HIGHER-ORDER SINGULARITIES IN VEC-VARIETIES

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Abstract. A **Vec**-variety is a functor from the category of vector spaces over a field K to the category of affine algebraic K-varieties, and many varieties arising in the geometry of tensors naturally fit into this framework. We study higher-order singularities for **Vec**-varieties. We show that, for every $m \geq 1$, the locus of points of embedding codimension at least m is eventually represented by a unique closed **Vec**-subvariety. We use these loci to describe the core of an irreducible **Vec**-variety and give a sufficient condition for the core to coincide with the singular locus. We also prove an analogous eventual functoriality result for loci defined by vanishing of higher-order derivatives.

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1. Introduction

A vector space W comes equipped with a natural structure as an affine space. If W is a G-representation for some group G, we can consider affine algebraic varieties embedded in W which are stable under the action of G. In particular, we are concerned with the case when V is another vector space and W is a $\mathrm{GL}(V)$-representation. It turns out that many examples of such varieties actually behave functorially with respect to the vector space V. We call such an object a **Vec**-variety.

It is natural to wonder whether classical properties of affine algebraic varieties–such as irreducibility, topological noetherianity, and singularity–are compatible with this functorial structure. These topics have been extensively studied in the past few years, and in many cases there are natural ways to translate these concepts into the language of **Vec**-varieties. We provide an introduction to some of this theory before focusing on our primary question of study: how do finer invariants of singularities of these **Vec**-varieties behave as we vary the initial vector space V? We first discuss properties such as embedding codimension at a point and vanishing of higher-order derivatives, before explaining how we can make sense of these properties for **Vec**-varieties and proving a series of short results about their behavior.

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Throughout this paper, unless otherwise stated, we fix K to be an algebraically closed field of characteristic 0.

1.1. **Vec-Varieties.** We begin with a few brief introductory remarks on the theory of **Vec**-varieties. Fixing K to be a field of characteristic 0, we let **Vec** and **Var** be the categories of vector spaces and affine algebraic varieties over K respectively. We say that a functor

$$P : \mathbf{Vec} \to \mathbf{Vec}$$

is a *polynomial functor* if $P(V)$ is a quotient of a direct sum of tensor powers of V. We then define a **Vec**-variety to be a functor

$$X : \mathbf{Vec} \to \mathbf{Var}$$

such that, for every vector space V, $X(V)$ is an affine algebraic variety embedded in the affine space $P(V)$.

Example 1.1. Let P be the polynomial functor given by $P(V) = V \otimes V$. Note that $P(V)$ is just the space of $\dim(V) \times \dim(V)$ matrices over K, that is, as an affine space, $P(V) \cong \mathbb{A}^{\dim(V)^2}$. Then, for any $r \in \mathbb{Z}_{\geq 0}$ we obtain a **Vec**-variety by taking

$$X_r(V) = \{M \in V \otimes V \mid \operatorname{rank}(M) \leq r\}.$$

Notice that linear maps on V induce linear maps on $V \otimes V$. Since the induced linear maps cannot increase the rank of tensors, it follows that X_r is indeed stable under the action of $\mathrm{GL}(V)$. ♫

The **Vec**-variety X_r will provide an excellent example moving forward, and in many cases its behavior provided motivation for our results. We will provide a more detailed introduction to the theory of **Vec**-varieties in Section 2. For additional background, we strongly recommend [CDD25]. We will also briefly present the equivalent perspective of **GL**-varieties, which will occasionally be preferable. One can read more about **GL**-varieties in [CDS26].

1.2. **Higher Order Singularities.** The notion of singularity plays a fundamental role in algebraic geometry. It is only natural to ask what the singular points of a **Vec**-variety look like for each $V \in \mathbf{Vec}$. In the case of Example 1.1, it is well known that, for V with $\dim(V) > r$, the singular locus of $X_r(V)$ is precisely $X_{r-1}(V)$. In fact, it has been shown that for any **Vec**-variety X, the singular locus of $X(V)$ admits functorial behavior when $\dim V \gg 0$. More precisely, we have the following result:

Theorem 1.2 ([CDD25])**.** *Let X be a* **Vec***-variety. Then there exists a unique closed* **Vec***-subvariety X_{Sing} of X such that $X_{\mathrm{Sing}}(V) = \mathrm{Sing}(X(V))$ for all $V \in \mathbf{Vec}$ with $\dim(V) \gg 0$. This X_{Sing} satisfies $X_{\mathrm{Sing}}(V) \supset \mathrm{Sing}(X(V))$ for all $V \in \mathbf{Vec}$.*

We generalized this result to show that higher-order singular loci exhibit a similar functorial behavior. Let $\mathrm{Sing}_m(X(V))$ be the set of points $x \in X(V)$ such that $\operatorname{embcodim}_x(X(V)) \geq m$ (for detailed definition, see Definition 2.4). In particular, $\mathrm{Sing}_1(X(V))$ coincides with the usual singular locus of $X(V)$. We show that the methods used to prove Theorem 1.2 extend naturally to the following result.

Theorem A. Let X be an irreducible **Vec**-variety. Then, for any $m \in \mathbb{Z}_{\geq 0}$ there exists a unique closed **Vec**-variety $X_{\mathrm{Sing}_m} \subset X$ such that $\mathrm{Sing}_m(X(V)) \subseteq X_{\mathrm{Sing}_m}(V)$ for all $V \in \mathbf{Vec}$, and for V such that $\dim(V) \gg 0$ we have that

$$X_{\mathrm{Sing}_m}(V) = \mathrm{Sing}_m(X(V)).$$

Notice that here we assume X to be irreducible. This assumption is imposed to ensure that X is equidimensional– when X is reducible, the locus $\mathrm{Sing}_m(X(V))$ need not be closed (see Remark 2.10). However, we have shown that the following result does hold.

Theorem B. Let X be a **Vec**-variety. Then for any $m \in \mathbb{Z}_{\geq 0}$, there exists a unique closed **Vec**-subvariety X_{Sing_m} of X such that $\overline{\mathrm{Sing}_m(X(V))} \subseteq X_{\mathrm{Sing}_m}(V)$ for all $V \in \mathbf{Vec}$, and for V such that $\dim V \gg 0$ we have that

$$X_{\mathrm{Sing}_m}(V) = \overline{\mathrm{Sing}_m(X(V))}$$

Just like the situation in Theorem 1.2, this behavior is illustrated by Example 1.1.

Example 1.3. Given $M \in X_r(V)$ with $\dim(V) = n \geq r$, we can compute that $\operatorname{embcodim}_M(X_r(V))$ has the following value:

$$\operatorname{embcodim}_M(X_r) = \begin{cases} 0, & \text{if } \operatorname{rank}(M) = r \\ (n-r)^2, & \text{if } \operatorname{rank}(M) \leq r-1 \end{cases}$$

In particular, if n is sufficiently large so that $(n-r)^2 \geq m$, then $\mathrm{Sing}_m(X_r(V))$ coincides with $X_{r-1}(V) \subseteq X_r(V)$. ♫

1.3. **Connection to the Core.** Let X be an irreducible **Vec**-variety and $X_{\mathrm{Sing}_m} \subset X$ be as in Theorem A. We then define the *core* of a **Vec**-variety to be

$$X_{\mathrm{core}} = \bigcap_{m=1}^{\infty} X_{\mathrm{Sing}_m}.$$

We show that this definition is equivalent to that originally given for **GL**-varieties in [CDS26]. Moreover, we give a sufficient criterion for when $X_{\mathrm{core}} = X_{\mathrm{Sing}}$, that is, when the core and singular locus coincide.

Theorem C. Let P and Q be polynomial functors such that $Q(\{0\}) = \{0\}$, and let $\alpha : Q \to P$ be a natural transformation such that for every $V \in \mathbf{Vec}$ with $\dim V \gg 0$ we have $\mathrm{Im}(\alpha_V) = X(V)$. Then,

$$X_{\mathrm{core}} = X_{\mathrm{Sing}}.$$

1.4. **A Higher-Derivative Locus.** Let $X \subseteq \mathbb{A}^n$ be an affine algebraic K-variety with defining ideal $I = (f_1, ..., f_r)$ where $f_1, ..., f_r \in K[x_1 ..., x_n]$. Then, for $m \in \mathbb{Z}_{\geq 0}$ we define $\partial_m I \subset K[x_1, ..., x_n]$ to be the ideal generated by all the $\leq m$ partial derivatives of the f_i. We then define the closed subvariety $\mathrm{Mult}_m(X)$ to be the vanishing locus of $\partial_{m-1} I$. Given a **Vec**-variety X, we wish to investigate how $\mathrm{Mult}_m(X(V))$ behaves as we vary $V \in \mathbf{Vec}$. Once again, we refer to Example 1.1 for motivation.

Example 1.4. For the **Vec**-variety X_r we have that $\mathrm{Mult}_m(X_r(V)) = X_{r-m+1}(V)$ whenever $r \geq m-1$ and $\mathrm{Mult}_m(X_r(V)) = \{0\} = X_0$ for all $r > m-1$. ♫

This example suggests that the locus $\mathrm{Mult}_m(X)$ may also behave functorially. We have shown that this is precisely the case.

Theorem D. For $m \in \mathbb{Z}_{\geq 0}$, there exists a unique **Vec**-subvariety $X_{\mathrm{Mult}_m} \subset X$ such that for all $V \in \mathbf{Vec}$ we have that $X_{\mathrm{Mult}_m}(V) \subseteq \mathrm{Mult}_m(X(V))$, and for V such that $\dim(V) \gg 0$, we have that

$$X_{\mathrm{Mult}_m}(V) = \mathrm{Mult}_m(X(V)).$$

We also give an explicit sharp lower bound for the dimension of V required for equality to hold.

2. Background

We assume familiarity with basic concepts in classical algebraic geometry, as well as with the basics of scheme theory. We begin by briefly recalling some equivalent notions of singularity in classical algebraic geometry, since we will be using these extensively throughout future sections. We also provide an introduction to **Vec**-varieties, state a number of important results about these objects, and explain the closely related notion of **GL**-variety.

2.1. **Notation.** We list some important notations:

- K: The coefficient field of characteristic 0.
- $\mathbf{Vec}_K$/**Vec**: The category of finite dimensional vector spaces over the field K.
- $\mathbf{AffSch}_K^{\mathrm{ft}}$: The category of affine schemes of finite type over the field K.
- **V**: Infinitemensional K-vector space with basis $\{e_i\}_{i\geq 1}$.
- **GL**: The group of automorphisms of **V** fixing almost all basis $\{e_i\}_{i\geq 1}$.
- $G(n)$: The subgroup of **GL** fixing $e_1, \ldots, e_n$ and mapping $\mathrm{span}\{e_i\}_{i>n}$ into itself.
- $\mathcal{O}_{X,x}$: the local ring of a scheme theoretic point x in a scheme X.
- Ω_X, Ω_R: Let $X = \mathrm{Spec}(R)$ be a **GL**-variety or **Vec**-variety, we use both Ω_X and Ω_R to denote the Kähler differentials of X over K, depending on the context.

2.2. **Singularities in Finite-Dimensional Algebraic Geometry.** Let $X = \mathrm{Spec}(A)$ be a reduced affine scheme over K. We refer to X as a *variety* to mean the closed points of X. Let $\{\mathfrak{q}_i\}_{i\in I}$ be the minimal primes of A for some suitable index set I. Let x be a scheme theoretic point of X. If A is of finite-type, the *dimension of X at x* is the maximal Krull-dimension $\dim_{\mathrm{Krull}}(A/\mathfrak{q}_i)$ for all the minimal primes $\mathfrak{q}_i$ contained in the associated prime ideal $\mathfrak{p}$ of x. If x is a closed point, the dimension of X at x coincides with the height of x.

Definition 2.1. Let $x \in X$ be a scheme-theoretic point, and let $(\mathcal{O}_{X,x}, \mathfrak{m}_x)$ be the stalk at x. The *tangent cone* of X at x is the spectrum of the graded k-algebra

$$\mathrm{gr}(\mathcal{O}_{X,x}) = \bigoplus_{i\geq 0} \mathfrak{m}_x^i/\mathfrak{m}_x^{i+1}.$$

♪

Assuming A Noetherian, it is a standard result that the Krull-dimension of $\mathrm{gr}(\mathcal{O}_{X,x})$ coincides with the Krull-dimension of the local ring $\mathcal{O}_{X,x}$, see [Eis95, Exercise 13.8]. If x is a closed point, then $\dim \mathcal{O}_{X,x} = \max_{x\in X_i} \dim X_i$, where the maximum is taken over the irreducible components X_i of X containing x. Thus the dimension of the tangent cone at x equals the maximum of the dimensions of the irreducible components of X passing through x.

Definition 2.2. Let $x \in X$ be a scheme-theoretic point, and let $(\mathcal{O}_{X,x}, \mathfrak{m}_x)$ be the stalk at x. The *tangent space* of X at x is the spectrum of the k-algebra:

$$\mathrm{Sym}(\mathfrak{m}_x/\mathfrak{m}_x^2).$$

♪

Let $\Omega_{X/K}$ be the Kähler differentials of X over K.

Proposition 2.3 (Proposition 8.7, [Har77])**.** *Let $x \in X$ be a scheme-theoretic point. Let $(\mathcal{O}_{X,x}, \mathfrak{m}_x)$ be the stalk at x and let $\kappa(x) = \mathcal{O}_{X,x}/\mathfrak{m}_x$ be the residue field of x. Then there is a natural exact sequence of K-modules:*

$$\mathfrak{m}_x/\mathfrak{m}_x^2 \longrightarrow \Omega_{X/K,x} \otimes_{\mathcal{O}_{X,x}} \kappa(x) \longrightarrow \Omega_{\kappa(x)/K} \longrightarrow 0.$$

In particular, if x is a K-rational point, then the first map is a canonical isomorphism $\mathfrak{m}_x/\mathfrak{m}_x^2 \cong \Omega_{X/K,x} \otimes_{\mathcal{O}_{X,x}} \kappa(x)$.

There is a natural map $\Phi : \mathrm{Sym}(\mathfrak{m}_x/\mathfrak{m}_x^2) \to \mathrm{gr}(\mathcal{O}_{X,x})$ that corresponds to the inclusion of the tangent cone in the tangent space.

Definition 2.4. Let x be a scheme-theoretic point of X. The *embedding codimension* of X at x is the height of the kernel of Φ. ♪

For finite-dimensional varieties, the above definition is analogous to computing the difference between the dimension of X at x (namely, the dimension of $\mathrm{gr}_x(\mathcal{O}_{X,x})$) and the dimension of the tangent space of X at x. The equality $\dim_{\mathrm{Krull}}(\mathrm{Sym}(\mathfrak{m}_x/\mathfrak{m}_x^2)) = \mathrm{ht}(\ker(\Phi)) + \dim_{\mathrm{Krull}}(\mathrm{gr}(\mathcal{O}_{X,x}))$ holds as these are all K-algebras of finite-type.

Definition 2.5. Let x be a scheme-theoretic point of X. We say that x has *order of singularity* m if its embedding codimension is m. We denote by $\mathrm{Sing}_m(X)$ the subset of points of X with order of singularity at least m. In particular, $\mathrm{Sing}_1(X)$ is the usual singular locus of X, and we simply call a point p of $\mathrm{Sing}_1(X)$ *singular*. ♪

Note that it follows immediately from this definition that $\mathrm{Sing}_{m'}(X) \subseteq \mathrm{Sing}_m(X)$ whenever $m' \geq m$. Next, we recall an equivalent definition of embedding codimension.

Definition 2.6. If $X = \mathrm{Spec}(A)$ where $A \cong K[x_1, ..., x_n]/I$ is a finitely-generated K-algebra, we can find generators $f_1, ..., f_r$ for I. Then, given a closed point $p \in X \subseteq \mathbb{A}^n$, we define the *Jacobian matrix* of X at p to be

$$\mathcal{J}_X(p) = \left(\frac{\partial f_i}{\partial x_j}(p)\right)_{\substack{1\leq i\leq r\\ 1\leq j\leq n}}.$$

♪

For irreducible affine algebraic varieties, the Jacobian allows us to give a characterization of embedding codimension in terms of polynomial equations.

Proposition 2.7. *If X is irreducible, then, for a closed point $p \in X$ as above, the embedding codimension of X at p is given by* $\mathrm{embcodim}_x(X) = n - \dim_p(X) - \mathrm{rank}(\mathcal{J}_p(X))$. *This is a classical fact; see [Har77, Theorem 5.1].*

We thus have that the locus $\mathrm{Sing}_m(X)$ is given by the vanishing loci of certain minors of J_X. In particular, we have the following corollary.

Corollary 2.8. *For an irreducible algebraic variety X, the locus $\mathrm{Sing}_m(X)$ is a closed subvariety.*

Remark 2.9. For $m = 1$, we have that the locus $\mathrm{Sing}_1(X)$ is always closed. This is due to the fact that, if we decompose X into irreducible components $X = X_1 \cup \cdots \cup X_r$ such that $X_i \not\subset X_j$ for any $i \neq j$, then $X_i \cap X_j \subseteq \mathrm{Sing}_1(X)$ whenever $i \neq j$. We can thus write

$$\mathrm{Sing}_1(X) = \mathrm{Sing}_1(X_1) \cup \cdots \cup \mathrm{Sing}_1(X_r) \cup \bigcup_{1\leq i<j\leq r} X_i \cap X_j.$$

All the above subsets are closed in X, so we have that $\mathrm{Sing}_1(X)$ is itself closed. ♪♫

Remark 2.10. Note that for $m \geq 2$, $\mathrm{Sing}(Z)$ is not necessarily closed for a variety Z. Let $m = 2$. Consider the morphism $K[x, y, z] \to K[t]$ sending $x \mapsto t^3, y \mapsto t^4, z \mapsto t^5$, and let C be the corresponding curve whose defining ideal is $I = (xz - y^2, x^3 - yz, x^2y - z^2)$. One can check that the origin is a singular point of order of singularity 2. In $\mathbb{A}^4$, consider the union Z of $C \times \mathbb{A}^1$ with $\mathbb{A}^3 \times \{0\}$. The point $(\underline{0}, s) \in Z \subset \mathbb{A}^3 \times \mathbb{A}^1$ has embedding codimension one when $s = 0$, while it has embedding codimension 2 for $s \neq 0$. This shows that the locus $\mathrm{Sing}_2(Z)$ is not closed in Z. See [Chi26, Proposition 3.10] for a result on arbitrary finite-type schemes. ♪♫

We can also ask when $\text{embcodim}_p(X)$ is the worst it could possibly be; that is, at what points $p \in X$ do we have that $T_pX \cong \mathbb{A}^n$. This motivates the following definition.

