see [2]), we can therefore conclude that $\det(D_{n-1}) = cP_n(x)$ were c is some constant. It follows from this not $p_n(x)$ will exist if and only if z is not a root of a Legendre polynomial as desired. □

3. Recurrence Relations for Orthogonal Polynomials

Theorem 3. *Suppose that a function $w(x)$ is chosen so that the set of OPs $\{P_k\}_{k=0}^{n+1}$ satisfy $\deg P_k = k$ for $k = 0, 1, \cdots, n+1$. Then there exists sequences complex numbers $\{\alpha_k\}$ and $\{\beta_k\}$ such that*

$$xP_n = P_{n+1} + \alpha_n P_n + \beta_n P_{n-1}.$$

In our case α_n and β_n are meromorphic functions of z. The first few recurrence coeffients are given by

n	α_n	β_n
1	$\frac{5-3z^2}{15z-45z^3}$	$\frac{1}{3} - \frac{1}{9z^2}$
2	$\frac{-15+6z^2+5z^4}{35z(3-14z^2+15z^4)}$	$\frac{12z^2(-3+5z^2)}{25(1-3z^2)^2}$
3	$\frac{-81+45z^2-15z^4+35z^6}{63(-9+105z^2-255z^4+175z^6)}$	$\frac{3(-1+3z^2)(3-30z^2+35z^4)}{49z^2(3-5z^2)^2}$
4	$\frac{-405+108z^2+210z^4-364z^6+315z^8}{99z(3-30z^2+35z^4)(15-70z^2+63z^4)}$	$\frac{80z^2(-3+5z^2)(15-70z^2+63z^4)}{81(3-30z^2+35z^4)^2}$

What's more there are some formulas for α_n and β_n in terms of orthogonal polynomials. The first set of formulas give α_n and β_n in terms of inner products of orthogonal polynomials

$$\alpha_n = \frac{\langle xP_n^q, P_n^q\rangle_w}{\langle P_n^q, P_n^q\rangle_w}$$

and

$$\beta_n = \frac{\langle xP_n^q, P_{n-1}^q\rangle_w}{\langle P_{n-1}^q, P_{n-1}^q\rangle_w},$$

where $\langle\cdot,\cdot\rangle_w$ is defined in (1). In addition to this formula we have that

$$\alpha_n = \kappa_{n-1}^{(n)} - \kappa_n^{(n+1)} \tag{4}$$

$$\beta_n = \kappa_{n-2}^{(n)} - \kappa_{n-1}^{(n)}\kappa_n^{(n+1)} + (\kappa_{n-1}^{(n)})^2 - \kappa_{n-1}^{(n+1)}. \tag{5}$$

4. String Equations

The idea of determine equations by computing the integrals

$$\int_{-1}^{1} [(1-x^2)(P_n^q(x))^2 q(x)]' \mathrm{d}x$$

and

$$\int_{-1}^{1} [(1-x^2)P_n^q(x)P_{n-1}^q(x)q(x)]' \mathrm{d}x$$

was first introduced by Magnus in [4]. Using this method Celsus et al. [7] found that recurrence coefficients for OPs with weight $q(x) = \mathrm{e}^{\mathrm{i}\omega x}$ which they called kissing polynomials satisfy a string equation similiar to the equations found by Magnus with goal of using them to compute OPs efficiently. This method can also be applied to OPs with weight $q(x) = x - z$.

The first way of evaluating our integrals is with the fundamental theorem of calculus. Applying the fundamental theorem of calculus to the first integral gives us

$$I = \int_{-1}^{1} [(1-x)^2 [P_n^q(x)]^2 q(x)]' \mathrm{d}x = (1-x)^2 [P_n^q(x)]^2 q(x) \Big|_{-1}^{1} = 0.$$

What's more applying the fundamental theorem of calculus to second integral gives us

$$J = \int_{-1}^{1} [(1-x^2) P_n^q(x) P_{n-1}^q(x) q(x)]' \mathrm{d}x = (1-x^2) P_n^q(x) P_{n-1}^q(x) q(x) \Big|_{-1}^{1} = 0.$$

Now if we expand the derivative in first integral we get

$$I = \int_{-1}^{1} \left[(1-x^2)[P_n(x)^q]^2 q(x)\right]' \mathrm{d}x = I_1 + I_2 + I_3$$

where

$$\begin{aligned} I_1 &= \int_{-1}^{1} -2x[P_n^q(x)]^2 q \mathrm{d}x \\ I_2 &= \int_{-1}^{1} (1-x^2) P_n^q(x) \frac{\mathrm{d}P_n^q}{\mathrm{d}x}(x) q(x) \mathrm{d}x \\ I_3 &= \int_{-1}^{1} (1-x^2)[P_n^q(x)]^2 \mathrm{d}x. \end{aligned}$$

Now we can compute I_1 and I_2 using a method that is analogous to the methods used in [7] to get[1]

$$\begin{aligned} I_1 &= -2 \int_{-1}^{1} (P_{n+1}^q + \alpha_n P_n^q + \beta_n P_{n-1}^q) P_n^q \mathrm{d}x \\ &= -2\alpha_n \chi_n. \\ I_2 &= \int_{-1}^{1} -2x P_n^q x \frac{\mathrm{d}P_n^q}{\mathrm{d}x} q \mathrm{d}x \\ &= -2 \int_{-1}^{1} (P_{n+1}^q + \alpha_n P_n^q + \beta_n P_{n-1}^q)(n P_n^q - \kappa_{n-1}^n P_{n-1}^q + \cdots) \mathrm{d}x \\ &= -2n\alpha_n \chi_n + 2\beta_n \kappa_{n-1}^n \chi_{n-1}. \end{aligned}$$

[1]Because the notation $P_i^q(x)$ is somewhat cumbersome we will will instead write P_i^q from now on.

Where $\chi_k := \langle P_k^q, P_q^k \rangle_q$. However the final integral I_3 requires some modifications to compute consider the expression[2]

$$\begin{aligned}
U &= \frac{\mathrm{d}}{\mathrm{d}z}\int_{-1}^{1}(1-x^2)(P_n^q)^2 q\mathrm{d}x \\
&= \dot{\chi}_n - \frac{\mathrm{d}}{\mathrm{d}z}\int_{-1}^{1} x^2 (P_n^q)^2 q\mathrm{d}x \\
&= \dot{\chi}_n - \frac{\mathrm{d}}{\mathrm{d}z}\int_{-1}^{1} (P_{n+1}^q + \alpha_n P_n^q + \beta_n P_{n-1}^q)^2 q\mathrm{d}x \\
&= \dot{\chi}_n - \dot{\chi}_{n+1} - \frac{\mathrm{d}}{\mathrm{d}z}(\alpha_n^2 \chi_n + \beta_n^2 \chi_{n-1}).
\end{aligned}$$

