Derived Relative Geometry and Applications to Complex Geometry

Benjamin Singer*

August 2026

Abstract

We develop a framework for derived geometry relative to an arbitrary derived algebraic context and apply it to complex-analytic geometry through the lens of bornological functional analysis, following foundational work of Ben-Bassat–Kelly–Kremnizer. Working over a symmetric monoidal stable ∞-category equipped with a compatible t-structure and a suitable class of compact projective generators, we assemble the associated six-functor formalism of quasi-coherent sheaves in this relative setting via the extension formalism of Heyer–Mann. To pass from algebra to analysis, we take as our context the derived category of complete bornological vector spaces over $\mathbb{C}$. Within it, we introduce derived dagger affinoid and Stein algebras, construct derived dagger and complex analytic spaces, and develop their six-functor formalisms of quasi-coherent sheaves. This yields theories of derived complex analytic and dagger geometry whose quasi-coherent sheaves satisfy descent, shoring up the main deficiency of classical analytic geometry. As applications, we develop analytification for both classes of derived analytic spaces, prove a GAGA-type comparison for quasi-coherent sheaves, and establish a Tannaka duality theorem recovering maps of derived analytic spaces from their categories of quasi-coherent sheaves.

*benjamin.d.singer.27@dartmouth.edu

Contents

1 Introduction **3**

1.1 Motivation: Issues of Descent . . . 3

1.2 What We Do . . . 5

1.3 Notation and Conventions . . . 9

1.4 Remarks and Acknowledgements . . . 10

2 DAG Review **11**

2.1 Animation . . . 11

2.2 $\mathbb{E}_\infty$-Algebras . . . 13

2.3 Modules Over Animated Rings and Quasi-Coherent Sheaves . . . 17

2.3.1 The Module ∞-Category . . . 18

2.3.2 Faithfully Flat Descent . . . 19

3 Relative Geometry **20**

3.1 Derived Algebraic Contexts and Relative Geometry Tuples . . . 21

3.2 The Descendable Topology and Quasi-Coherent Sheaves . . . 23

3.2.1 Quasi-Coherent Sheaves in the Descendable Topology . . . 25

4 Six-Functor Formalisms **25**

4.1 The Category of Correspondences . . . 27

4.1.1 Foundations . . . 27

4.1.2 Three and Six-Functor Formalisms . . . 28

4.1.3 Constructing Six-functor Formalisms . . . 31

4.2 Extending Six-Functor Formalisms, and the Six-Functor Formalism on $\mathcal{C}$Alg 33

4.3 The Category of Kernels, Cohomologically Smooth Morphisms, and Suave/Prim Morphisms . . . 35

5 Bornological Spaces and $D(\mathrm{CBorn}_{\mathbb{C}})$ **41**

5.1 Bornological Spaces, a Crash Course . . . 41

5.1.1 Bornological Sets . . . 41

5.1.2 Complete Bornological Spaces . . . 43

5.1.3 Bornological Tensor Products . . . 53

5.2 Categories of Complexes and Derived Categories of Quasi-Abelian Categories 55

5.2.1 Complexes and the Projective Model Structure . . . 55
5.2.2 The Derived ∞-Category, Stability and t-Structure . . . 57
5.3 Animated Bornological Algebras . . . 62

6 Derived Complex Analytic and Dagger Spaces 65
6.1 Classical Dagger Affinoid Spaces, Stein Spaces, and Dagger Stein Spaces . . . 66
6.2 Derived Dagger Affinoid and Stein Spaces . . . 69
6.3 Nonsense with Grothendieck Topologies . . . 77
6.4 Compatibility of Sheaves . . . 80

7 Analytification 84
7.1 Analytification as a Geometric Morphism . . . 85
7.2 GAGA, à la Clausen-Scholze . . . 91
7.3 The Six-Functor Formalism on $\mathrm{dAn}^{\dagger}_{\mathbb{C}}$ and New Morphisms . . . 95
7.4 Derived GAGA . . . 99
7.5 Tannaka Duality and Algebraization . . . 102

1 Introduction

1.1 Motivation: Issues of Descent

Of the various theories of geometry in existence – smooth, complex, algebraic, etc. – analytic geometry is the most subtle from an algebraic perspective. The main reason is that analytic geometry's sheaves and rings of functions are often at odds with the algebra governing them. There are several instances of this problem, but we will focus on a primary one: analytic geometry's general failure to have a good theory of quasi-coherent sheaves.

Much of modern algebraic geometry rests on the fact that quasi-coherent sheaves satisfy descent in all relevant topologies (Zariski, étale, fppf, fpqc). This allows us to associate a stack of quasi-coherent sheaves to an arbitrary scheme/stack X, yielding many pleasant consequences, such as:

1. Torsors, Cohomology, and Galois Descent. The fact that quasi-coherent sheaves satisfy descent is relevant for cohomological interpretations of geometric objects making up substacks of $\mathrm{QCoh}(X)$. For example, it can be used to show the bijection

$$\{\text{Rank } n \text{ Vector Bundles on X}\} \leftrightarrow H^1_{\mathrm{fppf}}(X, GL_n)$$

and in Grothendieck's Hilbert 90

$$\mathrm{Pic}(X) \cong H^1_{\mathrm{fppf}}(X, \mathbb{G}_m) \cong H^1_{\text{ét}}(X, \mathbb{G}_m) \cong H^1_{zar}(X, \mathbb{G}_m).$$

More classically, in the case that $X = \mathrm{Spec}(K)$, it can recover several instances of Galois descent, for example

$$\{\text{Central Simple Algebras of Degree } n\} \leftrightarrow H^1(G, \mathrm{PGL_n}).$$

This is vital for class field theory, as it is central in showing that a field K with absolute Galois group G has Brauer group $\mathrm{Br}(K) \cong H^2(G, \mathbb{G}_m)$.

2. Gluing Schemes. To give an affine morphism of schemes $X \to Y$ is to give a quasi-coherent $\mathcal{O}_X$-algebra $\mathcal{A}$ (as we can recover Y as the relative spectrum $\underline{\mathrm{Spec}}(\mathcal{A})$). Hence, descent for quasi-coherent sheaves allows us to glue schemes together fppf locally, which is critical in the construction of many moduli spaces.

3. Grothendieck's Six Operations. On affine schemes $X = \mathrm{Spec}(R)$, there are 6 operations on the derived categories of quasi-coherent sheaves (which is equivalent to the derived category $D(R)$). Each of these comes in an adjoint pair:

$$\mathbb{L}f^* \dashv \mathbb{R}f_*, - \otimes^{\mathbb{L}} \mathcal{F} \dashv \mathcal{H}om(\mathcal{F}, -), f_! \dashv f^!,$$

all satisfying classical computation results on sheaves (base change formulas, projection formula, etc.). Since quasi-coherent sheaves satisfy descent, we can glue these together into 6 operations on quasi-coherent sheaves on an arbitrary scheme X. This has many immediate consequences, the most important of which being the re-packaging of several duality theorems (Verdier duality, Poincaré duality, Serre duality, etc.) under the umbrella of Grothendieck duality. It has further applications in the development of many modern cohomology theories, thus underlying the resolutions of many significant problems in modern algebraic/arithmetic geometry (namely, the Weil conjectures and geometric Langlands).

4. The ability to define quasi-coherent sheaves on stacks, and quasi-coherent sheaves forming a stack. Many important objects in the subject arise as quasi-coherent sheaves on certain stacks (e.g. $\mathcal{D}$-modules or local systems), a perspective also vital to the Langlands program.

This all fails badly in the analytic case. For this discussion, we fix a nonarchimedean field K. The naïve definition of a quasi-coherent sheaf in rigid analytic geometry is the following:

Definition 1.1 ([Con06])**.** Let S be a rigid analytic space. An $\mathcal{O}_S$-module $\mathcal{F}$ is **quasi-coherent** if, locally on S, $\mathcal{F}$ is a direct limit of coherent sheaves. Concretely speaking, S admits an admissible covering $\{S_i\}_{i\in I}$ such that $\mathcal{F}_{|S_i} \cong \varinjlim \mathcal{F}_{i,\alpha}$ for a directed system $\{\mathcal{F}_{i,\alpha}\}_{\alpha\in A}$ of coherent $\mathcal{O}_{S_i}$ modules.

This definition can be carried over to other analytic settings, exchanging admissible coverings for coverings in the relevant topologies in those theories. However, based on this definition, it is not hard to find sheaves that fail to satisfy descent:

Example 1.2 (Gabber)**.** We will construct a quasi-coherent sheaf $\mathcal{F}$ on an affinoid space such that $H^1(X, \mathcal{F}) \neq 0$. Let $X = \mathrm{Sp}(K\langle t\rangle)$ be the closed unit disk over K, and let $x', x'' \in X(K)$ be two distinct K-rational points. Define $U' = X \setminus \{x'\}$, $U'' = X \setminus \{x''\}$, and let $U = U' \cap U''$. Let

$$\mathcal{F}' = \bigoplus_{n\in\mathbb{Z}} \mathcal{O}_{U'} e'_n, \qquad \mathcal{F}'' = \bigoplus_{n\in\mathbb{Z}} \mathcal{O}_{U''} e''_n$$

be free sheaves of countable rank over $\mathcal{O}_{U'}$ and $\mathcal{O}_{U''}$ respectively. Both of these sheaves are direct limits of coherent sheaves on an admissible covering of X. Hence, if we can produce a gluing $\mathcal{F}$ of the two with no nonzero global sections, it is quasi-coherent. Furthermore, we have an exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}/t\mathcal{F} \longrightarrow 0$$

where the map $\mathcal{F} \to \mathcal{F}$ is multiplication by t. The sheaf $\mathcal{F}/t\mathcal{F}$ is a nonzero skyscraper sheaf at the origin, so it has a global section. Furthermore, by construction, $\mathcal{F}$ has no global sections, so we get an injection $H^0(X, \mathcal{F}/t\mathcal{F}) \to H^1(X, \mathcal{F})$, i.e. $H^1(X, \mathcal{F}) \neq 0$.

To produce the gluing, choose $h \in \mathcal{O}_X(U)$ with essential singularities at x and x'. We identify $\mathcal{F}'_{|U}$ and $\mathcal{F}''_{|U}$ with the free sheaf $\bigoplus_{n\in\mathbb{Z}} \mathcal{O}_U e_n$ via defining the relations

$$\begin{aligned} e_{2m} &= e'_{2m|U} = e''_{2m|U} + he''_{2m+1|U} \\ e_{2m+1} &= e''_{2m+1|U} = e'_{2m+1|U} + he'_{2m+2|U}. \end{aligned}$$

Let $f \in \mathcal{F}(X)$ be a global section. For V an open affinoid subset, write $f_{|V} = \sum_n f_{n,V} e_{n|V}$ for $f_{n,V} \in \mathcal{O}_X(V)$ which vanish for all but finitely many n. Since U is a connected and normal set, and $\mathcal{F}_{|U}$ is a direct sum of copies of $\mathcal{O}_U$, the restriction map $\mathcal{F}(U) \to \mathcal{F}(V)$ is injective. This implies that $f = \sum_n f_n e_n$ for $f_n \in \mathcal{O}_X(U)$ vanishing for all but finitely many n.

By definition of $\mathcal{F}$, the restriction map $\mathcal{F}(X) \to \mathcal{F}(U)$ is again injective, and its image consists of those finite sums $f = \sum_n f_n e_n$ such that:

1. f_n is analytic at x' for even n, and at x'' for odd n.
2. $f_n + hf_{n-1}$ is analytic at x' for odd n, and x'' for even n.

However, since $f = \sum_n f_n e_n$ is a finite sum, there exists a maximum n_0 such that $f_{n_0} \neq 0$. This implies that f_{n_0} and hf_{n_0} are both analytic at a common point amongst x', x''. Thus, $h = \frac{hf_{n_0}}{f_{n_0}}$ is meromorphic at x' or x'', which is a contradiction since we assumed h to have essential singularities.

In abstraction, this is really a statement about the failure of local finite presentation to see essential singularities, a purely analytic phenomenon present over non-noetherian rings. It is a situation one has to face in any theory of analytic geometry, meaning that similar constructions go through elsewhere. In fact, in the complex analytic case, the above example can be rephrased verbatim.

1.2 What We Do

With these problems in mind, the goals of a modern theory of analytic geometry should be:

1. To be general enough to reproduce preexisting theories.

2. To produce a theory of quasi-coherent sheaves satisfying descent, along with a six-functor formalism.

Currently, there are two distinct methods for achieving this goal. The first is via the theory of condensed mathematics developed by Clausen and Scholze [Sch26]. The second is the approach we will be using. However, the two approaches do the same thing motivationally; they just approach it with respect to different functional analytic categories. In short, the approach is a vast generalization of the functor of points approach to algebraic geometry and a categorical recasting of descent, which we review now. Recall that for any scheme S, one can consider its functor of points

$$h^S \colon \mathrm{AffSch}^{op} \to \mathrm{Sets}, \qquad \mathrm{Spec}(R) \mapsto \mathrm{Hom}_{\mathrm{Sch}}(\mathrm{Spec}(R), S).$$

Usually, we call $\mathrm{Hom}_{\mathrm{Sch}}(\mathrm{Spec}(R), S)$ the R-valued points of S and denote it $S(R)$. It is a classical theorem that h^S is a sheaf in the topologies on schemes relevant to our discussion (Zariski, étale, fppf, fpqc). This allows us to recast schemes (and similarly, stacks) as the subcategory of $\mathrm{Shv}(\mathrm{AffSch}, \tau)$ given by locally representable sheaves. This was further developed by Deligne [Del07], building a notion of algebraic geometry relative to an arbitrary symmetric monoidal category.

Although this perspective is less natural/geometrically oriented than the usual definition of a scheme (that being of a locally ringed space with prescribed affine charts), it allows for a clear recasting of many fundamental concepts in algebraic geometry. The best example is of quasi-coherent sheaves, which we can reformulate in the following way:

Definition 1.3. Given an affine scheme $X = \mathrm{Spec}(R)$, we define the category of **quasi-coherent sheaves** on X to be $\mathrm{QCoh}(X) = \mathrm{Mod}_R$.

Theorem 1.4 (Gluing for Quasi-Coherent Sheaves)**.** QCoh *is a sheaf on* AffSch *in any of the major topologies. Hence, by Kan extension, it glues to a functor*

$$\mathrm{QCoh} \colon \mathrm{Sch} \to \mathrm{Cat}$$

which is a sheaf in the Zariski, étale, fppf, and fpqc topologies on Sch *induced by* AffSch*.*

This result fundamentally relies on **faithfully flat descent**:

Definition 1.5. Let $f \colon X \to Y$ be a morphism of schemes. A **descent datum** is a pair $(\mathcal{F}, \phi)$ of a quasi-coherent sheaf $\mathcal{F} \in \mathrm{QCoh}(X)$ and an isomorphism $\phi \colon p_1^*\mathcal{F} \xrightarrow{\sim} p_2^*\mathcal{F}$ on $X \times_Y X$, satisfying the **cocycle condition** on $X \times_Y X \times_Y X$, i.e.

$$p_{02}^*\phi = p_{01}^*\phi \circ p_{12}^*\phi$$

where p_{ij} is the projection $X \times_Y X \times_Y X$ to the product of ith and jth factor. We call the category of descent data $\mathrm{Desc}(X/Y)$.

Theorem 1.6 (Faithfully Flat Descent [BLR12])**.** *Let $f \colon X \to Y$ be a faithfully flat morphism of schemes. Then, there is an equivalence of categories* $\mathrm{QCoh}(Y) \simeq \mathrm{Desc}(X/Y)$*.*

This boils down to showing the equivalence on the level of affines, where it becomes the following statement:

Lemma 1.7 (Faithfully Flat Descent for Affines)**.** *Let $A \to B$ be a faithfully flat morphism of rings. Then, A can be realized as the limit of the diagram of A-modules*

$$A \to B \rightrightarrows B \otimes_A B$$

where the top arrow is given by $b \mapsto b \otimes 1$, and the bottom arrow is given by $b \mapsto 1 \otimes b$.

This precisely picks out why sheaves fail to satisfy descent in analytic geometry: the topologies aren't flat, so this theorem fails to remain true!

So, how can we generalize descent to a context that includes coordinate rings of analytic varieties? The answer is to reexamine what faithfully flat descent *really* is. Although faithfully flat descent can be proven purely algebraically, it is best thought of as a categorical consequence of the properties of base change for faithfully flat morphisms. Indeed, if $A \to B$ is a faithfully flat morphism of rings, then it is immediate by the Barr-Beck theorem that the adjunction between extension of scalars $\mathrm{Mod}_A \to \mathrm{Mod}_B$ and restriction of scalars $\mathrm{Mod}_B \to \mathrm{Mod}_A$ is comonadic. This allows us to invoke the following crucial category theory result, which becomes Lemma 1.7 when applied to extension and restriction of scalars.

Theorem 1.8. *Let T be a comonad on a category $\mathcal{C}$. Then, for all objects $A \in \mathcal{C}$, A can be written as the limit of the diagram*

$$A \to T(A) \rightrightarrows T(T(A)).$$

So, in the analytic setting, the goal should be to work in a setting that allows for a looser notion of descent. In other words, we should be using homotopy theory and higher algebra. Indeed, in this setting, one has access to the powerful Barr–Beck–Lurie theorem:

Theorem 1.9 (Barr–Beck–Lurie, [Lur09])**.** *Let $F \dashv G$ be an adjoint pair of functors, with $F\colon \mathcal{D} \to \mathcal{C}$ and $G\colon \mathcal{C} \to \mathcal{D}$. Then, the adjunction is comonadic if and only if:*

1. *F is conservative, i.e. it reflects isomorphisms.*
2. *$\mathcal{D}$ admits and F preserves totalizations of F-split cosimplicial objects, i.e. for every F-cosplit cosimplicial object $Y^\bullet\colon \Delta \to \mathcal{D}$, the totalization $\mathrm{Tot}(Y^\bullet) = \lim_\Delta Y^\bullet$ exists in $\mathcal{D}$, and the canonical comparison map*

$$F\left(\lim_{\Delta} Y^\bullet\right) \longrightarrow \lim_{\Delta}\left(F \circ Y^\bullet\right)$$

is an equivalence in $\mathcal{C}$.