Definition 2.11. Let $X \subseteq \mathbb{A}^n$ be an affine algebraic variety with defining ideal $I(X) \subseteq K[x_1, ..., x_n]$. For $m \in \mathbb{Z}_{\geq 1}$ we define the mth differential ideal of X, denoted by $\partial_m I(X)$, to be the ideal generated by the set

$$\left\{ \frac{\partial^{|\alpha|} f}{\partial x^\alpha} \mid f \in I(X) \text{ and } \alpha \in \mathbb{Z}^m_{\geq 0}, |\alpha| \leq m-1 \right\}.$$

Then, for $m \in \mathbb{A}_{\geq 1}$, we define the *multiplicity* $\geq m$ *locus* of X, denoted by $\text{Mult}_m(X)$ to be the vanishing locus of the ideal $\partial_{m-1} I(X)$. We define $\text{Mult}_0(X) := \mathbb{A}^n$. ♪

In other words, we say a point $p \in X$ is in $\text{Mult}_m(X)$ if, for all polynomials $f \in I(X)$, all kth partial derivatives of f vanish at p for $0 \leq k \leq m-1$. It is immediate from this definition that $\text{Mult}_m(X)$ is closed in X. Moreover, we can easily see that

$$\text{Mult}_{m'}(X) \subseteq \text{Mult}_m(X)$$

for all $m' \geq m$.

We first give a few examples of Mult_m for some varieties

Example 2.12. When $m = 1$ we have that $\text{Mult}_1(Y) = Y$. When $m = 2$ we have that $\text{Mult}_2(Y)$ consists of points $p \in Y$ where $T_pY \cong \mathbb{A}^n$. Notice that, if $Y \subsetneq \mathbb{A}^n$ then $\text{Mult}_2(Y) \subseteq \text{Sing}(Y)$ and if Y is a hypersurface, then this is in fact an equality. Moreover, the latter automatically gives us that $\text{Mult}_m(Y) \subseteq \text{Sing}(Y)$ for all $m \geq 2$. ♫

Example 2.13. Fix $n \geq 2, r < n$ and $X_r \subset \mathbb{A}^{n^2}$ be the closed variety of $n \times n$ matrices of rank $\leq r$. One can easily check that, for $m \leq r+1$,

$$\text{Mult}_m(X_r) = X_{r-m+1}.$$

♫

As we will see, this example motivates our interest in the behavior of Mult_m for **Vec**-varieties.

Definition 2.14. If $p \in X$ and $m \in \mathbb{Z}_{\geq 0}$ such that $p \in \text{Mult}_m(X)$ but $p \notin \text{Mult}_{m+1}(X)$ then we say that the point p has *multiplicity* m in X. ♪

Remark 2.15. There is an easy characterization for when a point $p \in X$ is in $\text{Mult}_m(X)$. We first notice that when $p = (0, ..., 0) \in \mathbb{A}^n$, then $p \in \text{Mult}_m(X)$ if and only if $\deg(\text{in}(f)) \geq m$ for all $f \in I(X)$ where $\text{in}(f)$ is the lowest-degree term of f. If $f = 0$ then we use the convention that $\deg(\text{in}(f)) = +\infty$. A change of coordinates gives us that a point $p = (p_1, ..., p_n)$ is in $\text{Mult}_m(X)$ if and only if $\deg(\text{in}(f_p)) \geq m$ for all $f \in I(X)$ where

$$f_p(x_1, ..., x_n) = f(x_1 + p_1, ..., x_n + p_n).$$

♪♫

The following lemma follows immediately from the Leibniz Rule.

Lemma 2.16. *Let $X \subseteq \mathbb{A}^n$ be an affine algebraic variety such that $I(X) = (f_i)_{i \in I}$ for some indexing set I and $f_i \in k[x_1, ..., x_n]$. Then, for $m \in \mathbb{Z}_{\geq 1}$ if $p \in \text{Mult}_{m-1}(X)$ we have that $p \in \text{Mult}_m(X)$ if and only if*

$$\frac{\partial^{|\alpha|} f_i}{\partial x^\alpha}(p) = 0$$

for all i and $\alpha \in \mathbb{Z}^m_{\geq 0}, |\alpha| = m-1$.

In particular, it suffices to check the criterion for $p \in \text{Mult}_m(X)$ on just the generators for $I(X)$. We will make use of this fact in section 5.

2.3. **Vec-varieties.** We now introduce our main objects of study, **Vec**-varieties, and state some fundamental results which we will make use of throughout the rest of the article. As before, K is an algebraically closed field of characteristic 0 and we write **Vec** for the category of vector spaces over K, **Var** for the category of affine algebraic varieties over K, and $\mathbf{AffSch}_K$ for the category of affine schemes over K.

2.3.1. *Polynomial Functors.* We first recall some basic tools from representation theory. In particular, given a vector space $V \in \mathbf{Vec}$, we can naturally view V as a $\mathrm{GL}(V)$-representation. Moreover, this induces an action on direct sums and tensor powers of V, giving $\otimes^d V$ and $\oplus^d V$ the structures of $\mathrm{GL}(V)$ representations as well for any $d \in \mathbb{Z}_{\geq 1}$.

Moreover, recall that given any $d \in \mathbb{Z}_{\geq 1}$ and any partition λ of $\{1, ..., d\}$, we get an irreducible sub-representation of $\otimes^d V$ denoted by

$$\mathbb{S}_\lambda(V).$$

The functor $\mathbb{S}_\lambda : \mathbf{Vec} \to \mathbf{Vec}$ which maps V to $\mathbb{S}_\lambda(V)$ is called the *Schur functor* associated to λ. We say that the Schur functor S_λ has *degree* d. For our purposes here, we also include constant functors as a type of Schur functor: that is, given some vector space $W \in \mathbf{Vec}$ we get a functor

$$\mathbb{S}_W : \mathbf{Vec} \to \mathbf{Vec}$$

where $\mathbb{S}_W(V) = W$ and $\mathbb{S}_W(\varphi) = \mathrm{Id}_W$ for any vector space V and any K-linear map φ. We say that $\mathbb{S}_W$ is a *degree* 0 Schur functor.

We will not give a further introduction to Schur functors here, so we encourage anyone not familiar with the topic to refer to [FH04, Chapter 6.1]. This brings us to our first main definition.

Definition 2.17. A *polynomial functor* is a functor $P : \mathbf{Vec} \to \mathbf{Vec}$ that is isomorphic to a finite direct sum of Schur functors. For $V \in \mathbf{Vec}$ and a linear map $\varphi : V \to W$, we write $P(V)$ for the image of V and $P(\varphi) : P(V) \to P(W)$ for the induced linear map. ♪

In particular, P is a functor on $\mathbf{Vec}$ that takes V to a sub-$\mathrm{GL}(V)$-representation of

$$W \oplus \bigoplus_{i=1}^{n} \bigotimes^{d_i} V$$

for some $n \in \mathbb{Z}_{\geq 0}$, $d_1, ..., d_n \in \mathbb{Z}_{\geq 1}$ and some fixed $W \in \mathbf{Vec}$. Notice that P can be uniquely decomposed as

$$P_0 \oplus P_1 \oplus \cdots \oplus P_d$$

where P_i is the direct sum of all degree i Schur sub-functors of P and we assume P_d is nontrivial. We then say that P_i is the *ith degree component* of P and that P has degree d. Moreover, we say that P is *homogeneous* if $P = P_d$.

2.3.2. *A partial order on polynomial functors.* The notion of degree allows us to define a well-founded partial order on polynomial functors. Namely, given two polynomial functors $P, Q : \mathbf{Vec} \to \mathbf{Vec}$, we say that $Q \prec P$ if Q_e is a quotient of P_e, where $e \in \mathbb{Z}_{\geq 0}$ is the largest nonnegative integer such that $Q_e \ncong P_e$.

2.3.3. *Examples of polynomial functors.* We now give a few basic examples of polynomial functors.

Example 2.18. As previously mentioned, given any vector space $W \in \mathbf{Vec}$ we get a constant polynomial functor $\mathbb{S}_W$ which assigns to every vector space W and to every K-linear map Id_W. Again, we note that this polynomial functor is homogeneous of degree 0. ♫

We say that a polynomial functor is *pure* if it has no degree 0 component. The following are both pure, homogeneous polynomial functors.

Example 2.19. The identity functor $\mathrm{Id}_{\mathbf{Vec}} : \mathbf{Vec} \to \mathbf{Vec}$ which assigns to every $V \in \mathbf{Vec}$ itself and to any $\varphi \in \mathrm{Hom}(V, W)$ itself, is a polynomial functor of degree 1. ♫

Example 2.20. The functor which takes V to $V \otimes V$ is a degree 2 polynomial functor. Recall that this is the polynomial functor which appears in Example 1.1, and will be referred to throughout the remainder of this and future sections. ♫

Now, let P be a polynomial functor and observe the following: given $V, W \in \mathbf{Vec}$ and $\varphi \in \mathrm{Hom}(V, W)$ such that φ is injective, we can construct another map $\pi \in \mathrm{Hom}(W, V)$ such that $\pi \circ \varphi = \mathrm{Id}_V$. We thus have that

$$P(\pi) \circ P(\varphi) = P(\pi \circ \varphi) = P(\mathrm{Id}_V) = \mathrm{Id}_{P(V)}.$$

In particular, this means that $P(\varphi)$ is also injective. A very similar argument shows that if φ is surjective, then $P(\varphi)$ must also be surjective. We will use both these facts liberally in the discussions to come. Moreover, this means that P restricts to a group homomorphism

$$P : \mathrm{GL}(V) \to \mathrm{GL}(P(V))$$

for any $V \in \mathbf{Vec}$, thus giving us that $P(V)$ is a $\mathrm{GL}(V)$-representation. We say that P is *irreducible* if each $P(V)$ is an irreducible $\mathrm{GL}(V)$-representation.

2.3.4. *Shifting.* Given a fixed polynomial functor P and a fixed vector space $U \in \mathbf{Vec}$, we can construct another polynomial functor known as the *shift* of P by U.

Definition 2.21. Let U be a finite-dimensional vector space, and let $P : \mathbf{Vec} \to \mathbf{Vec}$ be a polynomial functor. The *shift* of P by U, denoted $\mathrm{Sh}_U P$, is the polynomial functor defined by

$$(\mathrm{Sh}_U P)(V) = P(U \oplus V)$$

for every $V \in \mathbf{Vec}$, and

$$(\mathrm{Sh}_U P)(\varphi) = P(\mathrm{Id}_U \oplus \varphi)$$

for every linear map $\varphi \in \mathrm{Hom}(V, W)$. ♪

This construction will come up a number of times, as taking shifts can often help us simplify polynomial functors and, more importantly, their subfunctors.

2.3.5. *Definition of a* **Vec***-variety.* Finally we are ready to define **Vec**-varieties. We first recall that, given a vector space $V \in \mathbf{Vec}$ we can give V the structure of an affine space with coordinate ring $K[\mathrm{Sym}(V^*)]$. This will obviously be compatible with any linear maps, which become algebraic morphisms. In particular, this means that, given a polynomial functor P, we can view P as a functor

$$\mathbb{A}(P) : \mathbf{Vec} \to \mathbf{AffSch}_K$$

where

$$\mathbb{A}(P)(V) = \mathrm{Spec}(K[\mathrm{Sym}(P(V)^*)]) \cong \mathbb{A}^{\dim(P(V)}.$$

We now have the structures in place to make the following definitions.

Definition 2.22. Fix a polynomial functor P. A sub-functor $X : \mathbf{Vec} \to \mathbf{AffSch}_K$ of $\mathbb{A}(P)$ is called a **Vec**-*scheme*. That is, for every $V \in \mathbf{Vec}$ we have a natural map $\alpha : X(V) \to P(V)$ such that α is an embedding. If α is also closed we say that X is a closed **Vec**-scheme. If we take reduced structures, we can view $\mathbb{A}(P)$ as a functor from **Vec** to **Var** and, in this case, we call the subfunctor X a (resp. closed) **Vec**-*variety*. Given a linear map $\varphi \in \mathrm{Hom}(V, W)$ we denote by $X(\varphi) = P(\varphi)|_{X(V)}$ the induced morphism of algebraic varieties from $X(V)$ to $X(W)$. ♪

For the majority of this paper we will be concerned primarily with **Vec**-varieties. Notice that in this setup, polynomial functors act as affine spaces, and **Vec**-varieties as the closed algebraic sets sitting inside of them. Almost any operation one can imagine done in the classical setting (taking finite unions, intersections, looking at sub-varieties, etc.) can be done with **Vec**-varieties by looking at what happens on each individual vector space V.

2.3.6. *Basic Properties of* **Vec***-varieties.* Recall from our discussion in the previous section that polynomial functors preserve injectivity and surjectivity of linear maps. We have an analogous statement for **Vec**-varieties:

Proposition 2.23 ([CDD25, Proposition 2.16])**.** *Let X be a closed* **Vec***-variety in a polynomial functor P, and let $\varphi \in \mathrm{Hom}(V, W)$. Then:*

1. *If φ is injective, then $P(\varphi)$ restricts to a closed embedding $X(V) \hookrightarrow X(W)$.*
2. *If φ is surjective, then $P(\varphi)$ restricts to a surjective morphism $X(V) \twoheadrightarrow X(W)$.*
3. *If $V = W$ and φ is a linear isomorphism, then $P(\varphi)$ restricts to an automorphism $X(V) \xrightarrow{\sim} X(V)$. Moreover, the map*

$$\mathrm{GL}(V) \times X(V) \longrightarrow X(V), \qquad (\varphi, p) \longmapsto P(\varphi)(p),$$

is an algebraic group action.

Just as in the finite-dimensional case, one can define an analog to the Zariski topology on a polynomial functor P by taking the **Vec**-subvarieties in P to be a basis of closed sets. It is easy to check that this does indeed give rise to a topology. As in the classical setting, a fundamental, but highly nontrivial result, is that this topology makes P into a Noetherian topological space.

Theorem 2.24 ([Dra19, Theorem 1])**.** *Let X be a* **Vec***-variety in a polynomial functor P. Then every descending chain of closed* **Vec***-subvarieties*

$$X = X_0 \supseteq X_1 \supseteq X_2 \supseteq \cdots$$

stabilizes. That is, there exists an integer $N \geq 0$ such that $X_n = X_{n+1}$ for every $n \geq N$.

Equivalently, for every closed **Vec***-subvariety $Y \subseteq X$, there exist finite-dimensional vector spaces $U_1, \ldots, U_k \in \mathbf{Vec}$ and regular functions $f_i \in K[X(U_i)], 1 \leq i \leq k$, such that, for every $V \in \mathbf{Vec}$,*

$$Y(V) = \left\{ p \in X(V) : f_i\big(X(\varphi(p)\big) = 0 \text{ for every } 1 \leq i \leq k \text{ and every } \varphi \in \mathrm{Hom}_{\mathbf{Vec}}(V, U_i) \right\}.$$

Moreover, this topology also gives us an analog to the notion of irreducibility.

Definition 2.25. Let X be a **Vec**-variety. We say that X is *irreducible* if, whenever $X = Y \cup Z$ for closed **Vec**-subvarieties $Y, Z \subseteq X$, one has $X = Y$ or $X = Z$. ♪

This definition is compatible with the classical notion of irreducibility at every level in the following sense.

Proposition 2.26. *Let X be a* **Vec***-variety. Then X is irreducible if and only if $X(V)$ is irreducible for every $V \in$* **Vec**.

Proof. Suppose first that $X(V)$ is irreducible for every $V \in$ **Vec**. Assume that $X = Y \cup Z$ for two closed **Vec**-subvarieties $Y \subseteq X$ and $Z \subseteq X$. We show that either $Y = X$ or $Z = X$. If both are proper, then there are vector spaces $U, V \in$ **Vec** such that $Y(U) \neq X(U)$ and $Z(W) \neq X(W)$. Take $V = U \oplus W$. Then $X(V) = Y(V) \cup Z(V)$. Since $X(V)$ is irreducible, either $Y(V) = X(V)$ or $Z(V) = X(V)$. Without loss of generality we assume that $Y(V) = X(V)$ and take $p_U : V = U \oplus W \to U$ to be the natural projection. By Proposition 2.23 we know that $X(p_U) = P(p_U)|_{X(V)} = P(p_U)|_{Y(V)} = Y(p_U)$ is surjective. In particular, this means that

$$X(U) = P(p_U)(X(V)) = P(p_U)(Y(V)) = Y(U).$$

This provides a contradiction, thus completing this direction of the proof.

