Diophantine Properties of Two-Dimensional Mahler Vectors

Youyi Zhang

Mentor: Reynold Fregoli

Abstract

We study the irrationality exponent of two-dimensional vectors obtained by evaluating vector-valued functions satisfying a Mahler system. In particular, we establish an upper bound for the irrationality exponent of such functions at reciprocals of positive integers, under a non-vanishing condition on the associated sequence of Hankel determinants. Further, we construct a class of two-dimensional Mahler vectors whose irrationality exponent equals $\frac{3}{2}$, the minimum possible exponent. The proofs combine type I Hermite–Padé approximation with Hankel-determinants. To the best of our knowledge, this is the first application of type I Hermite–Padé approximation to the irrationality exponents of multidimensional Mahler vectors.

1. Introduction

1.1. **Mahler Numbers and Diophantine Approximation.** Let $\mathbb{Q}[[z]]$ denote the ring of formal power series in the variable z and coefficients in $\mathbb{Q}$ and let $\mathbb{Q}(z)$ denote the corresponding field of rational functions.

In its simplest form, a Mahler equation is a functional equation of the type

$$f(z) = a(z)f(z^d) + b(z), \tag{1.1}$$

with coefficients a, b in $\mathbb{Q}(z)$ and indeterminate f in $\mathbb{Q}[[z]]$. Such equations were first introduced by Mahler [Mah29] as a tool to study power series f with lacunary terms and, specifically, to establish the irrationality of numbers of the form $f(p/q)$ $(p, q \in \mathbb{N})$. Mahler proved that for many such series f the numbers $f(\alpha)$ are always transcendental whenever α (with $0 < |\alpha| < 1$) is algebraic [Mah29, IV]. A very useful survey about Mahler's foundational work in transcendence theory and what later became known as Mahler's method can be found in [Ada19] (see Theorem 1.1 and Section 3 ibidem for a precise formulation and a proof of Mahler's result). We remark that (1.1) is in fact a simplified form of Mahler equation and that Mahler's method applies in much greater generality (see [Ada19, Subsection 2.1]).

A surprising connection exists between Mahler equations and the theory of automatic sequences. These are, in vague terms, sequences of non-negative integers that can be produced by ideal machines known as finite automata (see [AS03, Sections 4 and 5] and [BHWY16, page 2] for a precise definition). Real numbers whose base-b expansion is an automatic sequence a_n are known in the literature as *automatic numbers* and may be viewed as those numbers whose digits behave "deterministically". It was shown by Cobham [Cob68, page 57], [Ada19, Section 7] that the generating function $f(z) =$

$a_0 + a_1 z + a_2 z^2 + \cdots$ of an automatic sequence a_n always satisfies a Mahler equation (slightly more general than (1.1)), so that when α can be written as $a_0.a_1a_2\cdots$ in base b it must be $\alpha = f(1/b)$ for f essentially as in (1.1). This observation together with a modern re-formulation of Mahler's method ultimately led to a proof of the striking fact that all automatic numbers are either rational or transcendental [AF17, Corollaire 1.8] [1].

Given the extensive amount of work on the transcendence of automatic numbers and, more generally, of numbers of the form $f(p/q)$, with f as in (1.1) (sometimes called *Mahler numbers*), it appears natural to further investigate Diophantine properties of these special types of numbers. For example, the irrationality exponent of a real number ξ is defined as

$$\mu(\xi) := \sup\left\{\mu > 0 : \left|\xi - \frac{p}{q}\right| < q^{-\mu} \text{ for infinitely many rationals } \frac{p}{q}\right\}.$$

By Dirichlet's approximation theorem, every irrational number satisfies $\mu(\xi) \geq 2$. For transcendental numbers, determining the irrationality exponent can be an extremely challenging task. In fact, the precise value of the exponent is known just in a handful of cases, such as $\mu(e) = 2$. Nonetheless, we now know that plenty of automatic numbers have exponent 2, providing the first large class of transcendental numbers whose irrationality exponent is minimal. This was established in [Bug11] for the Thue-Morse constant (a well-known automatic number) and in [BHWY16] for large class of Mahler numbers.

In [BZ15], the Thue-Morse constant (together with some other Mahler numbers) is further shown to be well-approximable and in the same paper, the authors pose two interesting open problems. The second of these is the starting point of this paper. Let us briefly recall it.

In place of (1.1), one may consider a more general functional equation in several indeterminates, known as a Mahler system. This is system of the form

$$\underline{f}(z) = A(z)\underline{f}(z^d) + B(z), \tag{1.2}$$

where

$$\underline{f}(z) \in (\mathbb{Q}[[z]])^{k\times 1}, \quad A(z) \in (\mathbb{Q}(z))^{k\times k}, \quad \text{and} \quad B(z) \in (\mathbb{Q}(z))^{k\times 1},$$

with $k, d \geq 1$ fixed integers. Then the following is [BZ15, Open Problem 2].

Problem 1.1. Let $\underline{f}$ be an analytic solution to (1.2) and assume that its components f_i are transcendental over $\mathbb{Q}(z)$ [2]. What is the irrationality exponent of the numbers $f_i(r)$ for $i = 1, \ldots, k$ and $r \in \mathbb{Q}$? Are these numbers badly approximable?

In this paper, we revisit this problem from a different angle, studying instead Diophantine properties of $\underline{f}(r)$ as a vector. Namely, we ask:

[1] It should be remarked that the first proof of this fact appears in [AB07] and, in fact, the result is also claimed by Cobham [Cob68, page 57], without proof.

[2] By this we mean that each of the f_i's does not satisfy a polynomial equation whose coefficients are in $\mathbb{Q}(z)$.

Problem 1.2. Let $\underline{f}(z) \in (\mathbb{Q}[[z]])^k$ be as in Problem 1.1. What are the Diophantine properties of the vector $\underline{f}(r)$, where $r \in \mathbb{Q}$? What are its simultaneous and dual Diophantine exponents (see below for the definition)? Is it a badly approximable vector?

1.2. **Main Results.** In order to to make our main results precise, we require a few definitions. The simultaneous Diophantine exponent of a vector $\underline{\xi} \in \mathbb{R}^s$ is defined as

$$\omega(\underline{\xi}) = \sup\left\{\omega > 0 : \|q\underline{\xi} - \underline{p}\| < q^{-\omega} \text{ for infinitely many } (q, \underline{p}) \in \mathbb{N} \times \mathbb{Z}^s\right\},$$

where $\|\cdot\|$ denotes the supremum norm. For $(A_0, A_1, \ldots, A_s) \in \mathbb{Z}^{s+1}$, put

$$H(A_0, \ldots, A_s) = \max\{|A_0|, \ldots, |A_s|\}.$$

The dual Diophantine exponent of $\underline{\xi} \in \mathbb{R}^s$ is defined as

$$\omega^*(\underline{\xi}) = \sup\Big\{\omega > 0 : |A_0 + A_1\xi_1 + \cdots + A_s\xi_s| < H(A_0, \ldots, A_s)^{-\omega} \\ \text{for infinitely many } (A_0, \ldots, A_s) \in \mathbb{Z}^{s+1} \setminus \{\underline{0}\}\Big\}.$$

Remark 1.3. It is a well-known fact that for all $\underline{\xi} \in \mathbb{R}^s$ it holds that $\omega(\underline{\xi}) \geq \frac{1}{s}$ and $\omega^*(\underline{\xi}) \geq s$. The irrationality exponent of $\underline{\xi}$ is then

$$\mu(\underline{\xi}) = \omega(\underline{\xi}) + 1.$$

Note that $\mu(\underline{\xi}) \geq 1 + \frac{1}{s}$ for all $\underline{\xi} \in \mathbb{R}^s$.

In order to state our result, we require to introduce the notion of Hankel determinant. This is a classical tool in the study of continued fractions for power series (see [BHWY16]). For the sake of compactness, we introduce this definition and ultimately state our main result in the case $s = 2$, although our techniques may be used to extend the statement to general $s \geq 2$ with relatively little effort.

Definition 1.4 (Type I Hankel determinants)**.** Let $\underline{f} = (f_1, f_2) \in (\mathbb{Q}[[z]])^2$. Write

$$f_1(z) = \sum_{k=0}^{\infty} c_k z^k \quad \text{and} \quad f_2(z) = \sum_{k=0}^{\infty} d_k z^k.$$

The Type I Hankel determinant of $\underline{f}$ of order k is the determinant

$$H_k(f_1, f_2) = \begin{vmatrix} c_0 & c_1 & \cdots & c_k & d_0 & d_1 & \cdots & d_k \\ c_1 & c_2 & \cdots & c_{k+1} & d_1 & d_2 & \cdots & d_{k+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{2k} & c_{2k+1} & \cdots & c_{3k} & d_{2k} & d_{2k+1} & \cdots & d_{3k} \\ c_{2k+1} & c_{2k+2} & \cdots & c_{3k+1} & d_{2k+1} & d_{2k+2} & \cdots & d_{3k+1} \end{vmatrix}.$$

Our first main result reads as follows (compare with [BHWY16, Theorem 4.1]).