We can then recover a homotopical upgrade of Theorem 1.8:

Theorem 1.10. *Let $F \dashv G$ be a comonadic adjunction of ∞-categories, with $F\colon \mathcal{D} \to \mathcal{C}$, $G\colon \mathcal{C} \to \mathcal{D}$, and resulting comonad $T = FG$ on $\mathcal{C}$. Then, for all objects $A \in \mathcal{C}$, A can be written as the limit of the diagram*

$$A \simeq \lim_{[m]\in\Delta} T^{m+1}(A).$$

With this in mind, our mantra is:

Find a theory of higher algebra for rings of analytic origin that allows us to use homotopical methods to show descent for particular classes of morphisms. Then, form a theory of relative derived analytic geometry via putting a Grothendieck topology on the opposite category corresponding to these morphisms and passing to sheaves of anima. Such a theory will naturally lead to a good theory of quasicoherent sheaves satisfying descent, along with a suitable six-functor formalism.

This perspective already exists with derived algebraic geometry (see [TV05] [TV08] in particular for a relative approach similar to ours), where faithfully flat descent is shown for commutative algebras over the stable ∞-category $D(R)$ for a ring R. It is then bootstrapped to more abstract settings through Mathew's general notion of **descendability**, which shows that the pullback functor is comonadic for much more general classes of morphisms. In the purely analytic setting, though most categories of algebraic objects with analytic structure are not abelian, some are close enough to the point that they still yield derived categories as strong as $D(R)$ for a ring R (i.e stable presentable symmetric monoidal ∞-categories). In particular, our analytic theory will be based on *quasi-abelian* categories of certain general functional analytic objects called **complete bornological spaces**, which are simply nice directed systems of Banach spaces. As observed by Ben-Bassat–Kremnizer [BK17] and Ben-Bassat–Kelly–Kremnizer [BKK24], the setting of complete bornological spaces is general enough to recover many of the classes of rings one considers in analytic geometry, thus allowing our hypothetical relative theory to inherit the classical objects of analytic geometry.

Remark 1.11. We note that this is where the difference between the bornological and condensed approaches lies. On one hand, the condensed approach makes the underlying categories fully abelian, thus making the derived stable ∞-category relative to which geometry is done much easier to construct. On the other hand, the bornological approach shortcuts the pain point of fully fixing the deficiencies on the 1-categorical level, in exchange for the construction of the stable ∞-category relative to which geometry is done being more delicate. No matter what approach is taken, care must be exercised at some point in the relevant constructions; each approach simply redistributes where it is.

The goal of this document is to serve as an introduction to the bornological approach to derived analytic geometry, with applications to the complex analytic setting later on as a proof of concept. Sections 2-5 are devoted to laying the groundwork for the remainder of the project, covering the foundations of derived algebraic geometry, relative geometry, six functor formalisms, and the finer details of bornological spaces and their homological theory. The

tools developed within this document can realize analytic geometry in settings far beyond the ones explored here, such as rigid geometry [Soo25; Soo26] and analytic geometry over the integers [BKK24]. Sections 6 onward are the meat of the research, focused on the complex analytic setting. Some exposition is laced throughout to help give motivation for certain constructions.

Our main results, informally, are the following:

Theorem 1.12 (Derived Complex Analytic and Dagger Spaces, Section 6)**.** *There exist* ∞*-categories of derived complex analytic spaces and derived analytic spaces* $\mathrm{dAn}_{\mathbb{C}}$ *and* $\mathrm{dAn}_{\mathbb{C}}^{\dagger}$*. Furthermore, there is a natural inclusion* $i_! \colon \mathrm{dAn}_{\mathbb{C}} \hookrightarrow \mathrm{dAn}_{\mathbb{C}}^{\dagger}$ *gotten as a restriction of a geometric morphism of* ∞*-topoi. Each category comes with a presentable six-functor formalism* QCoh*.*

Theorem 1.13 (Analytification and Quasi-Coherent Derived GAGA, Section 7)**.** *There exists a complex analytification functor* $(-)^{an} \colon \mathrm{dSch}_{\mathbb{C}}^{\mathrm{lfp}} \to \mathrm{dAn}_{\mathbb{C}}$ *and dagger analytification functor* $(-)^{an,\dagger} \colon \mathrm{dSch}_{\mathbb{C}}^{\mathrm{lfp}} \to \mathrm{dAn}_{\mathbb{C}}^{\dagger}$*, where* $\mathrm{dSch}_{\mathbb{C}}^{\mathrm{lfp}}$ *is the category of derived* $\mathbb{C}$*-schemes locally of finite presentation, again gotten as a restriction of a geometric morphism of* ∞*-topoi. Furthermore, there is an equivalence* $i_!(Y^{an}) \simeq Y^{an,\dagger}$ *for all locally finitely presented derived* $\mathbb{C}$*-schemes* Y*. This yields equivalences*

$$\mathrm{QCoh}(X^{an,\dagger}) \simeq \mathrm{QCoh}(X^{an}) \simeq \mathrm{QCoh}(X)$$

for proper derived $\mathbb{C}$*-schemes* X*.*

Theorem 1.14 (Tannaka Duality for Derived Dagger Spaces, Section 7)**.** *The functor* QCoh *is fully faithful on* $\mathrm{dAn}_{\mathbb{C}}^{\dagger}$*.*

Theorem 1.15 (Algebraization of Analytic Maps, Section 7)**.** *For proper derived schemes* $X, Y \in \mathrm{dSch}$*, there is an equivalence*

$$\mathrm{Map}_{\mathrm{dAn}_{\mathbb{C}}^{\dagger}}(X^{an,\dagger}, Y^{an,\dagger}) \simeq \mathrm{Map}_{\mathrm{dAn}_{\mathbb{C}}}(X^{an}, Y^{an}) \simeq \mathrm{Map}_{\mathrm{dSch}_{\mathbb{C}}}(X, Y).$$

1.3 Notation and Conventions

Throughout, we will fix the following notation, vocabulary, and conventions:

1. All rings are commutative rings with 1.
2. The ∞-category of anima is denoted Anima. Note that these have many other names in the literature (spaces, ∞-groupoids, Kan complexes), we use anima to line up with the most recent nomenclature.
3. The stable ∞-category of spectra is denoted Spt.
4. The category $\mathrm{CommAlg}_R$ will denote the category of usual commutative R-algebras.
5. The category AniAlg_R will denote the category of animated R-algebras.

6. The category $\mathcal{C}\mathrm{Alg}(\mathcal{C})$ will denote the category of $\mathbb{E}_\infty$ algebras over a symmetric monoidal ∞-category $\mathcal{C}$, see 2.10.

7. The category Pr^L will denote the category of presentable ∞-categories with left adjoints. Pr^R will denote the category of presentable ∞-categories with right adjoints.

8. For a 1-category $\mathcal{C}$, $N(\mathcal{C})$ will denote its nerve, considered as an ∞-category.

9. For a 1-category $\mathcal{C}$, $\mathrm{sInd}(\mathcal{C})$ will denote the sifted cocompletion of $\mathcal{C}$.

10. For a 1-category $\mathcal{C}$, $\mathcal{C}_\Delta$ will denote the simplicial objects in $\mathcal{C}$.

11. For an ∞-category $\mathcal{C}$, $\mathcal{P}_\Sigma(\mathcal{C})$ will denote the sifted cocompletion of $\mathcal{C}$. Equivalently, this is the full subcategory of $\mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Anima})$ spanned by functors which preserve finite products (see Proposition 5.5.8.10 in [Lur09]).

12. In the presence of an ∞-category $\mathcal{C}$ with a Grothendieck topology τ, we may use (pre)stack as a shorthand for a (pre)sheaf of anima on $(\mathcal{C}, \tau)$.

1.4 Remarks and Acknowledgements

Thank you to Arun Soor for working with me on this project, and for trusting my ability to handle such deep and technical mathematics. I have grown immensely as a mathematician because of it. Much of the latter half of this document is an adaptation of the techniques in [Soo25; Soo26] to the complex setting. Much ends up changing because of the sheer difference between the complex, dagger, and rigid settings, but an immense amount of intellectual debt is still owed!

This document is the product of the Summer 2026 REU at the University of Michigan. Thank you so much to the Department of Mathematics at UMich and to the REU director, Alex Wright, for accepting me and for providing a collaborative research environment. Thank you in particular to Matthias Jonsson, whose NSF grant DMS-2452797, *Non-Archimedean Methods in Complex Analysis and Geometry*, funded me. Thank you as well to the Jack Byrne Scholars Program in Math and Society at Dartmouth for further funding.

Finally, this manuscript is an evolving set of research notes. Comments and corrections are very welcome.

Statement on AI and Technology Usage: I engaged with Claude Opus 4.8 and Fable 5 in the very early stages of the project to help with conceptualizing higher category theory and higher algebra, as well as initially outlining the correct results to state in Section 2.3 (I was not comfortable enough with ∞-operads at the time to fully navigate the relevant sections of Higher Algebra myself). Later on, I also used them to help search the literature for correct citations I needed (in particular Higher Topos Theory and Higher Algebra, as well as various papers of Bambozzi, Ben-Bassat, Kelly, and Kremnizer), and to check for typos/notational inconsistencies in the editing process. Beyond that, all of the organization of and writing in the expository sections are my own, and I claim full responsibility for that. All of the new ideas and writing in the original research sections are either my own or were suggested by Arun Soor. Anyone interested is welcome to email me for further details!

2 DAG Review

We begin with a review of derived algebraic geometry as motivation/setup for future analytic contexts. Our goal for this section will be threefold:

1. To define the ∞-category of animated algebras AniAlg_R over a ring R, define the Zariski site τ_{zar} on AniAlg_R, and define derived schemes over R. What we call animated algebras are often called simplicial commutative rings in the classical setting.

2. To define $\mathbb{E}_\infty$-algebras over a symmetric monoidal stable ∞-category, and show the equivalence between connected objects in the usual t-structure on $D(R)$ and animated algebras over R when R contains $\mathbb{Q}$.

3. To build a notion of modules over commutative algebras in an arbitrary presentable symmetric monoidal ∞-category, and to use this to build a theory of quasicoherent sheaves for derived schemes.

These three items form the backbone of derived algebraic geometry.

2.1 Animation

The goal of animation is to associate a "non-abelian derived ∞-category" to a suitably behaved (presentable, compact projectively generated) category $\mathcal{C}$. The terminology was first introduced by Clausen and Scholze, though the concept at its core – freely adjoining sifted colimits to a category – heavily predates the terminology.

We begin by recalling the following definition:

Definition 2.1. Let $\mathcal{C}$ be a cocomplete category and let S be a collection of objects of $\mathcal{C}$. We say that S is a set of **compact projective generators for** $\mathcal{C}$ if the following conditions are satisfied:

1. Each element of S is a compact projective object of $\mathcal{C}$.

2. The full subcategory of $\mathcal{C}$ spanned by the elements of S is essentially small.

3. The set S generates $\mathcal{C}$ under small colimits.

We note that the same definition goes through in the ∞-categorical context.

Part of the motivation for studying animation comes from the following powerful result, showing that we can recover a category by sifted cocompletion of compact projective generators:

Proposition 2.2 ([Lur09], Proposition 5.5.8.25)**.** *Let $\mathcal{C}$ be a cocomplete ∞-category and let S be a set of compact projective generators for $\mathcal{C}$. Let $\mathcal{C}^0 \subseteq \mathcal{C}$ be the full subcategory spanned by finite coproducts of the objects S, let $\mathcal{D} \subseteq \mathcal{C}^0$ be a minimal model for $\mathcal{C}^0$, and let $F\colon \mathcal{P}_\Sigma(\mathcal{D}) \to \mathcal{C}$ be the left Kan extension of the inclusion $\mathcal{D} \subseteq \mathcal{C}$. Then, F is an equivalence of ∞-categories. In particular, $\mathcal{C}$ is a compactly generated presentable ∞-category.*

A similar result goes through in the 1-categorical context, see [ČS24]. Later on, we will also need the universal property of sifted cocompletion, so we state it here:

Theorem 2.3. *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories. Then F factors through $\mathcal{P}_\Sigma(\mathcal{C})$ via a sifted colimit-preserving extension if and only if $\mathcal{D}$ has sifted colimits and F preserves finite coproducts.*

We define animation as the following:

Definition 2.4. Let $\mathcal{C}$ be a category. We define the **animation of** $\mathcal{C}$, denoted $\mathrm{Ani}(\mathcal{C})$, to be the ∞-category $\mathcal{P}_\Sigma(N(\mathcal{C}))$.

Though we can define animation for arbitrary categories, we care about it with respect to presentable compact projectively generated categories. This is due to the following results, showing that animation can be connected to simplicial objects:

Proposition 2.5. *Let $\mathcal{C}$ be a presentable category generated under sifted colimits by a small full subcategory of compact projective generators $\mathcal{C}^{cp}$. Then, there is an equivalence $\mathrm{Ani}(\mathcal{C}^{cp}) \cong \mathcal{C}_\Delta[W^{-1}]$, where the right hand side is the Dwyer-Kan localization of $\mathcal{C}$-valued simplicial objects in the usual model structure.*

Proof. By [Lur09, Corollary 5.5.9.3], if $\mathbb{A}$ denotes the subcategory of functors $\mathcal{C}^{cp,op} \to \mathrm{Sets}_\Delta$ which preserve finite products, then there is a simplicial model structure on $\mathbb{A}$ (with fibrant-cofibrant objects $\mathbb{A}^\circ$) yielding an equivalence $N(\mathbb{A}^\circ) \simeq \mathrm{Ani}(\mathcal{C}^{cp})$. The result then follows by showing $N(\mathbb{A}^\circ)[H^{-1}] \simeq N(\mathbb{A})[W^{-1}]$, where H denotes path homotopy equivalences [Cis19]. □

Corollary 2.6. *We can identify* $\mathrm{Anima} \cong \mathrm{Ani}(\mathrm{FinSet})$.

Proof. Sets is compactly projectively generated by finite sets. Hence, by Proposition 2.5, we have an equivalence $\mathrm{Ani}(\mathrm{FinSet}) \cong \mathrm{Sets}_\Delta[W^{-1}]$. However, the Dwyer-Kan localization of simplicial sets in the Quillen model structure is precisely the category of anima. □

That animation generalizes derived categories to a nonabelian setting rests on the following corollary in particular. Recall that for an abelian category $\mathcal{A}$, we can associate a stable ∞-category $D(\mathcal{A})$ with the following properties:

1. The homotopy category of $D(\mathcal{A})$ is the usual triangulated derived category $D(\mathcal{A})$.
2. There is an embedding $\mathcal{A} \hookrightarrow D(\mathcal{A})$, roughly corresponding to thinking of objects of $\mathcal{A}$ as 1-term complexes up to quasi-isomorphism.
3. $D(\mathcal{A})$ admits derived functors; any left exact functor $F\colon \mathcal{A} \to \mathcal{B}$ admits a right Kan extension $\mathbf{R}F\colon D(\mathcal{A}) \to D(\mathcal{B})$, and any right exact functor $G\colon \mathcal{A} \to \mathcal{B}$ admits a left Kan extension $\mathbf{L}G\colon D(\mathcal{A}) \to D(\mathcal{B})$.

4. In particular, in the case that $\mathcal{A}$ is a closed symmetric monoidal category with unit $\mathbf{1}$, D inherits a structure as a closed symmetric monoidal category via the derived tensor product $- \otimes^{\mathbb{L}} -$ (see [NS18, Appendix A], or [Hin13] for a discussion).

5. $D(\mathcal{A})$ admits a right-complete t-structure $(D(\mathcal{A})_{\leq 0}, D(\mathcal{A})_{\geq 0}, D(A)^{\heartsuit})$ corresponding to the usual t-structure on $D(\mathcal{A})$ given by complexes concentrated in positive and negative degrees. We may also write $D(\mathcal{A})_{\geq 0}$ as $D(\mathcal{A})^{cn}$, short for **connective objects.**

In the particular case of a ring R, we write $D(R)$ for $D(\mathrm{Mod}_R)$.

Corollary 2.7. *For a ring R, we can identify $D(R)^{cn} \cong \mathrm{Ani}(\mathrm{Mod}_R^{f.free})$, where $\mathrm{Mod}_R^{f.free}$ is the category of finite free modules.*

Proof. Recall that for a ring R, there is a monadic adjunction $R\langle - \rangle \dashv F$ between Mod_R and Sets given by free R-modules and the forgetful functor. Since $R\langle - \rangle$ is a left adjoint and hence preserves compact projective objects, the compact projective objects of Mod_R are given by $R\langle P \rangle$ for the compact projective objects of Sets. These are precisely $R\langle S \rangle$ for finite sets S, i.e. finite free R-modules. Hence, by Proposition 2.5, we get an identification $\mathrm{Ani}(\mathrm{Mod}_R^{f.free}) \simeq \mathrm{Mod}_{R,\Delta}[W^{-1}]$. However, by the Dold-Kan correspondence, $\mathrm{Mod}_{R,\Delta}[W^{-1}]$ corresponds to the localization $\mathrm{Ch}_{\geq 0}(R)[W^{-1}]$ at quasi-isomorphisms. One way to define $D(R)^{cn}$ is as this localization, so we are done. □

Proposition 2.2 and Corollary 2.7 give us the intuition that to form a "nonabelian derived category" of R-algebras for a ring R, we should animate the compact projective generators of Alg_R. By the exact same argument as for modules in the proof of Corollary 2.7, the compact projective generators of Alg_R are precisely the free algebras $R[S]$ for finite sets S, i.e. the polynomial rings $R[x_1, \cdots, x_n]$ for $n \in \mathbb{N}$. We formally define this as the following:

Definition 2.8. Let Poly_R be the category of polynomial algebras over R, i.e. the full subcategory of Alg_R generated by finite type polynomial algebras $R[x_1, \cdots, x_n]$ for $n \in \mathbb{N}$. We define the category of **animated algebras over** R, denoted AniAlg_R, to be the category $\mathrm{Ani}(\mathrm{Poly}_R)$.