Conversely, suppose $X(V)$ is reducible for some $V \in$ **Vec**. Then we can write $X(V) = C_1 \cup C_2$ for two proper closed subsets $C_1, C_2 \subsetneq X(V)$. Define

$$Y_1(U) = \{x \in X(U) : X(\varphi)(x) \in C_1 \text{ for every } \varphi \in \mathrm{Hom}(U, V)\}$$

and

$$Y_2(U) = \{x \in X(U) : X(\varphi)(x) \in C_2 \text{ for every } \varphi \in \mathrm{Hom}(U, V)\}$$

for any $U \in$ **Vec**. It is easy to check that both Y_1 and Y_2 are closed **Vec**-subvarieties of X. Moreover, taking $\varphi = \mathrm{Id}_V \in \mathrm{Hom}(V, V)$ we can see that Y_1 and Y_2 must be proper, as otherwise this would contradict the assumption on C_1 and C_2. We claim that $X = Y_1 \cup Y_2$. Given a vector space U and $x \in X(U)$, let $\alpha_x : \mathrm{Hom}(U, V) \to X(V)$ be given by $\alpha_x(\varphi) = X(\varphi)(x)$. Viewing $\mathrm{Hom}(U, V)$ as an affine space, α_x is the restriction of a polynomial map $\mathrm{Hom}(U, V) \to P(V)$ and thus an algberiac morphism. Since $X(V) = C_1 \cup C_2$, $\mathrm{Hom}(U, V) = \alpha_x^{-1}(C_1) \cup \alpha_x^{-1}(C_2)$. Since $\mathrm{Hom}(U, V)$ is irreducible, one of them must equal $\mathrm{Hom}(U, V)$. Say $\alpha_x^{-1}(C_1) = \mathrm{Hom}(U, V)$. Then $X(\varphi)(x) \in C_1$ for every $\varphi \in \mathrm{Hom}(U, V)$. Thus $x \in Y_1(U) \cup Y_2(U)$. Hence X is not irreducible. □

Remark 2.27. An immediate consequence of this topological neotherianity is that every **Vec**-variety admits a unique decomposition into irreducible closed **Vec**-subvarieties,

$$X = X_1 \cup X_2 \cup \cdots \cup X_r$$

such that $X_i \not\subset X_j$ for $i \neq j$. ♪♫

2.3.7. *Embedding theorem.* We now recall an important result: the embedding theorem for **Vec**-varieties. This will allow us to embed an open subset of a **Vec**−variety X into a smaller polynomial functor (where smaller refers to the order defined in the previous sub-section). This will prove a basis for induction arguments in the next section.

Definition 2.28. Let X be a closed **Vec**-subvariety of a polynomial functor P, and let $h \in K[P(0)]$. Define the polynomial functor Q by $Q(V) = K \oplus P(V)$. The *localization* of X at h, denoted $X[1/h]$, is the closed **Vec**-subvariety of Q defined by

$$X[1/h](V) = \{(t, x) \in K \times X(V) \mid t\, h(x) = 1\},$$

where h is regarded as a regular function on $P(V)$ via pullback along the linear map $P(0_V) : P(V) \to P(0)$. We regard $X[1/h]$ as the principal open **Vec**-subvariety of X. ♪

Theorem 2.29 ([CDD25, Theorem 2.30])**.** *Let P be a polynomial functor and X be a proper* **Vec***-variety in P. Let R be an irreducible polynomial subfunctor of P of degree > 0 and let $\pi : P \to P/R$ be the projection transformation. Let X' be the closure of X along π. Then one of the following holds:*

(1) $X = \pi^{-1}(X') \cong X' \times R$; or
*(2) there exists a finite-dimensional vector space $U \in$ **Vec** and a function $h \in k[P(U)]$ that does not vanish identically on $X(U)$ such that the projection $\mathrm{Sh}_U P \to (\mathrm{Sh}_U P)/R$ restricts to a closed embedding*

$$\mathrm{Sh}_U X\left[\frac{1}{h}\right] \to ((\mathrm{Sh}_U P)/R)\left[\frac{1}{h}\right].$$

Furthermore, h can be chosen as the directional derivative $\frac{\partial f}{\partial r}$ for some vector $r \in R(U)$ and some function $f \in k[P(U)]$ that vanishes identically on $X(U)$, and for any h constructed in this manner the map above is a closed embedding.

2.3.8. *Dimension.* We also briefly state a couple of results regarding how the dimension of $X(V)$ varies with V, as these will prove useful to us later. The following result for polynomial functors is well-known.

Proposition 2.30. *Let P be a polynomial functor. Then there exist $f_P \in \mathbb{Q}[n]$ of degree equal to $\deg P$ such that $\dim(P(V)) = f_p(\dim V)$ for every $V \in \mathbf{Vec}$.*

Since P is a direct sum of Schur functors, it suffices to show that the statement is true for the latter. This follows from [FH04, Theorem 6.3]. A much more nontrivial result states that the dimensions of **Vec**-varieties behave in largely the same fashion.

Definition 2.31. Let X be a **Vec**-variety. Define the *dimension function* to be

$$d_X(n) \coloneqq \dim X(K^n).$$

♪

Proposition 2.32 ([CDD25, Proposition 2.38])**.** *Let X be a* **Vec***-variety. Then there exists $p_X(n) \in \mathbb{Q}[n]$ such that $p_X(n) = d_X(n)$ for all $n \gg 0$.*

Note that every vector space $V \in \mathbf{Vec}$ of dimension n is isomorphic, via a choice of basis, to K^n. By Proposition 2.23 this isomorphism induces an isomorphism $X(V) \cong X(K^n)$. So Proposition 2.32 tells us that for $V \in \mathbf{Vec}$ such that $\dim(V) \gg 0$, the dimension of $X(V)$ agrees with a polynomial in the dimension of V.

2.4. **The GL Perspective.** Let P be a polynomial functor and consider some $V \in \mathbf{Vec}$ with $\dim(V) = n$. As just mentioned, choosing a basis for V will give us an isomorphism of V with K^n which induces an isomorphism of $P(V)$ with $P(K^n)$ and of $X(V)$ with $X(K^n)$. That is, there is essentially no information lost by just considering $X(K^n)$.

Moreover, letting $\pi_n : K^n \to K^{n-1}$ be the projection onto the first $n-1$ coordinates, we obtain an inverse system

$$\{0\} \xleftarrow{\pi_1} K \xleftarrow{\pi_2} K^2 \longleftarrow \cdots \longleftarrow K^{n-1} \xleftarrow{\pi_n} K^n \longleftarrow \cdots$$

of vector spaces. We denote the group $\mathrm{GL}(K^n)$ by $G(n)$ and notice that $G(n)$ includes into $G(n+m)$ via a map $\iota_{n,m}$ given by

$$M \mapsto \begin{bmatrix} M & 0 \\ 0 & \mathrm{Id}_m \end{bmatrix}.$$

We thus have that the system

$$\{0\} \hookrightarrow G(1) \hookrightarrow G(2) \hookrightarrow \cdots \hookrightarrow G(n-1) \hookrightarrow G(n) \hookrightarrow \cdots$$

acts on our system of vector spaces. Taking projective and direct limits respectively, we see that the infinite general linear group

$$\mathbf{GL} := \bigcup_{n=0}^{\infty} G(n)$$

acts on the vector space $\mathbf{V} := K^\infty$. Now, applying the polynomial functor P gives us another inverse system of vector spaces

$$P(\{0\}) \xleftarrow{P(\pi_0)} P(K) \xleftarrow{P(\pi_2)} P(K^2) \longleftarrow \cdots \longleftarrow P(K^{n-1}) \xleftarrow{P(\pi_n)} P(K^n) \longleftarrow \cdots$$

where each $P(K^n)$ is a polynomial representation of $G(n)$. Letting $\mathbf{P}$ be the projective limit of this system, we have a natural action of **GL** on $\mathbf{P}$. We then say that $\mathbf{P}$ is a **GL**-*representation* and the induced algebraic structure we discussed given by

$$K[\mathrm{Sym}(P(\{0\})^*)] \hookrightarrow K[\mathrm{Sym}(P(K)^*)] \hookrightarrow \cdots \hookrightarrow K[\mathrm{Sym}(P(K^{n-1})^*)] \hookrightarrow K[\mathrm{Sym}(P(K^n)^*)] \hookrightarrow \cdots$$

is called a **GL**-*algebra*.

Now, suppose that we have a **Vec**-variety X inside our polynomial functor P. This is where it will be helpful to think of the scheme-theoretic structure of X. In particular, we have that X is a functor from **Vec** to $\mathbf{AffSch}_K$. We know from Proposition 2.23 that the inverse systems above will induce an inverse system of affine algebraic varieties

$$X(K^0) \xleftarrow{\pi_1} X(K) \xleftarrow{\pi_2} X(K^2) \longleftarrow \cdots \longleftarrow X(K^{n-1}) \xleftarrow{\pi_n} X(K^n) \longleftarrow \cdots$$

with projective limit $\mathbf{X} \in \mathbf{AffSch}$. We call this $\mathbf{X}$ a **GL**-*variety*. Notice that $\mathbf{X}$ comes with a natural algebraic group action by **GL**.

In general, a polynomial representation of **GL** is defined to be a subquotient of a direct sum of tensor powers of $\mathbf{V}$ (notice that this coincides with our definition of a polynomial functor). A **GL**-algebra is a K-algebra A with an

action of **GL** by automorphisms such that it forms a polynomial representation. We say that a **GL**-algbera A is *finitely generated* if it is generated, as a K-algebra, by the **GL**-orbits of finitely many elements.

Now, given a **GL**-algebra A, a **GL**-*ideal* of A is an ideal that is **GL**-stable. A **GL**-ideal is *finitely* **GL**-*generated* if it is generated, as an ideal, by the **GL**-orbits of finitely many elements. A **GL**-*variety* is an affine scheme $\mathbf{X}$ of the form $\mathrm{Spec}(A)$, where A is a reduced, finitely **GL**-generated **GL**-algebra. It can be shown that the categories of **GL**-representations and polynomial functors are equivalent, as are those of **Vec**-varieties and **GL**-varieties. For these results and more on **GL**-varieties we refer the reader to [CDS26].

One advantage of this approach is that it allows us to talk about points of X, a notion which, in our previous setup, was very ambiguous. In particular, a point x of a **GL**-variety $\mathbf{X}$ is a tuple $(x_0, x_1, x_2, ...)$ such that $x_{n-1} = P(\pi_n)(x_n)$ for all $n \in \mathbb{Z}_{\geq 0}$. Notice that if Y is another **Vec**-variety in our polynomial functor P such that $x_n \in Y(K^n)$ for all $n \gg 0$, then, since Y must be preserved by all the $P(\pi_i)$'s, we automatically have that $x_i \in Y(K^n)$ for all n and thus that $x \in \mathbf{Y}$. This is precisely why, in general, we care about proving results for $X(V)$ when $\dim(V) \gg 0$. Moreover, we say that a point $x \in \mathbf{X}$ has a certain property (e.g. being singular) if x_n has that property for all $n \gg 0$.

Virtually all properties of **Vec**-varieties can be restated equivalently in the **GL**-setting and vice versa. In some cases, we will see that one approach is easier to work with than the other, hence why we wish to have both at our disposal. Given a **Vec**-variety X, we denote by $\mathbf{X}$ the **GL**-variety associated to X and by $\mathbf{X}\{K^n\}$ the evaluation of $\mathbf{X}$ at K^n when opting to use this point of view. We now state a few results about **GL**-varieties which we will reference later.

2.4.1. *Generic freeness.*

Theorem 2.33 ([CDS26, Theorem 2.2])**.** *Let A be an integral* **GL**-*algebra and let M be a finitely* **GL**-*generated A-module. There there exists $n \geq 0$, a nonzero $G(n)$-invariant element $h \in A$, and a finite length polynomial representation V of $G(n)$ such that $M[1/h]$ is isomorphic to $V \otimes A[1/h]$ as a $G(n)$ equivariant $A[1/h]$-module.*

2.4.2. *Flat locus.* The following gives a description of the flat locus of a module on a **GL**-variety.

Theorem 2.34 ([CDS26, Theorem 6.1])**.** *Let R be a reduced finitely* **GL**-*generated* **GL**-*algebra, let $\mathbf{X} = \mathrm{Spec}(R)$, and let M be a finitely* **GL**-*generated R-module. Let $\mathbf{U} \subseteq \mathbf{X}$ be the set of prime ideals $\mathfrak{p}$ such that $M_\mathfrak{p}$ is a flat $R_\mathfrak{p}$-module. Then:*

(a) The set $\mathbf{U}$ is open and dense in $\mathbf{X}$.
(b) For each $p \in \mathbf{U}$, the R_p-module M_p is free.
(c) Let $\mathbf{Z} = \mathbf{X} \setminus \mathbf{U}$ be the non-flat locus, and let $\mathbf{W}_n \subseteq \mathbf{X}\{K^n\}$ be the non-flat locus of $M\{K^n\}$. Then

$$\mathbf{W}_n \subseteq \mathbf{Z}\{K^n\}$$

with equality for $n \gg 0$.

2.5. **The Core.** In this section, we discuss the notion of the core for both **GL**-varieties and **Vec**-varieties. There are many equivalent characterizations of the core. A notable result proved in [CDS26] is that the core coincides with those points with infinite embedding codimension. This characterization motivates our study of the core and, in particular, its relationship with the singular locus.

We begin by recalling some basic properties of the core of a **GL**-variety. For detailed discussion, see [CDS26].

2.5.1. *The core of a* **GL**-*variety.* Let K be a algebraically closed field of characteristic 0. Given a finite length polynomial representation V of **GL**, denote by $\mathbb{A}(V) \coloneqq \mathrm{Spec}(\mathrm{Sym}(V))$, which is a **GL**-variety. We say that a **GL**-variety is *elementary* if it is of the form $B \times \mathbb{A}(V)$, where B is a finite dimension variety with trivial **GL**-action and V is a finite length polynomial representation of **GL**.

Definition 2.35. Let $\mathbf{X}$ be a **GL**-variety. A point $x \in \mathbf{X}$ is called *elementary* if there exist an integer n and an open elementary $G(n)$-subvariety $\mathbf{U} \subseteq \mathbf{X}$ such that $x \in \mathbf{U}$. We denote the set of elementary points of $\mathbf{X}$ by $\mathbf{X}_{\mathrm{elem}}$ The *core* of $\mathbf{X}$ is defined to be the complement of the elementary locus: $\mathbf{X}_{\mathrm{core}} \coloneqq \mathbf{X} \setminus \mathbf{X}_{\mathrm{elem}}$. ♪

If we cover $\mathbf{X}$ by open elementary subvarieties, then the core is precisely the closed locus that remains. The shift theorem ensures that every nonempty **GL**-variety contains a nonempty open elementary $G(n)$-subvariety for some n. Moreover, the core has several properties. For example, [CDS26, Proposition 4.5] shows that $\mathbf{X}_{\mathrm{core}}$ is a proper, closed, nowhere-dense **GL**-subvariety of $\mathbf{X}$.

Moreover, there is an alternative characterization of the core, in terms of embedding codimension, which will prove particularly interesting to us.

Theorem 2.36 ([CDS26, Theorem C])**.** *Let $\mathbf{X}$ be a* **GL**-*variety. Then a point $x \in \mathbf{X}$ belongs to $\mathbf{X}_{\mathrm{core}}$ if and only if the embedding codimension of $\mathbf{X}$ at x is infinite.*

Let $\mathbf{X}$ be an irreducible $\mathbf{GL}$-variety, and let $\mathbf{X}_{\mathrm{Sing}_m}$ be as in Theorem A. Then we see that the core is always contained in the singular locus of order at least m.

Corollary 2.37. *Let $\mathbf{X}$ be an irreducible $\mathbf{GL}$-variety, then $\mathbf{X}_{\mathrm{core}} \subseteq \mathbf{X}_{\mathrm{Sing}_m}$.*

We now relate this characterization to the higher-order singular loci constructed in Theorem A. For each $n \geq 0$, let $P(\pi_n)\colon \mathbf{X}\{K^n\} \longrightarrow \mathbf{X}\{K^{n-1}\}$ be the morphism induced by the projection $K^n \to K^{n-1}$ to the first $n-1$ coordinates. Recall that a point $x \in \mathbf{X}$ is a tuple of the form $(x_0, x_1, x_2, ...)$ with each $x_n \in \mathbf{X}\{K^n\}$ such that $x_{n-1} = P(\pi_n)(x_n)$ for all n. If $\mathbf{Z} \subseteq \mathbf{X}$ is a closed $\mathbf{GL}$-subvariety, then $x \in \mathbf{Z}$ if and only if $x_n \in \mathbf{Z}\{K^n\}$ for all sufficiently large n. This characterization leads to the following definition:

Definition 2.38. Let $\mathbf{X}$ be an irreducible $\mathbf{GL}$-variety and let $x \in \mathbf{X}$. Then we say that

$$\mathrm{embcodim}(\mathcal{O}_{\mathbf{X},x}) = \infty.$$

if and only if, for every $m \geq 1$, there exists an integer N_m such that

$$\mathrm{embcodim}_{x_n}\left(\mathbf{X}\{K^n\}\right) \geq m.$$

for every $n \geq N_m$. That is, for all $n \geq N_m$, $x_n \in \mathrm{Sing}_m(\mathbf{X}\{K^n\})$. ♪

Now, suppose that $\mathbf{X}$ is irreducible and let $\mathbf{X}_{\mathrm{Sing}_m} \subseteq \mathbf{X}$ be as in Theorem A. We then have that $x_n \in \mathrm{Sing}_m(\mathbf{X}\{K^n\})$ for all n large enough if and only if $x \in \mathbf{X}_{\mathrm{Sing}_m}$. This gives us precisely that

$$\mathbf{X}_{\mathrm{core}} = \bigcap_{m=0}^{\infty} \mathbf{X}_{\mathrm{Sing}_m}.$$

2.5.2. *The core of a* **Vec***-variety.* Let K be an algebraically closed field of characteristic 0 and let X be a **Vec**-variety inside a polynomial functor P.

Definition 2.39. Let $Y \subseteq X$ be a **Vec**-subvariety. We say that Y is *elementary* if there exist a finite-dimensional affine algebraic variety B and a polynomial functor Q such that

$$Y \cong B \times Q$$

as **Vec**-varieties. ♪

Needless to say, elementary **Vec**-varieties have, essentially, the simplest possible structure, making their properties much easier to study. The following important application of Theorem 2.29, known as the Shift Theorem, tells us that, given a fixed **Vec**-variety X, we can always find, up to shift of a finite dimensional vector space, an elementary open **Vec**-subvariety of X.