On the other hand we also have that

$$\begin{aligned}
U &= \int_{-1}^{1} 2(1-x^2) P_n^q \dot{P}_n^q q\mathrm{d}x - I_3 \\
&= 2\int_{-1}^{1} x^2 P_n^q \dot{P}_n^q q\mathrm{d}x - I_3 \\
&= 2\int_{-1}^{1} (P_{n+1}^q + \alpha_n P_n^q + \beta_n P_{n-1}^q)(\dot{\kappa}_{n-1}^{(n)} P_n^q + (\dot{\kappa}_{n-2}^{(n)} - \dot{\kappa}_{n-1}^{(n)}\kappa_{n-2}^{(n)})P_{n-1}^q + \cdots)\mathrm{d}x - I_3 \\
&= 2\alpha_n \dot{\kappa}_{n-1}^{(n)}\chi_n + 2(\dot{\kappa}_{n-2}^{(n)} - \dot{\kappa}_{n-1}^{(n)}\kappa_{n-2}^{(n)})\chi_{n-1} - I_3.
\end{aligned}$$

Now putting all of these together we get the string equation

$$\begin{aligned}
-2\alpha_n\chi_n - 2n\alpha_n\chi_n + 2\beta_n\kappa_{n-1}^{(n)}\chi_{n-1} + \dot{\chi}_n - \dot{\chi}_{n+1} + \frac{\mathrm{d}}{\mathrm{d}z}(\alpha_n^2\chi_n + \beta_n^2\chi_{n-1}) \\
+2\alpha_n\dot{\kappa}_{n-1}^{(n)}\chi_n + 2(\dot{\kappa}_{n-2}^{(n)} - \kappa_{n-1}^{(n)} - \kappa_{n-1}^{(n)}\kappa_{n-2}^{(n)})\chi_{n-1} = 0.
\end{aligned}$$

Next if we expand out the second integral we get

$$J = J_1 + J_2 + J_3 + J_4$$

where

$$\begin{aligned}
J_1 &= -2\int_{-1}^{1} x P_n^q P_{n-1}^q q\mathrm{d}x \\
J_2 &= \int_{-1}^{1}(1-x^2)\frac{\mathrm{d}P_n^q}{\mathrm{d}x}P_{n-1}^q q\mathrm{d}x \\
J_3 &= \int_{-1}^{1}(1-x^2)P_n^q\frac{\mathrm{d}P_{n-1}^q}{\mathrm{d}x} q\mathrm{d}x \\
J_4 &= \int_{-1}^{1}(1-x^2)P_n^q P_{n-1}^q \mathrm{d}x.
\end{aligned}$$

[2]We use a . above a variable to denote differentiation with respect to the parameter z.

If we apply our recurrence formula to J_1 we get

$$\begin{aligned}J_1 &= -2\int_{-1}^{1}(P_{n+1}^q+\alpha_nP_n^q+\beta_nP_{n-1}^q)P_{n-1}^q q\mathrm{d}x\\ &= -2\beta_n\chi_{n-1}.\end{aligned}$$

Next we need to evaluate J_2 by expanding $x^2\frac{\mathrm{d}P_n^q}{\mathrm{d}x} = nP_{n-1}^q + ((n-1)\kappa_{n-1}^{(n)} - n\kappa_n^{(n+1)})P_n^q + ((n-2)\kappa_{n-2}^{(n)} - n\kappa_{n-1}^{(n+1)} + (n-1)\kappa_{n-1}^{(n)})P_{n-1}^q + \cdots$ we get

$$\begin{aligned}J_2 &= \int_{-1}^{1}\frac{\mathrm{d}P_n^q}{\mathrm{d}x}P_{n-1}^q q\mathrm{d}x - \int_{-1}^{1}x^2\frac{\mathrm{d}P_n^q}{\mathrm{d}x}P_{n-1}^q q\mathrm{d}x\\ &= n\chi_{n-1} - ((n-2)\kappa_{n-2}^{(n)} - n\kappa_{n-1}^{(n+1)} + (n-1)\kappa_{n-1}^{(n)})\chi_{n-1}.\end{aligned}$$

J_3 can be evaluated directly using the orthogonality conditions to get $J_3 = -(n-1)\chi_n$. Finally evaluating J_4 requires a special trick that is similar to trick that was used to evaluated I_3. We will start by defining

$$V \coloneqq \frac{\mathrm{d}}{\mathrm{d}z}\int_{-1}^{1}(1-x^2)P_n^qP_{n-1}^q q\mathrm{d}x.$$

If we directly evaluate V using the recurrence relation for P_n^q we get

$$\begin{aligned}V &= -\frac{\mathrm{d}}{\mathrm{d}z}\int_{-1}^{1}(P_{n+1}^q+\alpha_nP_n^q+\beta_nP_{n-1}^q)(P_n^q+\alpha_{n-1}P_{n-1}^q+\beta_{n-1}P_{n-2}^q)q\mathrm{d}x\\ &= -\frac{\mathrm{d}}{\mathrm{d}z}\left[\alpha_n\chi_n+\beta_n\alpha_{n-1}\chi_{n-1}\right].\end{aligned}$$

If we instead evaluate V by moving $\mathrm{d}/\mathrm{d}z$ inside our integral we get

$$\begin{aligned}V = \int_{-1}^{1}(1-x^2)\dot{P}_n^qP_{n-1}^q q\mathrm{d}x + \int_{-1}^{1}(1-x^2)P_n^q\dot{P}_{n-1}^q q\mathrm{d}x - \int_{-1}^{1}(1-x^2)P_n^qP_{n-1}^q\mathrm{d}x&\\ = \dot{\kappa}_{n-1}^{(n)}\chi_{n-1} - \int_{-1}^{1}x^2\dot{P}_n^qP_{n-1}^q q\mathrm{d}x - \int_{-1}^{1}x^2P_n^q\dot{P}_{n-1}^q q\mathrm{d}x - J_4&\\ = \dot{\kappa}_{n-1}^{(n)}\chi_{n-1} - \int_{-1}^{1}\Big\{\dot{\kappa}_{n-1}^{(n)}P_{n+1}^q + (\dot{\kappa}_{n-2}^{(n)} - \dot{\kappa}_{n-1}^{(n)}\kappa_n^{(n+1)})P_n^q&\\ +(\dot{\kappa}_{n-3}^{(n)} - \dot{\kappa}_{n-1}^{(n)}\kappa_{n-1}^{(n+1)} - \dot{\kappa}_{n-2}^{(n)}\kappa_{n-1}^{(n)})P_{n-1}^q + \cdots\Big\}P_{n-1}^q q\mathrm{d}x - \dot{\kappa}_{n-2}^{(n-1)}\chi_n - J_4&\\ = \dot{\kappa}_{n-1}^{(n)}\chi_{n-1} - (\dot{\kappa}_{n-3}^{(n)} - \dot{\kappa}_{n-1}^{(n)}\kappa_{n-1}^{(n+1)} - \dot{\kappa}_{n-2}^{(n)}\kappa_{n-1}^{(n)})\kappa_{n-1} - \dot{\kappa}_{n-2}^{(n-1)}\chi_n - J_4.&\end{aligned}$$