By Proposition 2.5, we get a simplicial presentation for AniAlg_R:

Proposition 2.9. *We can identify $\mathrm{AniAlg}_R \cong \mathrm{CommAlg}_{R,\Delta}[W^{-1}] \cong N(\mathrm{CommAlg}^{\circ}_{R,\Delta})$, where $\circ$ means the fibrant-cofibrant objects.*

2.2 $\mathbb{E}_\infty$-Algebras

We've defined animated algebras. However, in order to fully form derived algebraic geometry, we need a good notion of homotopy groups of animated algebras. We will accomplish this with $\mathbb{E}_\infty$ algebras, homotopy coherent commutative algebras, in $D(R)$. Before stating the definition, recall that a symmetric monoidal ∞-category $(\mathcal{C}, \otimes, \mathbf{1})$ can be given as a cocartesian fibration of ∞-categories $p\colon \mathcal{C}^{\otimes} \to N(\mathrm{FinSet}_*)$, where FinSet_* is the category of finite sets with basepoint.

Definition 2.10. Given a symmetric monoidal ∞-category $p\colon \mathcal{C}^{\otimes} \to N(\mathrm{FinSet}_*)$, an **$\mathbb{E}_\infty$-algebra** in $\mathcal{C}$ is a section of p sending inert morphisms to cocartesian morphisms. We denote the category of $\mathbb{E}_\infty$-algebra objects in $\mathcal{C}$ as $\mathcal{C}\mathrm{Alg}(\mathcal{C})$.

By evaluating at the set $\langle 1 \rangle$, we obtain a continuous functor

$$F\colon \mathcal{C}\mathrm{Alg}(\mathcal{C}) \to \mathcal{C}$$

recovering the underlying object of $\mathcal{C}$ corresponding to an $\mathbb{E}_\infty$-algebra A. This allows us to give the following definition:

Definition 2.11. If $\mathcal{C}$ is a stable ∞-category with a choice of t-structure $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0}, \mathcal{C}^{\heartsuit})$, then we say that an $\mathbb{E}_\infty$-algebra A in $\mathcal{C}$ is **connective** if the underlying object of $\mathcal{C}$, $F(A)$, is. We denote the category of connective $\mathbb{E}_\infty$-algebras by $\mathcal{C}\mathrm{Alg}(\mathcal{C})^{cn}$.

Furthermore, F has a left adjoint, which we call the **free $\mathbb{E}_\infty$ algebra** and denote $\mathrm{Sym}^\bullet(A)$. In the case that $\mathcal{C} = D(R)$, for $M \in \mathcal{C}$, it is explicitly given by the connective $\mathbb{E}_\infty$-algebra

$$\mathrm{Sym}^\bullet(M) = \bigoplus_n \mathrm{Sym}^n(M) := \bigoplus_n (M^{\otimes^{\mathbb{L}} n})_{hS_n},$$

where the subscript denotes the homotopy co-invariants of S_n acting on $M^{\otimes n}$ through the braiding operation.

We are now ready to prove the following fundamental theorem:

Theorem 2.12. *If $R \supseteq \mathbb{Q}$, then there is an equivalence of ∞-categories $\Theta\colon \mathrm{AniAlg}_R \cong \mathcal{C}\mathrm{Alg}(D(R))^{cn}$.*

To prove the theorem, we will need a few standard facts. The first is the following result from group homology:

Proposition 2.13. *Suppose that $R \supseteq \mathbb{Q}$, let G be a finite group, and let M be an $R\langle G \rangle$-module. Then, $H_i(G, M) = 0$ for $i > 0$.*

Proof. We want to show that the functor $M \to M_G \cong M \otimes_{R\langle G \rangle} R$ is exact. To do this, we observe that since R contains $\mathbb{Q}$ and thus has arbitrary denominators, the augmentation map

$$\begin{aligned} R\langle G \rangle &\longrightarrow R \\ \sum_{i=1}^n r_i g_i &\longmapsto \sum_{i=1}^n r_i \end{aligned}$$

has a G-equivariant splitting

$$\begin{aligned} R &\longrightarrow R\langle G \rangle \\ r &\longmapsto \frac{1}{|G|} \sum_{g \in G} g(r). \end{aligned}$$

Hence, R is a direct summand of the free $R\langle G \rangle$-module $R\langle G \rangle$, so it is projective, and the result follows. $\square$

The next is the following fundamental fact about the category of $\mathbb{E}_\infty$-algebras that we do not prove:

Proposition 2.14 ([Lur17], Proposition 7.4.2.27)**.** *The ∞-category $\mathcal{C}\mathrm{Alg}(D(R))$ is generated by compact projective objects. Furthermore, a connective $\mathbb{E}_\infty$-algebra A is a compact projective object if and only if it retracts onto $R[x_1, \cdots, x_n]$.*

Finally, we need a result on monadic adjunctions of ∞-categories:

Proposition 2.15. *Let $F \dashv G$ be a monadic adjunction of ∞-categories, with $F\colon \mathcal{C} \to \mathcal{D}$ a functor preserving sifted colimits, $G\colon \mathcal{D} \to \mathcal{C}$ and monad $T := GF$. Then, $\mathcal{D} \cong \mathrm{Alg}_T(\mathcal{C})$. Furthermore, if $\mathcal{C}$ is generated by compact projective objects $\{P_i\}$, then $\mathcal{D}$ is generated by $F(P_i)$ under colimits.*

Proof. It suffices to show that any object $Y \in \mathcal{D}$ can be written as a sifted colimit of compact projective objects. I claim that we can write Y as the colimit of the simplicial object

$$B\colon \Delta^{op} \to \mathcal{C}, B[n] = FT^nG(Y)$$

Since G is part of a monadic adjunction, it is conservative and preserves colimits of G-split simplicial objects. Hence, it suffices to check this on $G(Y)$. However, $G \circ B$ is simply the bar resolution of $G(Y)$. Furthermore, since $\mathcal{C}$ is generated by compact projective objects, each $T^nG(Y)$ can be identified as a colimit of compact projective objects in $\mathcal{C}$. Hence, Y can as well after applying F, since F preserves compact projective objects. □

Proof of Theorem 2.12. Constructing the functor is fairly simple. We define a functor

$$\theta_0\colon \mathrm{Poly}_R \longrightarrow \mathcal{C}\mathrm{Alg}(D(R))$$

by sending $R[x_1, \cdots, x_n]$ to $\mathrm{Sym}^\bullet(R^{\oplus n})$, where $R^{\oplus n}$ is the object of $D(R)$ corresponding to the finite free R-module $R^{\oplus n}$. Since coproducts in $\mathcal{C}\mathrm{Alg}(R)$ are computed by derived tensor product over R and polynomial algebras are flat, this means that the derived tensor product does not affect the homotopy groups of $\theta_0(R[x_1, \cdots, x_n]) \otimes^{\mathbb{L}} \theta_0(R[x_1, \cdots, x_m])$. This yields an identification

$$\theta_0(R[x_1, \cdots, x_n]) \otimes^{\mathbb{L}} \theta_0(R[x_1, \cdots, x_m]) \cong \theta_0(R[x_1, \cdots, x_{n+m}]),$$

i.e. θ_0 preserves finite coproducts. Furthermore, $D(R)$ has sifted colimits. Hence, by universal property of sifted cocompletion, we get an induced functor $\Theta\colon \mathrm{AniAlg}_R \to \mathcal{C}\mathrm{Alg}(D(R))$ which preserves small colimits.

Since $H_i(S_n, M^{\otimes n}) = 0$ for all n and i by Proposition 2.13, the higher homotopy groups of $\mathrm{Sym}^\bullet_R(M)$ vanish. Hence, $\mathrm{Sym}^\bullet_R(M)$ is discrete. Furthermore, we can identify $\pi_0(\mathrm{Sym}^\bullet_R(R^{\oplus m}))$ with the classical symmetric algebra $\mathrm{Sym}(R^{\oplus m}) \cong R[x_1, \cdots, x_m]$. Hence, by Proposition 2.14, the essential image of θ_0 consists of compact projective objects of $\mathcal{C}\mathrm{Alg}(D(R))$, which generate $\mathcal{C}\mathrm{Alg}(D(R))$ under colimits by Proposition 2.15. So, Θ is an equivalence of categories by Proposition 2.2. □

Now that we have an equivalence between animated R-algebras and $\mathbb{E}_\infty$-algebras over R, we can define homotopy groups of animated rings in terms of the corresponding $\mathbb{E}_\infty$-algebra. Put the standard right complete t-structure on $D(R)$. Recall that for an $\mathbb{E}_\infty$-algebra A in $D(R)$, the homotopy groups $\pi_n(A)$ are the homotopy groups of the underlying complex, given by $\pi_n(A) = H_n(A)$. This is well-defined since it is a composition of functors of ∞-categories:

Definition 2.16. Let $A \in \mathrm{AniAlg}_R$ be an animated R-algebra. We define the n**th homotopy group**, $\pi_n(A)$, to be $\pi_n(\Theta(A))$.

Armed with a good theory of homotopy coherences for animated algebras, we can now move toward constructing the affines of our theory of derived geometry and the Zariski site on them:

Definition 2.17. The category of **derived affine schemes** over R, denoted dAff_R, is defined to be AniAlg_R^{op}. We write $\mathrm{Spec}(A)$ for the object of dAff corresponding to an animated algebra A.

To define the Zariski topology on dAff_R, we first note that the notion of π_0 on dAff_R is well-defined. Indeed, we have the usual functor $\pi_0 \colon \mathrm{CommAlg}_{R,\Delta} \to \mathrm{CommAlg}_R$, which sends weak equivalences to isomorphisms by definition of weak equivalence on simplicial commutative R-algebras. Hence, we get a unique induced functor $\pi_0 \colon \mathrm{AniAlg}_R \to \mathrm{CommAlg}_R$ by the universal property of Dwyer-Kan localization. Hence, by passing to the opposite categories and denoting $\mathrm{CommAlg}_R^{op}$ by affine R-schemes Aff_R, we get an induced functor $\pi_0^{op} \colon \mathrm{dAff}_R \to \mathrm{Aff}_R$. We then define the **Zariski topology** on dAff_R, τ_{zar}, to be the pullback of the Zariski topology on Aff_R by π_0^{op}. Concretely speaking, for $\mathrm{Spec}(A) \in \mathrm{dAff}_R$, a family $\{\mathrm{Spec}(A_i) \to \mathrm{Spec}(A)\}_i$ is a Zariski covering if and only if:

1. $\pi_0(A) \to \pi_0(A_i)$ is a localization of ordinary commutative R-algebras and corresponds to open immersions of affine schemes.
2. For all $i \geq 0$, $\pi_i(A_i) \cong \pi_i(A) \otimes_{\pi_0(A)} \pi_0(A_i)$.
3. The family $\{\mathrm{Spec}(\pi_0(A_i)) \to \mathrm{Spec}(\pi_0(A))\}$ covers $\mathrm{Spec}(\pi_0(A))$.

Since the Zariski topology on affine schemes is subcanonical, the Zariski topology on dAff is subcanonical.

Remark 2.18. We define a morphism $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ to be a **Zariski open immersion** if it satisfies conditions 1 and 2 of the above. Hence, a Zariski covering is a family of Zariski open immersions covering the underlying affine scheme.

Morally, a derived scheme should be a sheaf of anima on $(\mathrm{dAff}, \tau_{zar})$ that is "locally representable", similarly to how an R-scheme is locally $\mathrm{Spec}(A)$ for a commutative R-algebra A. In the classical case, for a scheme X, the open covering is a family of Zariski open immersions $\{\mathrm{Spec}(A_i) \to X\}_i$. Hence, in the derived setting, we should give an analogous definition building off Remark 2.18:

Definition 2.19. Let $U, X \in \mathrm{Shv}(\mathrm{dAff}_R, \tau_{zar})$. A morphism $j\colon U \to X$ is a **Zariski open immersion** if for every derived affine scheme $T = \mathrm{Spec}(A)$ and every morphism $T \to X$, the fiber product $U \times_X T$ is a Zariski open subfunctor of T. In other words, there is an open subset $W \subseteq \mathrm{Spec}(\pi_0(A))$ with

$$U \times_X T \simeq \mathrm{Spec}(A)_W := \mathrm{colim}_{D(f)\subseteq W}\ \mathrm{Spec}\Big(A[1/f]\Big),$$

the colimit ranging over the distinguished opens $D(f) \subseteq W$ (equivalently, over those $f \in \pi_0(A)$ with $D(f) \subseteq W$) and formed in $\mathrm{Shv}(\mathrm{dAff}_R, \tau_{zar})$.

Now that our open immersions are defined, we use the usual notion of covering in ∞-categories – effective epimorphisms – to get our notion of derived schemes:

Definition 2.20. A **derived R-scheme** is a sheaf $X \in \mathrm{Shv}(\mathrm{dAff}_R, \tau_{zar})$ such that there exists a small family of representable objects $U_i = \mathrm{Spec}(A_i)$, each with Zariski open immersions $U_i \to X$, such that the map $\coprod_i U_i \to X$ gotten by the universal property of coproduct is an effective epimorphism. A morphism of derived R-schemes is a morphism of sheaves $X \to Y$, and the resulting ∞-category is denoted dSch_R.

Our last step is to show

Proposition 2.21. *The Zariski topology τ_{zar} is subcanonical, i.e. the Yoneda embedding $\iota\colon \mathrm{dAff}_R \to \mathrm{PreSh}(\mathrm{dAff}_R)$ restricts to $\iota\colon \mathrm{dAff}_R \to \mathrm{dSch}$.*

To prove this, we will need the following definition:

Definition 2.22. A map of animated R-algebras $A \to B$ is **faithfully flat** if it is faithfully flat on π_0, and $\pi_n(B) \cong \pi_n(A) \otimes_{\pi_0(A)} \pi_0(B)$ for all $n \geq 0$.

Proof. The sheaf condition boils down to showing that for a Zariski cover of derived affine schemes $\{\mathrm{Spec}(A_i) \to \mathrm{Spec}(A)\}_i$, we can write A as the limit of the Čech nerve

$$A \simeq \lim \check{C}^\bullet\left(A \to \prod_i A_i\right).$$

However, since $A \to A_i$ is a localization on π_0, it is faithfully flat on π_0. Furthermore, the condition on the homotopy groups baked into the definition of Zariski cover ensures that $A \to A_i$ is faithfully flat. Hence, the map $A \to \prod_i A_i$ by the universal property of product is faithfully flat. So, the base change $- \otimes^{\mathbb{L}}_A \prod_i A_i$ is comonadic by Barr–Beck–Lurie (see the proof of Proposition 2.29 for further detail). The above presentation of A then holds by the existence of the bar resolution. □

2.3 Modules Over Animated Rings and Quasi-Coherent Sheaves

Now that we have a good theory of $\mathbb{E}_\infty$-algebras, we can attach an ∞-category of modules to each of them. This is in an effort toward defining quasi-coherent sheaves on derived schemes.

We make the definition at the level of an arbitrary symmetric monoidal ∞-category instead of for $D(R)$ alone, as the definition then applies to any general setting for derived algebra. This is something we will need later, as we will be working over a derived category of Ind-Banach spaces which is beyond $D(R)$ for a ring R. We black box most of the results in this section, as many of the proofs rely on material not developed in these notes.

Throughout this subsection, we let $\mathcal{C}$ denote an object of $\mathcal{C}\mathrm{Alg}(\mathrm{Pr}^L)$ – a presentable symmetric monoidal ∞-category whose tensor product preserves colimits separately in each variable. One can take $\mathcal{C} = D(R)$, which is such a category by our earlier work.

2.3.1 The Module ∞-Category

Definition 2.23 ([Lur17], Section 3.3, Section 4.5)**.** Let $A \in \mathcal{C}\mathrm{Alg}(\mathcal{C})$. The **$\infty$-category of A-modules** $\mathrm{Mod}_A(\mathcal{C})$ is the ∞-category of A-module objects of $\mathcal{C}$ in the sense of Lurie, i.e. the fiber over A of the ∞-category $\mathrm{Mod}(\mathcal{C})$ of pairs (A, M) with A an $\mathbb{E}_\infty$-algebra and M an A-module. We write

$$\theta\colon \mathrm{Mod}(\mathcal{C}) \longrightarrow \mathcal{C}\mathrm{Alg}(\mathcal{C}), \qquad (A, M) \longmapsto A,$$

for the projection, so that $\theta^{-1}(A) = \mathrm{Mod}_A(\mathcal{C})$. When $\mathcal{C} = D(R)$ we abbreviate $\mathrm{Mod}_A := \mathrm{Mod}_A(D(R))$.

The following are some important results on module categories.

Proposition 2.24. *Let $A \in \mathcal{C}\mathrm{Alg}(\mathcal{C})$. Then:*

1. *$\mathrm{Mod}_A(\mathcal{C})$ is presentable. Furthermore, if $\mathcal{C}$ is stable, then so is $\mathrm{Mod}_A(\mathcal{C})$.*
2. *The forgetful functor $U\colon \mathrm{Mod}_A(\mathcal{C}) \to \mathcal{C}$ is conservative and preserves all small limits and colimits. Furthermore, it is monadic, with left adjoint the **free module** functor $A \otimes (-)$.*
3. *If $\mathcal{C}$ has a t-structure compatible with $\otimes$ and A is connective, then $\mathrm{Mod}_A(\mathcal{C})$ inherits a t-structure making U t-exact. Hence, an A-module is connective if and only if the underlying object of $\mathcal{C}$ is.*

Theorem 2.25 ([Lur17], Section 4.5)**.** *For $A \in \mathcal{C}\mathrm{Alg}(\mathcal{C})$ the ∞-category $\mathrm{Mod}_A(\mathcal{C})$ carries a canonical symmetric monoidal structure, the **relative tensor product** $\otimes_A$, with unit A. This gives $\mathrm{Mod}_A(\mathcal{C}) \in \mathcal{C}\mathrm{Alg}(\mathrm{Pr}^L)$.*

The assignment $A \mapsto \mathrm{Mod}_A(\mathcal{C})$ is functorial in A, allowing us to define good notions of base change:

Proposition 2.26 ([Lur17], Section 3.4)**.** *The projection $\theta\colon \mathrm{Mod}(\mathcal{C}) \to \mathcal{C}\mathrm{Alg}(\mathcal{C})$ is both a Cartesian and a coCartesian fibration. For a morphism $f\colon A \to B$ of $\mathbb{E}_\infty$-algebras:*

- *the Cartesian pullback is **restriction of scalars** $f_*\colon \mathrm{Mod}_B(\mathcal{C}) \to \mathrm{Mod}_A(\mathcal{C})$, which exists unconditionally;*

- *the coCartesian pushforward is* ***extension of scalars*** $f^* = B \otimes_A (-) \colon \mathrm{Mod}_A(\mathcal{C}) \to \mathrm{Mod}_B(\mathcal{C})$*, which is symmetric monoidal and left adjoint to* f_**.*

Unstraightening θ *yields a symmetric monoidal functor*

$$\mathrm{Mod}_{(-)}(\mathcal{C}) \colon \mathcal{C}\mathrm{Alg}(\mathcal{C}) \longrightarrow \mathcal{C}\mathrm{Alg}(\mathrm{Pr}^L), \qquad A \mapsto \mathrm{Mod}_A(\mathcal{C}), \qquad (f \colon A \to B) \mapsto \big(B \otimes_A (-)\big).$$

Specialization to Animated Rings

Over a base ring $R \supseteq \mathbb{Q}$, the equivalence $\Theta \colon \mathrm{AniAlg}_R \xrightarrow{\sim} \mathcal{C}\mathrm{Alg}(D(R))^{\mathrm{cn}}$ of Theorem 2.12 lets us transport the definition of the category of modules.