Theorem 2.40 ([BDES23, Theorem 5.1])**.** *Let X be a* **Vec***-variety. Then there exists a finite-dimensional vector space $U \in$* **Vec***, a nonzero function $h \in K[X(U)]$, a polynomial functor Q and a finite-dimensional affine variety B over K such that*

$$\mathrm{Sh}_U X[1/h] \cong B \times Q.$$

Such a construction need not be unique, and many such choices of U and h yielding isomorphisms to different elementary **Vec**-varieties could exist. However, we very well may not be able to cover X by such open sets. This motivates the following definition; once again, thinking of X as the $\mathbf{GL}$-variety $\mathbf{X}$ will be helpful here.

We have now obtained a characterization of the core that also makes sense for irreducible **Vec**-varieties.

Definition 2.41. Let X be an irreducible **Vec**-variety. We define the *core* of X by

$$X_{\mathrm{core}} \coloneqq \bigcap_{m=0}^{\infty} X_{\mathrm{Sing}_m}.$$

♪

Remark 2.42. It is easy to see from the preceding discussion that this notion of core is the same as what we've defined in Definition 2.35, according to Theorem 2.36. ♪♫

With this we conclude our background discussion, and proceed to the proofs of our results.

3. Higher Order Singular Loci

Let K be an algebraically closed field of characteristic 0. In this section, we prove Theorem A and Theorem B. Let X be a **Vec**-variety inside a polynomial functor P. Fix $M \in \mathbb{Z}_{\geq 1}$. We can define Sing_m just as we did in the finite-dimensional setting: for each $V \in$ **Vec**, we write

$$\mathrm{Sing}_m(X(V)) := \{p \in X(V) \mid \mathrm{embcodim}_p(X(V)) \geq m\}.$$

3.1. **Irreducible Case.** We first prove Theorem A, doing so in a nearly identical fashion to that of Theorem 1.2, simply adapting the details to match our new condition.

Going forward, we assume $m \in \mathbb{Z}_{\geq 1}$ is fixed. Our goal, recall, is to find a **Vec**-subvariety of X such that $X_{\mathrm{Sing}_m}(V) = \mathrm{Sing}_m(X(V))$ for all V such that $\dim(V) \gg 0$. We first notice that uniqueness of this X_{Sing_m}, if it exists, follows immediately just as in Section 3.1 of [CDD25].

Next, we adapt the first two lemmas in Section 3.2 of [CDD25]. The proofs will follow almost identically.

Lemma 3.1. *Let X be an irreducible* **Vec***-variety , $U \in$ **Vec**, and $p \in \mathrm{Sing}_m(X(U))$. Then, for any $V \in$ **Vec** and any injective linear map $\iota \in \mathrm{Hom}(U, V)$ we have that*

$$X(\iota)(p) \in \mathrm{Sing}_m(X(V)).$$

Proof. Just as in the aforementioned article, let $\pi : V \to U$ be a surjective linear map such that $\pi \circ \iota = \mathrm{Id}_U$. We thus have maps $X(\pi) : X(V) \to X(U)$ and $X(\iota) : X(U) \to X(V)$ such that $X(\pi) \circ X(\iota) = \mathrm{Id}_{X(U)}$. Notice that this means that $X(\pi)$ must be surjective, and hence $\dim(\mathrm{Im}(X(\pi))) = \dim(X(U))$.

These maps induce linear maps on tangent spaces $T_{\iota(p)}X(\pi) : T_{\iota(p)}X(V) \to T_pX(U)$ and $T_pX(\iota) : T_pX(U) \to T_{\iota(p)}X(V)$ such that

$$T_{\iota(p)}X(\pi) \circ T_pX(\iota) = \mathrm{Id}_{T_pX(U)}.$$

In particular, $T_{\iota(p)}X(\pi)$ is surjective and thus $\dim(\mathrm{Im}(T_{\iota(p)}X(\pi))) = \dim(T_pX(U))$.

Combining these observations with remark 6.8 in [CFD22], we obtain

$$\begin{aligned}
\mathrm{embcodim}_{\iota(p)}(X(V)) &= \dim(T_{\iota(p)}X(V)) - \dim_p(X(V)) \\
&\geq \dim\big(\mathrm{Im}(T_{\iota(p)}X(\pi))\big) - \dim\big(\mathrm{Im}(X(\pi))\big) \\
&= \dim(T_pX(U)) - \dim(X(U)) \\
&= \mathrm{embcodim}_p(X(U)) \\
&\geq m.
\end{aligned}$$

We thus conclude that $\iota(p) \in \mathrm{Sing}_m(X(V))$. □

Lemma 3.2. *Let $X_{\mathrm{Sing}_m} \subseteq X$ be a closed* **Vec***-subvariety such that $X_{\mathrm{Sing}_m}(V) = \mathrm{Sing}_m(X(V))$ for all $V \in$ **Vec** with $\dim(V) \gg 0$. Then, for every $V \in$ **Vec**, we have $\mathrm{Sing}_m(X(V)) \subseteq X_{\mathrm{Sing}_m}(V)$.*

Proof. Fix $U \in$ **Vec**, and choose $V \in$ **Vec** with $\dim(V) \gg 0$ together with an injective linear map $\iota : U \to V$. Let $\pi \in \mathrm{Hom}(V, U)$ be as in the previous proof. Then

$$X(\iota)\big(\mathrm{Sing}_m(X(U))\big) \subseteq \mathrm{Sing}_m(X(V)) = X_{\mathrm{Sing}_m}(V).$$

Moreover, since $\dim(V) \gg 0$, we have

$$X_{\mathrm{Sing}_m}(U) = X(\pi)\big(X_{\mathrm{Sing}_m}(V)\big) = X(\pi)\big(\mathrm{Sing}_m(X(V))\big).$$

Apply $X(\pi)$ to the first inclusion, we obtain

$$\mathrm{Sing}_m(X(U)) \subseteq X(\pi)(\mathrm{Sing}_m(X(V))) = X_{\mathrm{Sing}_m}(U).$$

□

We can now proceed to the actual proof of the theorem. As in Section 3.5 of [CDD25], we proceed by induction on the order of P. If $\deg(P) = 0$ then $X(V) = X(0)$ for all $V \in$ **Vec**. We may therefore set $X_{\mathrm{Sing}_m}(V) \coloneqq \mathrm{Sing}_m(X(0))$ for all $V \in$ **Vec**. Recall, since X is irreducible, by Proposition 2.26 we have that $X(0)$ must be irreducible. Hence the subset $\mathrm{Sing}_m(X(0))$ is closed.

Now, let $d = \deg(P)$ and suppose that $d > 0$. Assume that Theorem A holds for all polynomial functors $Q \prec P$. Let R be an irreducible sub-functor of the degree d part of P, and let $\pi : P \to P' \coloneqq P/R \prec P$ be the natural projection. We define a closed **Vec**-variety $X' \subset P'$ by setting

$$X'(V) \coloneqq \overline{\pi_V(X)} \subseteq P'.$$

Then, by Theorem 2.29, there are two possible cases. We first consider the case in which

$$X(V) \cong X'(V) \times R(V).$$

In this case, it follows immediately that

$$\mathrm{Sing}_m(X(V)) = \mathrm{Sing}_m(X'(V)) \times R(V).$$

By the induction hypothesis, there exists some closed **Vec**-subvariety $X'_{\mathrm{Sing}_m} \subseteq X'$ such that $X'_{\mathrm{Sing}_m}(V) = \mathrm{Sing}_m(X'(V))$ for all $V \in$ **Vec** with $\dim(V) \gg 0$. Therefore, setting $X_{\mathrm{Sing}_m} \coloneqq X'_{\mathrm{Sing}_m} \times R$, we obtain a closed **Vec**-subvariety of X with the desired property.

On the other hand, suppose that there is no irreducible subfunctor $R \subseteq P_d$ for which the preceding case applies. In this case, we aim to embed an open subset of a shift of X into a smaller polynomial functor. However, there is a closed locus on which such an embedding is not available. We first analyze the behavior of this locus.

As in [CDD25], we define a closed **Vec**-subvariety $Z \subset X$ as follows: For each $V \in \mathbf{Vec}$, let $I_X(V)$ denote the defining ideal of $X(V)$. We define Z to be the largest closed **Vec**-subvariety of X such that, for every $V \in \mathbf{Vec}$, all polynomials in the set

$$\left\{ \frac{\partial f}{\partial r} \;\middle|\; f \in I_X(V),\ R \text{ an irreducible subfunctor of } P_d,\ r \in R(V) \right\}$$

vanish on $Z(V)$.

Lemma 3.3. *For every $V \in \mathbf{Vec}$ with $\dim(V) \gg 0$ we have $Z(V) \subseteq \mathrm{Sing}_m(X(V))$.*

Proof. This lemma is an analog of Lemma 3.7 in [CDD25], and its proof follows with essentailly the same argument. Indeed, we first observe that, since P_d is the sum of its irreducible subfunctors, for all $V \in \mathbf{Vec}$ we have $\frac{\partial f}{\partial r}$ vanish on $Z(V)$ for all $f \in I_X(V)$ and all $r \in P_d(V)$.

Next, as in the proof of the original lemma, let $c(V)$ denote the comension of $X(V)$ in $P(V)$. For $\dim(V) \gg 0$, the function $c(V)$ is given by a polynomial of degree d in $\dim(V)$. Consequently, $c(V) - m + 1$ is also given by a polynomial of degree d in $\dim V$ for $\dim(V) \gg 0$.

We can then construct the Jacobian matrix for $X(V)$ as in the original lemma: with the first $n_{<d} = n_{<d}(V)$ columns corresponding to coordinates on $P_{<d}(V)$ and the last n_d columns corresponding to coordinates on $P_d(V)$. Recall from Proposition 2.30 that $n_{<d}$ and n_d grow as polynomials in degrees $< d$ and d respectively. It follows that for $\dim(V)$ large enough, any $(c(V) - m + 1) \times (c(V) - m + 1)$ minor of the Jacobian matrix must intersect at least one of the last n_d columns. Thus, $Z(V) \subseteq \mathrm{Sing}_m(V)$ as claimed. □

We now replicate the construction in [CDD25] to build X_{Sing_m} on $X \setminus Z$. To do this, we first note that by Topological Noetherianity, there exist finitely many $U_i \in \mathbf{Vec}$ and $h_i = \frac{\partial f_i}{\partial r_i}$ that cut out Z, where $f_i \in I_X(U_i)$ and $r_i \in R_i$ for $R_1, ..., R_l$ irreducible subfunctors of P. We can then apply Theorem 2.29 to obtain, for each i, an isomorphism

$$\mathrm{Sh}_{U_i} P\left[\frac{1}{h_i}\right] \cong Z_i$$

where $Z_i \subseteq P_i''$ an irreducible closed **Vec**-variety and $P_i'' := K^1 \oplus (\mathrm{Sh}_{U_i} P)/R_i$. Then, since $P_i'' \prec P$ the inductive hypothesis yields a closed **Vec**-subvariety $Y'_{m_i} \subseteq Z_i$ such that

$$Y'_{m_i}(V) = \mathrm{Sing}_m(Z_i(V))$$

whenever $\dim(V) > M_i$ for some $M_i \in \mathbb{N}$.

Then, we can use the isomorphism $\mathrm{Sh}_{U_i} P[\frac{1}{h_i}] \cong Z_i$ to identify Y'_{m_i} with a locally closed subset of $X(U_i \oplus V)$. Since $\mathrm{GL}(U_i \oplus V)$ acts on $X(U_i \oplus V)$ by automorphisms ([CDD25, Lemma 2.16]), it follows from Lemma 3.1 that this action preserves the locus $\mathrm{Sing}_m(X(U_i \oplus V))$. Now fix some $p \in Y'_{m_i}(V) \subseteq X(U_i \oplus V)$ and observe that, for $\dim(V) \geq M_i$, the map

$$\mathrm{GL}(U_i \oplus V) \longrightarrow X(U_i \oplus V) \quad \varphi \longmapsto \varphi(p).$$

maps $\mathrm{GL}(U_i \oplus V)$ into $\mathrm{Sing}_m(X(U_i \oplus V))$. Indeed, for $\dim(V) \geq M_i$, $Y'_{m_i}(V)$ coincides with $\mathrm{Sing}_m(Z_i(V))$ and Sing_m will be preserved by the automorphism $\mathrm{Sh}_{U_i} P[\frac{1}{h_i}] \cong Z_i$.

We now define $Y_i \subset X$ as follows: for each $W \in \mathbf{Vec}$, choose $V \in \mathbf{Vec}$ such that $\dim(V) \geq \max\{M_i, \dim(W) - \dim(U_i)\}$, and let $\varphi : V \oplus U_i \twoheadrightarrow W$ be any surjective linear map. We then set

$$Y_{m_i}(W) := \overline{P(\varphi)(Y'_{m_i}(V))}.$$

It now remains to prove that, Y_{m_i}, defined in this way, is in fact a closed **Vec**-subvariety of X and that it satisfies the desired conditions. To do this, we establish an analogue of Lemma 3.8 in [CDD25].

Lemma 3.4. *This Y_{m_i} defines a closed **Vec**-subvariety of X which satisfies $Y_{m_i}(W) \subseteq \mathrm{Sing}_m(X(W))$ for all $W \in \mathbf{Vec}$ with $\dim(W) \geq M_i + \dim(U_i)$. In particular, for such W of sufficiently large dimension, $Y_{m_i}(W)$ is the closure in $X(W)$ of the set*

$$\{p \in \mathrm{Sing}_m(X(W)) \mid \text{there exists some } \psi \in \mathrm{Hom}(W, U_i) \text{ such that } h_i(P(\psi))(p) \neq 0\}.$$

Proof. To simplify notation, we write $U = U_i$, $h = h_i$, $X'_{\mathrm{Sing}_m} = Y'_{m_i}$, and $X_{\mathrm{Sing}_m} = Y_{m_i}$. We first note that X_{Sing_m} is well-defined, i.e. does not depend on our choice of V or φ, and that it defines a closed **Vec**-subvariety of X. Both statement follows by the same argument as in the proof of Lemma 3.8 in [CDD25].

Next, notice that if $\dim(W) \geq M_i + \dim(U)$, we can regard U as a subspace of W via some injection, and then take $V = W/U$ and φ to be an isomorphism. This follows that

$$X_{\mathrm{Sing}_m}(W)\left[\frac{1}{h}\right] = P(\varphi)(X'_{\mathrm{Sing}_m}(W/U)) = X'_{\mathrm{Sing}_m}(W/U) = \mathrm{Sing}_m(X(W))\left[\frac{1}{h}\right].$$

Taking closures in $X(W)$, we obtain last two statements of the lemma. □

To complete the proof of the theorem, we let

$$X_{\mathrm{Sing}_m}(V) = Y_1(V) \cup \cdots \cup Y_l(V) \cup Z(V)$$

We claim that, for all $V \in \mathbf{Vec}$ with $\dim(V) \gg 0$,

$$\mathrm{Sing}_m(X(V)) = Y(V).$$

The previous two lemmas immediately give us the inclusion $X_{\mathrm{Sing}_m}(V) \subseteq \mathrm{Sing}_m(X(V))$. For the reverse inclusion, consider any point $p \in \mathrm{Sing}_m(X(V))$. If $p \in Z(V)$ then there's nothing to prove. Otherwise, by the choice of the h_i, there exist some i and some linear map $\psi : V \to U_i$ such that $h_i(P(\psi)(p)) \neq 0$. So, we have that $p \in \mathrm{Sing}_m(X(V))[\frac{1}{h}] = Y_i(V)[\frac{1}{h}] \subseteq Y_i(V)$. We therefore conclude that $\mathrm{Sing}_m(X(V)) = X_{\mathrm{Sing}_m}(V)$ for all $V \in \mathbf{Vec}$ with $\dim V \gg 0$, and the statement of Theorem A follows.

3.2. **Reducible Case.** We first explain why the reducible case is trickier in the situation when $m > 1$. For $m = 1$, Remark 2.9 shows that if X is a reducible variety, then the intersection of any two distinct irreducible components of X is contained in $\mathrm{Sing}_1(X)$. However, no analogous statement is known for higher embedding codimension. Indeed, it is not true in general that if $p \in X_i \cap X_j$, and $\mathrm{embcodim}_p X_i(V) \geq m$, $\mathrm{embcodim}_p X_j(V) \geq m$, then $\mathrm{embcodim}_p\big((X_i \cup X_j)(V)\big) \geq m + 1$. In particular, there is no obvious decomposition of $\mathrm{Sing}_m X(V)$ that we can apply here.

Instead, in this section, we show that we can obtain a convenient stratification of a **Vec**-variety, which will allow us to understand the locus Sing_m up to closure. In particular, our goal will be to prove Theorem B. The proof relies upon the following key assertion.

Proposition 3.5. *Let X be a* **Vec***-variety. Then there exist a finite index set Λ and a family of pairs*

$$(Z_\alpha, Y_\alpha), \qquad \alpha \in \Lambda,$$

of closed **Vec***-subvarieties of X satisfying the following properties:*

(1) Y_α is irreducible for every $\alpha \in \Lambda$.
(2) $Z_\alpha \subsetneq Y_\alpha \subseteq X$ for every $\alpha \in \Lambda$.
(3) For all V with $\dim V \gg 0$,

$$X(V) = \bigsqcup_{\alpha \in \Lambda} \big(Y_\alpha(V) \setminus Z_\alpha(V)\big).$$

(4) For every $\alpha \in \Lambda$ and all V with $\dim V \gg 0$, exactly one of the following holds:

$$Y_\alpha(V) \setminus Z_\alpha(V) \subseteq \mathrm{Sing}_m(X(V)),$$

or

$$\big(Y_\alpha(V) \setminus Z_\alpha(V)\big) \cap \mathrm{Sing}_m(X(V)) = \varnothing.$$

Notice that a crucial condition here is that $(Y_\alpha(V) \setminus Z_\alpha(V)) \cap (Y_{\alpha'}(V) \setminus Z_{\alpha'}(V)) = \emptyset$ if $\alpha \neq \alpha'$. This effectively eliminates the issue posed by intersections of different components described above.