Theorem 1.5. *Let $d \geq 2$ and $b \geq 2$ be integers. Let*

$$\underline{f}(z) = (f_1(z), f_2(z)) \in \mathbb{Z}[[z]]^2$$

such that both f_1 and f_2 converge inside the unit disk, and suppose that $\underline{f}$ satisfies the Mahler system

$$\underline{f}(z) = \frac{C(z)}{D(z)}\underline{f}(z^d) + \frac{A(z)}{B(z)},$$

where

$$B(z), D(z) \in \mathbb{Q}[z], \quad C(z) \in (\mathbb{Q}[z])^{2\times 2}, \quad \text{and} \quad A(z) \in (\mathbb{Q}[z])^{2\times 1}.$$

Assume that

$$B\left(\frac{1}{b^{d^m}}\right) D\left(\frac{1}{b^{d^m}}\right) \det C\left(\frac{1}{b^{d^m}}\right) \neq 0 \qquad \text{for } m \geq 0$$

and that there exists a strictly increasing sequence $n_1 < n_2 < \cdots$ such that

$$H_{n_i}(f_1, f_2) \neq 0 \quad \text{for } i \geq 1,$$

with

$$\limsup_{i\to\infty} \frac{n_{i+1}}{n_i} = \rho < \infty. \tag{1.3}$$

Then $1, f_1(1/b), f_2(1/b)$ are linearly independent over $\mathbb{Q}$, and

$$\omega^*(\underline{f}(1/b)) \leq 2\rho^2.$$

From the above result, if $\rho = 1$, $\omega^*(\underline{f}(1/b)) = 2$ for any $b \geq 2$. Furthermore, according to Khintchine's transference inequalities

$$\frac{\omega^*(\underline{\xi})}{(s-1)\omega^*(\underline{\xi}) + s} \leq \omega(\underline{\xi}) \leq \frac{\omega^*(\underline{\xi}) - s + 1}{s}.$$

where $1, \xi_1, \ldots, \xi_s$ are linearly independent over $\mathbb{Q}$ [Khi26, Satz VI, Eq. (32)], we can get a simple consequence:

Corollary 1.6. *Let $\underline{f}$ be as in Theorem 1.5 and assume that $\rho = 1$. Then for any $b \geq 2$*

$$\omega(\underline{f}(1/b)) = \frac{1}{2} \quad \text{and} \quad \mu(\underline{f}(1/b)) = \frac{3}{2}.$$

Remark 1.7. Observe that any vector of the form $\underline{f}(1/b)$, where $\underline{f}$ satisfies the hypotheses of Theorem 1.5 must have transcendental components that are linearly independent over $\mathbb{Q}$, which follows directly from the upper bound on the dual exponent. And both components are transcendental by that both functions $f_1(z)$ and $f_2(z)$ are transcendental and the use of Corollary 2 in [Nis90]. But it is non-trivial to show that the functions $f_i(z)$ are algebraically independent.

In view of Corollary 1.6, it is becomes particularly interesting to construct Mahler systems such that consition (1.3) holds with $\rho = 1$. We are able to construct such examples, by carefully choosing the matrix C.

A typical example of a multidimensional Mahler system is the following.

$$\underline{f}(z) = \begin{pmatrix} 1 & -z \\ 1 & z \end{pmatrix} \underline{f}(z^2).$$

However, even for this simple system, it is unclear if Condition (1.3) with $\rho = 1$ holds. To circumvent this obstruction, we consider a slightly more general system, given by

$$\underline{f}(z) = \begin{pmatrix} 1 & -\alpha z \\ 1 & \beta z \end{pmatrix} \underline{f}(z^2), \tag{1.4}$$

with α, β real numbers. Our second main result reads as follows.

Theorem 1.8. *Let p be a prime number and let $\alpha, \beta \in \mathbb{Z}$ be such that $p \nmid \alpha$ and $p \mid \beta$. Let $\underline{f} \in \mathbb{Z}[[z]]^2$ be a nonzero solution to the linear Mahler system in* (1.4) *with this choice of α and β. Then Condition* (1.3) *holds with $\rho = 1$. In particular, for any integer $b \geq 2$ it holds that*

$$\mu(\underline{f}(1/b)) = \frac{3}{2}.$$

2. Preliminary Results

We begin with a simple observation.

Lemma 2.1. *Let $\underline{f} \in \mathbb{Z}[[z]]^2$ and assume that the sequence $H_k(\underline{f})$ is not eventually null. Then $1, f_1(z), f_2(z)$ are linearly independent over $\mathbb{C}(z)$.*

Proof. Suppose $1, f_1(z), f_2(z)$ are not linearly independent over $\mathbb{C}(z)$. Then there exists $P_0, P_1, P_2 \in \mathbb{C}(z)$ not all zero such that

$$P_0(z) + P_1(z)f_1(z) + P_2(z)f_2(z) = 0.$$

Denote

$$P_0(z) = \sum_{i=0}^{N_0} \alpha_i z^i \qquad P_1(z) = \sum_{i=0}^{N_1} \beta_i z^i \qquad P_2(z) = \sum_{i=0}^{N_2} \gamma_i z^i.$$

Then for all $n > \max_{j=0,1,2} N_j$,

$$\beta_{N_1} c_{n-N_1} + \cdots + \beta_0 c_n + \gamma_{N_2} d_{n-N_2} + \cdots + \gamma_0 d_n = 0.$$

Then for all $k > \max_{j=0,1,2} N_j$,

$$\begin{aligned} H_k(f_1, f_2) &= \begin{vmatrix} c_0 & \cdots & c_{k-N_1} & \cdots & c_k & d_0 & \cdots & d_{k-N_2} & \cdots & d_k \\ c_1 & \cdots & c_{k+1-N_1} & \cdots & c_{k+1} & d_1 & \cdots & d_{k+1-N_2} & \cdots & d_{k+1} \\ \vdots & & \vdots & & \vdots & \vdots & & \vdots & & \vdots \\ c_{2k+1} & \cdots & c_{3k+1-N_1} & \cdots & c_{3k+1} & d_{2k+1} & \cdots & d_{3k+1-N_2} & \cdots & d_{3k+1} \end{vmatrix} \\ &= \begin{vmatrix} c_0 & \cdots & c_{k-N_1} & \cdots & c_k & d_0 & \cdots & d_{k-N_2} & \cdots & s_0 \\ c_1 & \cdots & c_{k+1-N_1} & \cdots & c_{k+1} & d_1 & \cdots & d_{k+1-N_2} & \cdots & s_1 \\ \vdots & & \vdots & & \vdots & \vdots & & \vdots & & \vdots \\ c_{2k+1} & \cdots & c_{3k+1-N_1} & \cdots & c_{3k+1} & d_{2k+1} & \cdots & d_{3k+1-N_2} & \cdots & s_{2k+1} \end{vmatrix}, \\ &= 0, \end{aligned}$$

where

$$s_r = \sum_{i=0}^{N_1} \beta_i c_{k+r-i} + \sum_{i=0}^{N_2} \gamma_i d_{k+r-i} = 0 \qquad (r \geq 0).$$

This contradicts the assumption that the sequence $H_k(\underline{f})$ is not eventually null. Hence $1, f_1(z), f_2(z)$ are linearly independent over $\mathbb{C}(z)$ □

The following is a generalization of Lemma 4.1 in [AR09].

Lemma 2.2. *Let*

$$\underline{\xi} = (\xi_1, \cdots, \xi_s)$$

and assume that $1, \xi_1, \cdots, \xi_s$ *are linearly independent over* $\mathbb{Q}$.

Suppose that for all sufficiently large n *there are* $s+1$ *integer linear forms*

$$L_{n,j}(\underline{\xi}) = p_{n,j,0} + p_{n,j,1}\xi_1 + \cdots + p_{n,j,s}\xi_s, \qquad j = 0, 1, \cdots, s,$$

that satisfy the following:

(1) $\det \begin{pmatrix} \ell_{n,0} \\ \ell_{n,1} \\ \vdots \\ \ell_{n,s} \end{pmatrix} \neq 0$, *where* $\ell_{n,j} = (p_{n,j,0}, p_{n,j,1}, \cdots, p_{n,j,s})$.

(2) *There exists an increasing sequence* $H_n \geq \max_{0\leq j\leq s} \|\ell_{n,j}\|_\infty$ *with* $\lim_{n\to\infty} H_n = \infty$, *and there exist constants* $c_0, c_1 > 0$, $\theta \geq 1$, *and* $\delta > s-1$ *such that*
   (a) $H_n < H_{n+1} \leq c_0 H_n^\theta$.
   (b) $\max_{0\leq j\leq s} |L_{n,j}(\underline{\xi})| \leq c_1 H_n^{-\delta}$.

Then

$$\omega^*(\underline{\xi}) \leq \frac{s\theta}{\delta - s + 1}.$$

Proof. Take an arbitrary integer linear form

$$L(\underline{\xi}) = p_0 + p_1\xi_1 + \cdots + p_s\xi_s,$$

and put

$$\ell = (p_0, p_1, \cdots, p_s) \neq \mathbf{0}, \qquad H = \|\ell\|_\infty = \max\{|p_0|, |p_1|, \cdots, |p_s|\}.$$

and $L(\underline{\xi}) \neq 0$ because $1, \xi_1, \cdots, \xi_s$ are linearly independent over $\mathbb{Q}$.