Definition 2.27. For $R \supseteq \mathbb{Q}$ and $A \in \mathrm{AniAlg}_R$, the ∞-category of **A-modules** is $\mathrm{Mod}_A := \mathrm{Mod}_{\Theta(A)}(D(R))$, and we set $\pi_n(M) := \pi_n(U(M))$ for $M \in \mathrm{Mod}_A$, the homotopy groups of the underlying object of $D(R)$.

Remark 2.28. By Proposition 2.24, part 3), a module M is connective exactly when $\pi_n(M) = 0$ for $n < 0$. Furthermore, the heart of Mod_A for connective A is the abelian category of $\pi_0(A)$-modules. One can check that the connective modules defined this way agree with simplicial π_0-modules via the Dold–Kan correspondence, recovering simplicial objects!

2.3.2 Faithfully Flat Descent

We now discuss gluing properties for $\mathrm{Mod}_{(-)}$, which allow us to define quasi-coherent sheaves on arbitrary derived schemes.

Proposition 2.29. *Let* $f \colon A \to B$ *be a faithfully flat morphism in* $\mathcal{C}\mathrm{Alg}(\mathcal{C})$*. Then extension of scalars* $f^* = - \otimes_A B$ *is comonadic. Hence,*

$$\mathrm{Mod}_A(\mathcal{C}) \simeq \lim_{[n] \in \Delta} \mathrm{Mod}_{B^{\otimes_A (n+1)}}(\mathcal{C}),$$

is the limit gotten by applying $\mathrm{Mod}_{(-)}(\mathcal{C})$ *to the Čech conerve* $A \rightrightarrows B \rightrightarrows B \otimes_A B \cdots$*.*

Remark 2.30. This can be reinterpreted through Mathew's notion of a **descendable** morphism [Mat16], condition in the non-flat setting, as we will see later.

This notion immediately geometrizes, given our prebuilt formalism.

Definition 2.31. For $\mathrm{Spec}(A) \in \mathrm{dAff}_R$, the ∞-category of **quasicoherent sheaves** on $\mathrm{Spec}(A)$ is $\mathrm{QCoh}(A) := \mathrm{Mod}_A$. By Proposition 2.26, this yields a functor

$$\mathrm{QCoh} \colon \mathrm{dAff}_R^{\mathrm{op}} = \mathcal{C}\mathrm{Alg}(D(R))^{\mathrm{cn}} \longrightarrow \mathcal{C}\mathrm{Alg}(\mathrm{Pr}^L), \qquad \mathrm{Spec}(A) \mapsto \mathrm{Mod}_A.$$

Corollary 2.32. QCoh *is a sheaf for the Zariski topology* τ_{zar} *on* dAff_R*.*

Proof. Let $\{\mathrm{Spec}(A_i) \to \mathrm{Spec}(A)\}_i$ be a τ_{zar}-cover and put $B := \prod_i A_i$. As in the proof of Proposition 2.21, $A \to B$ is faithfully flat, so Proposition 2.29 gives $\mathrm{Mod}_A \simeq \lim_\Delta \mathrm{Mod}_{B^{\otimes_A \bullet+1}}$, which is the Čech-descent condition for QCoh along the cover. Hence QCoh satisfies descent. □

Because QCoh is a τ_{zar}-sheaf valued in an ∞-category with all limits, it extends from dAff_R to derived schemes in the following way:

Definition 2.33. For a derived R-scheme $X \in \mathrm{dSch}_R$, the ∞-category of **quasicoherent sheaves** on X is

$$\mathrm{QCoh}(X) := \lim_{(\mathrm{Spec}(A) \to X) \in (\mathrm{dAff}_R)_{/X}} \mathrm{Mod}_A,$$

the limit taken over the ∞-category of derived affine schemes mapping to X. Equivalently, QCoh is the right Kan extension of Definition 2.31 along $\mathrm{dAff}_R^{\mathrm{op}} \hookrightarrow \mathrm{dSch}_R^{\mathrm{op}}$.

By Corollary 2.32, this limit is given by the Čech nerve of an affine Zariski cover of X. For $X = \mathrm{Spec}(A)$, this recovers Mod_A.

Remark 2.34. This already gives us two of the operations we will want in our six functor formalism: the fiberwise tensor product $\otimes$ of Theorem 2.25 and the pullback f^* of Proposition 2.26. By Corollary 2.32, both of these descend, giving the right adjoint pushforward f_*. We will discuss the exceptional pair $(f_!, f^!)$ later.

3 Relative Geometry

We've now constructed a good theory of derived algebraic geometry by considering polynomial algebras over R within the category of animated R-algebras, categories of modules over animated R-algebras, along with sheaves on the Zariski site. In order to do more general derived geometry, we'd like to abstract the properties of this setup into a general setting. There are three steps to doing this:

1. Stripping the setting of polynomial algebras embedded within $\mathcal{C}\mathrm{Alg}(R)^{cn}$ down to its conceptual core, yielding a general definition of "derived algebraic context".
2. Defining a suitable notion of derived stacks via a Grothendieck topology on a derived algebraic context.
3. Finding a suitable notion of "local representability" for sheaves, yielding to a subcategory of derived schemes within derived stacks relative to a derived algebraic context.

The third item will be handled on a case by case basis, as the local geometry will differ slightly in various settings. However, the first two can be given in a fully abstract manner. We carry this out in the first section, and then move toward general definitions of the descendable topology and quasicoherent sheaves in the next.

3.1 Derived Algebraic Contexts and Relative Geometry Tuples

The first step is the following definition originally due to Raksit:

Definition 3.1 ([Rak20])**.** A **derived algebraic context** is a tuple $(\mathcal{C}, \mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0}, \mathcal{C}^0)$ consisting of a stable presentable symmetric monoidal ∞-category $\mathcal{C}$, a choice of right complete t-structure $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$, and a subcategory $\mathcal{C}^0 \subseteq \mathcal{C}^\heartsuit$ such that:

1. $\mathcal{C}^0$ is a symmetric monoidal subcategory of $\mathcal{C}$.
2. $\mathcal{C}^0$ is closed under finite coproducts, and its objects form a set of compact projective generators for $\mathcal{C}_{\geq 0}$.
3. $\mathcal{C}^0$ is closed under the formation of 1-categorical symmetric powers computed in $\mathcal{C}^\heartsuit$; for $X \in \mathcal{C}^0$ and $n \geq 0$, we have $\mathrm{Sym}^n_{\mathcal{C}^\heartsuit}(X) \in \mathcal{C}^0$.

We define **derived commutative algebras** relative to a derived algebraic context $(\mathcal{C}, \mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0}, \mathcal{C}^0)$ as $\mathcal{C}\mathrm{Alg}(\mathcal{C})^{cn}$, and **derived affine schemes** as $\mathrm{Aff}_{\mathcal{C}} := \mathcal{C}\mathrm{Alg}(\mathcal{C})^{cn,op}$. For a derived commutative algebra A, we denote the corresponding derived affine scheme by $\mathrm{Spec}(A)$.

Example 3.2. The tuple of $(D(R), D(R)_{\geq 0}, D(R)_{\leq 0}, \mathrm{Mod}_R^{f.free})$ forms a derived algebraic context by our work in the last section, with commutative algebra objects $\mathcal{C}\mathrm{Alg}(R)^{cn}$. In fact, it can be shown that $(D(\mathbb{Z}), D(\mathbb{Z})_{\geq 0}, D(\mathbb{Z})_{\leq 0}, \mathrm{Mod}_{\mathbb{Z}}^{f.free})$ is initial in the ∞-category of derived algebraic contexts. This is analogous to the fact that the category of Spectra is initial in the ∞-category of stable ∞-categories.

Remark 3.3.

1. It can be shown that a derived algebraic context $(\mathcal{C}, \mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0}, \mathcal{C}^0)$ is uniquely determined by $\mathcal{C}^0$. Indeed, by Proposition 2.2 and the assumption that $\mathcal{C}^0$ is a set of compact projective generators for $\mathcal{C}_{\geq 0}$, we can identify $\mathcal{C} \simeq \mathcal{P}_\Sigma(\mathcal{C}^0)$. Furthermore, by Remarks C.3.1.5 and C.1.5.9 of [Lur17], we can identify $\mathcal{C}$ with the category of $\mathcal{C}_{\geq 0}$ valued spectra, and $\mathcal{C}^\heartsuit$ with the category of product preserving functors $\mathcal{C}^{0,op} \to \mathrm{Sets}$.
2. We note that in most works making use of derived algebraic contexts, derived algebras are not defined as $\mathbb{E}_\infty$-algebra contexts. They are instead defined as algebras over an $\mathbb{L}\mathrm{Sym}$-monad. This is to account for the case of positive characteristic, where the free $\mathbb{E}_\infty$-algebra is no longer discrete. However, we do not need this extra machinery since we primarily work over $\mathbb{C}$-algebras later on, where the free $\mathbb{E}_\infty$-algebra is discrete by our proof of Theorem 2.12. Hence, we use the simpler definition.

The second item is comprised of the definitions below, due to Ben-Bassat–Kelly–Kremnizer [BKK24] and Kelly–Kremnizer–Mukherjee [KKM22]. Given a derived algebraic context $(\mathcal{C}, \mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0}, \mathcal{C}^0)$, the idea is to form a category of stacks relative to a Grothendieck topology τ on $\mathcal{C}\mathrm{Alg}(\mathcal{C})$, further keeping track of a class of formally "smooth" morphisms. Additionally, in the case of derived analytic geometry, we may not want to consider just derived affine schemes, but instead derived affine schemes corresponding to certain nice subcategories of $\mathcal{C}\mathrm{Alg}(\mathcal{C})^{cn}$. For example, in derived complex analytic geometry, we may not want to consider all algebras over $D(\mathrm{Ind}(\mathrm{Ban}_{\mathbb{C}}))^{cn}$, but only those algebras whose π_0 are Stein algebras with appropriate finiteness conditions on higher homotopy groups, or those whose π_0 are Dagger affinoid algebras with appropriate finiteness conditions on higher homotopy groups:

Definition 3.4. A **relative $(\infty,1)$-geometry tuple** is a tuple $(\mathrm{Aff}_{\mathcal{C}},\tau,\mathbf{P},\mathbf{A})$ consisting of the derived affine scheme objects given by a derived algebraic context $(\mathcal{C},\mathcal{C}_{\geq 0},\mathcal{C}_{\leq 0},\mathcal{C}^0)$, a Grothendieck topology τ on $\mathcal{C}\mathrm{Alg}(\mathcal{C})$, a class of "smooth" morphisms $\mathbf{P}\subseteq \mathrm{Mor}(\mathrm{Aff}_{\mathcal{C}})$, and a subcategory $\mathbf{A}\subseteq \mathrm{Aff}_{\mathcal{C}}$ of "affine objects", such that:

1. If $\{\mathrm{Spec}(B_i)\to \mathrm{Spec}(A)\}_i$ is a covering in τ, then each morphism $\mathrm{Spec}(B_i)\to\mathrm{Spec}(A)\in \mathbb{P}$.
2. If $f\colon \mathrm{Spec}(B)\to\mathrm{Spec}(A)$ is a morphism in $\mathrm{Aff}_{\mathcal{C}}$ and $\{\mathrm{Spec}(B_i)\to\mathrm{Spec}(B)\}_i$ is a covering in τ such that the compositions $\mathrm{Spec}(B_i)\to\mathrm{Spec}(A)$ are in $\mathbf{P}$ for all i, then $f\in\mathbf{P}$.
3. If $f\colon \mathrm{Spec}(C)\to\mathrm{Spec}(A)\in\mathbf{P}\cap\mathbf{A}$ and $\mathrm{Spec}(B)\to\mathrm{Spec}(A)$ is a morphism in $\mathrm{Aff}_{\mathcal{C}}$ with $\mathrm{Spec}(B)\in\mathbf{A}$, then the pullback $\mathrm{Spec}(B)\times_{\mathrm{Spec}(A)}\mathrm{Spec}(C)$ is in $\mathbf{A}$.
4. The restriction $\tau_{|\mathbf{A}}$ is subcanonical; in other words, for all $X\in\mathbf{A}$, the functor
$$\mathrm{Map}(-,X)\colon \mathbf{A}^{op}\to \mathrm{Anima}$$
is a sheaf in $\tau_{|\mathbf{A}}$.

Furthermore, we say that a relative geometry tuple is **strong** if whenever $\{\mathrm{Spec}(A_i)\to\mathrm{Spec}(A)\}_i$ is a covering in τ, and $\mathrm{Spec}(B)\to\mathrm{Spec}(A)$ is a morphism with $\mathrm{Spec}(B)\in\mathbf{A}$, then $\{\mathrm{Spec}(B\otimes^{\mathbb{L}}_A A_i)\to\mathrm{Spec}(A)\}$ is a covering in $\tau_{|\mathbf{A}}$.

Definition 3.5. Given a relative geometry tuple $(\mathrm{Aff}_{\mathcal{C}},\tau,\mathbf{P},\mathbf{A})$, we define **stacks relative to $\mathbf{A}$** to be the ∞-topos $\mathrm{Shv}(\mathbf{A},\tau_{|\mathbf{A}})$.

We will mostly be concerned with strong relative geometry tuples, due to the following result:

Lemma 3.6 ([KKM22])**.** *Let $(\mathrm{Aff}_{\mathcal{C}},\tau,\mathbf{P},\mathbf{A})$ be a strong relative geometry tuple. Then, there is a fully faithful embedding $\mathrm{Shv}(\mathbf{A},\tau_{|\mathbf{A}})\hookrightarrow\mathrm{Shv}(\mathrm{Aff}_{\mathcal{C}},\tau)$.*

Remark 3.7. It's possible to form an analogue of relative geometry tuples in the 1-categorical setting, where it is usually called a "functor of points" approach to geometry. In this setting, we fix a symmetric monoidal 1-category $(\mathcal{C},\otimes,\mathbf{1})$ generated under 1-sifted colimits by compact projective objects (this generalizes Alg_R and its generation by Poly_R). In the same fashion as Definition 2.20, we then consider schemes relative to $\mathcal{C}$ as "locally representable" sheaves of sets in a given subcanonical Grothendieck topology τ, and stacks relative to $\mathcal{C}$ as sheaves of groupoids in τ.

1. For a ring R, the category of schemes relative to $(\mathrm{Alg}_R,\otimes_R,R)$ in the Zariski topology is the usual category of R-schemes Sch_R. This approach to algebraic geometry, where an R-scheme S corresponds to the sheaf
$$\begin{aligned}\mathrm{Alg}_R &\longrightarrow \mathrm{Sets}\\ A &\longmapsto \mathrm{Hom}(\mathrm{Spec}(A),S),\end{aligned}$$
is the classical functor of points approach to algebraic geometry.

2. The category of stacks relative to $(\mathrm{Alg}_R, \otimes_{\mathbb{Z}}, R)$ in the étale topology is the usual category of stacks over R.

3. For a nonarchimedean field K of arbitrary characteristic, letting $\mathrm{AfndAlg}_K$ denote the category of K-affinoid algebras, the category of schemes relative to $(\mathrm{AfndAlg}_K, \widehat{\otimes}_K, K)$ in the G-topology is the category of rigid analytic spaces.

4. Letting SteinAlg denote the category of Stein algebras over $\mathbb{C}$, the category of schemes relative $(\mathrm{SteinAlg}, \widehat{\otimes}_{\mathbb{C}}, \mathbb{C})$ is the usual category of complex analytic spaces. We leave this and the above rigid space example vague for now, as we will explore them more deeply in later sections.

3.2 The Descendable Topology and Quasi-Coherent Sheaves

We now come to describing quasi-coherent sheaves with respect to arbitrary derived algebraic contexts. For affine objects, this is not hard to define. Throughout this section, we fix a derived algebraic context $(\mathcal{C}, \mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0}, \mathcal{C}^0)$:

Definition 3.8. Given $\mathrm{Spec}(A) \in \mathrm{Aff}_{\mathcal{C}}$, we define the **category of quasi-coherent sheaves on** $\mathrm{Spec}(A)$ to be $\mathrm{Mod}_A(\mathcal{C})$.

Given a derived algebraic context, there is a canonical topology we can put on the associated derived affine schemes such that quasi-coherent sheaves always satisfy the sheaf condition, the descendable topology. We will be concerned with this situation at several points later on. This requires a general definition:

Definition 3.9 ([Mat16], Section 3)**.**

1. An object $A \in \mathcal{C}\mathrm{Alg}(\mathcal{C})$ is **descendable** if the thick $\otimes$-ideal generated by A is equal to $\mathcal{C}$. In other words, $\mathcal{C}$ is generated under finite (co)limits, retracts, and tensor products by A.