Remark 3.6. This stratification is not unique in general. Consider our example $X_r(V) = \{M \in V \otimes V \mid \mathrm{rank}(M) \leq r\}$. It follows from Example 1.3 that we can decompose it in two ways when $\dim V \gg 0$:

$$X_r(V) = \bigsqcup_{i=0}^{r} \big(X_i(V) \setminus X_{i-1}(V)\big)$$

$$X_r(V) = \big(X_r(V) \setminus X_{r-1}(V)\big) \bigsqcup \big(X_{r-1}(V) \setminus \varnothing\big)$$

where we use the convention that $X_{-1}(V) = \varnothing$. ♪♫

We first show that *Proposition* 3.5 is enough to prove Theorem B.

Proof of Theorem B. By Proposition 3.5, there exists a finite indexed set Λ and finte number of pairs of closed **Vec**-subvarieties (Z_α, Y_α) such that

(1) Y_α is irreducible for every $\alpha \in \Lambda$.
(2) $Z_\alpha \subsetneq Y_\alpha \subseteq X$ for every $\alpha \in \Lambda$.

(3) For all V with $\dim V \gg 0$,
$$X(V) = \bigsqcup_{\alpha\in\Lambda} \big(Y_\alpha(V) \setminus Z_\alpha(V)\big).$$
(4) For every $\alpha \in \Lambda$ and all V with $\dim V \gg 0$, exactly one of the following holds:
$$Y_\alpha(V) \setminus Z_\alpha(V) \subseteq \operatorname{Sing}_m(X(V)),$$
or
$$\big(Y_\alpha(V) \setminus Z_\alpha(V)\big) \cap \operatorname{Sing}_m(X(V)) = \varnothing.$$

Let $\Gamma \subset \Lambda$ be the subset of all indices β for which $Y_\beta(V) \setminus Z_\beta(V) \subseteq \operatorname{Sing}_m X(V)$. Let $X_{\operatorname{Sing}_m} = \bigcup_{\beta\in\Gamma} Y_\beta$. We claim that $X_{\operatorname{Sing}_m}(V) = \overline{\operatorname{Sing}_m X(V)}$ for $\dim V \gg 0$. By our construction, for all $\dim V \gg 0$, we can write
$$\operatorname{Sing}_m X(V) = \bigcup_{\beta\in\Gamma} (Y_\beta(V) \setminus Z_\beta(V))$$
Since $Y_\beta(V)$ is irreducible for all V, $\overline{(Y_\beta(V) \setminus Z_{Y,\beta}(V))} = Y_\beta(V)$ for all $\beta \in \Lambda$. Thus, we get
$$\overline{\operatorname{Sing}_m X(V)} = \bigcup_{\beta\in\Gamma} \overline{Y_\beta(V) \setminus Z_\beta(V)} = \bigcup_{\beta\in\Gamma} Y_\beta(V) = X_{\operatorname{Sing}_m}(V).$$
The fact that this $X_{\operatorname{Sing}_m}$ is unique follows from the same argument as that in the irreducible case. □

Remark 3.7. Proposition 3.5 actually gives a stronger conclusion than that in Theorem B. In general, a stratum $Y_\alpha \setminus Z_\alpha$ need not be a **Vec**-subvariety, since it need not be preserved under arbitrary linear maps. For example, the locus of matrices of rank exactly r is not preserved by the map induced by the zero linear map, which sends every matrix to a matrix of rank zero.

Nevertheless, each $Y_\alpha \setminus Z_\alpha$ is a locally closed **GL**-stable subset. Hence the proposition shows that the order at least m singular locus is, in sufficiently large dimensions, represented by a **GL**-constructible subset, i.e. a finite union of locally closed **GL**-stable subsets. ♪♫

The remainder of this section will be dedicated to proving Proposition 3.5. Given the **Vec**-variety X, let
$$X = X_1 \cup \cdots \cup X_r$$
be the decomposition of X into its irreducible **Vec**-subvarieties. Fix an irreducible closed **Vec**-subvariety $Y \subseteq X$.

Definition 3.8. For the fixed irreducible closed **Vec**-subvariety $Y \subseteq X$, define its *component index set* by
$$\mathcal{C}_X(Y) := \{i \in \{1, \dots, r\} : Y \subseteq X_i\}$$
In other words, $\mathcal{C}_X(Y)$ is the set of indices of the irreducible components of X that contain Y. ♪

Definition 3.9. Define the *component incidence locus* by
$$Z_Y^{\text{inc}} := \bigcup_{i\notin\mathcal{C}_Y(X)} (Y \cap X_i)$$
♪

Proposition 3.10. *The locus Z_Y^{in} is a proper closed* **Vec***-subvariety of Y. Moreover, for every $V \in$* **Vec***, every $p \in Y(V) \setminus Z_Y^{\text{in}}(V)$, and every $i \in \{1, \dots, r\}$,*
$$p \in X_i(V) \iff i \in \mathcal{C}_X(Y)$$

Proof. To prove the first assertion, we see that if $j \notin \mathcal{C}_X(Y)$, then $Y \nsubseteq X_j$ and thus $Y \cap X_j \subsetneq Y$. Therefore, $Z_Y^{\text{in}} = \bigcup_{j\notin\mathcal{C}_X(Y)}(Y \cap X_j)$ is a finite union of proper closed **Vec**-subvarieties of Y. Since Y is irreducible, it follows that Z_Y^{in} is a proper closed **Vec**-subvariety of Y. The second assertion immediately follows from the definition of Z_Y^{in} and the choice of p.

□

This allows us to prove the following result about the local dimension at points in $Y(V) \setminus Z_Y^{\text{in}}(V)$.

Proposition 3.11. *Let Y be an irreducible closed* **Vec***-subvariety of X. Then for all $V \in$* **Vec** *with $\dim V \gg 0$, and all $p \in Y(V) \setminus Z_Y^{\text{in}}(V)$, there exists a polynomial $d_Y(n)$ with rational coefficients, such that*
$$\dim_p X(K^n) = d_Y(n)$$
More explicitly, $d_Y(n) = \max_{i\in\mathcal{C}_X(Y)} \dim X_i(K^n)$.

Proof. By Proposition 3.10, the irreducible components of X passing through p are exactly the X_i for $i \in \mathcal{C}_X(Y)$. Therefore, $\dim_p X(K^n) = \max_{i \in I(Y)_X} \dim X_i(K^n)$. By Proposition 2.32, for each $i \in \mathcal{C}_X(Y)$, $\dim X_i(K^n)$ is a polynomial for $n \gg 0$. Hence their maximum is also a polynomial for $n \gg 0$. □

For notational convenience, we formulate the remainder of the proof in the language of **GL**-varieties. Let R be a reduced finitely **GL**-generated **GL**-algebra over k such that $\mathbf{X} = \mathrm{Spec}(R)$ is the **GL**-variety corresponding to X, the previous **Vec**-variety. Let $\mathbf{Y} \subseteq \mathbf{X}$ be a closed, irreducible **GL**-subvariety corresponding to an ideal $I_Y \subseteq R$. Set $S = R/I_Y$, so that $\mathbf{Y} = \mathrm{Spec}(S)$. Let $i : \mathbf{Y} \hookrightarrow \mathbf{X}$ be the natural closed embedding. Denote both Ω_R and Ω_X as the Kähler differentials of $\mathbf{X}$. We define

$$M_Y := i^*\Omega_X$$

to be the restriction of the module of Kähler differentials of $\mathbf{X}$ to the closed **GL**-subvariety Y. By [Har77, Chapter II, Proposition 5.2(e)], $M_Y = i^*\Omega_X = \Omega_R \otimes_R S$.

Lemma 3.12. *Let R be a finitely* **GL***-generated* **GL***-algebra. Then Ω_R is a finitely* **GL***-generated R-module. Moreover, if $I \subseteq R$ is a* **GL** *ideal, and let $S = R/I$, then $M_Y = \Omega_R \otimes_R S$ is a finitely* **GL***-generated S-module.*

Proof. By definition of finitely generated **GL**-generated **GL**-algebra, there exists a finite length polynomial representation W and a surjective homomorphism of **GL**-algebras

$$\pi : \mathrm{Sym}(W) \twoheadrightarrow R$$

Let $P = \mathrm{Sym}(W)$, then we get a quotient map $P \twoheadrightarrow R$. Let J be the kernel. There is an exact sequence

$$J/J^2 \to \Omega_P \otimes_P R \to \Omega_R \to 0$$

Hence the natural map $\Omega_P \otimes_P R \to \Omega_R$ is surjective. For the free polynomial **GL**-algebra $P = \mathrm{Sym}(W)$, the module of Kähler differentials is free with basis dW, and $\Omega_P = P \otimes W$. Therefore, Ω_R is a quotient of $(P \otimes W) \otimes_P R \cong W \otimes R$. Since $W \otimes R$ is finitely **GL**-generated as an R-module, we obtain that Ω_R is finitely **GL**-generated as an R-module.

From the surjection $W \otimes R \to \Omega_R$, we get a surjection $(W \otimes R) \otimes_R S \twoheadrightarrow \Omega_R \otimes S$. Since $(W \otimes R) \otimes_R S \cong W \otimes S$, we see that $\Omega_R \otimes S$ is a quotient of $W \otimes S$, hence is finitely **GL**-generated as an S-module. □

Lemma 3.13. *Let $\mathbf{Y} = \mathrm{Spec}(S)$ and M_Y be as above. Let $\mathbf{U} \subseteq \mathbf{Y}$ be the flat locus of M in $\mathbf{Y}$, and let $\mathbf{Z} = \mathbf{Y} \setminus \mathbf{U}$ be the non-flat locus of M in $\mathbf{Y}$. Then there exist a polynomial t_M with rational coefficients, such that for $n \gg 0$, and all $p \in \mathbf{Y}\{K^n\} \setminus \mathbf{Z}\{K^n\}$, one has*

$$\dim_{\kappa(p)} (M\{K^n\} \otimes \kappa(p)) = t_M(n)$$

Proof. By Theorem 2.34, $\mathbf{U}$ is open, dense, and **GL**-stable, and for $n \gg 0$, $M_Y\{K^n\}$ is flat on $\mathbf{U}\{K^n\}$. For all $p \in \mathbf{Y}\{K^n\} \setminus \mathbf{Z}\{K^n\} = \mathbf{U}\{K^n\}$, $M_Y\{K^n\}$ is flat at p. Since M_Y is finitely **GL**-generated as an S-module, there exists a finite-length polynomial representation W, and a surjective quotient map $S \otimes W \twoheadrightarrow M_Y$. By the compatibility of exactness and tensor of evaluation, we get a surjection

$$S\{K^n\} \otimes W\{K^n\} \twoheadrightarrow M_Y\{K^n\}$$

Since W is of finite length, $W\{K^n\}$ is a finite dimensional vector space. Thus $S\{K^n\} \otimes W\{K^n\}$ is a finite free $S\{K^n\}$-module and so $M\{K^n\}$ is finitely generated over the finite-type algebra $S\{K^n\}$. Now Theorem 2.34 gives, for $n \gg 0$, that the flat locus of $M_Y\{K^n\}$ is exactly $\mathbf{U}\{K^n\} = Y\{K^n\} \setminus Z\{K^n\}$. Therefore $M_Y\{K^n\}\big|_{\mathbf{U}\{K^n\}}$ is flat over $\mathbf{U}\{K^n\}$. Therefore $M_Y\{K^n\}|_{\mathbf{U}\{K^n\}}$ is a finitely presented flat module over $\mathbf{U}\{K^n\}$. Since a finitely presented flat module is locally free, it follows that $M_Y\{K^n\}|_{\{\mathbf{U}\{K^n\}\}}$ is finite locally free. Therefore, on $\mathbf{U}\{K^n\}$, the function

$$p \to \dim_{\kappa(p)}(M_Y\{K^n\} \otimes \kappa(p))$$

is locally constant. Since $\mathbf{Y}$ is irreducible, $\mathbf{U}\{K^n\}$ is irreducible for n sufficiently large. In particular, it is connected, and so $\dim_{\kappa(p)}(M_Y\{K^n\} \otimes \kappa(p))$ must be constant on $\mathbf{U}\{K^n\}$. So there is an integer $t_M(n)$ such that $\dim_{\kappa(p)}(M_Y\{K^n\} \otimes \kappa(p)) = t_M(n)$ for all $p \in \mathbf{U}\{K^n\}$ for every n.

It remains to show that $t_M(n)$ is a polynomial for n large enough. By Theorem 2.33, there exists $\ell \geq 0$, a nonzero $G(\ell)$-invariant element $h \in S$, and a finite length polynomial representation V of $G(\ell)$, such that

$$M_Y[1/h] \cong V \otimes S[1/h]$$

as a $G(\ell)$-equivariant $S[1/h]$-module. For $n \geq \ell$, evaluate at $K^{n-\ell}$. Under the shift convention, denote $h_n \in S\{K^n\}$ as the image of h. Since evaluation is compatible with localization and tensor products, the isomorphism gives

$$M_Y\{K^n\}[1/h_n] \cong V\{K^{n-\ell}\} \otimes S\{K^n\}[1/h_n].$$

Therefore, $M_Y\{K^n\}[1/h_n]$ is a free $S\{K^n\}[1/h_n]$ module of rank $d_n = \dim V\{K^{n-\ell}\}$. Equivalently, the restriction of $M_Y\{K^n\}$ to the principal open subset $\mathbf{Y}\{K^n\}[1/h_n] \subseteq \mathbf{Y}\{K^n\}$ is free of rank $d_n = \dim V\{K^{n-\ell}\}$.

For $n \gg 0$, h_n is nonzero on $S\{K^n\}$. Moreover, since $\mathbf{Y}\{K^n\}$ is irreducible and reduced for $n \gg 0$, h_n is not nilpotent and so $\mathbf{Y}\{K^n\}[1/h_n]$ is nonempty. Since $M_Y[1/h]$ is free over $S[1/h]$, it is flat on $\mathbf{Y}\{K^n\}[1/h_n]$. So $\mathbf{Y}\{K^n\}[1/h_n] \subseteq \mathbf{U}$, and thus $\mathbf{Z}\{K^n\}$ is contained in the vanishing locus of h_n. So, we conclude that for $n \gg 0$, $Z\{K^n\}$ is contained in the vanishing locus of h.

For $p \in \mathbf{Y}\{K^n\}[1/h]$, its residue field $\kappa(p)$ is naturally an $S\{K^n\}[1/h_n]$-algebra. We get

$$M_Y\{K^n\} \otimes_{S\{K^n\}} \kappa(p) \cong \kappa(p)^{d_n}$$

So $\dim_{\kappa(p)} M_Y\{K^n\} \otimes_{S\{K^n\}} \kappa(p) = d_n = \dim V\{K^{n-\ell}\}$ which is a polynomial for $n \gg 0$. □

Proposition 3.14. *Let $\mathbf{Z}$ be the non-flat locus of $M_Y = i^*\Omega_X$ in $\mathbf{Y}$. Then there exists a polynomial $t_Y(n)$ with rational coefficients, such that for $n \gg 0$, and all $p \in \mathbf{Y}\{k^n\} \setminus \mathbf{Z}\{k^n\}$, we have*

$$\dim T_p\mathbf{X}\{k^n\} = t_Y(n)$$

Proof. By Lemma 3.12, M_Y is finitely $\mathbf{GL}$-generated as an S-module.

Since the evaluation functor M_Y on polynomial representations is exact and compatible with tensor products and Kähler differentials, we get

$$M_Y\{k^n\} \cong \Omega_{R\{k^n\}} \otimes_{R\{k^n\}} S\{k^n\} = \Omega_{\mathbf{X}\{k^n\}} \otimes_{\mathcal{O}(\mathbf{X}\{k^n\})} \mathcal{O}(\mathbf{Y}\{k^n\}).$$

Thus, for $p \in \mathbf{Y}(k^n)$, we get

$$M_Y\{k^n\} \otimes_{\mathcal{O}(\mathbf{Y}\{k^n\})} \kappa(p) \cong \Omega_{\mathbf{X}\{k^n\}} \otimes_{\mathbf{X}\{k^n\}} \kappa(p) \cong \Omega_{\mathcal{O}_{\mathbf{X}\{k^n\},p}} \otimes_{\mathcal{O}_{\mathbf{X}\{k^n\},p}} \kappa(p)$$

Since the Zariski tangent space at p is dual to the cotangent space:

$$T_p\mathbf{X}\{k^n\} \cong \mathrm{Hom}_{\kappa(p)}(\Omega_{\mathcal{O}_{\mathbf{X}\{k^n\},p}} \otimes_{\mathcal{O}_{\mathbf{X}\{k^n\},p}} \kappa(p), \kappa(p))$$

we get

$$\dim T_p\mathbf{X}\{k^n\} = \dim_{\kappa(p)}(M_Y\{k^n\} \otimes \kappa(p))$$

Applying Lemma 3.13, we get the desired result. □

This allows us to prove the following result, which will be the key step in constructing the desired stratification.

Proposition 3.15. *Let $Y \subseteq X$ be an irreducible closed* **Vec***-subvariety. Then there exists a proper closed* **Vec***-subvariety $Z \subsetneq Y$ such that exactly one of the following alternatives holds for every $V \in \mathbf{Vec}$ with $\dim V \gg 0$:*

(1) $\big(Y(V) \setminus Z(V)\big) \cap \mathrm{Sing}_m(X(V)) = \varnothing$.

(2) $Y(V) \setminus Z(V) \subseteq \mathrm{Sing}_m(X(V))$.