Since for any sufficiently large n, $\det \begin{pmatrix} \ell_{n,0} \\ \ell_{n,1} \\ \vdots \\ \ell_{n,s} \end{pmatrix} \neq 0$, the rows $\ell_{n,0}, \ldots, \ell_{n,s}$ form a basis of $\mathbb{Q}^{s+1}$. Hence, at least one of the $s+1$ determinants where ℓ replaces the j-th row

$$\Delta_{n,j}(L) = \det \begin{pmatrix} \ell_{n,0} \\ \vdots \\ \ell \\ \vdots \\ \ell_{n,s} \end{pmatrix}, \qquad j = 0, \ldots, s,$$

is nonzero. For such a j, since all entries are integers,

$$|\Delta_{n,j}(L)| \geq 1.$$

Perform a column operation and we deduce

$$\Delta_{n,j}(L) = \begin{vmatrix} L_{n,0}(\underline{\xi}) & p_{n,0,1} & \cdots & p_{n,0,s} \\ \vdots & \vdots & & \vdots \\ L(\underline{\xi}) & p_1 & \cdots & p_s \\ \vdots & \vdots & & \vdots \\ L_{n,s}(\underline{\xi}) & p_{n,s,1} & \cdots & p_{n,s,s} \end{vmatrix}.$$

Expanding the determinant along the first column gives

$$|\Delta_{n,j}(L)| \le c_2 \left(|L(\underline{\xi})| H_n^s + H H_n^{s-1} \max_{0 \le r \le s} |L_{n,r}(\underline{\xi})| \right),$$

where $c_2 > 0$ depends only on s.

Using

$$\max_{0 \le r \le s} |L_{r,n}(\underline{\xi})| \le c_1 H_n^{-\delta},$$

we get

$$1 \le c_2 |L(\underline{\xi})| H_n^s + c_2 c_1 H H_n^{s-1-\delta}.$$

Choose $K > 0$ large enough so that $c_2 c_1 K^{-1} \le \frac{1}{2}$.

We require that H be sufficiently large, and since H_n is strictly increasing with $\lim_{n\to\infty} H_n = \infty$, there exists some n such that

$$H_{n-1}^{\delta-s+1} < KH \le H_n^{\delta-s+1}.$$

From

$$KH \le H_n^{\delta-s+1},$$

we obtain

$$c_2 c_1 H H_n^{s-1-\delta} \le \frac{1}{2}.$$

Hence

$$1 \le c_2 |L(\underline{\xi})| H_n^s + \frac{1}{2},$$

and so

$$|L(\underline{\xi})| \ge \frac{1}{2c_2} H_n^{-s}.$$

From

$$H_{n-1}^{\delta-s+1} < KH,$$

we get

$$H_{n-1} < (KH)^{1/(\delta-s+1)}.$$

Combining the growth condition

$$H_n \le c_0 H_{n-1}^{\theta},$$

we obtain

$$H_n \le c_0 (KH)^{\theta/(\delta-s+1)}.$$

Therefore

$$H_n^{-s} \ge c_0^{-s} K^{-s\theta/(\delta-s+1)} H^{-s\theta/(\delta-s+1)}.$$

Consequently, there exists a constant $c > 0$, independent of L, such that

$$|L(\underline{\xi})| \geq cH^{-s\theta/(\delta-s+1)}.$$

Thus every nonzero integer linear form

$$L(\underline{\xi}) = p_0 + p_1\xi_1 + \cdots + p_s\xi_s$$

with sufficiently large $H = \max\{|p_0|, \ldots, |p_s|\}$, satisfies the lower bound

$$|p_0 + p_1\xi_1 + \cdots + p_s\xi_s| \geq cH^{-s\theta/(\delta-s+1)}.$$

By the definition of the dual Diophantine exponent,

$$\omega^*(\underline{\xi}) \leq \frac{s\theta}{\delta - s + 1}.$$

□

Remark 2.3. When $s = 2$, the upper bound one can get for $\omega^*(\underline{\xi})$ is then $\frac{2\theta}{\delta-1}$.

Remark 2.4. The above lemma is based on ideas from Nesterenko's linear independence criterion [Nes85] according to the reformulation in [Fis15, Proposition 1]. Here is how Proposition 1 (b) in [Fis15] can derive the same result as 2.2 when $\theta = 1$:
Let $\underline{\xi} \in \mathbb{R}^s$ and all assumptions in Proposition 1 along with $Q_{n+1} = Q_n^{1+o(1)}$ hold. Suppose

$$\omega(\underline{\xi}) > \frac{1}{\delta}.$$

Choose $\omega(\underline{\xi}) > \omega_0 > \frac{1}{\delta}$. Then there are infinitely many integers $q \geq 1$ and $p_1, \ldots, p_s \in \mathbb{Z}$ such that

$$\left\| q\underline{\xi} - \underline{p} \right\|_\infty < q^{-\omega_0}.$$

Let $e = (1, \underline{\xi})$. Write $v = (q, p_1, \ldots, p_s) = qe + r$, where $r = (0, p_1 - \xi_1 q, \ldots, p_s - \xi_s q)$. Then

$$||r|| < \sqrt{s}q^{-\omega_0}.$$

One can also write $v = \lambda e + u$ where $\lambda \in \mathbb{R}, u \in e^\perp$. Then

$$||u|| \leq ||r|| < \sqrt{s}q^{-\omega_0},$$

since u is the orthogonal projection of r onto $e^\perp$. Also

$$\lambda = \frac{v \cdot e}{||e||^2} = q + \frac{r \cdot e}{||e||^2}.$$

So, $|\lambda - q| \leq \frac{||r||}{||e||} \leq q$ for sufficiently large q. Then $|\lambda| \leq 2q$.

Take $\varepsilon > 0$ sufficiently small such that $\delta - 2\varepsilon > 0$ and $\omega_0(\delta - 3\varepsilon) > 1 + \varepsilon$. Let

$$Q = q^{\frac{1}{\delta-2\varepsilon}}.$$

Then $q = Q^{\omega_0(\delta-2\varepsilon)}$. For sufficiently large q, and thus sufficiently large Q, we have

$$||u|| < \sqrt{s}q^{-\omega_0} = \sqrt{s}Q^{-\omega_0(\delta-2\varepsilon)} \leq Q^{-\omega_0(\delta-3\varepsilon)} < Q^{-1-\varepsilon},$$

and

$$|\lambda| \leq 2q = 2Q^{(\delta-2\varepsilon)} \leq Q^{(\delta-\varepsilon)}.$$

Therefore $v = \lambda e + u$ satisfies the conditions in Proposition 1 (b) but v is nonzero, contradicting the conclusion of Proposition 1 (b). Hence we obtain that

$$\omega(\underline{\xi}) \leq \frac{1}{\delta}.$$

From now on, we focus on the case where dimension $s = 2$.

Bugeaud, Han, Wen and Yao (see [BHWY16]) used a tool called Padé approximation to obtain preliminary approximants to $f(z)$. A Padé approximant is defined as follows:

Definition 2.5 (Padé approximant)**.** Let $f \in \mathbb{F}[[z]]$. A Padé approximant of order k to f is a rational fraction $\frac{P(z)}{Q(z)} \in \mathbb{F}(z)$ such that

$$\deg P \leq k-1, \qquad \deg Q \leq k,$$

and

$$f(z) - \frac{P(z)}{Q(z)} = O(z^{2k}).$$

One might be naturally inspired to attempt using a tool called the Type II Hermite-Padé approximation, in the form of

$$\frac{(P_{k,1}(z), P_{k,2}(z))}{Q_k(z)}$$

in the two-dimensional problem. However, it could not lead to desirable results. It turns out that the following techniques using the Type I Hermite-Padé approximation provide a solution.

Definition 2.6 (Type I Hermite–Padé approximant)**.** Let

$$f_1(z) = \sum_{n\geq 0} c_n z^n, \qquad f_2(z) = \sum_{n\geq 0} d_n z^n$$

be two formal power series with coefficients in a field $\mathbb{F}$. Fix an integer $k \geq 1$.

A Type I Hermite–Padé approximant of order k to (f_1, f_2) is a triple of polynomials

$$(P_{k,0}(z), P_{k,1}(z), P_{k,2}(z)) \in \mathbb{F}[z]^3$$

such that

$$\deg P_{k,0} \leq k_0, \qquad \deg P_{k,1} \leq k_1, \qquad \deg P_{k,2} \leq k_2,$$

with $k_0 + k_1 + k_2 = 3k$, and

$$P_{k,0}(z) + P_{k,1}(z)f_1(z) + P_{k,2}(z)f_2(z) = O(z^{3k+2}).$$

Remark 2.7. We learn from [KVAW05, Theorem 5.1] a way to construct three triples of type I Hermite–Padé approximants of order k as in the following lemma 2.8. Then one can use the triples of polynomials to obtain linearly independent linear forms when z is evaluated at a certain value.

Lemma 2.8. *Let*

$$f_1(z) = \sum_{n\geq 0} c_n z^n, \qquad f_2(z) = \sum_{n\geq 0} d_n z^n$$

be non-trivial formal power series over a field $\mathbb{F}$. Fix $k \geq 1$ large enough such that at least one of $c_0, \ldots, c_{2k}$ is nonzero and at least one of $d_0, \ldots, d_{2k}$ is nonzero.

Assume that

$$H_k(f_1, f_2) \neq 0.$$

Then there exist unique triples

$$l_j = (P_{j,0}, P_{j,1}, P_{j,2}) \in \mathbb{F}[z]^3, \qquad j = 0, 1, 2,$$

with the following properties:

- $L_j(z) := P_{j,0}(z) + P_{j,1}(z)f_1(z) + P_{j,2}(z)f_2(z) = O(z^{3k+2}), \qquad j = 0, 1, 2.$
- $\deg P_{0,0} = k, \qquad \deg P_{0,1} \leq k, \qquad \deg P_{0,2} \leq k, \qquad [z^k]P_{0,0}(z) = 1;$
  $\deg P_{1,0} \leq k-1, \qquad \deg P_{1,1} = k+1, \qquad \deg P_{1,2} \leq k, \qquad [z^{k+1}]P_{1,1}(z) = 1;$
  $\deg P_{2,0} \leq k-1, \qquad \deg P_{2,1} \leq k, \qquad \deg P_{2,2} = k+1, \qquad [z^{k+1}]P_{2,2}(z) = 1.$

Moreover,

$$\det \begin{pmatrix} P_{0,0}(z) & P_{0,1}(z) & P_{0,2}(z) \\ P_{1,0}(z) & P_{1,1}(z) & P_{1,2}(z) \\ P_{2,0}(z) & P_{2,1}(z) & P_{2,2}(z) \end{pmatrix} = z^{3k+2}.$$

Proof. We first prove the existence and uniqueness of the three forms.