Therefore it follows that we have

$$J_4 = \frac{\mathrm{d}}{\mathrm{d}z}\left[\alpha_n\chi_n+\beta_n\alpha_{n-1}\chi_{n-1}\right] + \dot{\kappa}_{n-1}^{(n)}\chi_{n-1} - (\dot{\kappa}_{n-3}^{(n)} - \dot{\kappa}_{n-1}^{(n)}\kappa_{n-1}^{(n+1)} - \dot{\kappa}_{n-2}^{(n)}\kappa_{n-1}^{(n)})\kappa_{n-1} - \dot{\kappa}_{n-2}^{(n-1)}\chi_n.$$

Our second string equation is given by

$$\begin{aligned}n\chi_{n-1} - ((n-2)\kappa_{n-2}^{(n)} - n\kappa_{n-1}^{(n+1)} + (n-1)\kappa_{n-1}^{(n)})\chi_{n-1} + \frac{\mathrm{d}}{\mathrm{d}z}\left[\alpha_n\chi_n+\beta_n\alpha_{n-1}\chi_{n-1}\right] + \dot{\kappa}_{n-1}^{(n)}\chi_{n-1}&\\ -(\dot{\kappa}_{n-3}^{(n)} - \dot{\kappa}_{n-1}^{(n)}\kappa_{n-1}^{(n+1)} - \dot{\kappa}_{n-2}^{(n)}\kappa_{n-1}^{(n)})\chi_{n-1} - \dot{\kappa}_{n-2}^{(n-1)}\chi_n = 2\beta_n\chi_{n-1} + (n-1)\kappa_{n-2}^{(n-1)}\chi_n&\end{aligned}$$

5. A Different Method to Find String Equations

This new method starts with an integral very similar to integral from the previous method except with q replaced by q^2 using this method gives us

$$U = \int_{-1}^{1} \left[(1-x^2)(P_n^q)^2(x-z)^2\right]' \mathrm{d}x = 0.$$

On the hand we also have that

$$U = 2\Big(\int_{-1}^{1} (1-x^2)(P_n^q)^2(x-z)\mathrm{d}x - \int_{-1}^{1} x(x-z)(P_n^q)^2(x-z)\mathrm{d}x \\ + \int_{-1}^{1} (1-x^2)(x-z)P_n^q(P_n^q)'(x-z)\mathrm{d}x\Big).$$

If we can evaluate each of the three seperately to get

$$\int_{-1}^{1} (1-x^2)(P_n^q)^2(x-z)\mathrm{d}x = \chi_n - \chi_{n+1} - \alpha_n^2\chi_n - \beta_n^2\chi_{n-1}.$$

For the next term we get

$$\int_{-1}^{1} (x^2-zx)(P_n^q)^2(x-z)\mathrm{d}x = -z\alpha_n\chi_n + \chi_{n+1} + \alpha_n^2\chi_n + \beta_n^2\chi_{n-1}.$$

And for the final term we have

$$\begin{aligned} &\int_{-1}^{1} (-x^3 - zx^2 + x - z)P_n^q(P_n^q)'(x-z)\mathrm{d}x \\ &= \int_{-1}^{1} (-x^2 - zx + 1)(P_{n+1}^q + \alpha_n P_n^q + \beta_n P_{n-1}^q)(P_n^q)'(x-z)\mathrm{d}x \\ &= \int_{-1}^{1} (P_{n+1}^q + \alpha_n P_n^q + \beta_n P_{n-1}^q)(c_1 P_{n+1}^q + c_2 P_n^q + c_3 P_{n-1}^q + \cdots)(x-z)\mathrm{d}x \\ &= c_1\chi_{n+1} + c_2\alpha_n\chi_n + c_3\beta_n\chi_{n-1}, \end{aligned}$$

where

$$\begin{aligned} c_1 &= -n \\ c_2 &= (1-n)\kappa_{n-1}^{(n)} + zn \\ c_3 &= n + z(n-1)\kappa_{n-1}^{(n)} - n\kappa_{n-2}^{(n)}. \end{aligned}$$

If we combine these three terms we end up with the equation

$$-(2+n)\chi_{n+1} + (1 + z\alpha_n - 2\alpha_n^2 + (n-1)\kappa_{n-1}^{(n)} + zn)\chi_n \\ +(n + z(n-1)\kappa_{n-1}^{(n)} - n\kappa_{n-2}^{(n)} - 2\beta_n^2)\chi_{n-1} = 0.$$

We can apply similar reasoning to

$$V = \int_{-1}^{1} [(1-x^2)P_n^q P_{n-1}^q (x-z)^2]'\mathrm{d}x,$$

we get

$$V = 2\int_{-1}^{1}(-x^2+zx)P_n^q P_{n-1}^q(x-z)\mathrm{d}x + \int_{-1}^{1}(1-x^2)(x-z)(P_n^q)'P_{n-1}^q(x-z)\mathrm{d}x$$
$$+\int_{-1}^{1}(1-x^2)(x-z)P_n^q(P_{n-1}^q)'(x-z)\mathrm{d}x + 2\int_{-1}^{1}(1-x^2)P_n^q P_{n-1}^q(x-z)\mathrm{d}x.$$

For the first term in our integral we have

$$2\int_{-1}^{1}(-x^2+zx)P_n^q P_{n-1}^q(x-z)\mathrm{d}x =$$
$$2\Big(\int_{-1}^{1} -(P_{n+1}^q+\alpha_n P_n^q+\beta_n P_{n-1}^q)(P_n^q+\alpha_{n-1}P_{n-1}^q+\beta_{n-1}P_{n-2}^q)(x-z)\mathrm{d}x$$
$$+z\int_{-1}^{1}(P_{n+1}^q+\alpha_n P_n^q+\beta_n P_{n-1}^q)P_{n-1}^q(x-z)\mathrm{d}x\Big)$$
$$= -2\alpha_n\chi_n + 2\beta_n(z-\alpha_{n-1})\chi_{n-1}.$$

Now applying fact that our orthogonal polynomials will form a basis when there is no degeneration to our second term we can write $(-x^3+zx^2+x-z)(P_n^q)'$ as linear combination of orthogonal polynomials

$$\int_{-1}^{1}(-x^3+zx^2+x-z)(P_n^q)'P_{n-1}^q(x-z)\mathrm{d}x$$
$$= \int_{-1}^{1}(c_4P_{n+2}^q+c_5P_{n+1}^q+c_6P_n^q+c_7P_{n-1}^q+\cdots)P_{n-1}^q(x-z)\mathrm{d}x$$
$$= c_7\chi_{n-1}.$$

were

$$c_4 = -n$$
$$c_5 = n\kappa_{n+1}^{(n+2)} + zn + (1-n)\kappa_{n-1}^{(n)}$$
$$c_6 = -c_4\kappa_n^{(n+2)} - c_5\kappa_n^{(n+1)} + n + (n-2)\kappa_{n-2}^{(n)} + (n-1)z\kappa_{n-1}^{(n)}$$
$$c_7 = -c_4\kappa_{n-1}^{(n+2)} - c_5\kappa_{n-1}^{(n+1)} - c_6\kappa_{n-1}^{(n)} - zn + (n-3)\kappa_{n-3}^{(n)} + (n-2)z\kappa_{n-2}^{(n)} + (n-1)\kappa_{n-1}^{(n)}.$$