Proof. Fix Y. Take $Z = Z_Y^{\mathrm{inc}} \cup Z_Y^{\mathrm{flat}}$ inside Y, where Z_Y^{inc} is defined as in Definition 3.9, and Z_Y^{flat} is the non-flat locus of $M_Y = i^*\Omega_X$. Then by Proposition 3.11 and Proposition 3.14, for all $V \in \mathbf{Vec}$ with $\dim V \gg 0$, and all $p \in Y(V) \setminus Z(V)$, the embedding codimension of p can we written as

$$\mathrm{embcodim}_p X(V) = t_Y(n) - d_Y(n)$$

where $t_Y(n)$ and $d_Y(n)$ are polynomials with rational coefficients taking integer values for $n \gg 0$. Since $\mathrm{embcodim}_p X(V)$ is a polynomial expression, for all $V \in \mathbf{Vec}$ with $\dim V \gg 0$, either $\big(Y(V) \setminus Z(V)\big) \subseteq \mathrm{Sing}_m X(V)$, or $\big(Y(V) \setminus Z(V)\big) \cap (\mathrm{Sing}_m X(V)) = \varnothing$. □

Finally we are ready to complete the proof of Proposition 3.5

Proof of Proposition 3.5. We argue by Noetherian induction, which is applicable by Theorem 2.24. Thus we may assume that every proper closed **Vec**-subvariety of X admits a stratification of the form asserted in Proposition 3.5. Let

$$X = X_1 \cup X_2 \cup \cdots \cup X_r$$

be the decomposition of X into irreducible components. Then, for each X_i, applying Proposition 3.15 gives us a proper closed **Vec**-subvariety $C_i \subseteq X_i$ such that either $\big(X_i(V) \setminus C_i(V)\big) \cap \mathrm{Sing}_m(X(V))\big) = \varnothing$ or $\big(X_i(V) \setminus C_i(V)\big) \subseteq \mathrm{Sing}_m(X(V))$ whenever $\dim(V) \gg 0$.

Now, for each $i = 1, ..., r$ we define another proper closed **Vec**-subvariety of X_i by

$$Z_i := C_i \cup \big(\bigcup_{j \neq i} X_i \cap X_j\big) \subsetneq X_i.$$

Notice that all of the $X_i \setminus Z_i$'s will be disjoint, and, since $X_i \setminus Z_i \subseteq X_i \setminus C_i$, the former will still satisfy the desired condition when $\dim(V) \gg 0$. Moreover, letting $Z = \bigcup_{i=1}^r Z_i$ we have that

$$X = Z \sqcup \bigsqcup_{i=1}^{r} (X_i \setminus Z_i)$$

where $Z \subsetneq X$.

Since Z has the desired stratification by assumption, we get the desired stratification for X. □

Remark 3.16. If our **Vec**-variety X is irreducible, then we already know that $\overline{\mathrm{Sing}_m(X(V))} = \mathrm{Sing}_m(X(V))$ for all $V \in \mathbf{Vec}$. So, in that case, Theorem B gives precisely the statement of Theorem A. ♪♫

4. The Core and Singular Locus

Let X be an irreducible **Vec**-variety. Recall the definition of the core:

$$X_{\mathrm{core}} = \bigcap_{m=1}^{\infty} X_{\mathrm{Sing}_m}$$

Recall that the X_{Sing_m}'s give us a decreasing chain of **Vec**-subvarieties. So, by Theorem 2.24, we know that there must exist some $M \in \mathbb{Z}_{\geq 0}$ such that $X_{\mathrm{core}} = X_{\mathrm{Sing}_M}$. The question of what this M is, and particularly when this $M = 1$, is what motivated Theorem C.

4.1. **Proof of Theorem C.** We first recall the setup in Theorem C: we are given an irreducible **Vec**-variety X, a pure polynomial functor Q and a surjective natural transformation $\alpha : \mathbb{A}(Q) \to X$. Our goal is to show that the core of X as defined above is precisely the singular locus Y_1.

Example 4.1. We begin with an example to illustrate why the hypothesis $Q_0 = \{0\}$ is necessary. Let P be the polynomial functor $P(V) = K^2 \otimes V$ and Q the polynomial functor $Q(V) = K \oplus V$. Then, take the affine curve $C = \{(t^2, t^3) : t \in K\} \subset K^2$ and define a **Vec**-variety $X(V) := C \times V \subset P(V)$. Since X is elementary, we know that $X_{\mathrm{core}} = \emptyset$. It is easy to see that for each $V \in \mathbf{Vec}$, $Y_1 = \mathrm{Sing}((X(V)) = \{0, 0\} \times V$ which evidently does not coincide with the core. However, taking α to be the polynomial transformation with $\alpha_V : Q(V) \to P(V)$ given by $\alpha_V(t, v) = (t^2, t^3, v)$, we see that its image is precisely X. ♫

We now begin the proof of Theorem C. Our strategy will be to consider the behavior of the codimension of X_{core} as a closed subvariety of X. In particular, we have that the following lemma holds.

Lemma 4.2. *Let X be an irreducible* **Vec***-variety and suppose that*

$$\mathrm{codim}_{X(K^n)} \mathrm{Sing}(X(K^n)) \to \infty \quad \text{as } n \to \infty.$$

Then $X_{\mathrm{Sing}} = X_{\mathrm{core}}$.

Proof. From Corollary 2.37, we already have $X_{\mathrm{core}} \subseteq X_{\mathrm{Sing}}$. Suppose that $X_{\mathrm{Sing}} \neq X_{\mathrm{core}}$. Then, for some $N \in \mathbb{Z}_{\geq 0}$, there exists some $x \in X_{\mathrm{Sing}}(K^N) \setminus X_{\mathrm{core}}(K^N)$.

We first extend x_N to a point of the associated **GL**-variety. For $n \geq N$, let $\iota_{N,n} \colon K^N \longrightarrow K^n$ be the standard inclusion, and set $x_n \coloneqq X(\iota_{N,n})(x_N)$. For $n < N$, define x_n using the standard projections. It is easy to see that the resulting sequence $\mathbf{x} = (x_0, x_1, x_2, \ldots)$ defines a point of the **GL**-variety $\mathbf{X}$ associated to $\mathbf{X}$, and $\mathbf{x} \in \mathbf{X}_{\mathrm{Sing}} \setminus \mathbf{X}_{\mathrm{core}}$. Hence, $\mathbf{x}$ is elementary. That is, there exists a finite dimensional vector space U, an open **Vec**-subvariety $O \subseteq \mathrm{Sh}_U(X)$, a finite dimensional variety B, and a polynomial functor R such that

$$O \cong B \times \mathbb{A}(R)$$

and $\mathbf{x} \in O$. Since X is irreducible, B is irreducible.

After choosing an identificition $U \cong K^d$. we see that $x_{d+n} \in O(K^n)$ for all $n \geq 0$. Under the isomorphism, write $x_{n+d} = (b, r_n)$, where $b \in B$ and $r_n \in \mathbb{A}(R)(K^n)$. For sufficiently large n, we have that $x_{n+d} \in \mathrm{Sing}(O(K^n))$. By [CFD22, Proposition 7.6], we have that $\mathrm{embcodim}\, \mathcal{O}_{B,b} = \mathrm{embcodim}\, \mathcal{O}_{O,x_{n+d}}$. Taking n sufficiently large, we see that $x_{n+d} \in \mathrm{Sing}(O(K^n))$ and hence $b \in \mathrm{Sing}(B)$. Since $O(K^n) \cong B \times \mathbb{A}(R)(K^n)$ and $\mathbb{A}(R)(K^n)$ is smooth, we obtain

$$\mathrm{Sing}(O(K^n)) \cong \mathrm{Sing}(B) \times \mathbb{A}(R)(K^n)$$

for every $n \geq 0$. In particular, since $\mathrm{Sing}(B)$ is nonempty, $\mathrm{Sing}(O(K^n))$ is nonempty for every $n \geq 0$.

Now, since X is irreducible and $O(K^n)$ is an open subset of $X(U \oplus K^n)$, we have that $\dim X(U \oplus K^n) = \dim O(K^n) = \dim B + \dim R(K^n)$. Since $\mathrm{Sing}(O(K^n)) = \mathrm{Sing}(X(U \oplus K^n)) \cap O(K^n)$, we conclude that

$$\begin{aligned}
\mathrm{codim}_{X(U\oplus K^n)} \mathrm{Sing}(X(U \oplus K^n)) &= \dim(X(U \oplus K^n)) - \dim(\mathrm{Sing}(X(U \oplus K^n))) \\
&\leq (\dim(B) + \dim(R(K^n))) - (\dim(\mathrm{Sing}(B)) + \dim(R(K^n))) \\
&= \dim(B) - \dim(\mathrm{Sing}(B)) \\
&= \mathrm{codim}_B \mathrm{Sing}(B).
\end{aligned}$$

Since $\mathrm{codim}_B \mathrm{Sing}(B) < \infty$ is a fixed finite number, we see that $\mathrm{codim}_{X(U\oplus K^n)} \mathrm{Sing}(X(U \oplus K^n))$ is bounded as $n \to \infty$. □

We wish to show that X, under the hypotheses of Theorem C, satisfies the condition in Lemma 4.2. To that end, we state and prove two more general results about codimension.

Lemma 4.3. *Let Q be a polynomial functor with $Q_0 = \{0\}$. Let $Z \subsetneq Q$ be a proper closed* **Vec***-subvariety. Then*

$$\operatorname{codim}_{Q(V)} Z(V) \to \infty \quad \text{as } \dim V \to \infty$$

Proof. If $Z = \varnothing$, then the above statement is trivial, so suppose that $Z \neq \varnothing$. Then, since Z is proper, there exists some $U \in \mathbf{Vec}$ such that $Z(U) \subsetneq Q(U)$. Let $c = \operatorname{codim}_{Q(U)} Z(U) > 0$. Fix $m \geq 1$, and set

$$V = U^{\oplus m}.$$

Let $i_j : U \to V$, $\pi_j : V \to U$ be the natural injections and projections onto the jth coordinates, so that $\pi_j i_j = \mathrm{id}_U$ and $\pi_k i_j = 0$ for $j \neq k$. Define $\Phi_V : Q(V) \to Q(U)^m$ by

$$q \mapsto (Q(\pi_1)(q), \dots, Q(\pi_m)(q))$$

We claim that Φ_V is surjective. Indeed, define $\Psi_V : Q(U)^m \to Q(V)$ by

$$(q_1, \dots, q_m) \mapsto \sum_{j=1}^{m} Q(i_j) q_j$$

We know that $Q(\pi_j i_j) = \mathrm{Id}_{Q(U)}$ and, since $Q(0) = \{0\}$, we have that $Q(\pi_k i_j) = 0$ when $j \neq k$. So,

$$\Phi_V \Psi_V (q_1, \dots, q_m) = \left(\sum_{j=1}^{m} Q(\pi_1 i_j)(q_1), \sum_{j=1}^{m} Q(\pi_2 i_j)(q_2), \dots, \sum_{j=1}^{m} Q(\pi_m i_j)(q_m) \right) = (q_1, \dots, q_m).$$

Let $z \in Z(V)$, then $Q(\pi_j)(z) \in Z(U)$ for any j. So $Z(V) \subseteq \Phi_V^{-1}(Z(U)^m)$. Since Φ_V is surjective, we have that

$$Q(V) \cong Q(U)^m \oplus \ker(\Phi)$$

and thus that

$$\Phi^{-1}(Z(U)^m) \cong Z(U)^m \times \ker(\varphi).$$

Since $\ker(\varphi)$ is just a linear subspace, we conclude that

$$\operatorname{codim}_{Q(V)} \Phi_V^{-1}(Z(U)^m) = \operatorname{codim}_{Q(U)^m} Z(U)^m.$$

Hence $\operatorname{codim}_{Q(V)} Z(V) \geq \operatorname{codim}_{Q(U)^m} Z(U)^m = mc$. Taking $m \to \infty$, we see that $\operatorname{codim}_{Q(V)} Z(V) \to \infty$. □

Lemma 4.4. *Let $f : X \to Y$ be a surjective, finite type morphism between irreducible varieties. Let $T \subseteq Y$ be a closed subvariety, and set $W = f^{-1}(T)$, then*

$$\operatorname{codim}_Y T \geq \operatorname{codim}_X W.$$

Proof. Let $r = \dim X - \dim Y$. Let $T_0 \subseteq T$ be a irreducible component of maximal dimension, so that $\dim T = \dim T_0$. Since f is surjective, the restriction $f|_{f^{-1}(T_0)} : f^{-1}(T_0) \to T_0$ is surjective. For surjective morphisms between irreducible varieties, we have that

$$\dim f^{-1}(T_0) \geq \dim T_0 + r.$$

Hence $\dim W \geq \dim T + r$, and we compute that

$$\begin{aligned} \operatorname{codim}_X W &= \dim X - \dim W \\ &\leq \dim X - (\dim T + r) \\ &= \dim Y - \dim T \\ &= \operatorname{codim}_Y T. \end{aligned}$$

□

It now follows easily from these two lemmas that the hypothesis in Lemma 4.2 will be satisfied.

Proof of Theorem C. Let $Z = \alpha^{-1}(X_{\mathrm{Sing}})$. Since $X_{\mathrm{Sing}} \subsetneq X$ and α is surjective, we have that Z is a proper closed **Vec**-variety in Q. So, by Lemma 4.3, $\operatorname{codim}_{Q(V)} Z(V) \to \infty$ as $\dim(V) \to \infty$. Since each $\alpha_V : Q(V) \to X(V)$ is surjective, Lemma 4.4 gives us that $\operatorname{codim}_{X(V)}(\mathrm{Sing}(X(V)) \to \infty$ as $\dim(V) \to \infty$ as well. We then conclude by Lemma 4.2 that $X_{\mathrm{core}} = X_{\mathrm{Sing}}$.

□

We hypothesize that weaker conditions, analogous to those in Theorem C, can be placed on the polynomial functor Q to conclude that $X_{\mathrm{core}} = X_{\mathrm{Sing}_m}$ for larger $m \in \mathbb{Z}_{\geq 1}$. This is a direction we are currently exploring.

5. A Higher-derivative Locus

Let X a **Vec**-variety in a polynomial functor P. Given some $V \in \mathbf{Vec}$, we denote the defining ideal of $X(V)$ by $I_X(V)$. In this section, we wish to analyze the behavior of $\mathrm{Mult}_m(X(V))$ as we vary V.

5.1. **Proof of Theorem D.** In order to prove Theorem D, we will examine the behavior of $\mathrm{Mult}_m(X(V))$ under linear maps. Recall from Remark 2.15 that it is sufficient to consider the initial term of $f \in I_X(V)$ up to some translation. We will begin by examining how composition with linear maps affects the degree of this initial term. Recall that we use the convention $\deg(0) = \infty$.

Lemma 5.1. *Let $V, W \in \mathbf{Vec}$ and $\varphi \in \mathrm{Hom}(V, W)$. Let $P(\varphi) : P(V) \to P(W)$ be the induced linear map. Let $f \in I_X(W)$ be nonzero. If $f \circ P(\varphi) \neq 0$, then $\deg(\mathrm{in}(f \circ P(\varphi))) \geq \deg(\mathrm{in}(f))$.*

Proof. Let $\varphi \in \mathrm{Hom}(V, W)$ and consider the map $P(\varphi) : P(V) \to P(W)$. Now, fix bases $\{E_1, ..., E_r\}$ for $P(V)$ and $\{F_1, ..., F_s\}$ for $P(W)$ with respective coordinate functions $\{x_i\}_{1 \leq i \leq r}$ and $\{y_j\}_{1 \leq j \leq s}$.

Then, the map $P(\varphi)$ is given by

$$E_i \mapsto \sum_{j=1}^{s} a_{ij} F_j.$$

So, the induced map on coordinate rings $P(\varphi)^* : K[\mathbb{A}(P)(W)] \to K[\mathbb{A}(P)(V)]$ is given by

$$y_j \mapsto \sum_{i=1}^{r} a_{ij} x_i$$

for some $a_{ij} \in K$. In particular, for $f \in I_X(W)$ we can see that $f \circ P(\varphi)$ will indeed be a polynomial in the x_i's. More importantly, we notice that if $\mathrm{in}(f) \circ P(\varphi) \neq 0$ then $\mathrm{in}(f \circ P(\varphi)) = \mathrm{in}(f) \circ P(\varphi)$ and so, by linearity of the expressions above, we have that

$$\deg(\mathrm{in}(f \circ P(\varphi))) = \deg(\mathrm{in}(f)).$$

The one exception to this is if $\mathrm{in}(f) \circ P(\varphi) = 0$ and $f \circ P(\varphi) \neq 0$, which will give us that

$$\deg(\mathrm{in}(f \circ P(\varphi))) > \deg(\mathrm{in}(f)).$$

This gives us our result. □

Proposition 5.2. *In the setting of Lemma 5.1, suppose that $\varphi : V \to W$ is surjective, then for every $f \in I_X(W)$, we have $\deg(\mathrm{in}(f \circ P(\varphi)) = \deg(\mathrm{in}(f))$*

Proof. Suppose that $\varphi \in \mathrm{Hom}(V, W)$ is surjective. Then, there exists some $\iota \in \mathrm{Hom}(W, V)$ such that $\varphi \circ \iota = \mathrm{Id}_W$. By functoriality of P and X, we have $P(\varphi) \circ P(\iota) = \mathrm{Id}_{P(W)}$ and their restrictions $X(\varphi) \circ X(\iota) = \mathrm{Id}_{X(W)}$.

Now, let $R = K[\mathbb{A}(P)(V)]$ and $R' = K[\mathbb{A}(P)(W)]$ with $I = I_X(V) \subseteq R$ and $I' = I_X(W) \subseteq R'$. The induced pullback ring homomorphisms satisfy $P(\iota)^* \circ P(\varphi)^* = \mathrm{Id}_{R'}$ and $X(\iota)^* \circ X(\varphi)^* = \mathrm{Id}_{R'/I'}$. Hence the homomorphism $P(\varphi)^* : R' \to R$, $f \mapsto f \circ P(\varphi)$ is injective.

Let $f \in I_X(W)$ be nonzero. Then its initial degree term $\mathrm{in}(f)$ is nonzero. Since $P(\varphi)^*$ is injective, $P(\varphi)^*(\mathrm{in}(f)) = \mathrm{in}(f) \circ P(\varphi) \neq 0$. By Lemma 5.1, $\deg(\mathrm{in}(f \circ P(\varphi))) = \deg(\mathrm{in}(f))$ if and only if $\mathrm{in}(f) \circ P(\varphi) \neq 0$, so $\deg(f \circ P(\varphi)) = \deg(\mathrm{in}(f))$. □

It follows from Proposition 5.2 that $\mathrm{Mult}_m(X(V))$ is preserved by surjective linear maps.