First to construct l_0, let $(P_{0,0}, P_{0,1}, P_{0,2}) \in \mathbb{F}[z]^3$ with

$$\deg P_{0,0} \leq k, \qquad \deg P_{0,1}, \deg P_{0,2} \leq k, \qquad [z^k]P_{0,0}(z) = 1.$$

We can write

$$P_{0,0}(z) = z^k + \widetilde{P}_{0,0}(z),$$

where $\deg \widetilde{P}_{0,0} \leq k-1$, and

$$P_{0,1}(z) = \sum_{j=0}^{k} a_j z^j, \qquad P_{0,2}(z) = \sum_{j=0}^{k} b_j z^j.$$

Apply the condition

$$L_0(z) = P_{0,0}(z) + P_{0,1}(z)f_1(z) + P_{0,2}(z)f_2(z) = O(z^{3k+2})$$

and it is clear that the coefficients of $\widetilde{P}_{0,0}$ are determined uniquely by $P_{0,1}$ and $P_{0,2}$. Meanwhile for degrees $k, k+1, \ldots, 3k+1$, there is the system

$$\begin{cases} a_k c_0 + \cdots + a_0 c_k + b_k d_0 + \cdots + b_0 d_k = -1, \\ a_k c_1 + \cdots + a_0 c_{k+1} + b_k d_1 + \cdots + b_0 d_{k+1} = 0, \\ \vdots \\ a_k c_{2k+1} + \cdots + a_0 c_{3k+1} + b_k d_{2k+1} + \cdots + b_0 d_{3k+1} = 0. \end{cases}$$

The coefficient matrix's determinant is exactly $H_k(f_1, f_2)$. Since $H_k(f_1, f_2) \neq 0$, this system has a unique solution. Hence l_0 exists uniquely.

Next consider l_1. We require

$$\deg P_{1,0} \leq k-1, \qquad \deg P_{1,1} \leq k+1, \qquad \deg P_{1,2} \leq k, \qquad [z^{k+1}]P_{1,1}(z) = 1.$$

Write

$$P_{1,1}(z) = z^{k+1} + \sum_{j=0}^{k} a_j z^j, \qquad P_{1,2}(z) = \sum_{j=0}^{k} b_j z^j.$$

According to

$$L_1(z) = P_{1,0}(z) + P_{1,1}(z)f_1(z) + P_{1,2}(z)f_2(z) = O(z^{3k+2}),$$

$P_{1,0}$ is determined by the coefficients of $P_{1,1}$ and $P_{1,2}$, and the linear system

$$\begin{cases} a_k c_0 + \cdots + a_0 c_k + b_k d_0 + \cdots + b_0 d_k = 0, \\ a_k c_1 + \cdots + a_0 c_{k+1} + b_k d_1 + \cdots + b_0 d_{k+1} = -c_0, \\ \vdots \\ a_k c_{2k+1} + \cdots + a_0 c_{3k+1} + b_k d_{2k+1} + \cdots + b_0 d_{3k+1} = -c_{2k}. \end{cases}$$

gives a unique solution for $(a_k, \cdots, a_0, b_k, \cdots, b_0)$. Hence l_1 exists uniquely.
The construction of l_2 is identical and one can find it similar to say l_2 exists uniquely.

Now define

$$\Delta(z) = \det \begin{pmatrix} P_{0,0}(z) & P_{0,1}(z) & P_{0,2}(z) \\ P_{1,0}(z) & P_{1,1}(z) & P_{1,2}(z) \\ P_{2,0}(z) & P_{2,1}(z) & P_{2,2}(z) \end{pmatrix}.$$

In the determinant expansion, the only term with the highest degree $3k+2$ is

$$P_{0,0}(z)P_{1,1}(z)P_{2,2}(z).$$

Therefore,

$$\deg \Delta(z) \leq 3k+2, \qquad [z^{3k+2}]\Delta(z) = 1.$$

Meanwhile, with

$$L_j(z) = P_{j,0}(z) + P_{j,1}(z)f_1(z) + P_{j,2}(z)f_2(z),$$

we may use a column operation to see

$$\Delta(z) = \det \begin{pmatrix} L_0(z) & P_{0,1}(z) & P_{0,2}(z) \\ L_1(z) & P_{1,1}(z) & P_{1,2}(z) \\ L_2(z) & P_{2,1}(z) & P_{2,2}(z) \end{pmatrix}.$$

Expanding it along the first column, and using

$$L_j(z) = O(z^{3k+2}),$$

we obtain

$$\Delta(z) = O(z^{3k+2}).$$

With all its terms with degrees less than $3k+2$ vanishing,

$$\Delta(z) = z^{3k+2}.$$

This proves the lemma. □

Remark 2.9. In a similar manner, one can also construct $s+1$ independent Type I Hermite-Padé approximants $(P_0(z), P_1(z), \cdots, P_s(z)) \in \mathbb{F}[z]^{s+1}$ for $(f_1, \cdots, f_s)$, given a sufficiently large order k.

3. Proof of Theorem 1.5

Proof. First we observe

$$\begin{pmatrix} 1 \\ f_1(z) \\ f_2(z) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{[A(z)]_1}{B(z)} & \frac{[C(z)]_{1,1}}{D(z)} & \frac{[C(z)]_{1,2}}{D(z)} \\ \frac{[A(z)]_2}{B(z)} & \frac{[C(z)]_{2,1}}{D(z)} & \frac{[C(z)]_{2,2}}{D(z)} \end{pmatrix} \begin{pmatrix} 1 \\ f_1(z^d) \\ f_2(z^d) \end{pmatrix}.$$

According to [AF17, Theorem 1.4], if $1, f_1(\frac{1}{b}), f_2(\frac{1}{b})$ are not linearly independent over $\mathbb{Q}$, there exists a homogeneous linear polynomial

$$Q(z, X_0, X_1, X_2) = q_0(z)X_0 + q_1(z)X_1 + q_2(z)X_2 \in \overline{\mathbb{Q}}[z, X_0, X_1, X_2]$$

such that

$$Q(z, 1, f_1(z), f_2(z)) = 0.$$

Yet by Lemma 2.1, we know that $1, f_1(z), f_2(z)$ are linearly independent over $\overline{\mathbb{Q}}(z)$. By the contradiction, we deduce that the numbers $1, f_1(\frac{1}{b}), f_2(\frac{1}{b})$ are linearly independent over $\mathbb{Q}$.

Iterating the Mahler system gives, for every $m \geq 1$,

$$\underline{f}(z) = \frac{C_m(z)}{d_m(z)}\underline{f}(z^{d^m}) + \frac{A_m(z)}{b_m(z)}, \tag{3.1}$$

where

$$\begin{aligned} C_m(z) &= C(z)C(z^d)\cdots C(z^{d^{m-1}}), \\ d_m(z) &= D(z)D(z^d)\cdots D(z^{d^{m-1}}), \\ b_m(z) &= d_{m-1}(z)\prod_{r=0}^{m-1} B(z^{d^r}), \end{aligned}$$

and

$$A_m(z) = \sum_{j=0}^{m-1}\left(C_j(z)\,A(z^{d^j})\prod_{\substack{0\leq r\leq m-1 \\ r\neq j}} B(z^{d^r})\cdot\prod_{r=j}^{m-2} D(z^{d^r})\right),$$

with the convention

$$\prod_{r=m-1}^{m-2} D(z^{d^r}) = 1.$$

By assumption we have that

$$b_m(1/b)d_m(1/b)\det C_m(1/b) \neq 0.$$

Fix i such that n_i is large enough. Since $H_{n_i}(f_1, f_2) \neq 0$, we can construct three Type I Hermite-Padé approximants as in 2.8 that give linear forms

$$L_{i,j}(z) = P_{j,0}(z) + P_{j,1}(z)f_1(z) + P_{j,2}(z)f_2(z) = O(z^{3n_i+2}), \qquad j = 0, 1, 2, \tag{3.2}$$

with

$$\max_{0\leq r\leq 2}\deg P_{j,r} \leq n_i + 1,$$

and

$$\det \begin{pmatrix} P_{0,0}(z) & P_{0,1}(z) & P_{0,2}(z) \\ P_{1,0}(z) & P_{1,1}(z) & P_{1,2}(z) \\ P_{2,0}(z) & P_{2,1}(z) & P_{2,2}(z) \end{pmatrix} = z^{3n_i+2}.$$

Now for each $j = 0, 1, 2$, substitute z^{d^m} into (3.2):

$$P_{j,0}(z^{d^m}) + (P_{j,1}(z^{d^m}), P_{j,2}(z^{d^m}))\underline{f}(z^{d^m}) = O\left(z^{d^m(3n_i+2)}\right). \tag{3.3}$$