Next if we examine the third term we see that

$$\int_{-1}^{1}(1-x^2)(x-z)P_n^q(P_{n-1}^q)'(x-z)\mathrm{d}x$$
$$= \int_{-1}^{1}(-x^3+zx^2)P_n^q(P_{n-1}^q)'(x-z)\mathrm{d}x$$

Now if we can use the fact that orthogonal polynomials for a basis we can right $(-x^3+zx^2)(P_{n-1}^q)'$ in terms of orthogonal polynomials as

$$(-x^3+zx^2)(P_{n-1}^q)' = (1-n)P_{n+1}^q + [(n-1)\kappa_{n-1}^{(n-1)} + z(n-2)\kappa_{n-2}^{(n-1)} - (n-3)\kappa_{n-3}^{(n-1)}]P_n^q + \cdots.$$

If we substitute this into our third term we get

$$\int_{-1}^{1}(1-x^2)(x-z)P_n^q(P_{n-1}^q)'(x-z)\mathrm{d}x = [(n-1)\kappa_{n-1}^{(n-1)} + z(n-2)\kappa_{n-2}^{(n-1)} - (n-3)\kappa_{n-3}^{(n-1)}]\chi_n.$$

Finally looking at the fourth term we see that

$$2\int_{-1}^{1}(1-x^2)P_n^qP_{n-1}^q(x-z)\mathrm{d}x =$$
$$2\int_{-1}^{1}(P_{n+1}^q+\alpha_nP_n^q+\beta_nP_{n-1}^q)(P_n^q+\alpha_{n-1}P_{n-1}^q+\beta_{n-1}P_{n-2}^q)(x-z)\mathrm{d}x =$$
$$2\alpha_n\chi_n+2\chi_{n-1}\beta_n\alpha_{n-1}.$$

If we put all of these terms together we get the equation

$$\chi_{n-1}(2z\beta_n+c_7)=0.$$

If we fully write out c_6 and c_7 we get

$$c_6 = n\kappa_n^{(n+2)}-(n\kappa_{n+1}^{(n+2)}+zn+(1-n)\kappa_{n-1}^{(n)})\kappa_n^{(n+1)}+n+(n-2)\kappa_{n-2}^{(n)}+(n-1)z\kappa_{n-1}^{(n)}$$
$$c_7 = n\kappa_{n-1}^{(n+2)}-(n\kappa_{n+1}^{(n+2)}+zn+(1-n)\kappa_{n-1}^{(n)})\kappa_{n-1}^{(n+1)}-(n\kappa_n^{(n+2)}-(n\kappa_{n+1}^{(n+2)}+zn+(1-n)\kappa_{n-1}^{(n)})\kappa_n^{(n+1)}$$
$$+n+(n-2)\kappa_{n-2}^{(n)}+(n-1)z\kappa_{n-1}^{(n)})\kappa_{n-1}^{(n)}-zn+(n-3)\kappa_{n-3}^{(n)}+(n-2)z\kappa_{n-2}^{(n)}+(n-1)\kappa_{n-1}^{(n)}.$$

Therefore we have the string equation

$$2z\beta_n+n\kappa_{n-1}^{(n+2)}-(n\kappa_{n+1}^{(n+2)}+zn+(1-n)\kappa_{n-1}^{(n)})\kappa_{n-1}^{(n+1)}-(n\kappa_n^{(n+2)}-(n\kappa_{n+1}^{(n+2)}+zn+(1-n)\kappa_{n-1}^{(n)})\kappa_n^{(n+1)}$$
$$+n+(n-2)\kappa_{n-2}^{(n)}+(n-1)z\kappa_{n-1}^{(n)})\kappa_{n-1}^{(n)}-zn+(n-3)\kappa_{n-3}^{(n)}+(n-2)z\kappa_{n-2}^{(n)}+(n-1)\kappa_{n-1}^{(n)}=0.$$

In the future we hope to be able to use expression like these to compute αs and βs which we could use to compute OPs using the three term recurrence relation. We would also like to simplify these string equations in the future perhaps using (4) and (5). Although we explored computing some integrals inspired by Magnus there are other integral we could have computed for example

$$\int_{-1}^{1}[(1-x^2)(P_n^q(x))^2(x-z)^r]'\mathrm{d}x$$

were r is some positive integer.

6. The Riemann Hilbert Problem

In addition to the characterizations of as being polynomials that are orthogonal to all polynomials of degree n for some given. There is another way to characterize OPs. This method is called a Riemann-Hilbert Problem (RHP) and it was first used as formulation of OPs by Fokas, Its, and Kitaev in [3]. In addition to formulating the RHP they used to RHP to derive the recurrence relation (see equation (3.14) in the reference). The RHP for OPs has also been used to obtain asymptotic information about OPs, for example [5] discusses the asymptotics for Jacobi polynomials which have weight $w(x)=(1+x)^\alpha(1-x)^\beta$ for $\alpha,\beta>-1$. In addition to this RHPs have been applied to prove results for kissing polynomials (polynomials with weight $w(x)=\mathrm{e}^{\mathrm{i}\omega x}$, $\omega\in\mathbb{R}$) for more information about this application see [7]. The RHP is a type of boundary value for matrix valued functions in the complex plane. In particular the problem is to look for a function $Y(u)$ that satisfies the following conditions

(1) $Y(u)$ is analytic $u=\mathbb{C}\setminus[-1,1]$.