Corollary 5.3. *Let $V, W \in \mathbf{Vec}$, and let $\varphi \in \mathrm{Hom}(V, W)$ be a surjective linear map, and let $m \geq 0$. Then*

$$P(\varphi)(\mathrm{Mult}_m(X(V)) \subseteq \mathrm{Mult}_m(X(W))$$

Proof. Fix $p \in \mathrm{Mult}_m(X(V))$, and let $f \in I_X(W)$. To show that $P(\varphi)(p) \in \mathrm{Mult}_m(X(W))$, it suffices to show that $\deg\big(\mathrm{in}(f_{P(\varphi)(p)})\big) \geq m$.

Set $F \coloneqq f \circ P(\varphi) \in I_X(V)$. Consider the translation of F by p, we have $F_p(x) = F(x+p) = f\big(P(\varphi)(x+p)\big)$. By linearity of $P(\varphi)$, we get $F_p(x) = f\big(P(\varphi)(x) + P(\varphi)(p)\big) = f_{P(\varphi)(p)}(P(\varphi)(x))$. Hence $F_p = f_{P(\varphi)(p)} \circ P(\varphi)$.

Since φ is surjective, Proposition 5.2 gives $\deg\big(\mathrm{in}(F_p)\big) = \deg\big(\mathrm{in}(f_{P(\varphi)(p)})\big)$. Moreover, since $p \in \mathrm{Mult}_m(X(V))$ and $F \in I_X(V)$, we get $\deg\big(\mathrm{in}(F_p)\big) \geq m$. It follows that $\deg\big(\mathrm{in}(f_{P(\varphi)(p)})\big) \geq m$. Since this holds for all $f \in I_X(W)$, we see that $P(\varphi)(p) \in \mathrm{Mult}_m(X(W))$. □

Since every $\varphi \in \mathrm{GL}(V)$ is surjective, we have the following special case.

Corollary 5.4. *For every $V \in \mathbf{Vec}$, $\mathrm{Mult}_m(X(V) \subseteq X(V)$ is $\mathrm{GL}(V)$-stable.*

We now examine what happens in the case of injective linear maps. Our goal will be the prove the following proposition:

Proposition 5.5. *Fix $m \in \mathbb{Z}_{\geq 1}$, and let $V, W \in \mathbf{Vec}$ satisfy $\dim(V), \dim(W) \geq \deg(P)(m-1)$. If $\varphi \in \mathrm{Hom}(V,W)$ is injective, then the induced map $P(\varphi)$ preserves Mult_m. More precisely, $P(\varphi)(p) \in \mathrm{Mult}_m(X(W))$ for $p \in \mathrm{Mult}_m(X(V))$.*

Proving this proposition will require us to determine how the degree of initial terms is affected by composition with injective linear maps. To do this, we will need to explicitly use the structure of P as a direct sum of Schur functors: that is, as a quotient of a direct sum of tensor powers. We first consider what happens in the simplest case. From this point forward, m will always be a number in $\mathbb{Z}_{\geq 1}$.

Lemma 5.6. *Let $P_r \coloneqq \mathbf{Vec} \to \mathbf{Vec}$ be the polynomial functor given by the r th tensor power $P^r(V) = \bigotimes^r(V)$.*

Let $V, W \in \mathbf{Vec}$ with $\dim(V) \geq r(m-1)$, and let $\varphi \in \mathrm{Hom}(V,W)$ be injective. We have the following: for every $d < m$, if there exists $f \in I_X(W)$ such that $\deg(\mathrm{in}(f)) = d$, then there exists $f' \in I_X(W)$, not necessarily equal to f, such that

$$\deg\big(\mathrm{in}(f' \circ P_r(\varphi))\big) = d.$$

Proof. Fix a basis $\{E_1, ..., E_k\}$ for V, where $k = \dim V$. Since φ is injective, we may choose a basis $\{F_1, ..., F_n\}$ for W such that $\varphi(E_i) = F_i$ for all $1 \leq i \leq k$. We thus have bases $\{E_{i_1} \otimes \cdots \otimes E_{i_r} \mid 1 \leq i_1, ..., i_r \leq k\}$ and $\{F_{j_1} \otimes \cdots \otimes E_{j_r} \mid 1 \leq j_1, ..., j_r \leq n\}$ for $P(V)$ and $P(W)$ respectively. With respect to these bases, $P(\varphi)$ is given by

$$E_{i_1} \otimes \cdots \otimes E_{i_r} \mapsto F_{i_1} \otimes \cdots \otimes F_{i_r}.$$

Let $\{x_{i_1 \cdots i_r} \mid 1 \leq i_1, \ldots, i_r \leq k\}$ and $\{y_{j_1 \cdots j_r} \mid 1 \leq j_1, \ldots, j_r \leq n\}$ denote the coordinate functions on $\mathbb{A}(P)(V)$ and $\mathbb{A}(P)(W)$, respectively, to the bases $\{E_{i_1} \otimes \cdots \otimes E_{i_r}\}$ and $\{F_{j_1} \otimes \cdots \otimes F_{j_r}\}$. Then the map induced by $P(\varphi)$ on coordinate rings is given by

$$P(\varphi)^* : K[\mathbb{A}(P)(W)] \to K[\mathbb{A}(P)(V)]$$
$$y_{j_1 \cdots j_r} = \begin{cases} x_{j_1 \cdots j_r} & \text{if } j_1, ..., j_r \leq k, \\ 0 & \text{otherwise.} \end{cases}$$

Now, let $f \in I_X(W)$ satisfy $\deg(\mathrm{in}(f)) = d < m$. Then, $\mathrm{in}(f)$ will contain a monomial term of the form

$$a \cdot y_{j_{11} \cdots j_{r1}} \cdots y_{j_{1d} \cdots j_{rd}}$$

where $a \in K^\times$. There are at most rd distinct values among the indices j_{st} occurring in this monomial. Since $d \leq m-1$, we have $rd \leq r(m-1) \leq \dim(V)$.

We now use the action of $\mathrm{GL}(W)$ on $I_X(W)$ to construct a monomial which will not vanish under $P(\varphi)^*$. Since at most $rd \leq k$ distinct indices occur among the j_{st}, we may choose $\psi \in \mathrm{GL}(W)$ which permutes the basis $\{F_j\}$ in such a way that

$$\psi(F_{i_{st}}) = F_{j_{st}}.$$

where $1 \leq i_{st} \leq k = \dim(V)$ for all $1 \leq s \leq r$ and $1 \leq t \leq d$. The induced map $P(\psi)^* : K[\mathbb{A}(P)(W)] \to K[\mathbb{A}(P)(W)]$ on coordinate rings will be a ring automorphism satisfying

$$y_{j_{1t} \ldots j_{rt}} \mapsto y_{i_{1t} \ldots i_{rt}}.$$

for each $1 \leq t \leq d$.

Since $I_X(W)$ is invariant under the action of $\mathrm{GL}(W)$ we may define $f' \coloneqq f \circ P(\psi) = P(\psi)^*(f) \in I_X(W)$. As $P(\psi)^*$ is an automorphism, we saw in the proof of Corollary 5.3 that $\mathrm{in}(f') = \mathrm{in}(f) \circ P(\psi)$, and injectivity ensures that there will be no cancellations among the monomial terms. In particular, $\mathrm{in}(f')$ will contain the monomial term

$$a \cdot y_{i_{11} \cdots i_{r1}} \cdots y_{i_{1d} \cdots i_{rd}}.$$

Finally, consider the polynomial $f' \circ P(\varphi) \in I_X(V)$. The monomial above is sent to

$$a \cdot x_{i_{11} \cdots i_{r1}} \cdots x_{i_{1l} \cdots i_{rd}} \neq 0.$$

We thus conclude that $\deg(\mathrm{in}(f' \circ P(\varphi))) = d$. □

This lemma proves that the degree of at least one initial term will be preserved by $P(\varphi)^*$ when P is simply a tensor power. We can easily extend the statement to the case when P is a direct sum of tensor powers.

Lemma 5.7. *Let $P = P_{r_1} \oplus \cdots \oplus P_{r_N}$ be a polynomial functor, where each P_{r_i} is the r_ith tensor power functor and $r_1 \leq \cdots \leq r_N$. Let $V, W \in \mathbf{Vec}$ satisfy $\dim(V) \geq r_N(m-1)$, and let $\varphi \in \mathrm{Hom}(V,W)$ be injective. We have the following: for every $d < m$, if there exists $f \in I_X(W)$ such that $\deg(\mathrm{in}(f)) = d$, then there exists $f' \in I_X(W)$, not necessarily equal to f, such that*

$$\deg\big(\mathrm{in}(f' \circ P(\varphi))\big) = d.$$

Proof. The proof is analogous to that of the previous lemma. The only additional point is that we now look at a monomial arising from the direct summand of different tensor powers. We consider a monomial of the form

$$a\, y_{j_{11}\cdots j_{r_{l_1}1}}\cdots y_{j_{1d}\cdots j_{r_{l_d}d}},$$

where the integers $l_1,\dots,l_d$ need not be distinct. Since $r_{l_t}\leq r_N$ for every t, at most $r_N d$ distinct basis indices occur among the j_{st}. Since we chose $\dim(V)\geq r_N d$, we can construct a map $\psi\in \mathrm{GL}(W)$ exactly as in the previous lemma so that the initial term of $f'=f\circ P(\psi)$ contains the monomial

$$a\cdot y_{i_{11}\cdots i_{r_{i_1}1}}\cdots y_{i_{1d}\cdots i_{r_{i_d}d}}\neq 0,$$

where $1\leq i_{st}\leq \dim(V)$ for all s,t. We thus conclude that

$$\deg(\mathrm{in}(f'\circ P(\varphi)))=d.$$

□

We now consider what happens when our polynomial P is a quotient of a tensor power. The simplest case is when P is a Schur functor. If P is constant, then the action of any map is trivial. We therefore restrict to the case when $\deg(P)>0$, for which we have the following lemma.

Lemma 5.8. *Fix $r\in\mathbb{Z}_{\geq 1}$ and λ a partition of $\{1,\dots,r\}$. Let $P=S_\lambda$ be the Schur functor corresponding to the partition λ. Then, for any $V,W\in$ **Vec** such that $\dim(V)\geq r\cdot(m-1)$ and any injective linear map $\varphi\in\mathrm{Hom}(V,W)$, the conclusion of the previous two lemmas holds.*

Proof. Take bases $\{E_i\}$ and $\{F_j\}$ for V and W as in the proof of Lemma 5.6. It is well-known (Theorem 8.1, [Ful99]) that bases for $S_\lambda(V)$ and $S_\lambda(W)$ are given by the images of $E_{i_1}\otimes\cdots\otimes E_{i_r}$ and $F_{j_1}\otimes\cdots\otimes F_{j_r}$ under the quotient map $\pi_r: V^{\otimes^r}\twoheadrightarrow S_\lambda(V)$, where $\{i_1,\dots,i_r\}$ and $\{j_1,\dots,j_r\}$ correspond to semistandard labelings of the Young tableaux for λ. Here semistandard means that entries are weakly increasing across rows, and strictly increasing across columns.

Notice now that if we replace $j_1,\dots,j_r$ by new indexes $j'_1,\dots,j'_r$ while preserving the relative order and equality, that is: $j_a<j_b\iff j'_a<j'_b$, and $j_a=j_b\iff j'_a=j'_b$, then the new filling remains semistandard. Hence, we get another such basis element $F_{j'_1}\otimes\cdots\otimes F_{j'_r}$. This indicates that a proof strategy very similar to that of Lemma 5.6 canbe used here.

In particular, the proof proceeds in much the same way, except we now consider a monomial term of the form

$$a\cdot y_{j_{11}\cdots j_{r1}}\cdots y_{j_{1d}\cdots j_{rd}},$$

where $a\in K^\times$ and, for each $1\leq t\leq d$, the indices $j_{1t},\dots,j_{rt}$ correspond a semistandard labeling of the Young diagram of shape λ. As the observation above, if we replace these indexes by a different order-preserving set, then we will get another basis element, and thus another coordinate function in $K[\mathbb{A}(P)(W)]$. As in the previous lemmas, we see that there are at most rd distinct basis indices among the j_{st} appearing in the monomial. Since $d\leq m-1$, we have $rd\leq r(m-1)\leq\dim(V)$.

We now once again construct a map $\psi\in\mathrm{GL}(W)$ that permutes the $\{F_j\}$. However, we must ensure that this map preserves the relative order of the indices appearing in the chosen monomial. To do so, list the distinct indices as $j^{(1)}<j^{(2)}<\cdots j^{(q)}$. Since $q\leq rd\leq\dim(V)=k$, we may choose ψ so that $\psi(F_\ell)=F_{j^{(\ell)}}$ for $1\leq\ell\leq q$. We then extend this map to the entire basis of W.

We thus obtain $f'=f\circ P(\psi)\in I_X(W)$ such that $\mathrm{in}(f')$ contains the monomial

$$a\cdot y_{i_{11}\cdots i_{r1}}\cdots y_{i_{1d}\cdots i_{rd}}\neq 0,$$

where $1\leq i_{st}\leq\dim(V)$ for all s,t, and the indices $i_{1t},\dots,i_{rt}$'s gices a semistandard labeling of the Young diagram of shape λ. It follows that $P(\varphi)^*(f')$ has initial term containing the monomial

$$a\cdot x_{i_{11}\cdots i_{r1}}\cdots x_{i_{1d}\cdots i_{rd}}\neq 0,$$

which is a nonzero monomial in the coordinate ring $K[\mathbb{A}(S_\lambda)(V)]$. Consequently, $\deg\big(\mathrm{in}(f'\circ P(\varphi)\big)=d$. □

The previous three lemmas show that the result holds in full generality, since every polynomial functor decomposes as a direct sum of Schur functors.

Corollary 5.9. *Let $P=U\oplus S_{\lambda_1}\oplus\cdots\oplus S_{\lambda_N}$ be any polynomial functor, where each λ_i is a partition of $\{1,\dots,r_i\}$ and $r_1\leq\cdots\leq r_N$ and $U\in$ **Vec** is a fixed finite-dimensional vector space. Then, for any $V,W\in$ **Vec** such that $\dim(V)\geq r_N\cdot(m-1)$ and any injective linear map $\varphi\in\mathrm{Hom}(V,W)$, we have the following holds: for $d<m$, if there exists $f\in I_X(W)$ such that $\deg(\mathrm{in}(f))=d$, then there exists $f'\in I_X(W)$, not necessarily equal to f, such that*

$$\deg\big(\mathrm{in}(f'\circ P(\varphi))\big)=d.$$

Proof. The proof follows the same argument as Lemma 5.7, with the additional requirement that the map $\psi \in \mathrm{GL}(W)$ be chosen as in Lemma 5.8 so as to preserve orderings. The degree 0 component U does not change anything, since the action of φ on it is trivial. □

Finally, we apply this result to prove Proposition 5.5.

Remark 5.10. Recall from our previous computations that, for $f \in I_X(W)$, if we let $F = f \circ P(\varphi)$ and fix some $p \in X(V)$, then $F_p = f_{P(\varphi)(p)} \circ P(\varphi)$. Then, the same argument as above shows us that, if there exists some $f \in I_X(W)$ such that $\deg(\mathrm{in}(f_{P(\varphi)(p)})) = d$, then there exists some $f' \in I_X(W)$ such that

$$\deg\big(\mathrm{in}(f'_{P(\varphi)(p)} \circ P(\varphi))\big) = \deg\big(\mathrm{in}(f_{P(\varphi)(p)})\big) = d.$$

♪♫

Keeping this remark in mind, we now proceed to the proof of the proposition.

Proof of Proposition 5.5. First, we note the uniqueness of X_{Mult_m} satisfying this property follows by the same argument as that in Section 3.1 of [CDD25]. It remains to show existence.

Let $P = U \oplus S_{\lambda_1} \oplus \cdots \oplus S_{\lambda_N}$ be as in Corollary 5.9, and assume that $\dim(V), \dim(W) \geq r_N(m-1)$. Note here that r_N is precisely the degree of P.

Suppose that there exists some $p \in X(V)$ such that $P(\varphi)(p) \notin \mathrm{Mult}_m(X(W))$. Then there exist $d < m$ and $f \in I_X(W)$ such that $\deg\big(\mathrm{in}(f_{P(\varphi)(p)})\big) = d$. By the preceding remark, together with Corollary 5.9, there exists $f' \in I_X(W)$ such that

$$\deg(\mathrm{in}(F'_p)) = \deg\big(\mathrm{in}(f'_{P(\varphi)(p)} \circ P(\varphi))\big) = \deg\big(\mathrm{in}(f_{P(\varphi)(p)})\big) = d < m$$

for $F' = f' \circ P(\varphi) \in I_X(V)$. It follows that

$$p \notin \mathrm{Mult}_m(X(V)).$$

We conclude that $P(\varphi)(\mathrm{Mult}_m(X(V))) \subseteq \mathrm{Mult}_m(X(W))$. □

Since every linear map factors through a composition of injective and surjective linear maps, the following corollary is immediate.

Corollary 5.11. *Fix $m \in \mathbb{Z}_{\geq 1}$. Let $V, W \in \mathbf{Vec}$ satisfy $\dim(V), \dim(W) \geq \deg(P) \cdot (m-1)$, and let $\varphi \in \mathrm{Hom}(V, W)$ be such that $\dim(\mathrm{Im}(\varphi)) \geq \deg(P) \cdot (m-1)$. Then*

$$P(\varphi)(\mathrm{Mult}_m(X(V))) \subseteq \mathrm{Mult}_m(X(W))$$

We are finally ready to prove the theorem.

Proof of Theorem D. Firstly, suppose that P is a degree 0 polynomial functor, that is, if P and X are both constant functors, then set $X_{\mathrm{Mult}_m}(V) = \mathrm{Mult}_m(X(V))$ for all $V \in \mathbf{Vec}$. So, going forward, we will assume that $\deg(P) > 0$.