With $\det C_m(z) \not\equiv 0$, we may obtain from (3.1)

$$\begin{aligned} \underline{f}(z^{d^m}) &= d_m(z)C_m(z)^{-1}\left(\underline{f}(z) - \frac{A_m(z)}{b_m(z)}\right) \\ &= \frac{d_m(z)}{\det C_m(z)} \operatorname{adj} C_m(z)\left(\underline{f}(z) - \frac{A_m(z)}{b_m(z)}\right). \end{aligned} \tag{3.4}$$

In (3.3), substituting $\underline{f}(z^{d^m})$ by (3.4) and multiplying by $b_m(z)\det C_m(z)$ gives

$$\mathcal{C}_{i,j,m}(z) + \mathcal{A}_{i,j,m}(z)f_1(z) + \mathcal{B}_{i,j,m}(z)f_2(z) = b_m(z)\det C_m(z)O\left(z^{d^m(3n_i+2)}\right), \tag{3.5}$$

where

$$(\mathcal{A}_{i,j,m}(z), \mathcal{B}_{i,j,m}(z)) = b_m(z)d_m(z)(P_{j,1}(z^{d^m}), P_{j,2}(z^{d^m}))\operatorname{adj} C_m(z),$$

and

$$\mathcal{C}_{i,j,m}(z) = -d_m(z)(P_{j,1}(z^{d^m}), P_{j,2}(z^{d^m}))\operatorname{adj} C_m(z)A_m(z) + b_m(z)\det C_m(z)P_{j,0}(z^{d^m}).$$

Evaluating (3.5) at $z = 1/b$, we obtain three linear forms

$$\mathcal{C}_{i,j,m}(1/b) + \mathcal{A}_{i,j,m}(1/b)\xi_1 + \mathcal{B}_{i,j,m}(1/b)\xi_2 = b_m(1/b)\det C_m(1/b)O\left(b^{-d^m(3n_i+2)}\right),$$

where

$$\xi_1 = f_1(1/b), \qquad \xi_2 = f_2(1/b).$$

Let

$$\mathcal{M}_{i,m}(z) = \begin{pmatrix} \mathcal{C}_{i,0,m}(z) & \mathcal{A}_{i,0,m}(z) & \mathcal{B}_{i,0,m}(z) \\ \mathcal{C}_{i,1,m}(z) & \mathcal{A}_{i,1,m}(z) & \mathcal{B}_{i,1,m}(z) \\ \mathcal{C}_{i,2,m}(z) & \mathcal{A}_{i,2,m}(z) & \mathcal{B}_{i,2,m}(z) \end{pmatrix}.$$

Then

$$\mathcal{M}_{i,m}(z) = P_i(z^{d^m})T_m(z),$$

where

$$P_i(z) = \begin{pmatrix} P_{0,0}(z) & P_{0,1}(z) & P_{0,2}(z) \\ P_{1,0}(z) & P_{1,1}(z) & P_{1,2}(z) \\ P_{2,0}(z) & P_{2,1}(z) & P_{2,2}(z) \end{pmatrix},$$

and

$$T_m(z) = \begin{pmatrix} b_m(z)\det C_m(z) & \mathbf{0}^T \\ -d_m(z)\operatorname{adj} C_m(z)A_m(z) & b_m(z)d_m(z)\operatorname{adj} C_m(z) \end{pmatrix}.$$

Since

$$\det T_m(z) = b_m(z)^3 d_m(z)^2 (\det C_m(z))^2,$$

we have

$$\det \mathcal{M}_{i,m}(z) = z^{d^m(3n_i+2)} b_m(z)^3 d_m(z)^2 (\det C_m(z))^2.$$

Therefore, $\det \mathcal{M}_{i,m}(\frac{1}{b}) \neq 0$.

Since $B, D, \det C$ are fixed polynomials, they are bounded on the closed unit disk. Thus there exists $K_0 > 0$ such that

$$|B(z)|,\ |D(z)|,\ |\det C(z)| \leq K_0 \qquad (|z| \leq 1).$$

Then

$$\left|B\left(\frac{1}{b^{d^j}}\right)\right|, \left|D\left(\frac{1}{b^{d^j}}\right)\right|, \left|\det C\left(\frac{1}{b^{d^j}}\right)\right| \leq K_0 \qquad (j \geq 0).$$

Recall that

$$b_m(z) = \prod_{j=0}^{m-2} D(z^{d^j}) \prod_{j=0}^{m-1} B(z^{d^j}).$$

and

$$C_m(z) = C(z)C(z^d)\cdots C(z^{d^{m-1}}).$$

Therefore

$$|b_m(1/b) \det C_m(1/b)| \leq K_0^{3m}.$$

For fixed i, for any $\varepsilon_1 > 0$, there exists $N_{i,\varepsilon_1,1} > 0$ such that for all $m > N_{i,\varepsilon_1,1}$,

$$O\left(b^{-d^m(3n_i+2)}\right) \leq b^{-d^m(3n_i+2)+\frac{\varepsilon_1}{2}d^m},$$

and

$$|b_m(1/b) \det C_m(1/b)| \leq K_0^{3m} \leq b^{\frac{\varepsilon_1}{2}d^m}.$$

Hence,

$$|\mathcal{C}_{i,j,m}(1/b) + \mathcal{A}_{i,j,m}(1/b)\xi_1 + \mathcal{B}_{i,j,m}(1/b)\xi_2| \leq b^{-d^m(3n_i+2)+\varepsilon_1 d^m}.$$

Next we estimate the degrees of $\mathcal{A}_{i,j,m}(z), \mathcal{B}_{i,j,m}(z), \mathcal{C}_{i,j,m}(z)$. Set

$$\kappa := \max_{i=1,2} \deg A_i(z), \qquad \lambda := \deg B(z), \qquad \mu := \max_{1\leq a,b\leq 2} \deg C_{a,b}(z), \qquad \nu := \deg D(z).$$

Then

$$\max_{1\leq a,b\leq 2} \deg(C_m)_{a,b}(z) \leq \sum_{j=0}^{m-1} \mu d^j = \mu \frac{d^m - 1}{d-1} \leq \mu d^m,$$

and

$$\deg d_m(z) \leq \sum_{j=0}^{m-1} \nu d^j \leq \nu d^m.$$

Moreover,

$$\deg b_m(z) \leq \deg d_{m-1}(z) + \sum_{j=0}^{m-1} \deg B(z^{d^j}) \leq (\nu+\lambda) d^m,$$

and

$$\max_{i=1,2} \deg(A_m)_i(z) \leq (\kappa + \lambda + \mu + \nu) d^m.$$

Since $C_m(z)$ is a 2×2 matrix, the entries of $\operatorname{adj} C_m(z)$ are, up to signs, entries of $C_m(z)$. Hence

$$\max_{1\leq a,b\leq 2} \deg\big(\operatorname{adj} C_m\big)_{a,b}(z) \leq \mu d^m.$$

Recall that

$$\deg P_{j,0}, \deg P_{j,1}, \deg P_{j,2} \le n_i + 1.$$

Hence

$$\deg P_{j,r}(z^{d^m}) \le (n_i+1)d^m, \qquad r = 0, 1, 2.$$

First consider

$$(\mathcal{A}_{i,j,m}, \mathcal{B}_{i,j,m}) = b_m d_m (P_{j,1}(z^{d^m}), P_{j,2}(z^{d^m}))\,\mathrm{adj}\, C_m.$$

Therefore

$$\begin{aligned}\deg \mathcal{A}_{i,j,m}, \deg \mathcal{B}_{i,j,m} &\le (\nu+\lambda)d^m + \nu d^m + (n_i+1)d^m + \mu d^m \\ &= (n_i + 1 + \lambda + \mu + 2\nu)d^m.\end{aligned}$$

Now consider

$$\mathcal{C}_{i,j,m} = -d_m(P_{j,1}(z^{d^m}), P_{j,2}(z^{d^m}))\,\mathrm{adj}\, C_m A_m + b_m \det C_m\, P_{j,0}(z^{d^m}).$$

For the first term, its degree is bounded by

$$\nu d^m + (n_i+1)d^m + \mu d^m + (\kappa+\lambda+\mu+\nu)d^m.$$

For the second term, using

$$\deg \det C_m \le 2\mu d^m.$$

we have

$$\deg\left(b_m \det C_m P_{j,0}(z^{d^m})\right) \le (\nu+\lambda)d^m + 2\mu d^m + (n_i+1)d^m.$$

Thus

$$\deg \mathcal{C}_{i,j,m} \le (n_i + 1 + \kappa + \lambda + 2\mu + 2\nu)d^m.$$

Therefore, setting

$$\Gamma = \kappa + \lambda + 2\mu + 2\nu$$

and then

$$E_i = n_i + 1 + \Gamma,$$

we obtain, for every $j = 0, 1, 2$,

$$\deg \mathcal{A}_{i,j,m}, \deg \mathcal{B}_{i,j,m}, \deg \mathcal{C}_{i,j,m} \le E_i d^m.$$

Thus multiplying by $b^{E_i d^m}$ clears the denominators coming from the evaluation at $1/b$. We also allow rational denominators in the coefficients. For fixed i, for any $\varepsilon_1 > 0$, there exists $N_{i,\varepsilon_1,2} > 0$ such that for all $m > N_{i,\varepsilon_1,2}$, there is an integer $D_{i,m}$ such that

$$D_{i,m}\mathcal{A}_{i,j,m}(1/b), \quad D_{i,m}\mathcal{B}_{i,j,m}(1/b), \quad D_{i,m}\mathcal{C}_{i,j,m}(1/b) \in \mathbb{Z}$$

for all $j = 0, 1, 2$, and

$$D_{i,m} \le b^{E_i d^m + \varepsilon_1 d^m}.$$

Define

$$A_{i,j,m} = D_{i,m}\mathcal{A}_{i,j,m}(1/b), B_{i,j,m} = D_{i,m}\mathcal{B}_{i,j,m}(1/b), \text{ and } C_{i,j,m} = D_{i,m}\mathcal{C}_{i,j,m}(1/b).$$