(2) For $x \in (-1,1)$, $Y(u)$ admits boundary values $Y_\pm(x) = \lim_{\epsilon \to 0^\pm} Y(x + \mathrm{i}\epsilon)$, which are related by the jump condition

$$Y_+(x) = Y_-(x) \begin{bmatrix} 1 & x-z \\ 0 & 1 \end{bmatrix}.$$

(3) As $u \to \infty$ we have that[3]

$$Y(u) = (I + O(1/u))u^{n\sigma_3}$$

were we are using the notation

$$f(u)^{\sigma_3} = \begin{bmatrix} f(u) & 0 \\ 0 & 1/f(u) \end{bmatrix}.$$

(4) As $u \to \pm 1$, we have that

$$Y(u) = \begin{bmatrix} O(1) & O(\log|1 \mp u|) \\ O(1) & O(\log|1 \mp u|) \end{bmatrix}.$$

In the case were $n \neq 0$ the solution to the RHP is given by[4]

$$Y(u) = \begin{bmatrix} P_n^q(u) & \frac{1}{2\pi\mathrm{i}} \int_{-1}^{1} \frac{P_n^q(s)(s-z)}{s-u} \mathrm{d}s \\ -\frac{2\pi\mathrm{i}}{\chi_{n-1}} P_{n-1}^q(u) & -\frac{1}{\chi_{n-1}} \int_{-1}^{1} \frac{P_{n-1}^q(s)(s-z)}{s-u} \mathrm{d}s \end{bmatrix}. \tag{6}$$

In the $n = 0$ case we have the special solution

$$Y(u) = \begin{bmatrix} 1 & \frac{1}{2\pi\mathrm{i}} \int_{-1}^{1} \frac{w(s)}{s-u} \mathrm{d}s \\ 0 & 1 \end{bmatrix}. \tag{7}$$

For more information about the Riemann-Hilbert problem and for a proof that formulas given above are in fact solutions to Riemann Hilbert see [6].

7. Transformation to Constant Jump

In order to transform our problem into a Riemann-Hilbert problem with constant jumps we make the substitution

$$W(u) \coloneqq Y(u)(u-z)^{\sigma_3/2}.$$

Making this substitution complicates our problem slightly because we no longer have that $W(u)$ is analytic on $\mathbb{C} \setminus [-1,1]$ since $W(u)$ is not analytic on the branch cut of $\sqrt{u-z}$. If we choose the principle branch of $\sqrt{u-z}$ then our contours were we have jumps look like

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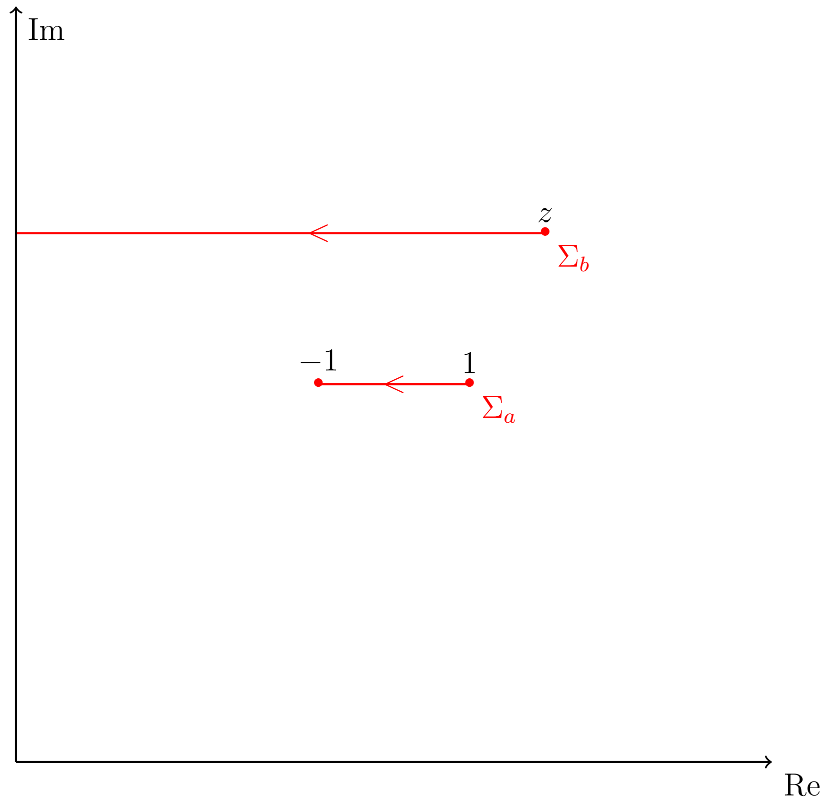
[3] $O(1/u)$ is what is know as landau o notation. We say that $f(x) = O(g(x))$ as $x \to x_0$ if for x sufficiently close to x_0 there exists a constant $M > 0$ such that $|f(x)| \leq Mg(x)$.

[4] This holds when $\deg P_n^q = n$ so $\chi_{n-1} \neq 0$ and $\int_{-1}^{1} \frac{P_{n-1}^q(s)(s-z)}{s-u} = \frac{1}{z^n}(1 + o(1))$.



Because we have two different contours we also have two different jump conditions. Across Σ_a we have the jump condition

$$W^+(x) = W^-(x) \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

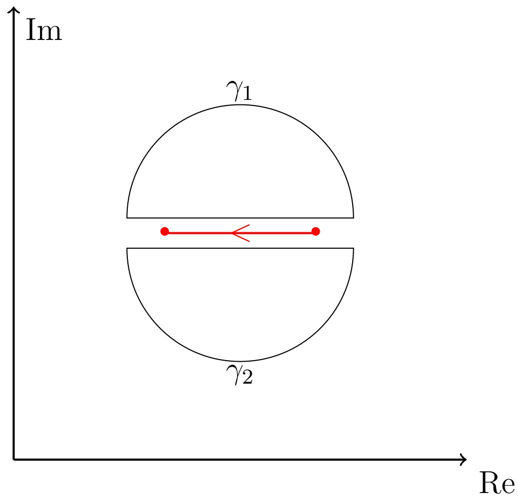
Across the contour Σ_b we have the jump condition

$$W^+(x) = -W^-(x).$$

8. Invertibility of $W(u)$

Theorem 4. *The matrix valued function $Y(u)$ satisfies* $\det(Y(u)) \equiv 1$.

Proof. To show this we start by showing that $\det(Y(u))$ is a bounded entire function and then use our boundary conditions to show that $\det(Y(u)) \to 1$ as $u \to \infty$. To show that $Y(u)$ is an entire function we can apply Morera's theorem. Because $Y(u)$ is analytic on $\mathbb{C} \setminus [-1, 1]$ we have that $\det(Y(u))$ is also analytic on $\mathbb{C} \setminus [-1, 1]$. There if $\gamma \subset \mathbb{C}$ is a closed contour that does not enclose any points from $[-1, 1]$ we have that $\int_\gamma \det(Y(u))\mathrm{d}u = 0$. Now suppose that γ is a closed contour that contains of points points of $Y(u)$. Then we can consider integrating over two different contours.



In this figure γ_1 has counter clockwise orientation and γ_2 has clockwise orientation. Because our jump matrix is upper triangular with 1s on the diagonal the limit of $\det(W(u))$ is the same weather u approaches some point $x \in (-1, 1)$ from above or from below. Therefore if we move the curves γ_1 and γ_2 closer to $[-1, 1]$ the contributions from the bottom of γ_1 and the top of γ_2 will cancel out and therefore we can see that integral of $Y(u)$ over any closed curve is zero and therefore $Y(u)$ is entire. From condition (3) of the RHP we can see that $\det(Y(u)) \to 1$ as $u \to \infty$ and therefore by Liouville's theorem we have that $\det(Y(u)) \equiv 1$ as desired. ☐

Now using this theorem we can write

$$\begin{aligned}\det(W(u)) &= \det(Y(u)(u-z)^{\sigma_3})\\ &= \det(Y(u))\det((u-z)^{\sigma_3})\\ &= 1.\end{aligned}$$

Therefore we also have that $\det(W(u)) \equiv 1$. Which implies that $[W(u)]^{-1}$ exists for all $u \in \mathbb{C}$.