2. A morphism $A \to B$ in $\mathcal{C}\mathrm{Alg}(\mathcal{C})$ is called **descendable** if the thick $\otimes$-ideal generated by B is equal to $\mathrm{Mod}_A(\mathcal{C})$.

Remark 3.10. We note that A is descendable if and only if it is descendable as the unit object of $\mathcal{C}\mathrm{Alg}(\mathrm{Mod}_A(\mathcal{C}))$.

There are a few important foundational properties of descendable morphisms that we quote here without proof:

Proposition 3.11 (Functoriality)**.** *Let $F\colon \mathcal{C} \to \mathcal{C}'$ be a symmetric monoidal functor of stable symmetric monoidal ∞-categories. If $A \in \mathcal{C}\mathrm{Alg}(\mathcal{C})$ is descendable, then $F(A)$ is as well.*

In particular, this shows that descendable objects and morphisms are stable under base change.

Proposition 3.12 (Stability Under Composition)**.** *Let $f\colon A \to B$, $g\colon B \to C$, and $g \circ f\colon A \to C$ be morphisms in $\mathcal{C}\mathrm{Alg}(\mathcal{C})$. If f and g are descendable, then so is $g \circ f$. If $g \circ f$ is descendable, then so is f.*

We can characterize descendability in terms of totalizations:

Proposition 3.13 ([Mat16])**.** *An object $A \in \mathcal{C}\mathrm{Alg}(\mathcal{C})$ is descendable if and only if the partial totalizations of the cobar resolution $\{\mathrm{Tot}_n A^{\otimes \bullet+1}\}_n$ form a constant* Pro*-object.*

Furthermore, as noted earlier, the analogue of Proposition 2.29 can be shown for descendable morphisms. This shows that the name "descendable" is well-suited:

Proposition 3.14. *Let $f\colon A \to B$ be a descendable morphism in $\mathcal{C}\mathrm{Alg}(\mathcal{C})$. Then extension of scalars $f^* = - \otimes_A B$ is comonadic, and thus*

$$\mathrm{Mod}_A(\mathcal{C}) \cong \lim_{[n]\in\Delta} \mathrm{Mod}_{B^{\otimes_A (n+1)}}(\mathcal{C})$$

is the limit of the cosimplicial ∞-category obtained by applying $\mathrm{Mod}_{(-)}(\mathcal{C})$ to the Čech conerve $A \rightrightarrows B \rightrightarrows B \otimes_A B \rightrightarrows \cdots$, with transition functors given by base change.

To prove this, we will need the following preliminary lemma:

Lemma 3.15. *If $A \in \mathcal{C}\mathrm{Alg}(\mathcal{C})$ is descendable, then tensoring with A is conservative: if $M \in \mathcal{C}$, and $M \otimes A \simeq 0$, then M is contractible.*

Proof. Consider the collection of all objects $N \in \mathcal{C}$ such that $M \otimes N \simeq 0$. This is clearly a thick $\otimes$-ideal. Since it contains A, it must contain the unit. Hence, $M \otimes \mathbf{1} \simeq M \simeq 0$, i.e. M is contractible. □

Proof of Proposition 3.14. By the relationship between descendable morphisms and descendable objects in Remark 3.10, it suffices to show this for the unit $\mathbf{1} \to A$. By 3.15, tensoring with A is conservative. Now, fix a cosimplicial object $X^\bullet\colon \Delta \to \mathcal{C}$ such that $A \otimes X^\bullet$ is split. We need to show that the map

$$A \otimes \mathrm{Tot}(X^\bullet) \to \mathrm{Tot}(A \otimes X^\bullet)$$

of totalizations is an equivalence. This follows if the pro-object defined by $X^\bullet$'s tower of partial totalizations is a constant pro-object. To see this, consider the collection of objects $M \in \mathcal{C}$ such that $M \otimes X^\bullet$ defines a constant pro-object. Since $A \otimes X^\bullet$ is split, it defines a constant pro-object. Hence, this collection contains A and is a thick $\otimes$-ideal. So, it contains $\mathbf{1} \otimes X^\bullet \simeq X^\bullet$, and we are done. □

This allows us to define:

Definition 3.16. The **descendable topology** on $\mathrm{Aff}_{\mathcal{C}}$ has sieves generated by finite families $\{\mathrm{Spec}(A_i) \to \mathrm{Spec}(A)\}_i$, where the map $A \to \prod_i A_i$ is descendable. We denote it by τ_{desc}.

Remark 3.17.

1. That the above class of sieves forms a topology follows from propositions 3.11 and 3.12. Another way to observe this is that those results imply that descendable morphisms form a pre-topology, and the above description is then the associated Grothendieck topology. This is the approach taken in [KM25].

3.2.1 Quasi-Coherent Sheaves in the Descendable Topology

Similarly to the case of derived schemes, we can show that quasi-coherent sheaves in an arbitrary derived algebraic context form a sheaf in the descendable topology. This uses analogous definitions to earlier.

Definition 3.18. For $\mathrm{Spec}(A) \in \mathrm{Aff}_{\mathcal{C}}$, the ∞-category of **quasicoherent sheaves** on $\mathrm{Spec}(A)$ is $\mathrm{QCoh}(A) := \mathrm{Mod}_A(\mathcal{C})$. This yields a functor

$$\mathrm{QCoh}\colon \mathrm{Aff}_{\mathcal{C}}^{op} := \mathcal{C}\mathrm{Alg}(\mathcal{C}) \longrightarrow \mathcal{C}\mathrm{Alg}(\mathrm{Pr}^L), \qquad \mathrm{Spec}(A) \mapsto \mathrm{Mod}_A.$$

By the exact same proof as the proof of Corollary 2.32, we get:

Proposition 3.19. QCoh *is a sheaf for the descendable topology* τ_{desc} *on* $\mathrm{Aff}_{\mathcal{C}}$.

By right Kan extension, this then allows us to define the category of quasi-coherent sheaves for any object $X \in \mathrm{Shv}(\mathrm{Aff}_{\mathcal{C}}, \tau_{\mathrm{desc}})$, similarly to how we did for derived schemes in Section 2.3.

4 Six-Functor Formalisms

So far, we have constructed an abstract formalism for derived geometry via considering commutative algebra objects over a derived algebraic context and placing the descendable topology on the derived affine schemes. This formed a well-behaved relative geometry tuple whose quasi-coherent sheaves satisfy descent. Now, for our next piece of formalism, we'd like to associate operations (pullback, pushforward, tensor product, internal Hom, etc.) on quasicoherent sheaves to our setup.

We recall the classical setting in topology and geometry for inspiration. Let X be a locally compact Hausdorff topological space. The category of sheaves of abelian groups on X, $\mathrm{Shv}(X)$ is abelian, so it comes with a triangulated (or, stable ∞) derived category which we will call $D(X)$. Sheaves already come with 4 natural operations. Given a sheaf $\mathcal{F}$ on X and a continuous map $f\colon X \to Y$, we can consider:

1. The sheaf tensor product $- \otimes \mathcal{F}\colon \mathrm{Shv}(X) \to \mathrm{Shv}(X)$.
2. The internal Sheaf Hom $\mathcal{H}om(\mathcal{F}, -)\colon \mathrm{Shv}(X) \to \mathrm{Shv}(X)$.
3. The pullback $f^*\colon \mathrm{Shv}(Y) \to \mathrm{Shv}(X)$.

4. The pushforward $f_*\colon \mathrm{Shv}(X) \to \mathrm{Shv}(Y)$.

These group into two adjoint pairs $- \otimes \mathcal{F} \dashv \mathcal{H}om(\mathcal{F}, -)$ and $f^* \dashv f_*$. Upon passing to the derived category, this is enhanced to derived adjoints $- \otimes^{\mathbb{L}} \mathcal{F}_\bullet \dashv \mathbb{R}\mathrm{Hom}(\mathcal{F}_\bullet, -)$ and $\mathbb{L}f^* \dashv \mathbb{R}f_*$. Furthermore, Verdier showed the existence of the **exceptional pushforward** $f_!\colon D(X) \to D(Y)$ and the **exceptional pullback** $f^!\colon D(Y) \to D(X)$, forming a third adjoint pair $f_! \dashv f^!$. Each of these adjoint pairs comes further equipped with the following relationships:

1. For all $f\colon X \to Y$ and $g\colon Y \to Z$ with composition $h\colon X \to Z$, there is a natural isomorphism $h_! \cong g_! f_!$ satisfying further associativity constraints.
2. For any pullback square

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

   there is a base change isomorphism

$$g^* f_! \cong f'_! g'^*$$

   compatible with horizontal and vertical composition.
3. For $f\colon X \to Y$, there is a projection formula

$$f_! \mathcal{F}_\bullet \otimes \mathcal{G}_\bullet \cong f_!(\mathcal{F}_\bullet \otimes f^* \mathcal{G}_\bullet)$$

   for $\mathcal{F}_\bullet \in D(X), \mathcal{G}_\bullet \in D(Y)$, which is compatible with composition, the base change isomorphisms, and further tensor products.

Such a setup can be replicated for abelian sheaves on locally compact Hausdorff spaces with some extra features:

1. For an open immersion $f\colon X \to Y$, the functor $f_!$ is left adjoint to f^*.
2. For a proper map $f\colon X \to Y$, the functor $f^!$ is equivalent to $f^* \otimes \omega_{X/Y}$, and right adjoint to f_*.
3. Furthermore, by Stone-Čech compactification, any morphism $g\colon X \to Y$ factors as hf, where f is an open immersion and h is proper (i.e. a compactification), and the functors $g_!$, $g^!$, g_*, and g^* are compatible with this factorization.

Nowadays, it is generally believed that a well-behaved algebro-geometric theory is one whose quasi-coherent sheaves inherit these six operations with all of these properties. For a long time, the language surrounding them was ill-defined due to the amount of consequences that should come from their existence, mostly serving as intuitive folklore. However, recently, it was more fully formalized by Liu–Zheng and Gaitsgory–Rozenblyum [LZ12; GR19], then made fully rigorous by Mann [Man22; Sch25] through **six-functor formalisms**. In the following, we will review the foundations of six-functor formalisms and employ them to construct a six-functor formalism on quasi-coherent sheaves on derived stacks associated to commutative algebras in a derived algebraic context. This will immediately give us a good setting to do computations with quasi-coherent sheaves later after we introduce the local geometry for our theory of derived complex analytic spaces. Our exposition will mostly build off [HM24; Sch25].

4.1 The Category of Correspondences

4.1.1 Foundations

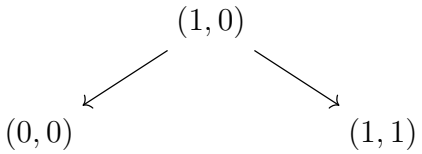
Definition 4.1. A **geometric setup** is a pair $(\mathcal{C}, E)$ consisting of a category $\mathcal{C}$ and a class E of morphisms (often called **!-able morphisms**) such that:

1. E contains all isomorphisms.
2. E is closed under composition and pullbacks along morphisms in $\mathcal{C}$.
3. For every morphism $Y \to X$ in E, the diagonal $\Delta\colon Y \to Y \times_X Y$ is in E.

Geometric setups form an obvious category, which we denote **GeomSetup**.

From now on, for simplicity's sake, we will assume that $\mathcal{C}$ is an ∞-category with all finite limits. We will view it as a symmetric monoidal ∞-category under products.

We can now define the crucial category for defining six-functor formalisms. In the following definition, we let $(\Delta^n_+)^2 \subseteq (\Delta^n)^{op} \times \Delta^n$ be the subset spanned by simplices $(i,j) \subseteq \{0,1,\cdots,n\}^2$ with $i \geq j$. For $n = 1$, this is the diagram

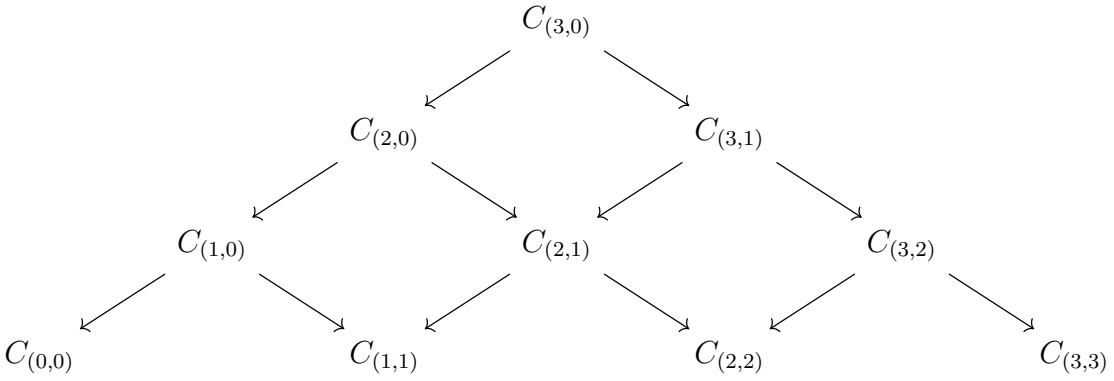


Definition 4.2. Given a geometric setup $(\mathcal{C}, E)$, the symmetric monoidal ∞-category of **correspondences** $\mathrm{Corr}(\mathcal{C}, E)$ has n-simplices given by maps $(\Delta^n_+)^2 \to \mathcal{C}$ such that:

1. Every arrow going down-right is in E.
2. All small squares are pullback squares.

Example 4.3.

1. The 0-simplices in $\mathrm{Corr}(\mathcal{C}, E)$ are simply the objects of $\mathcal{C}$.
2. If E is the class of all morphisms, then the 3-simplices in $\mathrm{Corr}(\mathcal{C}, E)$ are the diagrams



   where the squares are all pullback squares.
3. If E is the class of all isomorphisms, then $\mathrm{Corr}(\mathcal{C}, E) \cong \mathcal{C}^{op}$.

Remark. This is really a definition/theorem, as one does need to show that all inner horns can be filled in. However, this follows by the fact that E is stable under pullback and that $\mathcal{C}$ has all finite limits (including pullbacks). Furthermore, we have not described the symmetric monoidal structure, since it is somewhat technical. The fact that it exists is far more important for our goals than the details of its construction.

4.1.2 Three and Six-Functor Formalisms

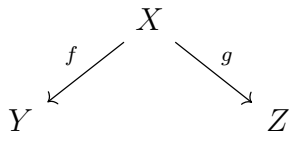
We are now able to define our main characters, starting with three-functor formalisms:

Definition 4.4. Given a geometric setup $(\mathcal{C}, E)$, a **three-functor formalism** on $(\mathcal{C}, E)$ is a lax symmetric monoidal functor of ∞-categories

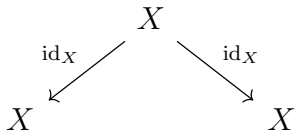
$$D\colon \mathrm{Corr}(\mathcal{C}, E)^{\otimes} \to \mathrm{Cat}_{\infty}.$$

If the image of D consists entirely of categories satisfying property P, we will say that D satisfies property P (e.g. if $D(X)$ is presentable for all X, we will say that D is presentable and write $D\colon \mathrm{Corr}(\mathcal{C}, E)^{\otimes} \to \mathrm{Pr}^{L}$).

It is essential to clarify how we can produce three functors from a three-functor formalism. Note that "morphisms" $Y \to Z$, i.e. 1-simplices in $\mathrm{Corr}(\mathcal{C}, E)$ with starting point Y and endpoint Z, are given by diagrams

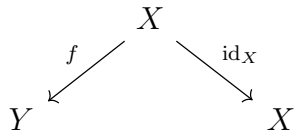


By functoriality, this induces a map $D(Y) \to D(Z)$ which we will call $g_! f^*$, such that the diagram

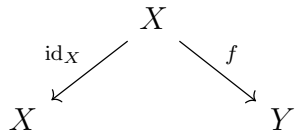


induces the identity $D(X) \to D(X)$. Hence, $\mathrm{id}_!$ and id^* act by the identity. From this data, the three functors become apparent:

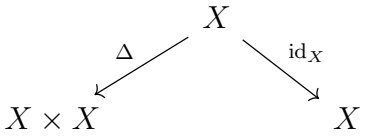
1. The pullback $f^*\colon D(Y) \to D(X)$ is given by applying D to the correspondence



2. The exceptional pushforward $f_!\colon D(X) \to D(Y)$ is given by applying D to the correspondence



3. The tensor product is given by applying D to the correspondence

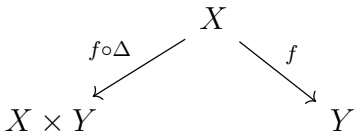


   after composing with the functor $D(X) \times D(X) \to D(X \times X)$ given by the lax symmetric monoidal structure of D. Here, $\Delta\colon X \to X \times X$ is the diagonal. This is inspired by the fact that for sheaves of $\mathcal{O}_X$-modules $\mathcal{F}$ and $\mathcal{G}$ on a scheme X, we can identify $\mathcal{F} \otimes \mathcal{G} \cong \Delta^*(p_1^*\mathcal{F} \otimes p_2^*\mathcal{G})$, where p_1 and p_2 are the projections from $X \times X$ to the first and second factor respectively. This yields a symmetric monoidal structure on $D(X)$. In practice, we will fix an object $\mathcal{F} \in D(X)$ and consider $- \otimes \mathcal{F}$.