Then

$$A_{i,j,m}, B_{i,j,m}, C_{i,j,m} \in \mathbb{Z},$$

and

$$\Lambda_{i,j,m}(\underline{\xi}) = C_{i,j,m} + A_{i,j,m}\xi_1 + B_{i,j,m}\xi_2.$$

Combining the previous estimates gives

$$|\Lambda_{i,j,m}(\underline{\xi})| \leq b^{E_i d^m + \varepsilon_1 d^m} b^{-d^m(3n_i+2)+\varepsilon_1 d^m}.$$

Therefore

$$|\Lambda_{i,j,m}(\underline{\xi})| \leq b^{-(3n_i+2-E_i)d^m + 2\varepsilon_1 d^m}.$$

Since

$$E_i = n_i + 1 + \Gamma,$$

we have

$$3n_i + 2 - E_i = 2n_i + 1 - \Gamma.$$

Thus

$$|\Lambda_{i,j,m}(\underline{\xi})| \leq b^{-(2n_i+1-\Gamma)d^m + 2\varepsilon_1 d^m}.$$

Fix some $\eta > 0$. Then choose $\varepsilon_1 < \eta$. Now let

$$\mathcal{H}_{i,m} = b^{(E_i+\eta)d^m}.$$

Define

$$\delta_{i,\eta} = \frac{2n_i + 1 - \Gamma}{E_i + \eta}.$$

Then

$$(2n_i + 1 - \Gamma)d^m = \delta_{i,\eta}(E_i + \eta)d^m.$$

Hence

$$b^{-(2n_i+1-\Gamma)d^m} = \mathcal{H}_{i,m}^{-\delta_{i,\eta}}.$$

Also,

$$b^{2\varepsilon_1 d^m} = \mathcal{H}_{i,m}^{\frac{2\varepsilon_1}{E_i+\eta}}.$$

Therefore

$$|\Lambda_{i,j,m}(\underline{\xi})| \leq \mathcal{H}_{i,m}^{-\delta_{i,\eta}+\frac{2\varepsilon_1}{E_i+\eta}}.$$

Given any $\varepsilon > 0$, choose $\varepsilon_1 > 0$ so small that

$$\frac{2\varepsilon_1}{E_i + \eta} < \varepsilon.$$

Then for all sufficiently large m,

$$|\Lambda_{i,j,m}(\underline{\xi})| \leq \mathcal{H}_{i,m}^{-\delta_{i,\eta}+\varepsilon}.$$

Since this is uniform for $0 \leq j \leq 2$, we get

$$\max_{0\leq j\leq 2} |\Lambda_{i,j,m}(\underline{\xi})| \leq \mathcal{H}_{i,m}^{-\delta_{i,\eta}+\varepsilon}. \tag{3.6}$$

Since

$$\limsup_{i\to\infty} \frac{E_{i+1}}{E_i} = \rho,$$

then for every $\varepsilon_2 > 0$, there exists i_0 such that for all $i \geq i_0$,

$$E_{i+1} \leq (\rho + \varepsilon_2)E_i. \tag{3.7}$$

For an integer ℓ, define a set of integers

$$\mathcal{A}_\ell = \{i > i_0 : \ d^{\ell-1} \le E_i < d^\ell\}.$$

Assume $\mathcal{A}_\ell \neq \varnothing$, and write

$$\mathcal{A}_\ell = \{i_1 < i_2 < \cdots < i_t\}.$$

For $1 \le l \le t$ and $m \ge M_\ell$, consider

$$\mathcal{H}_{i_l,m} = b^{(E_{i_l}+\eta)d^m}.$$

Arrange the distinct numbers $\mathcal{H}_{i_l,m}$, with $1 \le l \le t$ and $m \ge M_\ell$, in increasing order, and denote them by:

$$r_{\ell,0} < r_{\ell,1} < r_{\ell,2} < \cdots .$$

We prove that for every $\varepsilon > 0$, if ℓ is sufficiently large, then

$$r_{\ell,s+1} \le r_{\ell,s}^{\rho^2+\varepsilon}$$

for every $s \ge 0$.

Fix s, and write

$$r_{\ell,s} = \mathcal{H}_{i_l,m}.$$

There are two cases.

First suppose $l < t$. Then

$$\mathcal{H}_{i_{l+1},m}$$

is one of the admissible numbers larger than $\mathcal{H}_{i_l,m}$. Hence

$$r_{\ell,s+1} \le \mathcal{H}_{i_{l+1},m}.$$

Therefore

$$\frac{\log r_{\ell,s+1}}{\log r_{\ell,s}} \le \frac{(E_{i_{l+1}}+\eta)d^m}{(E_{i_l}+\eta)d^m} = \frac{E_{i_{l+1}}+\eta}{E_{i_l}+\eta}.$$

Since $i_{l+1} = i_l + 1$, we have

$$E_{i_{l+1}} \le (\rho+\varepsilon_2)E_{i_l}.$$

Thus

$$\frac{E_{i_{l+1}}+\eta}{E_{i_l}+\eta} \le \frac{(\rho+\varepsilon_2)E_{i_l}+\eta}{E_{i_l}+\eta} \le \rho+\varepsilon_2,$$

because $\rho+\varepsilon_2 \ge 1$. Hence

$$\frac{\log r_{\ell,s+1}}{\log r_{\ell,s}} \le \rho+\varepsilon_2.$$

Since $\rho \ge 1$, we can choose ε_2 small enough that

$$\rho+\varepsilon_2 \le \rho^2+\varepsilon.$$

Therefore

$$r_{\ell,s+1} \le r_{\ell,s}^{\rho^2+\varepsilon}.$$

Now suppose $l = t$. Since $d(E_{i_1}+\eta) \ge d^\ell + d\eta > E_{i_t}+\eta$,

$$r_{\ell,s+1} \le \mathcal{H}_{i_1,m+1} = b^{(E_{i_1}+\eta)d^{m+1}}.$$

Therefore

$$\frac{\log r_{\ell,s+1}}{\log r_{\ell,s}} \le \frac{(E_{i_1}+\eta)d^{m+1}}{(E_{i_t}+\eta)d^m} = d\frac{E_{i_1}+\eta}{E_{i_t}+\eta}.$$

Since i_1 is the first index in $\mathcal{A}_\ell$, we have

$$E_{i_1-1} < d^{\ell-1}.$$

Using (3.7),

$$E_{i_1} \le (\rho+\varepsilon_2)E_{i_1-1} < (\rho+\varepsilon_2)d^{\ell-1}.$$

Since i_t is the last index in the block, we have

$$E_{i_t+1} \ge d^\ell.$$

Again by (3.7),

$$E_{i_t+1} \le (\rho+\varepsilon_2)E_{i_t}.$$

Therefore

$$E_{i_t} \ge \frac{d^\ell}{\rho+\varepsilon_2}.$$

Consequently,

$$d\frac{E_{i_1}+\eta}{E_{i_t}+\eta} \le d\frac{E_{i_1}+\eta}{E_{i_t}} \le d\frac{(\rho+\varepsilon_2)d^{\ell-1}+\eta}{d^\ell/(\rho+\varepsilon_2)}.$$

Thus

$$d\frac{E_{i_1}+\eta}{E_{i_t}+\eta} \le (\rho+\varepsilon_2)^2 + \frac{d(\rho+\varepsilon_2)\eta}{d^\ell}.$$

Choose $\varepsilon_2 > 0$ small enough that

$$(\rho+\varepsilon_2)^2 < \rho^2 + \frac{\varepsilon}{2}.$$

Then choose ℓ sufficiently large that

$$\frac{d(\rho+\varepsilon_2)\eta}{d^\ell} < \frac{\varepsilon}{2}.$$

It follows that

$$\frac{\log r_{\ell,s+1}}{\log r_{\ell,s}} \le \rho^2+\varepsilon.$$

Therefore

$$r_{\ell,s+1} \le r_{\ell,s}^{\rho^2+\varepsilon}.$$

In both cases we have shown that

$$r_{\ell,s+1} \le r_{\ell,s}^{\rho^2+\varepsilon}.$$

We also have

$$r_{\ell,s} < r_{\ell,s+1}.$$

Therefore, after relabeling the ordered sequence $\{r_{\ell,s}\}$ as

$$\mathcal{H}_0 < \mathcal{H}_1 < \mathcal{H}_2 < \cdots,$$

we obtain

$$\mathcal{H}_s < \mathcal{H}_{s+1} \le \mathcal{H}_s^{\rho^2+\varepsilon} \tag{3.8}$$

for all sufficiently large s.

Recall that for every fixed i, for every $\varepsilon > 0$, and for all sufficiently large m, (3.6) holds with $\delta_{i,\eta} \to 2$ as $i \to \infty$. Fix numbers $1 < \delta < 2$ and $\varepsilon > 0$. Since $\delta_{i,\eta} \to 2$, we may choose ℓ sufficiently large such that

$$\delta_{i,\eta} - \varepsilon \geq \delta$$

for every $i \in \mathcal{A}_\ell$.