9. Existence and Uniqueness of the Solution

Theorem 5. *Formulas (6) and (7) give the unique solution to the RHP described above in the $n > 0$ and $n \neq 0$ cases respectively.*

Proof. Suppose that $Y_1(u)$ and $Y_2(u)$ both satisfy the RHP. Because from the previous theorem we have that $\det(Y_2(u)) = 1$ we can therefore consider the expression $Y_1(u)[Y_2(u)]^{-1}$. We can observe that $Y_1(u)[Y_2(u)]^{-1}$ is an entire function of u. What's more if we use our asymptotic condition as $u \to \infty$ we get

$$\begin{aligned}\lim_{u\to\infty} Y_1(u)[Y_2(u)]^{-1} &= \lim_{u\to\infty}(I + O(1/u))u^{n\sigma_3}u^{-n\sigma_3}(I + O(1/u))\\ &= I.\end{aligned}$$

Therefore we can apply Liouvell's theorem to get that $Y_1(u)[Y_2(u)]^{-1} \equiv I$. So we have that $Y_1 = Y_2$ and therefore it follows that our RHP has only one solution. For a proof that these are in fact the solutions see [6]. ☐

10. Asymptotic Expansion for $W(u)$

We because we have an explicit formula for $Y(u)$ we can expand $Y(u)$ into an asymptotic series by expanding each entry of $Y(u)$ into an asymptotic series

$$Y(u) = (I + \frac{A_n}{u} + \frac{B_n}{u^2} + \frac{C_n}{u^3} + O(1/u^4))u^{n\sigma_3}.$$

From this it follows that $W(u)$ has the asymptotic expansion

$$W(u) = (I + \frac{A_n}{u} + \frac{B_n}{u^2} + \frac{C_n}{u^3} + O(1/u^4))u^{n\sigma_3}(u-z)^{\sigma_3/2},$$

We can determine the values of A_n, B_n and C_n by expanding our solution for $n \neq 0$ a series

$$Y_{11}(u) = u^u(1 + \frac{\kappa_{n-1}^{(n)}}{u} + \frac{\kappa_{n-2}^{(n)}}{u^2} + \frac{\kappa_{n-3}^{(n)}}{u^3} + O(1/u^4)$$

$$Y_{12}(u) = -\frac{\chi_n}{2\pi \mathrm{i} u^{n+1}}\left(1 - \frac{\kappa_n^{(n+1)}}{u} + \frac{\kappa_{n+1}^{(n+2)}\kappa_n^{(n+1)} - \kappa_n^{(n+2)}}{u^2} + O(1/u^3)\right)$$

$$Y_{21}(u) = -\frac{2\pi \mathrm{i} u^{n-1}}{\chi_{n-1}}\left(1 + \frac{\kappa_{n-2}^{(n-1)}}{u} + \frac{\kappa_{n-3}^{(n-1)}}{u^2} + O(1/u^3)\right)$$

$$Y_{22}(u) = \frac{1}{u^n}\left(1 - \frac{\kappa_{n-1}^{(n)}}{u} + \frac{\kappa_n^{(n+1)}\kappa_{n-1}^{(n)} - \kappa_{n-1}^{(n+1)}}{u^2}\right.$$
$$\left. + \frac{\kappa_{n+1}^{(n+2)}\kappa_{n-1}^{(n+1)} + \kappa_n^{(n+2)}\kappa_{n-1}^{(n)} - \kappa_{n+1}^{(n+2)}\kappa_n^{(n+1)}\kappa_{n-1}^{(n)} - \kappa_{n-1}^{(n+2)}}{u^3} + O(1/u^4)\right)$$

$$A_n(z) = \begin{bmatrix} \kappa_{n-1}^{(n)} & -\frac{\chi_n}{2\pi \mathrm{i}} \\ -\frac{2\pi \mathrm{i}}{\chi_{n-1}} & -\kappa_{n-1}^{(n)} \end{bmatrix}$$
$$B_n(z) = \begin{bmatrix} \kappa_{n-2}^{(n)} & -\frac{\kappa_n^{(n+1)}\chi_n}{2\pi \mathrm{i}} \\ \frac{2\pi \mathrm{i}\kappa_{n-2}^{(n-1)}}{\chi_{n-1}} & \kappa_n^{(n+1)}\kappa_{n-1}^{(n)} - \kappa_{n-1}^{(n+1)} \end{bmatrix}$$
$$C_n(z) = \begin{bmatrix} \kappa_{n-3}^{(n)} & -\frac{\chi_n}{2\pi \mathrm{i}}(\kappa_{n+1}^{n+2}\kappa_n^{(n+1)} - \kappa_n^{(n+2)}) \\ -\frac{2\pi \mathrm{i}\kappa_{n-3}^{(n-1)}}{\chi_{n-1}} & \kappa_{n+1}^{(n+2)}\kappa_{n-1}^{(n+1)} + \kappa_n^{(n+2)}\kappa_{n-1}^{(n)} - \kappa_{n+1}^{(n+2)}\kappa_n^{(n+1)}\kappa_{n-1}^{(n)} - \kappa_{n-1}^{(n+2)} \end{bmatrix}.$$

What's more we can take the derivative of this asymptotic expansion to get

$$W'(u) = \frac{\partial}{\partial u}\left[(I + \frac{A_n(z)}{u} + \frac{B_n(z)}{u^2} + O(1/u^3)u^{n\sigma_3}(u-z)^{\sigma_3/2}\right]$$
$$= \left[-\frac{A_n(z)}{u^2} - \frac{2B_n(z)}{u^3} + O(1/u^4) + (I + \frac{A_n(z)}{u} + \frac{B_n(z)}{u^2} + O(1/u^3))(\frac{n\sigma_3}{u} + \frac{\sigma_3}{2(u-z)})\right]u^{n\sigma_3}(u-z)^{\sigma_3/2}$$
$$= \frac{1}{u}\left[\Gamma_0 + \frac{\Gamma_1}{u} + \frac{\Gamma_2}{u^2} + O(1/u^3)\right]u^{n\sigma_3}(u-z)^{\sigma_3/2},$$

were Γ_0, Γ_1 and Γ_2 are given by

$$\begin{aligned}
\Gamma_0 &= (n+\frac{1}{2})\sigma_3 \\
\Gamma_1 &= \frac{z\sigma_3}{2} + (n+\frac{1}{2})A_n\sigma_3 - A_n \\
\Gamma_2 &= \frac{z^2\sigma_3}{2} + \frac{A_n z\sigma_3}{2} + (n+\frac{1}{2})B_n\sigma_3 - 2B_n.
\end{aligned}$$

11. The Riemann-Hilbert Problem for $W'(u)$

Now using this asymptotic behavior we can determine that $W'(u)$ will satisfy the new Riemann-Hilbert problem.