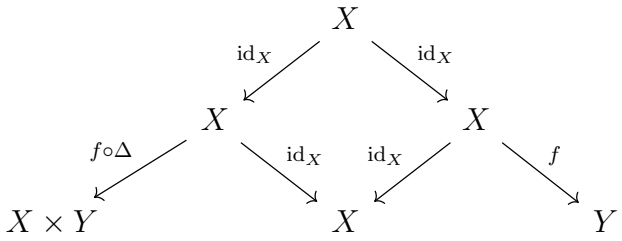
Furthermore, the fact that D is a lax symmetric monoidal functor of ∞-categories encodes the formulas and composition identifications that we sought after at the beginning. Instead of proving all of these, we give one example: the projection formula

$$f_!(\mathcal{F} \otimes f^*\mathcal{G}) \cong f_!\mathcal{F} \otimes \mathcal{G}$$

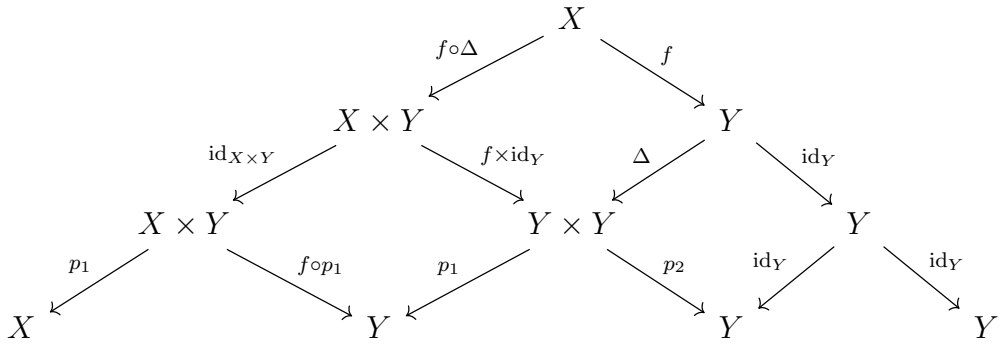
for $f\colon X \to Y$ in E, $\mathcal{F} \in D(X)$, and $\mathcal{G} \in D(Y)$. We can obtain this by expressing the correspondence



in two different ways. First is as the correspondence induced by the composition



The composition laws and lax symmetric monoidal structure of D yield $(A, B) \to A \otimes f^*B \to f_!(A \otimes f^*B)$ from $D(X \times Y) \to D(X) \to D(Y)$. Second is as the composition

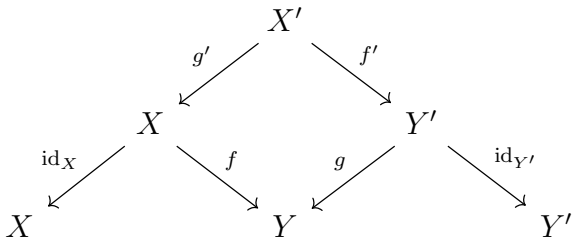


the composition laws and lax symmetric monoidal structure of D yield $(A, B) \to (f_!A, B) \to f_!A \otimes B$ from $D(X \times Y) \to D(Y \times Y) \to D(Y)$, thus providing an identification by uniqueness of composition up to natural isomorphism in ∞-categories.

Similarly, we can show the base change identification we sought after earlier. Let

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

be a pullback square in $\mathcal{C}$, with f (and thus f') in E. Then, we can form the composition of correspondences



By the fact that composition is unique up to homotopy, following the outer correspondence and the composition of the middle two yields the identification

$$g^* f_! \cong f'_! g'^*$$

we desired.

With three-functor formalisms spelled out in full, we can now define:

Definition 4.5. Given a geometric setup $(\mathcal{C}, E)$, a **six-functor formalism** on $(\mathcal{C}, E)$ is a three-functor formalism $D\colon \mathrm{Corr}(\mathcal{C}, E)^{\otimes} \to \mathcal{C}\mathrm{Alg}(\mathrm{Cat}_\infty)$ such that the functors $- \otimes \mathcal{F}$, f^*, and $f_!$ all admit right adjoints. We denote these right adjoints $\mathcal{H}om(\mathcal{F}, -)$, f_*, and $f^!$ respectively. In particular, $D(X)$ is a closed symmetric monoidal category for all X.

Remark 4.6. We note that for a morphism $f\colon X \to Y$ in E in a six-functor formalism (or more generally, any morphism in a three-functor formalism such that $f^!$ exists), there is a canonical natural transformation

$$f^!(\mathbf{1}_Y) \otimes f^*(-) \implies f^!(-).$$

Indeed, by the projection formula and counit of the adjunction $\epsilon\colon f_! f^! \implies \mathrm{id}$, for any $M \in D(Y)$ we have a natural map

$$f_!(f^!(\mathbf{1}_Y) \otimes f^*(M)) \cong f_! f^!(\mathbf{1}_Y) \otimes M \xrightarrow{\epsilon \otimes \mathrm{id}_M} M.$$

By applying $f^!$, we get a natural map

$$f^! f_!(f^!(\mathbf{1}_Y) \otimes f^*(M)) \to f^!(M).$$

Precomposing with the unit of the adjunction $\eta\colon \mathrm{id} \implies f^! f_!$ then gives a natural map

$$f^!(\mathbf{1}_Y) \otimes f^*(M) \xrightarrow{\eta} f^! f_!(f^!(\mathbf{1}_Y) \otimes f^*(M)) \to f^!(M),$$

thus defining our natural transformation.

4.1.3 Constructing Six-functor Formalisms

In practice, constructing three and six-functor formalisms can be difficult, since the correspondence category is hard to work with explicitly. However, it is often much easier to write functors

$$D_0\colon \mathcal{C}^{op} \to \mathcal{C}\mathrm{Alg}(\mathrm{Cat}_\infty)$$

which encode tensor products and pullbacks, i.e. a three-functor formalism on $(\mathcal{C}, \text{isomorphisms})$ (Here, $\mathcal{C}\text{Alg}(\text{Cat}_\infty)$ are symmetric monoidal ∞-categories). Inspired by the factorizations of morphisms into open immersions and proper morphisms we discussed earlier, we will show how to extend D_0 to a three-functor formalism on a class of morphisms E admitting analogous factorization properties.

By axiomatizing the properties of open immersions and proper morphisms, we get a list of mild assumptions we should make. Let E be a class of morphisms in $\mathcal{C}$ admitting two subclasses I and P such that:

1. I and P are stable under pullback, composition, and diagonals, and contain all isomorphisms. If $f \in I \cap P$, then f is n-truncated for some n. Any $f \in E$ is a composite $f = hg$ with $g \in I$ and $h \in P$.
2. For any $f \in I$, the functor f^* admits a left adjoint $f_!$ satisfying base change and the projection formula.
3. For any $f \in P$, the functor f^* admits a right adjoint f_* satisfying base change and the projection formula.
4. For any pullback square

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle h'}\downarrow & & \downarrow{\scriptstyle h} \\ Y' & \xrightarrow{g} & Y \end{array}$$

   with $g, g' \in I$ and $h, h' \in P$, the natural map $g_! h'_* \to h_* g'_!$ is an isomorphism.

With these assumptions, it is possible to show:

Theorem 4.7 ([Man22], Proposition A.5.10.)**.** *Under the above assumptions, there is a canonical extension of D_0 to a lax symmetric monoidal functor*

$$D \colon \text{Corr}(\mathcal{C}, E) \to \mathcal{C}\text{Alg}(\text{Cat}_\infty)$$

such that for $f \in I$, $f_!$ is the left adjoint of f^*, while for $f \in P$, $f_!$ is the right adjoint of f^* (i.e., it equals f_* if f^* admits a right adjoint in general).

In particular, we note the following consequences of this result:

1. For any $f \in I \cap P$, there is a natural isomorphism $f_! \cong f_*$.
2. If $f \colon X \to Y$ in E can be written in two ways as $f = hg$ and $f = h'g'$ with $g, g' \in I$ and h, h' in P, then there is a natural isomorphism

$$h_* g_! \cong h'_* g'_!$$

4.2 Extending Six-Functor Formalisms, and the Six-Functor Formalism on $\mathcal{C}\mathrm{Alg}$

Our goal is to define a six-functor formalism on $\mathcal{C}\mathrm{Alg}(\mathcal{C})^{op}$ for a derived algebraic context $\mathcal{C}$. This requires showing that when in the presence of a Grothendieck topology, a well-behaved six-functor formalism (i.e. one forming a sheaf) can be extended to the category of sheaves.

Definition 4.8. Let $(\mathcal{C}, E)$ be a geometric setup such that $\mathcal{C}$ is a site. A three-functor formalism $D\colon \mathrm{Corr}(\mathcal{C}, E) \to \mathcal{C}\mathrm{Alg}(\mathrm{Cat}_\infty)$ is called **sheafy** if the functor $D^*\colon \mathcal{C}^{op} \to \mathcal{C}\mathrm{Alg}(\mathrm{Cat}_\infty)$ is a sheaf of categories. A similar definition applies if $\mathcal{C}$ is a topos as well.

Definition 4.9. Let $D\colon \mathrm{Corr}(\mathcal{C}, E) \to \mathcal{C}\mathrm{Alg}(\mathrm{Cat}_\infty)$ be a six-functor formalism.

1. If D is a six-functor formalism, we say that a sieve $\mathcal{U} \subseteq (\mathcal{C}_E)_{/U}$ in $\mathcal{C}_E$ is a small $D^!$-cover if it is generated by a small family of maps $(U_i \to U)_i$ and $D^!$ descends along U.
2. We say that U is a small **universal** $D^!$-cover if for every map $V \to U$ in $\mathcal{C}$, the family $(U_i \times_U V \to V)$ generates a small $D^!$-cover (i.e. along every base change).

Definition 4.10. Let $D\colon \mathrm{Corr}(\mathcal{C}, E) \to \mathcal{C}\mathrm{Alg}(\mathrm{Cat}_\infty)$ be a six-functor formalism. Let S be a collection of maps in $\mathcal{C}$, and let $f\colon Y \to X$ be a fixed map.

1. We say that f is **$D^!$-locally on the source in** S if there is a universal small $D^!$-cover of Y such that the composition of f with every element in the cover lies in S.
2. We say that f is **$D^!$-locally on the target in** S if there is a universal small $D^!$-cover of X such that the pullback of f along every element in the cover lies in S.

If D is clear form the context, we will write !-local instead of $D^!$-local.

The powerful result we'll make use of is:

Theorem 4.11. *Let $D\colon \mathrm{Corr}(\mathcal{C}, E) \to \mathcal{C}\mathrm{Alg}(\mathrm{Pr}^L)$ be a sheafy presentable six-functor formalism, where $\mathcal{C}$ is a site with a subcanonical topology. Then, there is a collection of morphisms E' in the category of stacks $\mathcal{X} := \mathrm{Shv}(\mathcal{C})$ with the following properties:*

1. *The inclusion $\mathcal{C} \hookrightarrow \mathcal{X}$ under the Yoneda embedding defines a morphism of geometric setups $(\mathcal{C}, E) \to (\mathcal{X}, E')$, and D extends uniquely to a sheafy six-functor formalism D' on $(\mathcal{X}, E')$.*
2. *E' is $*$-local on the target. Concretely speaking, let $f\colon Y \to X$ be a map in $\mathcal{X}$ whose pullback to every object in $\mathcal{C}$ lies in E'. Then, $f \in E'$.*
3. *E' is !-local. Concretely speaking, let $f\colon Y \to X$ be a map in $\mathcal{X}$ which is !-locally on the source or target in E'. Then, $f \in E'$.*
4. *E' is tame. Concretely speaking, every map $f\colon Y \to X$ in E' with $X \in \mathcal{C}$ is !-locally on the source in E.*

By "in $\mathcal{C}$" we mean the image under the Yoneda embedding.

Remark 4.12. In practice, conditions 2 and 3 are the important ones, and condition 4 can usually be ignored. Condition 2 combined with condition 1 often allows one to reduce to the case that the target is in $\mathcal{C}$. The meat of showing that morphisms are in E' then usually comes down to condition 3, showing that the base change of a covering to an affine target is in E'.

We are now equipped to describe a six-functor formalism in relative algebraic geometry. Let $(\mathcal{C}, \mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0}, \mathcal{C}^0)$ be a derived algebraic context, and let $\mathcal{C}\mathrm{Alg}(\mathcal{C})$ be the category of commutative algebras in $\mathcal{C}$. Let τ be a subcanonical Grothendieck topology on $\mathrm{Aff}_{\mathcal{C}}$ such that $\mathrm{QCoh}\colon \mathrm{Aff}_{\mathcal{C}}^{op} := \mathcal{C}\mathrm{Alg}(\mathcal{C}) \to \mathcal{C}\mathrm{Alg}(\mathrm{Pr}^L)$ is a sheaf (for example, the descendable topology by Corollary 2.32). QCoh encodes the tensor product of modules since it is a symmetric monoidal functor. Furthermore, it sends a morphism $f\colon \mathrm{Spec}(A) \to \mathrm{Spec}(B)$ to the pullback

$$f^*\colon \mathrm{Mod}_B \to \mathrm{Mod}_A, \qquad C \mapsto A \otimes_B^{\mathbb{L}} C.$$

This functor has a forgetful right adjoint

$$f_*\colon \mathrm{Mod}_A \to \mathrm{Mod}_B.$$

By considering the geometric setup $(\mathrm{Aff}, \mathrm{all})$ consisting of all morphisms in all, we can show:

Proposition 4.13. QCoh *induces a presentable six-functor formalism*

$$\mathrm{QCoh}\colon \mathrm{Corr}(\mathrm{Aff}_{\mathcal{C}}, \mathrm{all}) \to \mathcal{C}\mathrm{Alg}(\mathrm{Pr}^L).$$

Proof. $(I, P) = (\text{equivalences}, \text{all})$ satisfies the assumptions of Theorem 4.7:

1. Equivalences and all morphisms are stable under pullback, composition, and diagonals, and contain all isomorphisms. equivalences are -1-truncated morphisms, and the fact that $P =$ all implies the factorization result trivially.
2. We already observed that extension of scalars has right adjoint given by restriction of scalars, so the assumptions on $f_!$ and f_* are immediate.
3. The base change formula follows from the properties of tensor product and restriction of scalars.

Furthermore, $f_! = f_*$ for all morphisms f since the class P consists of all morphisms, and $f_! = f_*$ for all morphisms in P by Theorem 4.7. Finally, $\mathcal{H}om$ is given by $\mathbb{R}\mathrm{Hom}$, and $f^!$ is given by $\mathbb{R}\mathrm{Hom}(A, -)$. □

Hence, by Theorem 4.11 and the fact that QCoh is a sheaf, we get:

Theorem 4.14. *There is a sheafy presentable six-functor formalism* $\mathrm{QCoh}\colon \mathrm{Corr}(\mathrm{Shv}(\mathrm{Aff}_{\mathcal{C}}, \tau), E') \to \mathcal{C}\mathrm{Alg}(\mathrm{Pr}^L)$ *extending the six-functor formalism* $\mathrm{QCoh}\colon \mathrm{Corr}(\mathrm{Aff}_{\mathcal{C}}, \tau) \to \mathcal{C}\mathrm{Alg}(\mathrm{Pr}^L)$ *such that* E' *is* $*$*-local on the target,* $!$*-local, and tame.*

Remark 4.15. 1. Later on, we will introduce different topologies on various subcategories of dAff for derived algebraic contexts $(\mathcal{C}, \mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0}, \mathcal{C}^0)$ corresponding to local data for analytic geometry (e.g. Stein algebras). Their sheaves will always embed into $\mathrm{Shv}(\mathrm{dAff}_{\mathcal{C}}, \tau_{\mathrm{desc}})$, meaning that we can pull the six-functor formalism back.

2. Much of our goal in showing Grothendieck duality lies in showing that certain morphisms of derived analytic spaces are in E'. We have not explicitly described how to construct E' since it is quite technical; we will introduce what we need later on.

4.3 The Category of Kernels, Cohomologically Smooth Morphisms, and Suave/Prim Morphisms

Before concluding, we will give a characterization of various morphisms relating $f^!$ and f^*. The first are the cohomologically smooth morphisms, which act like smooth morphisms on the level of sheaves:

Definition 4.16. Let $D\colon \mathrm{Corr}(\mathcal{C}, E) \to \mathcal{C}\mathrm{Alg}(\mathrm{Cat}_\infty)$ be a three-functor formalism. A morphism $f\colon X \to Y$ in E is **cohomologically smooth** if the following are satisfied:

1. The right adjoint $f^!$ of $f_!$ exists, and the natural transformation
$$f^!(\mathbf{1}_Y) \otimes f^*(-) \implies f^!(-)$$
is an equivalence.

2. The object $f^!(\mathbf{1}_Y)$ is $\otimes$-invertible.

3. For any morphism $g\colon Y' \to Y$ with base change $f'\colon X \times_Y Y' \to Y'$ of f and $g'\colon X \times_Y Y' \to X$ of g, properties 1 and 2 again hold for f', and the natural map
$$g'^* f^!(\mathbf{1}_Y) \implies f'^!(\mathbf{1}_{Y'})$$
is an equivalence.

Remark 4.17. All of these properties are the properties of a smooth morphism of schemes $f\colon X \to Y$ on the level of the derived category. However, this abstract definition does not address the problem of identifying $f^!\mathbf{1}_Y$.

Showing that a given morphism is cohomologically smooth is often difficult, but it is possible to show the following theorem giving looser criteria in the case that E is the class of all morphisms of $\mathcal{C}$:

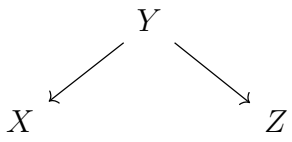
Theorem 4.18. *Assume that E is the class of all morphisms in $\mathcal{C}$, and let $f\colon X \to Y$ be a map to the final object in $\mathcal{C}$. Let $\Delta\colon X \to X \times_Y X$ be the diagonal. Then, f is cohomologically smooth if and only if there exists a $\otimes$-invertible object $L \in D(X)$ and maps*
$$\alpha\colon \Delta_! \mathbf{1}_X \to p_2^* L, \qquad \beta\colon f_! L \to \mathbf{1}_Y$$
such that:

1. *The composite*
$$\mathbf{1}_X \cong p_{1!}\Delta_! \mathbf{1}_X \xrightarrow{p_{1!}\alpha} p_{1!}p_2^* L \cong f^* f_! L \xrightarrow{f^*\beta} \mathbf{1}_X$$
is the identity where the middle isomorphism is base change.

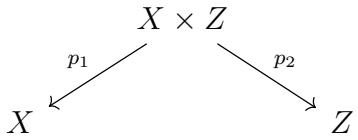
2. *The composite*
$$L \cong p_{2!}(p_1^* L \otimes \Delta_! \mathbf{1}_X) \xrightarrow{p_{2!}(p_1^* L \otimes \alpha)} p_{2!}(p_1^* L \otimes p_2^* L) \cong p_{2!}p_1^* L \otimes L \cong f^* f_! L \otimes L \xrightarrow{f^*\beta \otimes L} L$$
where the first isomorphism is obvious (since $p_{2!}\Delta_! \mathbf{1}_X) \cong \mathbf{1}_X$ and $p_{2!}(p_1^* L) \cong L$), the second isomorphism is projection formula, and the third isomorphism is base change.