For this $\mathcal{A}_\ell$, choose M_ℓ large enough so that (3.6) holds for every $i \in \mathcal{A}_\ell$ and every $m \geq M_\ell$. We then arrange $\mathcal{H}_{i,m}$, where $i \in \mathcal{A}_\ell, \quad m \geq M_\ell$, in increasing order:

$$\mathcal{H}_0 < \mathcal{H}_1 < \mathcal{H}_2 < \cdots .$$

If

$$\mathcal{H}_s = \mathcal{H}_{i,m},$$

we define

$$L_{j,s}(\underline{\xi}) := \Lambda_{i,j,m}(\underline{\xi}), \qquad j = 0, 1, 2.$$

By construction, the coefficient height of $L_{j,s}$ is bounded by

$$\mathcal{H}_s.$$

And we already know from (3.8) that

$$\mathcal{H}_s < \mathcal{H}_{s+1} \leq \mathcal{H}_s^{\rho^2+\varepsilon}.$$

Thus condition $(2)(a)$ of 2.2 is satisfied with

$$\theta = \rho^2 + \varepsilon.$$

Moreover, since $i \in \mathcal{A}_\ell$ and $m \geq M_\ell$, we have

$$\max_{0 \leq j \leq 2} |L_{j,s}(\underline{\xi})| = \max_{0 \leq j \leq 2} |\Lambda_{i,j,m}(\underline{\xi})| \leq \mathcal{H}_{i,m}^{-\delta_{i,\eta}+\varepsilon}.$$

But

$$\mathcal{H}_{i,m} = \mathcal{H}_s$$

and

$$\delta_{i,\eta} - \varepsilon \geq \delta.$$

Hence

$$\max_{0 \leq j \leq 2} |L_{j,s}(\underline{\xi})| \leq \mathcal{H}_s^{-\delta}.$$

Now we can apply 2.2 and get

$$\omega^*(\underline{\xi}) \leq \frac{2(\rho^2 + \varepsilon)}{\delta - 1}.$$

Since $1 < \delta < 2$ was arbitrary, we may let $\delta \to 2$. This gives

$$\omega^*(\underline{\xi}) \leq 2(\rho^2 + \varepsilon).$$

Finally, letting $\varepsilon \to 0$, we obtain

$$\omega^*(\underline{\xi}) \leq 2\rho^2.$$

□

4. Proof of Theorem 1.8

Proof. Let $\underline{f}(z) = \begin{pmatrix} 1 & -\alpha z \\ 1 & \beta z \end{pmatrix} \underline{f}(z^2)$ and $\underline{f}$ be nonzero. The coefficients satisfy

$$c_{2m} = c_m, c_{2m+1} = -\alpha d_m;$$
$$d_{2m} = c_m, d_{2m+1} = \beta d_m;$$
$$d_0 = c_0 = 1.$$

To verify convergence, set

$$M := \max\{1, |\alpha|, |\beta|\}.$$

For every $n \geq 1$,

$$|c_n|, |d_n| \leq M n^{\log_2 M}.$$

Consequently, for every $z \in \mathbb{R}$ with $|z| < 1$,

$$\sum_{n\geq 0} |c_n z^n| \leq 1 + M \sum_{n\geq 1} n^{\log_2 M} |z|^n < \infty,$$

and similarly,

$$\sum_{n\geq 0} |d_n z^n| < \infty.$$

Hence both $f_1(z)$ and $f_2(z)$ converge absolutely inside the open unit disk.

Specifically, first assume $\beta = 0$, so that

$$c_{2m} = c_m, \qquad c_{2m+1} = -\alpha d_m, \qquad d_{2m} = c_m, \qquad d_{2m+1} = 0. \tag{4.1}$$

For $j > i, k > i$, let

$$C^{(i)}_{j,k} = \begin{pmatrix} c_i & c_{i+1} & \cdots & c_k \\ c_{i+1} & c_{i+2} & \cdots & c_{k+1} \\ \vdots & \vdots & & \vdots \\ c_j & c_{j+1} & \cdots & c_{j-i+k} \end{pmatrix}, \qquad D^{(i)}_{j,k} = \begin{pmatrix} d_i & d_{i+1} & \cdots & d_k \\ d_{i+1} & d_{i+2} & \cdots & d_{k+1} \\ \vdots & \vdots & & \vdots \\ d_j & d_{j+1} & \cdots & d_{j-i+k} \end{pmatrix}.$$

Then the Hankel determinant $H_n = \begin{vmatrix} C^{(0)}_{2n+1,n} & D^{(0)}_{2n+1,n} \end{vmatrix}$.
And we define the following determinants for $n \geq 1$,

$$P_n = \begin{vmatrix} C^{(0)}_{2n,n} & D^{(0)}_{2n,n-1} \end{vmatrix},$$
$$Q_n = \begin{vmatrix} C^{(1)}_{2n+1,n} & D^{(0)}_{2n,n} \end{vmatrix},$$
$$R_n = \begin{vmatrix} C^{(1)}_{2n+2,n+1} & D^{(0)}_{2n+1,n} \end{vmatrix},$$
$$T_n = \begin{vmatrix} C^{(1)}_{2n,n} & D^{(1)}_{2n,n} \end{vmatrix},$$
$$U_n = \begin{vmatrix} C^{(1)}_{2n+1,n+1} & D^{(1)}_{2n+1,n} \end{vmatrix}.$$

The determinants satisfy the following system of absolute-value recurrences:

$$|H_{2n}| = |\alpha|^{2n+1} |P_n Q_n|,$$
$$|H_{2n+1}| = |\alpha|^{2n+2} |H_n R_n|, ,$$

$$|P_{2n}| = |\alpha|^{2n}|P_nR_{n-1}|, \qquad n \geq 1,$$

$$|P_{2n+1}| = |\alpha|^{2n+2}|H_nQ_n|,$$

$$|Q_{2n}| = |\alpha|^{2n}|P_nT_n|, \qquad n \geq 1,$$

$$|Q_{2n+1}| = |\alpha|^{2n+1}|H_nU_n|,$$

$$|R_{2n}| = |\alpha|^{2n}|P_nU_n|,$$

$$|R_{2n+1}| = |\alpha|^{2n+2}|H_nT_{n+1}|,$$

$$|T_{2n}| = |\alpha|^{2n}|R_{n-1}T_n|, \qquad n \geq 1,$$

$$|T_{2n+1}| = |\alpha|^{2n+1}|Q_nU_n|,$$

$$|U_{2n}| = |\alpha|^{2n+1}|Q_nT_n|, \qquad n \geq 1,$$

$$|U_{2n+1}| = |\alpha|^{2n+1}|R_nU_n|.$$

We first prove that

$$|H_{2n}| = |\alpha|^{2n+1}|P_nQ_n|.$$

By definition,

$$H_{2n} = \begin{vmatrix} c_0 & c_1 & \cdots & c_{2n} & d_0 & d_1 & \cdots & d_{2n} \\ c_1 & c_2 & \cdots & c_{2n+1} & d_1 & d_2 & \cdots & d_{2n+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{4n} & c_{4n+1} & \cdots & c_{6n} & d_{4n} & d_{4n+1} & \cdots & d_{6n} \\ c_{4n+1} & c_{4n+2} & \cdots & c_{6n+1} & d_{4n+1} & d_{4n+2} & \cdots & d_{6n+1} \end{vmatrix}.$$

$$|H_{2n}| = \left\| \begin{matrix} c_0 & c_2 & \cdots & c_{2n} & c_1 & \cdots & c_{2n-1} & d_0 & \cdots & d_{2n} & d_1 & \cdots & d_{2n-1} \\ c_2 & c_4 & \cdots & c_{2n+2} & c_3 & \cdots & c_{2n+1} & d_2 & \cdots & d_{2n+2} & d_3 & \cdots & d_{2n+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ c_{4n} & c_{4n+2} & \cdots & c_{6n} & c_{4n+1} & \cdots & c_{6n-1} & d_{4n} & \cdots & d_{6n} & d_{4n+1} & \cdots & d_{6n-1} \\ c_1 & c_3 & \cdots & c_{2n+1} & c_2 & \cdots & c_{2n} & d_1 & \cdots & d_{2n+1} & d_2 & \cdots & d_{2n} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ c_{4n+1} & c_{4n+3} & \cdots & c_{6n+1} & c_{4n+2} & \cdots & c_{6n} & d_{4n+1} & \cdots & d_{6n+1} & d_{4n+2} & \cdots & d_{6n} \end{matrix} \right\|.$$