(1) W' is analytic except on $[-1,1]$ and on the branch cut of $\sqrt{u-z}$.
(2) W' satisfies the same jump conditions as W.
(3) As $u \to \infty$ we have

$$W'(u) = \frac{1}{u}\Big(\Gamma_0 + \frac{\Gamma_1}{u} + \frac{\Gamma_2}{u^2} + O(1/u^3)\Big)u^{n\sigma_3}(u-z)^{\sigma_3/2}.$$

(4) As $u \to \pm 1$ we have the asymptotic behavior

$$W'(u) = \begin{bmatrix} O(1) & O(\frac{1}{|u\mp 1|}) \\ O(1) & O(\frac{1}{|u\mp 1|}) \end{bmatrix}.$$

12. Asymptotics for $[W(u)]^{-1}$

Before we can use the RHP to prove a u-differential equation we need an asymptotic expansion for $[W(u)]^{-1}$. Fortunately because $W(u)$ as an expansion in terms of inverse powers of u thing will work out nicely and we get that $W(u)$ has the asymptotic expansion

$$[W(u)]^{-1} = (u-z)^{-\sigma_3/2}u^{-n\sigma_3}(I + \frac{\Delta_1}{u} + \frac{\Delta_2}{u^2} + O(\frac{1}{u^3})), \tag{8}$$

where

$$\begin{aligned}
\Delta_1 &= -A_n \\
\Delta_2 &= A_n^2 - B_n.
\end{aligned}$$

13. Determining the u-differential equation

If we consider

$$\begin{aligned}
&(u-z)(1-u^2)W'(u)[W(u)]^{-1} = \\
&(-u^3+zu^2+u-z)\frac{1}{u}\Big(\Gamma_0 + \frac{\Gamma_1}{u} + \frac{\Gamma_2}{u^2} + O(1/u^3)\Big)\Big(I + \frac{\Delta_1}{u} + \frac{\Delta_2}{u^2} + O(1/u^3)\Big) \\
&= M(u) + O(1/u)
\end{aligned}$$

were

$$M(u) = (-u^2+zu+1)\Gamma_0 + (-u+z)\Gamma_1 - \Gamma_2 + (-u+z)\Gamma_0\Delta_1 - \Gamma_1\Delta_1 - \Gamma_0\Delta_2.$$

We can write out the entries of $M(u) = [M_{ij}(u)]$ explicitly

$$M_{11}(u) = \frac{1}{2} + n - (n+\frac{1}{2})u^2 + nzu + 2\kappa_{n-2}^{(n)} + (u-z)\kappa_{n-1}^{(n)} - (\kappa_{n-1}^{(n)})^2 + \frac{2(n+1)\chi_n}{\chi_{n-1}}$$

$$M_{12}(u) = \frac{\chi_n}{2\pi \mathrm{i}}\left[2(n+1)u - (1+2n)z + (1+2n)\kappa_{n-1}^{(n)} - (3+2n)\kappa_n^{(n+1)}\right]$$

$$M_{21}(u) = \frac{2\pi \mathrm{i}}{\chi_{n-1}}\left[2nu + (1-2n)z + (1-2n)\kappa_{n-2}^{(n-1)} - (1+2n)\kappa_{n-1}^{(n)}\right]$$

$$M_{22}(u) = -(\frac{1}{2}+n) + (1+2n)u^2 - 2nzu - 2(\kappa_{n-1}^{(n)})^2 - 4\kappa_{n-1}^{(n+1)} + \kappa_{n-1}^{(n)}(-2u + 2u + 4\kappa_n^{(n+1)}) + \frac{4n\chi_n}{\chi_{n-1}}.$$

Now if we subtract $M(u)$ from both sides we get

$$(u-z)(1-u^2)W'(u)[W(u)]^{-1} - M(u) = O(1/u).$$

Now because the left hand of this expression is an entire function we which decays to zero as $u \to \infty$ we can apply Liouville theorem to conclude the difference is a zero and therefore we have that

$$(u-z)(1-u^2)W'(u) = M(u)W(u).$$

Now we can use that $W(u) = Y(u)(u-z)^{\sigma_3/2}$ and therefore we have that

$$W'(u) = \left[Y'(u) + Y(u)\frac{\sigma_3}{2(u-z)}\right](u-z)^{\sigma_3/2}.$$

Looking at the first column of both sides we can see that we have the system of ODEs

$$\mathbf{y}' = X(u)\mathbf{y}$$

were $\mathbf{y} = (y_1, y_2)$, $y_1 = p_n(u)$ and $y_2 = -\frac{2\pi \mathrm{i}}{\chi_{n-1}}p_{n-1}(u)$ and were

$$X(u) = \frac{M(u)}{(u-z)(1-u^2)} - \frac{I}{2(u-z)}.$$

We can rewrite this system as a second order ODE that y_1 satisfies namely

$$y_1'' - \left(\frac{X_{12}'}{X_{12}} + X_{11} + X_{22}\right)y_1' + \left(\frac{X_{12}'X_{11}}{X_{12}} - X_{12}X_{21} + X_{11}X_{22}\right)y_1 = 0.$$

14. Asymptotics for $\dot{W}(u)$

To determine the asymptotic series for $\dot{W}(u)$ all that we need to do is differentiate the asymptotic series for $W(u)$ with respect to z doing this gives

$$\dot{W}(u) = \frac{\partial}{\partial z}\left[(I + \frac{A_n}{u} + \frac{B_n}{u^2} + \frac{C_n}{u^3} + O(1/u^4))u^{n\sigma_3}(u-z)^{\sigma_3/2}\right]$$

$$= (\frac{\dot{A}_n}{u} + \frac{\dot{B}_n}{u^2} + \frac{\dot{C}_n}{u^3} + O(1/u^4))u^{n\sigma_3}(u-z)^{\sigma_3/2} - (I + \frac{A_n}{u} + \frac{B_n}{u^2} + O(1/u^3))\frac{\sigma_3}{2(u-z)}u^{n\sigma_3}(u-z)^{\sigma_3/2}$$

$$= \frac{1}{u}(\dot{A}_n + \frac{\dot{B}_n}{u} + \frac{\dot{C}_n}{u^2} + O(1/u^3))u^{n\sigma_3}(u-z)^{\sigma_3/2}$$

$$-\frac{1}{2u}(I + \frac{A_n}{u} + \frac{B_n}{u^2} + O(1/u^3))\sigma_3(1 + \frac{z}{u} + \frac{z^2}{u^2} + O(1/u^3))u^{n\sigma_3}(u-z)^{\sigma_3/2}$$

$$= \frac{1}{u}(H_0 + \frac{H_1}{u} + \frac{H_2}{u^2} + O(1/u^3))u^{n\sigma_3}(u-z)^{\sigma_3/2}.$$

were H_0, H_1 and H_2 are given by

$$\begin{aligned} H_0 &= -\frac{\sigma_3}{2} + \dot{A}_n \\ H_1 &= -\frac{z\sigma_3}{2} - \frac{A_n\sigma_3}{2} + \dot{B}_n \\ H_2 &= -\frac{B_n\sigma_3}{2} - \frac{zA_n\sigma_3}{2} - \frac{z^2\sigma_3}{2} + \dot{C}_n. \end{aligned}$$