By stability under base change, this is enough to give a criterion for cohomological smoothness in most situations.

Though we will not give the proof of the Theorem, we will note that it is fundamentally reliant on a 2-categorical redressing of six-functor formalisms. This redressing gives rise to other classes of morphisms that relate $f^!$ to f^* and $f_!$ to f_*. It relies on relating any correspondence



to an abstraction of Fourier-Mukai transforms in the classical study of derived categories, i.e. functors induced by correspondences



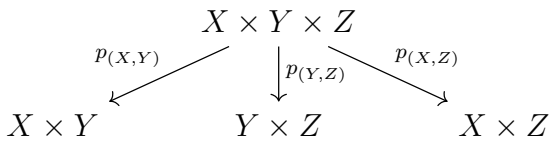
Definition 4.19. Let $D\colon \mathrm{Corr}(\mathcal{C}, E) \to \mathcal{C}\mathrm{Alg}(\mathrm{Cat}_\infty)$ be a three-functor formalism. The $(\infty, 2)$-category of **kernels** of D, denoted LZ_D, is the category whose objects are the objects of $\mathcal{C}$, and whose mapping ∞-categories are $\mathrm{Hom}(X, Y) = D(X \times Y)$. Identity objects are given by $\Delta_! \mathbf{1}_X$ in $D(X \times X)$ and composition

$$D(X, Y) \times D(Y, Z) \to D(X, Z)$$

is given by convolution

$$(\mathcal{F}, \mathcal{G}) \to \mathcal{F} \star \mathcal{G} = p_{(X,Z)!}(p^*_{(X,Y)}\mathcal{F} \otimes p^*_{(Y,Z)}\mathcal{G})$$

with respect to the diagram

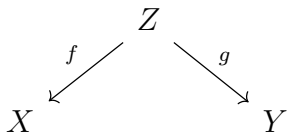


Remark 4.20. We again note that this is a definition/theorem, as one needs to show identifications between $(\mathcal{F} \star \mathcal{G}) \star \mathcal{H}$ and $\mathcal{F} \star (\mathcal{G} \star \mathcal{H})$ satisfying the pentagon axiom. This can be done by brute force but is quite cumbersome, so we omit it.

Any three-functor formalism $D\colon \mathrm{Corr}(\mathcal{C}, E) \to \mathcal{C}\mathrm{Alg}(\mathrm{Cat}_\infty)$ factors through its category of kernels via

$$\mathrm{Corr}(\mathcal{C}, E) \to \mathrm{LZ}_D \xrightarrow{\mathrm{FM}} \mathcal{C}\mathrm{Alg}(\mathrm{Cat}_\infty).$$

Here, the functor $\mathrm{Corr}(\mathcal{C}, E) \to \mathrm{LZ}_D$ is the identity on objects. On morphisms, it sends a correspondence



to $h_!(\mathbf{1}_Z) \in D(X \times Y)$, where $h\colon Z \to X \times Y$ is the map gotten by universal property of the product.

The functor $\mathrm{FM}\colon \mathrm{LZ}_D \to \mathcal{C}\mathrm{Alg}(\mathrm{Cat}_\infty)$ acts by sending $X \to D(X)$ on objects and $\mathcal{E} \in D(X \times Y)$ to the Fourier-Mukai functor with kernel $\mathcal{E}$

$$\Phi_{\mathcal{E}}\colon D(X) \to D(Y), \qquad \mathcal{F} \mapsto p_{2!}(p_1^*\mathcal{F} \otimes \mathcal{E}).$$

Since the morphisms in the category of kernels are themselves functors of categories, there are natural special classes in them: those that are adjoints. We give these the names below. From now on, we fix a six-functor formalism $D\colon \mathrm{Corr}(\mathcal{C}, E) \to \mathcal{C}\mathrm{Alg}(\mathrm{Cat}_\infty)$.

Definition 4.21. Let $Y \in \mathcal{C}$ be the final object, let $f\colon X \to Y$ be a morphism, and let $\mathcal{F} \in D(X)$.

1. We say that $\mathcal{F}$ is f-**suave** if $\mathcal{F}$ is a left adjoint when considered in $\mathrm{Hom}_{\mathrm{LZ}_D}(X, Y)$.
2. We say that $\mathcal{F}$ is f-**prim** if $\mathcal{F}$ is a left adjoint when considered in $\mathrm{Hom}_{\mathrm{LZ}_D}(Y, X)$.
3. We say that f is D-**suave** (or just **suave** if D is obvious) if $\mathbf{1}_X$ is f-suave.
4. We say that f is D-**prim** (or just **prim** if D is obvious) if $\mathbf{1}_X$ is f-prim.

We note that f-suave and f-prim objects are dual notions, by the fact that $\mathrm{LZ}_D \cong \mathrm{LZ}_D^{op}$ through the identification $D(X \times Y) \cong D(Y \times X)$.

Let's unpack this definition a bit. Firstly, since Y is a final object, $Y \times X \cong X \times Y \cong X$, so we have identifications $D(X \times Y) \cong D(Y \times X) \cong D(X)$, and thus $\mathrm{Hom}_{LZ_D}(X, Y) \cong \mathrm{Hom}_{LZ_D}(Y, X) \cong D(X)$. Thus, it makes sense to think of $\mathcal{F} \in D(X)$ as a morphism from X to Y (dually, from Y to X) in LZ_D. Secondly, the adjoint criteria can be made explicit by (co)unitality conditions. For example, $\mathcal{F}$ is f-suave if and only if there exists some $\mathcal{G} \in D(X) \cong \mathrm{Hom}_{LZ_D}(Y, X)$ with maps

$$\eta\colon \Delta_!(\mathbf{1}_X) \to p_1^*\mathcal{F} \otimes p_2^*\mathcal{G}, \qquad \epsilon\colon f_!(\mathcal{F} \otimes \mathcal{G}) \to \mathbf{1}_Y$$

such that the composite

$$\mathcal{G} \cong p_{2!}(p_1^*\mathcal{G} \otimes \Delta_!(\mathbf{1}_X)) \xrightarrow{\eta} p_{2!}(p_1^*\mathcal{G} \otimes p_1^*\mathcal{F} \otimes p_2^*\mathcal{G}) \cong p_{2!}p_1^*(\mathcal{G} \otimes \mathcal{F}) \otimes \mathcal{G} \cong f^*f_!(\mathcal{G} \otimes \mathcal{F}) \otimes \mathcal{G} \xrightarrow{\epsilon} f^*(\mathbf{1}_Y) \otimes \mathcal{G} \cong \mathcal{G}$$

is the identity, where the second isomorphism is projection formula, and the third is base change (likewise for the analogous composite with $\mathcal{F}$ and $\mathcal{G}$ exchanged). We note here that p_1 and p_2 are the projections $X \times X \to X$.

We will need a few results characterizing properties of f-suave and f-prim objects, as well as criteria for when morphisms are suave and prim, so we state and prove these here. The first is a characterization of f-suave objects:

Proposition 4.22. *Let $\mathcal{F} \in D(X)$ be f-suave with right adjoint $\mathcal{G} \in D(X)$. Then:*

1. *The object $\mathcal{G}$ is f-suave with right adjoint $\mathcal{F}$.*
2. *There is a natural isomorphism of functors*

   $$\mathcal{G} \otimes f^*(-) \cong \mathcal{H}om(\mathcal{F}, f^!(-)) \colon D(Y) \to D(X).$$

   *In particular, $\mathcal{G} \cong \mathcal{H}om(\mathcal{F}, f^!(\mathbf{1}_Y))$, the **Verdier dual** of $\mathcal{F}$. We denote it $\mathbb{D}_f(\mathcal{F})$ from now on.*
3. *The Verdier duality map*

   $$\mathbb{D}_f(\mathbb{D}_f(\mathcal{F})) \to \mathcal{F}$$

   is an isomorphism.
4. *The formation of Verdier dual $\mathcal{F}$ commutes with base change.*

Before proving this, we note that the second condition gives a relative identification between f^* and $f^!$ similar to cohomologically smooth morphisms. In fact, if $\mathcal{F} = \mathbf{1}_X$ is f-suave, then f is cohomologically smooth. Indeed, the identification in the second condition gives $\mathcal{G} \cong f^!(\mathbf{1}_Y)$ and thus an equivalence

$$f^!(\mathbf{1}_Y) \otimes f^*(-) \cong \mathcal{H}om(\mathbf{1}_Y, f^!(-)) \cong f^!(-).$$

Proof. 1 is true by the fact that $\mathrm{LZ}_D \cong \mathrm{LZ}_D^{op}$, exchanging $\mathcal{F}$ and $\mathcal{G}$'s places in the adjunction. For 2, we note that $\mathcal{G} \otimes f^*(-)$ is the right adjoint of $f_!(\mathcal{F} \otimes -)$. Indeed, the counit and unit of adjunction are the maps

$$\epsilon \colon f_!(\mathcal{F} \otimes \mathcal{G} \otimes f^*(-)) \cong f_!(\mathcal{F} \otimes \mathcal{G}) \otimes (-) \implies \mathrm{id}$$

and

$$\eta \colon \mathrm{id} \implies (\mathcal{G} \otimes \mathcal{F}) \otimes (-) \cong \mathcal{G} \otimes (\mathcal{F} \otimes (-)) \cong \mathcal{G} \otimes f^*(f_!(\mathcal{F} \otimes (-)))$$

respectively, which satisfy the (co)unit relations since $\mathcal{F}$ and $\mathcal{G}$ are adjoint. However, $f_!(A \otimes -)$ also has right adjoint $\mathcal{H}om(A, f^!(-))$. So, by uniqueness of adjoints up to equivalence, we get an equivalence $\mathcal{G} \otimes f^*(-) \cong \mathcal{H}om(\mathcal{F}, f^!(-))$ and an identification $\mathcal{G} \cong \mathcal{H}om(\mathcal{F}, f^!(\mathbf{1}_Y))$ by applying $\mathbf{1}_Y$ to the above.

For 3, we just apply 2 twice, and for 4, we use that formation of right adjoints commutes with any functor of 2-categories, of which base change is one. □

By taking $\mathcal{G}$ to be the Verdier dual, we automatically get the counit $\epsilon\colon f_!(\mathcal{F}\otimes\mathcal{G})\to \mathbf{1}_Y$. From here, we can characterize what is needed to get the unit $\eta\colon \Delta_!(\mathbf{1}_X)\to p_1^*\mathcal{F}\otimes p_2^*\mathcal{G}$:

Proposition 4.23. *Let $\mathcal{F}\in D(X)$. $\mathcal{F}$ is f-suave if and only if the map*

$$p_1^*(\mathcal{F})\otimes p_2^*\mathbb{D}_f(\mathcal{F})\to \mathcal{H}om(p_2^*\mathcal{F}, p_1^!\mathcal{F}),$$

which is the one adjoint to the natural map

$$p_1^*\mathcal{F}\otimes p_2^*\mathbb{D}_f(\mathcal{F})\otimes p_2^*\mathcal{F}\to p_1^*\mathcal{F}\otimes p_2^*f^!(\mathbf{1}_Y)\to p_1^*\mathcal{F}\otimes p_1^!(\mathbf{1}_X)\to p_1^!\mathcal{F},$$

is an isomorphism. In fact, it suffices to see that it induces an isomorphism on global sections (i.e. $\mathcal{H}om(\mathbf{1}_X,-)$) after applying $\Delta^!$.

Proof. If $\mathcal{F}$ is f-suave, then for any pullback square

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

and $\mathcal{G}\in D(Y')$, the map

$$f'^*\mathcal{G}\otimes g'^*\mathbb{D}_f(\mathcal{F})\to \mathcal{H}om(g'^*\mathcal{F}, f'^!\mathcal{G})$$

is an isomorphism by Proposition 4.22. If we apply this with $g\colon Y'\to Y$ given by $f\colon X\to Y$ and $\mathcal{G}=\mathcal{F}$, we have the desired result. □

The dual statements for f-prim objects are the following:

Proposition 4.24. *Let $\mathcal{F}\in D(X)$ be f-prim with right adjoint $\mathcal{G}\in D(X)$.*

1. *$\mathcal{G}$ is f-prim with right adjoint $\mathcal{F}$.*
2. *There is a natural isomorphism of functors*

$$f_!(\mathcal{G}\otimes -)\cong f_*\mathcal{H}om(\mathcal{F},-)\colon D(X)\to D(Y).$$

3. *The pairing*

$$\eta\colon p_1^*\mathcal{F}\otimes p_2^*\mathcal{G}\to \Delta_!(\mathbf{1}_X)$$

induces isomorphisms

$$\mathcal{G}\cong p_{2*}\mathcal{H}om(p_1^*\mathcal{F},\Delta_!(\mathbf{1}_X)),\qquad \mathcal{F}\cong p_{1*}\mathcal{H}om(p_2^*\mathcal{G},\Delta_!(\mathbf{1}_X)).$$

Proposition 4.25. *Let $\mathcal{F}\in D(X)$. $\mathcal{F}$ is f-prim if and only if the natural map*

$$f_!(\mathcal{F}\otimes p_{2*}\mathcal{H}om(p_1^*\mathcal{F},\Delta_!(\mathbf{1}_X)))\to f_*\mathcal{H}om(\mathcal{F},\mathcal{F})$$

is an isomorphism on global sections.

We will prove one important result on prim morphisms before proceeding.

Proposition 4.26. *Let $D\colon \mathrm{Corr}(\mathcal{C}, E) \to \mathcal{C}\mathrm{Alg}(\mathrm{Cat}_\infty)$ be a six-functor formalism induced by an I and P decomposition on $(\mathcal{C}, E)$ (in the sense of Theorem 4.7) such that P is left-cancellative. Then, any morphism $(f\colon X \to Y) \in P$ is prim.*

We will need the following result of Heyer-Mann:

Lemma 4.27 ([HM24], Corollary 4.5.5)**.** *$f\colon X \to Y$ is prim if and only if the natural map*

$$f_! p_{2*} \Delta_!(\mathbf{1}_X) \to f_*(\mathbf{1}_X)$$

is an isomorphism.

This will follow from the following lemma:

Lemma 4.28. *Let $D\colon \mathrm{Corr}(\mathcal{C}, E) \to \mathcal{C}\mathrm{Alg}(\mathrm{Cat}_\infty)$ be as above with decomposition (I, P). Let $f \in P$. Then, the diagonal $\Delta\colon X \to X \times_Y X$ is also in P.*

Proof of Lemma 4.28. We have the pullback diagram

$$\begin{array}{ccc} X \times_Y X & \xrightarrow{p_2} & X \\ \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle f} \\ X & \xrightarrow{f} & Y \end{array}$$

so p_1 and p_2 are base changes of f and thus in P. However, we then have that

$$\mathrm{id}_X = p_2 \circ \Delta,$$

hence Δ is in P by left-cancellativity. □

Proof of Proposition 4.26. Indeed, since $f \in P$, we know that $f_* \simeq f_!$. Since Δ is in P by Lemma 4.28, we also have $\Delta_* \simeq \Delta_!$. Hence, we have

$$f_! p_{2*} \Delta_!(\mathbf{1}_X) \simeq f_* p_{2*} \Delta_*(\mathbf{1}_X) \simeq f_* \mathrm{id}_{X*}(\mathbf{1}_X) \simeq f_*(\mathbf{1}_X)$$

since $\mathrm{id}_X = p_2 \circ \Delta$. We conclude by Lemma 4.27 □

In particular, this implies

Corollary 4.29. *Let $(\mathcal{C}, \mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0}, \mathcal{C}^0)$ be a derived algebraic context. Any morphism $f\colon A \to B$ in $\mathrm{Aff}_{\mathcal{C}}$ is prim.*

Proof. In Proposition 4.13, we showed that (equivalences, all) induces an I and P decomposition on $(\mathrm{Aff}_{\mathcal{C}}, \mathrm{all})$. For the second statement, the class of all morphisms is obviously left-cancellative, so we apply Proposition 4.26. □

5 Bornological Spaces and $D(\mathrm{CBorn}_{\mathbb{C}})$

The idea of derived analytic geometry is to consider stacks and schemes in a relative geometry tuple of analytic origin. A good setting for such a theory turns out to be complete bornological spaces (or the Ind-category of Banach spaces), an alternative approach to functional analysis via bounded subsets first noticed by Mackey [Mac45; Mac46] and the Bourbaki group [Bou55], and subsequently developed by Hogbe-Nlend [Hog66] and Waelbroeck [Wae67]. It turns out that although complete bornological spaces do not form an abelian category, they form a **quasi-abelian** category, a category good enough to still yield a triangulated derived category with canonical t-structure (as first observed in [Sch99]). Recent work by Kelly [Kel24] extends this to a symmetric monoidal stable ∞-category, similarly to the case for $D(R)$ of a ring R. Furthermore, this category has a clear collection of well-behaved compact projective generators, thereby forming a derived algebraic context and thus a theory of relative DAG. This differs from the condensed setting [CS26] [Sch26], where geometry is carried out relative to the derived category of liquid vector spaces, which form an abelian category.

5.1 Bornological Spaces, a Crash Course

The goal of this subsection is twofold:

1. To define bornological spaces, define a notion of completeness for them, and show that complete bornological spaces form a (co)complete closed symmetric monoidal quasi-abelian category.
2. To discuss the relationship between the category of complete bornological spaces and the Ind-category of Banach spaces. This often makes describing complete bornological spaces much easier.

Much of the results in this subsection were first proven either by Houzel [Hou06] or by Schneiders [Sch99]. Briefer discussions can be found in [Bam14; BB16].

5.1.1 Bornological Sets

Definition 5.1. Let X be a set. A **bornology** on X is a set of **bounded** subsets $\mathcal{B} \subseteq \mathcal{P}(X)$ such that:

1. $\mathcal{B}$ covers X, i.e. $\bigcup_{B\in\mathcal{B}} B = X$.
2. $\mathcal{B}$ is closed under containment: if $A \subseteq B$ and $B \in \mathcal{B}$, then $A \in \mathcal{B}$.
3. $\mathcal{B}$ is closed under finite union: If $A_1, \cdots, A_n \in \mathcal{B}$, then $\bigcup_i A_i \in \mathcal{B}$.