We claim that the following construction of X_{Mult_m} has the desired property as stated in the theorem: for $V \in \mathbf{Vec}$ such that $\dim(V) \geq \deg(P) \cdot (m-1)$, define

$$X_{\mathrm{Mult}_m}(V) \coloneqq \mathrm{Mult}_m(X(V)).$$

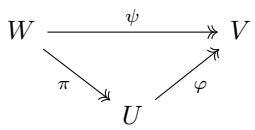
For $V \in \mathbf{Vec}$ with $\dim(V) < \deg(P) \cdot (m-1)$, choose a vector space U with $\dim(U) \geq \deg(P) \cdot (m-1)$ and a surjective map $\varphi \in \mathrm{Hom}(U, V)$. Define

$$X_{\mathrm{Mult}_m}(V) \coloneqq P(\varphi)(\mathrm{Mult}_m(X(U))).$$

The choices of U is not unique. It remains to verify that X_{Mult_m} is well-defined, independent of U, and defines a **Vec**-variety,

Lemma 5.12. *$X_{\mathrm{Mult}_m}(V)$ constructed above does not depend on the choice of $U \in \mathbf{Vec}$ with $\dim(U) \geq \deg(P) \cdot (m-1)$ and surjective $\varphi \in \mathrm{Hom}(U, V)$.*

Proof. Let W and $\psi \in \mathrm{Hom}(W, V)$ be another such pair, with ψ surjective. Without loss of generality, assume that $\dim(W) \geq \dim(V)$. Since φ, ψ are both surjective linear maps, we can construct a surjective map $\pi \in \mathrm{Hom}(W, U)$ such that the diagram commutes:



Thus, $\psi = \varphi \circ \pi$.

We claim that

$$P(\psi)(\mathrm{Mult}_m(X(W))) = P(\varphi)(\mathrm{Mult}_m(X(V))).$$

To see this, it suffices to prove that $P(\pi)(\mathrm{Mult}_m(X(W))) = \mathrm{Mult}_m(X(U))$.

Since π is surjective, there exists an injective linear map $\iota \in \mathrm{Hom}(U, W)$ such that $\pi \circ \iota = \mathrm{Id}_U$. First, π being surjective gives us that $P(\pi)$ preserves the multiplicity locus, thus giving us the $\subseteq$ inclusion.

For the reverse inclusion, let $q \in \mathrm{Mult}_m(X(U))$. Note that $\dim(U), \dim(W) \geq \deg(P) \cdot (m-1)$. By Proposition 5.5, $P(\iota)(q) \in \mathrm{Mult}_m(X(W))$. Moreover, $P(\pi) \circ P(\iota) = \mathrm{Id}_{P(U)}$, so $q = P(\pi)(P(\iota)(q))$. Thus, $\mathrm{Mult}_m(X(U)) \subseteq P(\pi)(\mathrm{Mult}_m(X(W))$, giving the $\supseteq$ inclusion. □

With this, we have shown that our X_{Mult_m} is well-defined. It remains to show that X_{Mult_m} is a closed **Vec**-subvariety of X. We begin by showing its functoriality.

Lemma 5.13. *Let $V_1, V_2 \in \mathbf{Vec}$ and $\varphi \in \mathrm{Hom}(V_1, V_2)$. Then $P(\varphi)$ maps $X_{\mathrm{Mult}_m}(V_1)$ into $X_{\mathrm{Mult}_m}(V_2)$. That is,*

$$P(\varphi)\big(X_{\mathrm{Mult}_m}(V_1)\big) \subseteq X_{\mathrm{Mult}_m}(V_2).$$

Proof. Fix some vector space $W \in \mathbf{Vec}$ such that $\dim(W) \geq \deg(P) \cdot (m-1)$, and set $W_1 \coloneqq V_1 \oplus W$ and $W_2 \coloneqq V_2 \oplus W$. Let $\pi_1 : W_1 \to V_1$, $\pi_2 : W_2 \to V_2$ be the natural projections onto V_1 and V_2 respectively. Since $\dim(W_1), \dim(W_2) \geq \deg(P) \cdot (m-1)$, we have that $X_{\mathrm{Mult}_m}(V_1) = P(\pi_1)(\mathrm{Mult}_m(X(W_1)))$ and $X_{\mathrm{Mult}_m}(V_2) = P(\pi_2)(\mathrm{Mult}_m(X(W_2)))$. Consider the commutative diagram

$$\begin{array}{ccc} W_1 & \xrightarrow{\varphi'} & W_2 \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_2} \\ V_1 & \xrightarrow[\varphi]{} & V_2, \end{array}$$

where $\varphi' \coloneqq V_1 \oplus W \to V_2 \oplus W$ is given by $\varphi \oplus \mathrm{Id}_W$. Since φ' restricts to the identity on W, we have $\dim(\mathrm{Im}(\varphi')) \geq \dim(W) \geq \deg(P) \cdot (m-1)$. So, by Corollary 5.11, we have

$$P(\varphi')(\mathrm{Mult}_m(X(W_1))) \subseteq \mathrm{Mult}_m(X(W_2)).$$

Now let $p \in X_{\mathrm{Mult}_m}(V_1) = P(\pi_1)(\mathrm{Mult}_m(X(W_1)))$. We know that there exists $q \in \mathrm{Mult}_m(X(V_1))$ such that $P(\pi_1)(q) = p$. Using the commutativity of the diagram and the functoriality of P, we have

$$P(\varphi)(p) = P(\varphi)(P(\pi_1)(q)) = P(\pi_2)(P(\varphi')(q)).$$

Since $P(\varphi')(q) \in \mathrm{Mult}_m(X(W_2))$, it follows that

$$P(\varphi)(p) \in P(\pi_2)(\mathrm{Mult}_m(X(V_2))) = X_{\mathrm{Mult}_m}(V_2).$$

Therefore,

$$P(\varphi)(X_{\mathrm{Mult}_m}(V_1)) \subseteq X_{\mathrm{Mult}_m}(V_2),$$

as desired. □

Finally, it remains to show that, for every $V \in \mathbf{Vec}$, the subset $X_{\mathrm{Mult}_m}(V)$ as we have defined is closed in $X(V)$. To this end, we give an alternative characterization of $X_{\mathrm{Mult}_m}(V)$ and prove that it is equivalent to our original definition.

Lemma 5.14. *Let $U, V \in \mathbf{Vec}$ satisfy $\dim(U) \leq \dim(V)$ and $\dim(V) \geq \deg(P) \cdot (m-1)$, and let $\varphi \in \mathrm{Hom}(U, V)$ be injective maps. Then*

$$X_{\mathrm{Mult}_m}(U) = P(\varphi)^{-1}(\mathrm{Mult}_m(X(V))).$$

In particular, $X_{\mathrm{Mult}_m}(U)$ is closed in $X(U)$, hence in $P(U)$, for all $U \in \mathbf{Vec}$.

Proof. By Lemma 5.13, $P(\varphi)$ preserves $X_{\mathrm{Mult}_m}(U)$. Since $\dim(V) \geq \deg(P) \cdot (m-1)$, we have $X_{\mathrm{Mult}_m}(V) = \mathrm{Mult}_m(X(V))$ and so we get that

$$X_{\mathrm{Mult}_m}(U) \subseteq P(\varphi)^{-1}(\mathrm{Mult}_m(X(V))).$$

To see the other inclusion, choose a surjective linear map $\pi : V \to U$ such that $\pi \circ \varphi = \mathrm{Id}_U$. Then, consider some $p \in P(U)$ such that $P(\varphi)(p) \in \mathrm{Mult}_m(X(V))$. By funcoriality of P,

$$p = P(\pi)(P(\varphi)(p)) \in P(\pi)(\mathrm{Mult}_m(X(V))) = X_{\mathrm{Mult}_m}(U).$$

This gives us the reverse inclusion and thus the fact that

$$X_{\mathrm{Mult}_m}(U) = P(\varphi)^{-1}(\mathrm{Mult}_m(X(V))).$$

Since $\mathrm{Mult}_m(X(V))$ is a closed subvariety of $X(V)$, and $P(\varphi)|_{X(U)}$ is a morphism, we see that $X_{\mathrm{Mult}_m}(U)$ must be closed in $X(U)$ as well. □

This completes the proof that $X_{\mathrm{Mult}_m} \subseteq X$ as we have defined satisfies the statement of the theorem. □

Remark 5.15. While we took $X \subseteq P$ to be a **Vec**-variety, so that each ideal $I_X(V)$ is radical, the argument above does not in fact use this radicality. More generally, one may take $X \subseteq P$ to be a functor to the category of affine schemes of finite type over K, then define $\mathrm{Mult}_m(X(V))$ the same way for each V. The same argument shows that there exists a **Vec**-*subscheme* $X_{\mathrm{Mult}_m} \subset X$ that coincides with $\mathrm{Mult}_m(X(V))$ when $\dim(V)$ is sufficiently large. ♪♫

5.2. **A Sharp Bound.** A natural question is whether the bound $\dim(V) \geq \deg(P) \cdot (m-1)$ is sharp. We will show that, in an appropriate sense, this is true. Before proving the general statement, we begin with a simple example.

Example 5.16. Let P be the polynomial functor given by $P(V) = V \otimes V$. Fix $W \cong K^2$ with the standard basis $e_1, e_2 \in K^2$. Then $\{e_1 \otimes e_1, e_1 \otimes e_2, e_2 \otimes e_1, e_2 \otimes e_2\}$ is a basis for $P(W)$, with corresponding coordinate functions $y_{11}, y_{12}, y_{21}, y_{22} \in K[\mathbb{A}(P)(W)]$.

Define a **Vec**-variety $X \subseteq P$ as follows:

$$X(V) = \{p \in V \otimes V \mid f(P(\varphi)(p)) = 0 \; \forall \, \varphi \in \mathrm{Hom}(V, W)\},$$

where $f = y_{12} - y_{21} \in K[\mathbb{A}(W \otimes W)]$.

Now, since $\deg(P) = 2$, our precious result shows that, for every V with $\dim(V) \geq 2$, we have that

$$X_{\mathrm{Mult}_2}(V) = \mathrm{Mult}_2(X(V)).$$

In particular, $X_{\mathrm{Mult}_2}(W) = \mathrm{Mult}_2(X(W))$. So, it only remains to see what happens in the case when $\dim(V) = 1$.

Let $U \cong K^1$, i.e. $U \in \mathbf{Vec}$ such that $\dim(U) = 1$, and let u_1 be a basis vector. Then, any linear map $\varphi \in \mathrm{Hom}(U, W)$ is given by

$$\varphi(u_1) = ae_1 + be_2$$

for some $a, b \in K$. So, the induced map on tensor products is given by

$$P(\varphi)(u_1 \otimes u_1) = a^2(e_1 \otimes e_1) + ab(e_1 \otimes e_2) + ab(e_2 \otimes e_1) + b^2(e_2 \otimes e_2).$$

Let x be the single coordinate function on $U \otimes U$. Then the induced map on coordinate rings is given by

$$y_{11} \mapsto a^2 x, \quad y_{12}, y_{21} \mapsto abx, \quad y_{22} \mapsto b^2 x$$

Consequently

$$f \circ P(\varphi) = abx - abx = 0.$$

Since this holds for any $\varphi \in \mathrm{Hom}(U, W)$, we conclude that $I_X(U) = (0)$, and hence $X(U) = U \otimes U = P(U)$. Therefore, $\mathrm{Mult}_2(X(U)) = U \otimes U$. In particular, $0 \in \mathrm{Mult}_2(X(U))$.

On the other had, since $f \in I_X(W)$, $\deg(\mathrm{in}(f)) = 1$. We have that $0 \notin \mathrm{Mult}_2(X(W)) = X_{\mathrm{Mult}_2}(W)$. Since X_{Mult_2} must be preserved by any map $\psi \in \mathrm{Hom}(U, W)$, and $\psi(0) = 0$, we would have $0 \notin X_{\mathrm{Mult}_2}(U)$. Since $0 \in \mathrm{Mult}_2(X(U))$, we conclude that

$$X_{\mathrm{Mult}_2}(U) \neq \mathrm{Mult}_2(X(U)).$$

That is, in this case, the bound on $\dim(V)$ is sharp. ♫

We can now generalize this example to show that, for any $m \in \mathbb{Z}_{\geq 1}$ and any degree of P, this bound can be made to be sharp.

Proposition 5.17. *Let $m, r \in \mathbb{Z}_{\geq 1}$ and set $N = r \cdot (m-1)$. Then, there exists a polynomial functor P of degree r and a* **Vec**-*variety $X \subset P$ such that, for $U \cong k^{N-1}$, we have*

$$X_{\mathrm{Mult}_m}(X(U)) \neq \mathrm{Mult}_m(X(U)).$$

That is, the inclusion $X_{\mathrm{Mult}_m}(X(U)) \subsetneq \mathrm{Mult}_m(X(U))$ is strict.

Proof. Let P be the rth tensor power polynomial functor. That is, $P(V) = \otimes^r V$. Define $P \coloneqq P_r^{\oplus(m-1)}$, so that $\deg(P) = r$. Set $N = r(m-1)$, and fix $W \cong K^N$ with basis $\{e_1, ..., e_N\}$. For each $1 \leq k \leq m-1$, the kth direct sum of $P^r(W)$ has basis

$$\{e_{j_1} \otimes \cdots \otimes e_{j_r} \mid 1 \leq j_1, \ldots, j_r \leq N\},$$

and we denote the corresponding coordinate functions by

$$y^{(k)}_{j_1 \cdots j_r} \in K[\mathbb{A}(P)(W)].$$

Define

$$f \coloneqq \sum_{\sigma \in S_N} \mathrm{sgn}(\sigma) \prod_{k=1}^{m-1} y^{(k)}_{\sigma((k-1)r+1) \cdots \sigma(kr)} \in K[\mathbb{A}(P)(W)].$$

It is easy to see that distinct permutations $\sigma \in S_N$ gives distinct monomials. Hence $f \neq 0$, and each monomial has degree $m-1$. Therefore, $\deg(\mathrm{in}(f)) = m-1$.

We define a **Vec**-variety $X \subseteq P$ as follows:

$$X(V) = \{p \in P(V) \mid f(P(\varphi)(p)) = 0 \; \forall \; \varphi \in \mathrm{Hom}(V, W)\},$$

In particular, we know that $f \in I_X(W)$ if we take $\varphi = \mathrm{Id}_W$. Since $\deg(\mathrm{in}(f)) = m-1 < m$, we have $0 \notin \mathrm{Mult}_m(X(W))$. Since $\dim(W) = N = r(m-1)$, it follows that $0 \notin X_{\mathrm{Mult}_m}(W)$.

Now let $U \cong K^{N-1}$. We claim that $X(U) = P(U)$. Fix a basis $u_1, ..., u_{N-1}$ of U. For each $1 \leq k \leq m-1$, let $x^{(k)}_{i_1 \cdots i_r}$ denote the coordinate function for the kth direct sum of $P_r(U)$ corresponding to the basis $\{u_{i_1} \otimes \cdots \otimes u_{i_r}$.

Next, fix $\varphi \in \mathrm{Hom}(U, W)$ and write $\varphi(u_i) = \sum_{j=1}^{N} a_{ij} e_j$ for some $a_{ij} \in K$. We will show that $f \circ P(\varphi) = 0$. For each $1 \leq k \leq m-1$, the induced map on coordinate rings is given by

$$P(\varphi)^*(y^{(k)}_{j_1 \cdots j_r}) = \sum_{i_1, \ldots, i_r = 1}^{N-1} a_{i_1 j_1} \cdots a_{i_r j_r} x^{(k)}_{i_1 \cdots i_r}$$

For $\sigma \in S_N$, we have

$$P(\varphi)^*\left(y^{(k)}_{\sigma((k-1)r+1) \cdots \sigma(kr)}\right) = \sum_{i_1, \ldots, i_r = 1}^{N-1} a_{i_1 \sigma((k-1)r+1)} \cdots a_{i_r \sigma(kr)} x^{(k)}_{i_1 \cdots i_r}$$

Consider a monomial of degree $m-1$ of the form $M = x^{(1)}_{i_{11} \cdots i_{r1}} \cdots x^{(m-1)}_{i_{1(m-1)} \cdots i_{r(m-1)}}$. Its coefficient in $f \circ P(\varphi)$ is

$$\sum_{\sigma \in S_N} \mathrm{sgn}(\sigma) \prod_{k=1}^{m-1} \prod_{s=1}^{r} a_{i_{sk}, \sigma((k-1)r+s)}$$

Set $\alpha_{(k-1)r+s} \coloneqq i_{sk}$ for $1 \leq s \leq r$ and $1 \leq k \leq m-1$. The coefficient above is

$$\det \begin{bmatrix} a_{\alpha_1, 1} & \cdots & a_{\alpha_1, N} \\ \vdots & & \vdots \\ a_{\alpha_N, 1} & \cdots & a_{\alpha_N, N} \end{bmatrix}$$

Since each α_ℓ satisfy $1 \leq \alpha_\ell \leq N$, and there are N rows, the determinant vanishes by Pigeonhole. Thus the coefficient of M in $f \circ P(\varphi)$ is zero. Since it holds for all such M, we conclude that $f \circ P(\varphi) = 0$. Hence $X(U) = P(U)$, and $\mathrm{Mult}_m(X(U)) = P(U)$. In particular we have $0 \in \mathrm{Mult}_m(X(U))$. But $0 \notin X_{\mathrm{Mult}_m}(W)$. If $0 \in X_{\mathrm{Mult}_m}(U)$, then functorility of X_{Mult_m} would give, for any $\psi \in \mathrm{Hom}(U, W)$, that $P(\psi)(0) = 0 \in X_{\mathrm{Mult}_m}(W)$, a contradiction. Therefore, $0 \notin X_{\mathrm{Mult}_m}(U)$. Since $0 \notin \mathrm{Mult}_m(X(U))$, it folows that $X_{\mathrm{Mult}_m}(U) \subsetneq \mathrm{Mult}_m(X(U))$.

□

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