15. The z-differential equation

To determine the z-differential equation apply our asymptotic expansion to get

$$\begin{aligned} (u-z)(1-u^2)\dot{W}(u)[W(u)]^{-1} &= \frac{(u-z)(1-u^2)}{u}(H_0 + \frac{H_1}{u} + \frac{H_2}{u^2} + O(\frac{1}{u^3}))(I + \frac{\Delta_1}{u} + \frac{\Delta_2}{u^2} + O(1/u^3)) \\ &= (-u^2+zu+1)H_0 + (-u+z)H_1 - H_2 + (-u+z)H_0\Delta_1 - H_1\Delta_1 - H_0\Delta_2. \end{aligned}$$

Once again it follows from Liouville's theorem that this result is exact and therefore we have that

$$(1-u^2)(u-z)\dot{W}(u) = A(u)W(u).$$

were $A(u) = [A_{ij}(u)]$ is given by

$$A(u) \coloneqq (-u^2+zu+1)H_0 + (-u+z)H_1 - H_2 + (-u+z)H_0\Delta_1 - H_1\Delta_1 - H_0\Delta_2.$$

We can also write our the entries of $A(u)$ explicitly using the notation $A_n = [p_{ij}]$, $B_n = [q_{ij}]$ and $C_n = [r_{ij}]$:

$$\begin{aligned} A_{11}(u) &= (-u^2+zu+1)(-\frac{1}{2} + \dot{p}_{11}) + (-u+z)(-\frac{z}{2} + \dot{p}_{11} + \dot{q}_{11}) + \frac{z^2}{2} + \frac{zp_{11}}{2} \\ &+\frac{q_{11}}{2} - \dot{r}_{11} + (-u+z)(-\frac{p_{11}}{2} - \dot{p}_{11}p_{11} - \dot{p}_{22}p_{21}) - \frac{zp_{11}}{2} - p_{11}^2 + p_{12}p_{21} + \dot{q}_{11}p_{11} + \dot{q}_{12}p_{21} \\ &-p_{11}^2 - p_{12}p_{21} + \dot{p}_{11}(p_{11}^2 + p_{12}p_{21}) + \dot{p}_{12}(p_{21}p_{11} + p_{22}p_{21}) + \frac{q_{11}}{2} - \dot{p}_{11}q_{11} - \dot{p}_{12}q_{21}, \\ A_{12}(u) &= (-u^2+zu+1)\dot{p}_{12} + (-u+z)(\frac{p_{12}}{2} + \dot{q}_{12}) - \frac{q_{12}}{2} - \frac{zp_{12}}{2} - \dot{r}_{12} \\ &+(-u+z)(\frac{p_{12}}{2} - \dot{p}_{11}p_{12} - \dot{p}_{12}p_{22}) - \frac{zp_{12}}{2} - p_{11}p_{12} + p_{12}p_{22} + \dot{q}_{11}p_{12} - p_{11}p_{12} - p_{12}p_{22} \\ &\dot{p}_{11}(p_{11}p_{12} + p_{12}p_{22}) + \dot{p}_{12}(p_{12}p_{21} + p_{22}^2) + \frac{q_{12}}{2} - \dot{p}_{11}q_{12} - \dot{p}_{12}q_{22}, \\ A_{21}(u) &= (-u^2+zu+1)\dot{p}_{21} + (-u+z)(-\frac{p_{21}}{2} + \dot{q}_{21}) + \frac{q_{21}}{2} + \frac{zp_{21}}{2} - \dot{r}_{21} \\ &+(-u+z)(-\frac{p_{21}}{2} - \dot{p}_{21}p_{11} - \dot{p}_{22}p_{21}) + \frac{zp_{21}}{2} - p_{21}p_{11} + p_{22}p_{21} + \dot{q}_{21}p_{11} + \dot{q}_{22}p_{21} \\ &+p_{11}p_{21} + p_{21}p_{22} + \dot{p}_{21}(p_{11}^2 + p_{12}p_{21}) + \dot{p}_{22}(p_{21}p_{11} + p_{22}p_{21}) - \frac{q_{21}}{2} - \dot{p}_{21}q_{11} - \dot{p}_{22}q_{21} \\ A_{22}(u) &= (-u^2+zu+1)(\frac{1}{2} + \dot{p}_{22}) + (-u+z)(\frac{z}{2} + \frac{p_{22}}{2} + \dot{q}_{22}) - \frac{z^2}{2} - \frac{q_{22}}{2} \\ &-\frac{zp_{22}}{2} - \dot{r}_{22} + (-u+z)(\frac{p_{22}}{2} - \dot{p}_{21}p_{12} - \dot{p}_{22}p_{22}) + \frac{zp_{21}}{2} - p_{21}p_{22} + p_{22}^2 + \dot{q}_{21}p_{12} + \dot{q}_{22}p_{22} \\ &-p_{22}^2 - p_{12}p_{21} + \dot{p}_{21}(p_{11}p_{12} + p_{12}p_{22}) + \dot{p}_{22}(p_{21}p_{12} + p_{22}^2) - \frac{q_{22}}{2} - \dot{p}_{21}q_{12} + \dot{p}_{22}q_{22}. \end{aligned}$$

were Using the same method as for the u-dfferiental equation we can determine that our OPs will satisfy

$$\ddot{y} - \left(\frac{\dot{M}_{12}}{M_{12}} + M_{11} + M_{22}\right)\dot{y} + \left(\frac{\dot{M}_{12}M_{11}}{A_{12}} - M_{12}M_{21} + M_{11}M_{22}\right)y = 0.$$

were $M = [M_{ij}]$ is given by

$$M = \frac{A(u)}{(1-u^2)(u-z)} - \frac{I}{2(u-z)}.$$

16. Concluding Remarks

Future work on the this project includes searching for formulas for α_n and β_n as functions of z. In particular we would like to understand the singularities of α_n and β_n. Looking at the table in section 3 it seems that singularities of α_n and β_n are roots of Legendre polynomials. Another future goal is to generalize these results for weights of the form $w(x) = (x-z_1)(x-z_2)\cdots(x-z_n)$ instead of $w(x) = x - z$. Finally we would like to look for a way to write our string equations in terms of recurrence coefficients without reference to the coefficients of our OPs.

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