Then one can apply the recurrence relation (4.1) and get that

$$
\begin{aligned}
|H_{2n}| &= \left\| \begin{array}{cccccccccccc}
c_0 & \cdots & c_n & -\alpha d_0 & \cdots & -\alpha d_{n-1} & c_0 & \cdots & c_n & 0 & \cdots & 0 \\
\vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
c_{2n} & \cdots & c_{3n} & -\alpha d_{2n} & \cdots & -\alpha d_{3n-1} & c_{2n} & \cdots & c_{3n} & 0 & \cdots & 0 \\
-\alpha d_0 & \cdots & -\alpha d_n & c_1 & \cdots & c_n & 0 & \cdots & 0 & c_1 & \cdots & c_n \\
\vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
-\alpha d_{2n} & \cdots & -\alpha d_{3n} & c_{2n+1} & \cdots & c_{3n} & 0 & \cdots & 0 & c_{2n+1} & \cdots & c_{3n}
\end{array} \right\| \\
&= \left\| \begin{array}{cccccccccccc}
c_0 & \cdots & c_n & -\alpha d_0 & \cdots & -\alpha d_{n-1} & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
c_{2n} & \cdots & c_{3n} & -\alpha d_{2n} & \cdots & -\alpha d_{3n-1} & 0 & \cdots & 0 & 0 & \cdots & 0 \\
-\alpha d_0 & \cdots & -\alpha d_n & c_1 & \cdots & c_n & \alpha d_0 & \cdots & \alpha d_n & c_1 & \cdots & c_n \\
\vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
-\alpha d_{2n} & \cdots & -\alpha d_{3n} & c_{2n+1} & \cdots & c_{3n} & \alpha d_{2n} & \cdots & \alpha d_{3n} & c_{2n+1} & \cdots & c_{3n}
\end{array} \right\| \\
&= \left\| \begin{array}{ccccccc}
c_0 & \cdots & c_n & -\alpha d_0 & \cdots & -\alpha d_{n-1} \\
\vdots & & \vdots & \vdots & & \vdots \\
c_{2n} & \cdots & c_{3n} & -\alpha d_{2n} & \cdots & -\alpha d_{3n-1}
\end{array} \right\| \left\| \begin{array}{cccccc}
\alpha d_0 & \cdots & \alpha d_n & c_1 & \cdots & c_n \\
\vdots & & \vdots & \vdots & & \vdots \\
\alpha d_{2n} & \cdots & \alpha d_{3n} & c_{2n+1} & \cdots & c_{3n}
\end{array} \right\| \\
&= |\alpha|^{2n+1} |P_n Q_n|.
\end{aligned}
$$

Next, we want to show $|H_{2n+1}| = |\alpha|^{2n+2} |H_n R_n|$. Similarly, we have

$$
\begin{aligned}
|H_{2n+1}| &= \left\| \begin{array}{cccccccc}
c_0 & c_1 & \cdots & c_{2n+1} & d_0 & d_1 & \cdots & d_{2n+1} \\
c_1 & c_2 & \cdots & c_{2n+2} & d_1 & d_2 & \cdots & d_{2n+2} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
c_{4n+2} & c_{4n+3} & \cdots & c_{6n+3} & d_{4n+2} & d_{4n+3} & \cdots & d_{6n+3} \\
c_{4n+3} & c_{4n+4} & \cdots & c_{6n+4} & d_{4n+3} & d_{4n+4} & \cdots & d_{6n+4}
\end{array} \right\| \\
&= \left\| \begin{array}{cccccccccccc}
c_0 & \cdots & c_{2n} & c_1 & \cdots & c_{2n+1} & d_0 & \cdots & d_{2n} & d_1 & \cdots & d_{2n+1} \\
\vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
c_{4n+2} & \cdots & c_{6n+2} & c_{4n+3} & \cdots & c_{6n+3} & d_{4n+2} & \cdots & d_{6n+2} & d_{4n+3} & \cdots & d_{6n+3} \\
c_1 & \cdots & c_{2n+1} & c_2 & \cdots & c_{2n+2} & d_1 & \cdots & d_{2n+1} & d_2 & \cdots & d_{2n+2} \\
\vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
c_{4n+3} & \cdots & c_{6n+3} & c_{4n+4} & \cdots & c_{6n+4} & d_{4n+3} & \cdots & d_{6n+3} & d_{4n+4} & \cdots & d_{6n+4}
\end{array} \right\| \\
&= \left\| \begin{array}{cccc}
C^{(0)}_{2n+1,n} & \alpha D^{(0)}_{2n+1,n} & 0 & 0 \\
\alpha D^{(0)}_{2n+1,n} & C^{(1)}_{2n+2,n+1} & \alpha D^{(0)}_{2n+1,n} & C^{(1)}_{2n+2,n+1}
\end{array} \right\| \\
&= \left\| \begin{array}{cccccc}
c_0 & \cdots & c_n & \alpha d_0 & \cdots & \alpha d_n \\
\vdots & & \vdots & \vdots & & \vdots \\
c_{2n+1} & \cdots & c_{3n+1} & \alpha d_{2n+1} & \cdots & \alpha d_{3n+1}
\end{array} \right\| \left\| \begin{array}{cccccc}
\alpha d_0 & \cdots & \alpha d_n & c_1 & \cdots & c_{n+1} \\
\vdots & & \vdots & \vdots & & \vdots \\
\alpha d_{2n+1} & \cdots & \alpha d_{3n+1} & c_{2n+2} & \cdots & c_{3n+2}
\end{array} \right\| \\
&= |\alpha|^{2n+2} |H_n R_n|.
\end{aligned}
$$

The remaining recurrences are obtained in the same way. For each determinant, we first permute the rows and columns so that entries with even indices precede those with odd indices. Since only absolute values of the determinants are considered, the signs of these permutations are irrelevant. We then apply the recurrences (4.1) entrywise. Finally, elementary block-column operations transform the resulting matrix into a block triangular matrix. Taking determinants and extracting the powers of α gives the asserted closed system of determinants in absolute value.

Given

$$|H_1| = |\alpha|^4, \qquad |P_1| = |Q_1| = |U_1| = |\alpha|^3,$$
$$|R_1| = |\alpha|^5, \qquad |T_1| = |\alpha|^2.$$

since every determinant on the right-hand side has smaller index and since $\alpha \neq 0$, this closed multiplicative system proves by induction that

$$H_n, P_n, Q_n, R_n, U_n \neq 0 \qquad (n \geq 0),$$

and

$$T_n \neq 0 \qquad (n \geq 1).$$

Write $H_n(\underline{f})$ as $H_n(\alpha, \beta)$, a function of α and β. In particular, for all $n \geq 0$,

$$H_n(\alpha, 0) = \sigma_n \alpha^{E_n} \neq 0,$$

where $\sigma_n \in \{-1, 1\}, E_n \in \mathbb{Z}^+$.

Let p be a prime number and let $\alpha, \beta \in \mathbb{Z}$ be such that $p \nmid \alpha$ and $p \mid \beta$. $H_n(\alpha, \beta) \equiv H_n(\alpha, 0) \equiv \sigma_n \alpha^{E_n} \not\equiv 0 \pmod{p}$. Therefore, for all $n \geq 1$, $H_n(\alpha, \beta) = H_n(\underline{f}) \neq 0$. Consequently, condition (1.3) holds with $\rho = 1$. Now apply 1.5 and we know, for any integer $b \geq 2$ it holds that

$$\mu(\underline{f}(1/b)) = \frac{3}{2}.$$

□

Acknowledgments

I would like to thank the Department of Mathematics at the University of Michigan for giving me the precious opportunity to participate in the Research Experiences for Undergraduates (REU) program. Most importantly, I am deeply grateful to my mentor, Dr. Reynold Fregoli, for his invaluable guidance and support throughout the project and in this specific report.

References

[AB07] Boris Adamczewski and Yann Bugeaud. On the complexity of algebraic numbers. I: Expansions in integer bases. *Ann. Math. (2)*, 165(2):547–565, 2007.

[Ada19] Boris Adamczewski. Mahler's method. *Doc. Math.*, Extra Vol.:95–122, 2019.

[AF17] Boris Adamczewski and Colin Faverjon. Mahler method: linear relations, transcendence and applications to automatic numbers. *Proc. Lond. Math. Soc. (3)*, 115(1):55–90, 2017.

[AR09] Boris Adamczewski and Tanguy Rivoal. Irrationality measures for some automatic real numbers. In *Mathematical Proceedings of the Cambridge Philosophical Society*, volume 147, pages 659–678. Cambridge University Press, 2009.

[AS03] Jean-Paul Allouche and Jeffrey Shallit. *Automatic sequences. Theory, applications, generalizations*. Cambridge: Cambridge University Press, 2003.

[BHWY16] Yann Bugeaud, Guo-Niu Han, Zhi-Ying Wen, and Jia-Yan Yao. Hankel determinants, Padé approximations, and irrationality exponents. *Int. Math. Res. Not.*, 2016(5):1467–1496, 2016.

[Bug11] Yann Bugeaud. On the rational approximation to the Thue-Morse-Mahler numbers. *Ann. Inst. Fourier*, 61(5):2065–2076, 2011.

[BZ15] Dzmitry Badziahin and Evgeniy Zorin. Thue-morse constant is not badly approximable. *Int. Math. Res. Not.*, 2015(19):9618–9637, 2015.

[Cob68] A. Cobham. On the Hartmanis–Stearns problem for a class of tag machines. In *Conference Record of 1968 Ninth Annual Symposium on Switching and Automata Theory*, pages 51–60, Schenectady, New York, 1968.

[Fis15] Stéphane Fischler. Nesterenko's linear independence criterion for vectors. *Monatsh. Math.*, 177(3):397–419, 2015.

[Khi26] Aleksandr Khintchine. Über eine klasse linearer diophantischer approximationen. *Rendiconti del Circolo Matematico di Palermo Series 2*, 50(2):170–195, 1926.

[KVAW05] ABJ Kuijlaars, Walter Van Assche, and Franck Wielonsky. Quadratic hermite–padé approximation to the exponential function: a riemann–hilbert approach. *Constructive approximation*, 21(3):351–412, 2005.

[Mah29] K. Mahler. Arithmetische Eigenschaften der Lösungen einer Klasse von Funktionalgleichungen. *Math. Ann.*, 101:342–366, 1929.

[Nes85] Yu. V. Nesterenko. On the linear independence of numbers. *Mosc. Univ. Math. Bull.*, 40(1):69–74, 1985.

[Nis90] Kumiko Nishioka. New approach in mahler's method. *Journal für die reine und angewandte Mathematik*, 1990(407):202–219, 1990.