We call the pair $(X, \mathcal{B})$ a **bornological set**.

Here are some important examples of bornological sets:

Example 5.2.

1. Let X be a topological vector space over a non-discretely valued field K. Then, the set of bounded subsets of X forms a bornology on X. We call this bornology the **Von Neumann bornology**.
2. The bornology consisting only of the empty set on X and X itself is called the **indiscrete bornology**.
3. Let $f\colon X \to Y$ be a map of sets, with $(Y, \mathcal{B})$ a bornological set. We define the **pullback bornology** on X to be the bornology $f^*(\mathcal{B})$ given by $f^{-1}(B)$ for all $B \in \mathcal{B}$.
4. Let $g\colon X \to Y$ be a map of sets, with $(X, \mathcal{B})$ a bornological set. We define the **pushforward bornology** on Y to be the bornology $g_*(\mathcal{B})$ given by $g(B)$ for all $B \in \mathcal{B}$.

The morphisms in the category of bornological sets are as follows:

Definition 5.3. Let $(X, \mathcal{B})$, $(Y, \mathcal{B}')$ be bornological sets. A map of sets $f\colon X \to Y$ is **bounded** if for all $B \in \mathcal{B}$, $f(B) \in \mathcal{B}'$. We denote the category of bornological sets, with objects bornological sets and morphisms bounded maps, by Born.

We can describe subspaces and quotients in the following way:

Definition 5.4.
1. An injective map $\iota\colon (X, \mathcal{B}) \to (Y, \mathcal{B}')$ of bornological sets identifies X as a **subspace** of Y if $\mathcal{B} = \iota^*(\mathcal{B}')$.
2. A surjective map $\pi\colon (X, \mathcal{B}) \to (Y, \mathcal{B}')$ of bornological sets identifies Y as a **quotient** of X if $\mathcal{B}' = \pi_*(\mathcal{B})$.

This allows us to define limits and colimits of bornological sets by way of (co)products and (co)equalizers:

Definition 5.5. Let $f, g\colon X \to Y$ be bounded maps of bornological sets. We define the **equalizer** of f and g, $\mathrm{eq}(f, g)$, to be the set theoretic equalizer endowed with the pullback bornology from the inclusion $\mathrm{eq}(f, g) \hookrightarrow X$. We define the **coequalizer** of f and g, $\mathrm{coeq}(f, g)$ to be the set theoretic coequalizer endowed with the pushforward bornology from the projection $Y \to \mathrm{coeq}(f, g)$.

Definition 5.6. Let $\{X_i\}_{i \in I}$ be a collection of bornological sets. The **product** is defined to be the set theoretic product $\prod_i X_i$ endowed with the bornology.

$$\left\{ (B_i)_{i \in I} \;\middle|\; B_i \in \mathcal{B}_i \;\forall\; i \in I \right\}$$

The **coproduct** is the set theoretic coproduct $\coprod_i X_i$ endowed with the bornology

$$\left\{ \coprod_{i \in I} B_i \;\middle|\; B_i = \varnothing \;\forall \text{ but finitely many } i \in I \right\}.$$

By the fact that any small (co)limit can be expressed through a series of (co)products and (co)equalizers, this yields:

Theorem 5.7. Born *is complete and cocomplete.*

5.1.2 Complete Bornological Spaces

Having described bornologies on sets, we now move toward describing bornologies on vector spaces. We will show that the category of complete bornological spaces over a Banach field is quasi-abelian and that it is equivalent to the Ind-category of Banach spaces. We will also describe the bornological tensor product and internal Hom, yielding a closed symmetric monoidal structure. Throughout, we fix a complete field K viewed with its von Neumann bornology:

Definition 5.8. A **bornological space** V over K is a K-vector space V equipped with a bornology $\mathcal{B}$ such that:

1. The addition map $m\colon V \times V \to V$ is bounded.
2. The scalar multiplication map $K \times V \to V$ is bounded.

Equivalently, V is a K-vector space object in the category of bornological sets. The morphisms in the category of bornological spaces are bounded linear maps, and we denote the resulting category $\mathrm{Born}(\mathrm{VS}_K)$.

Let's describe a few properties of $\mathrm{Born}(\mathrm{VS}_K)$. Firstly, we can describe finite biproducts via:

Definition 5.9. Let $(V, \mathcal{B})$ and $(W, \mathcal{B}')$ be bornological spaces over K. We define the **direct sum** of V and W to be the usual direct sum, endowed with the bornology given by (B, B') for $B \in \mathcal{B}$, $B' \in \mathcal{B}'$.

This description of the direct sum also upscales to infinite coproducts. Next, the 0 vector space with the indiscrete bornology is clearly a bornological space, hence the notion of zero morphism makes sense in $\mathrm{Born}(\mathrm{VS}_K)$. This implies that $\mathrm{Born}(\mathrm{VS}_K)$ is additive. Furthermore, products and equalizers remain unchanged in the category of bornological spaces due to the clear existence of an adjunction $\mathrm{Born}(\mathrm{VS}_K) \to \mathrm{VS}_K$. Hence, the notion of kernel is clear in the category of bornological spaces over K:

Definition 5.10. Let $f\colon V \to W$ be a bounded linear map of K-vector spaces. The **kernel** of f is defined to be the set-theoretic equalizer $\mathrm{eq}(f, 0_{VW})$ endowed with the pullback bornology.

Describing cokernels is not too difficult, given that we already have a good notion of quotient bornology:

Definition 5.11. Let $f\colon V \to W$ be a bounded linear map of K-vector spaces. The **cokernel** of f is defined to be $W/\mathrm{im}(f)$, equipped with the pushforward bornology.

That the cokernel is well-defined rests on the fact that we can endow $\mathrm{im}(f)$ with the subspace bornology. Hence, $\mathrm{Born}(\mathrm{VS}_K)$ is cocomplete. We will continue further discussions of abelian properties of bornological spaces later after restricting to complete bornological spaces. However, we note that this allows us to give an important example of a bornological vector space that we will use later:

Example 5.12. Let X be an infinite-dimensional vector space over $\mathbb{C}$. Then, writing $X = \varinjlim_\alpha X_\alpha$, where X_α are the finite dimensional subspaces, and equipping each X_α with its von Neumann bornology yields a bornology on X. We call this the **fine bornology** on X and denote it X^{fin}. A particularly important example later will be $\mathbb{C}[X]^{fin}$.

We now come to defining a notion of completeness for bornological spaces and inspecting its interaction with directed systems of Banach spaces. This relies on the following definitions relating bornological spaces to convexity:

Definition 5.13. Let V be a K-vector space. A subset $A \subseteq V$ is said to be **absolutely convex** (or a **disk**) if for all $a, b \in K$ with $|a| + |b| \leq 1$, $aA + bA \subseteq A$.

Given an absolutely convex subset of a K-vector space $A \subseteq V$, we can equip it with the **gauge seminorm** $\mu_A\colon V \to \mathbb{R}_{\geq 0}$, given by

$$\mu_A(x) = \inf_{\lambda \in K} \{|\lambda| \mid x \in \lambda A\}.$$

In the case that V is bornological, these give natural bornological subspaces of V by considering the von Neumann bornology. From this, we define:

Definition 5.14.

1. A bornological space V is **of convex type** if there exists a basis for the bornology consisting of absolute convex sets of V. In other words, letting $\mathcal{B}_V$ denote the set of absolutely convex subsets of V, we can write V as a filtered colimit

$$V \cong \varinjlim_{B \in \mathcal{B}_V} V_B$$

with respect to inclusion of absolutely convex subsets. We denote the category of bornological spaces of convex type by Born_K.

2. A bornological space V of convex type is said to be **separated** if in the colimit presentation

$$V \cong \varinjlim_{B \in \mathcal{B}_V} V_B,$$

each V_B corresponds to a normed vector space. We denote the category of separated bornological spaces by $\mathrm{SBorn}_K \subseteq \mathrm{Born}_K$.

3. A separated bornological space V is **complete** if the colimit

$$V \cong \varinjlim_{B \in \mathcal{B}_V} V_B$$

admits a final subsystem consisting of V_B's that are K-Banach spaces. We denote the full subcategory of complete bornological spaces by $\mathrm{CBorn}_K \subseteq \mathrm{SBorn}_K$

These are both reflective subcategories of Born_K:

Proposition 5.15. *The inclusion functors* $\mathrm{CBorn}_K \hookrightarrow \mathrm{SBorn}_K$ *and* $\mathrm{SBorn}_K \hookrightarrow \mathrm{Born}_K$ *both admit left adjoints, which we call* ***completion*** *and* ***separation*** *respectively. They are denoted* $\widehat{(-)}$ *and* $(-)^{sep}$*, and computed in the following manner: for* V *a bornological space of convex type, write*

$$V \simeq \varinjlim_{B \in \mathcal{B}_V} .$$

Then, we have

$$V^{sep} = \varinjlim_{B \in \mathcal{B}_V} (V_B / \overline{\{0\}}) = \varinjlim_{B \in \mathcal{B}_V} V_B^{sep}$$

and

$$\widehat{V} = \varinjlim_{B \in \mathcal{B}_V} \widehat{V_B^{sep}},$$

where the completion on the right-hand side is the completion of normed spaces.

From the above presentation, it's clear that there are functors between CBorn_K and the Ind-category of Banach spaces $\mathrm{Ind}(\mathrm{Ban}_K)$. They have a particular relationship with each other that is important to spell out:

Theorem 5.16. *There is an adjoint pair of functors*

$$\mathrm{CBorn}_K \underset{\text{diss}}{\overset{\text{colim}}{\longleftrightarrow}} \mathrm{Ind}(\mathrm{Ban}_K)$$

Furthermore, diss *is fully faithful, identifying* CBorn_K *with the full subcategory of* $\mathrm{Ind}(\mathrm{Ban}_K)$ *generated by essentially monomorphic* Ind*-systems, denoted* $\mathrm{Ind}^m(\mathrm{Ban}_K)$.

Proof. By earlier work, we know that CBorn_K is cocomplete, and that Ban_K embeds into CBorn_K via endowing a Banach space with the von Neumann bornology. The functor $\mathrm{Ind}(\mathrm{Ban}_K) \to \mathrm{CBorn}_K$ then follows by extending $\mathrm{Ban}_K \to \mathrm{CBorn}_K$ by colimits. The functor in the other direction follows from our earlier work. The (co)unit conditions obviously exist on colim $\circ$ diss and diss $\circ$ colim respectively from the presentation of $V \cong \varinjlim V_B$ for a bornological space V. Furthermore, the definition of a complete bornological space is that the counit of the adjunction is an isomorphism, i.e. diss is fully faithful.

For a complete bornological space V, the colimit is taken over the poset of absolutely convex subsets under inclusion. Hence, the system is monomorphic by construction, so diss naturally lands in the category of monomorphic Ind-systems of Banach spaces. Furthermore, if $\{V_i\}_{i \in I}$ is a monomorphic Ind-Banach system, then colim is clearly fully faithful. Hence,

the unit of the adjunction when restricted to $\mathrm{Ind}^m(\mathrm{Ban}_K)$ must be an isomorphism, i.e. $\mathrm{diss}(\mathrm{colim}(V_i)) \cong \{V_i\}_{i\in I}$. This implies that diss is essentially surjective onto $\mathrm{Ind}^m(\mathrm{Ban}_K)$, so we have an equivalence $\mathrm{CBorn}_K \cong \mathrm{Ind}^m(\mathrm{Ban}_K)$. □

Remark 5.17. Unfortunately, CBorn_K and $\mathrm{Ind}(\mathrm{Ban}_K)$ are not equivalent. The easiest way to see that this is to observe that CBorn_K is concrete (i.e. it admits a faithful functor to Sets) and $\mathrm{Ind}(\mathrm{Ban}_K)$ is not. This is because there are non-zero Ind-systems that have colimit 0. Although the two categories are not equivalent, we will show that they admit the same class of projective objects. This means that the resulting derived categories are equivalent, leading to the same theory of derived analytic geometry. In practice, both perspectives are helpful. As mentioned before, CBorn_K admits a faithful functor to Sets, making its objects much easier to work with once they are understood and categorical properties easier to understand. On the other hand, it may be easier to observe that certain spaces have Ind-Banach structures than complete bornological ones. For example, it is immediately obvious that for $r > 0$, the algebra

$$W_r = \left\{ f(z) = \sum_{n=0}^{\infty} a_n z^n \;\middle|\; f(z) \text{ converges on } D_\rho(0) \;\forall\; \rho > r \right\}$$

has an Ind-Banach structure as the colimit of the algebras of convergent functions on $D_\rho(0)$ for all $\rho > r$ under restriction.

By essentially the same proof, we can also identify Born_K and SBorn_K with $\mathrm{Ind}^m(\mathrm{sNrm}_K)$ and $\mathrm{Ind}^m(\mathrm{Nrm}_K)$, where sNrm_K and Nrm_K are the categories of seminormed and normed spaces respectively.

Another important aspect of bornological spaces of convex type is their interaction with topological vector spaces, specifically locally convex vector spaces. Let LCVS_K be the category of locally convex vector spaces over K. The fact that there is a functor $\mathrm{LCVS}_K \to \mathrm{Born}_K$ is clear by the existence of the von Neumann bornology. To get a functor in the opposite direction, we need to introduce a new notion of subspace for bornological spaces:

Definition 5.18. Let V be a bornological space.

1. Let $S \subseteq V$ be a subset. We say that a set A **absorbs** S if there exists $r > 0$ such that $cA \supseteq S$ for all scalars $c \in K$ with $|c| > r$.

2. A subset $A \subseteq V$ is **bornivorous** if it absorbs B for all $B \in \mathcal{B}$. We may also call A a **bornivore**.

Proposition 5.19. *Given a bornological space V over K, we define a topology on V by taking its basis to be the bornivorous sets. This yields a functor $(-)^t \colon \mathrm{Born}_K \to \mathrm{LCVS}_K$ and an adjoint pair*

$$\mathrm{Born}_K \underset{(-)^b}{\overset{(-)^t}{\leftrightarrows}} \mathrm{LCVS}_K$$

where the functor $(-)^b$ is endowing a locally convex vector space with its von Neumann bornology.

Proof. We first show that $V^t \in \mathrm{LCVS}_K$. Let $\mathfrak{U}_V$ denote the collection of bornivorous disks of V and let $\mathcal{B}$ denote the bornology. We show two closure properties. Firstly, we show that each $A \in \mathfrak{U}_V$ is absorbing. The bornology $\mathcal{B}_V$ covers V and is closed under subsets, so every singleton is bounded. Since A is bornivorous, it absorbs $\{x\}$, i.e. $x \in cA$ for some $c \in K^\times$. Hence, A is absorbing.

Next, we show that $\mathfrak{U}_V$ is closed under finite intersections. If $A_1, A_2 \in \mathfrak{U}_V$, then $A_1 \cap A_2$ is a disk. Given $B \in \mathcal{B}$, pick $c_i \in K^\times$ with $B \subseteq c_i A_i$ and let $c \in \{c_1, c_2\}$ have the larger absolute value. Since the A_i are balanced, $c_i A_i \subseteq cA_i$, so $B \subseteq cA_1 \cap cA_2 = c(A_1 \cap A_2)$. Hence, $A_1 \cap A_2$ is bornivorous.

Now let V^t be the locally convex topology on the underlying space of V generated by the family of seminorms $\{\mu_A \mid A \in \mathfrak{U}_V\}$. A family of seminorms always defines a unique locally convex topology, and since $\mathfrak{U}_V$ is closed under finite intersections and nonzero scalar multiples, the sets $A \in \mathfrak{U}_V$ themselves form a base of neighborhoods of 0. Hence, $V^t \in \mathrm{LCVS}_K$.

We now show that $(-)^t$ is functorial. Let W be a bornological space of convex type with bornology $\mathcal{B}'$, let $f\colon V \to W$ be a bounded linear map, and let $A \subseteq W$ be a bornivorous disk. Then $f^{-1}(A)$ is a disk, and it is bornivorous. Indeed, for $B \in \mathcal{B}$ we have $f(B) \in \mathcal{B}'$, so $f(B) \subseteq cA$ for some $c \in K^\times$, and thus $B \subseteq c\, f^{-1}(A)$. Hence, $f^{-1}(A)$ is a 0-neighborhood of V^t, so $f\colon V^t \to W^t$ is continuous. Since t does not change the underlying space or linear map, it preserves identities and composition, so $(-)^t\colon \mathrm{Born}_K \to \mathrm{LCVS}_K$ is a functor.

We now show that $(-)^b$ is a functor into Born_K. For $W \in \mathrm{LCVS}_K$, let W^b denote W equipped with its von Neumann bornology. This bornology is of convex type. Indeed, if B is bounded and $U \subseteq V$ is an absolutely convex neighborhood of 0, then $B \subseteq \lambda U$ for some λ, so the absolutely convex hull $\Gamma(B)$ satisfies $\Gamma(B) \subseteq \Gamma(\lambda U) = \lambda U$. Since the absolutely convex neighborhoods form a basis, $\Gamma(B)$ is bounded. Hence, the bounded disks form a basis of the bornology and $V^b \in \mathrm{Born}_K$. If $g\colon V \to W$ is a continuous linear map and $B \subseteq V$ is bounded, then for an absolutely convex neighborhood of 0 $U \subseteq W$, the set $g^{-1}(U)$ is a neighborhood of 0, so $B \subseteq \lambda g^{-1}(U)$ and $g(B) \subseteq \lambda U$. Hence, $g(B)$ is bounded and $g\colon V^b \to W^b$ is bounded. So, $(-)^b$ is a functor.

We finally show the adjunction $(-)^t \dashv (-)^b$. Both functors act as the identity on underlying vector spaces and linear maps, so for $V \in \mathrm{Born}_K$ and $W \in \mathrm{LCVS}_K$ the sets $\mathrm{Hom}_{\mathrm{LCVS}_K}(V^t, W)$ and $\mathrm{Hom}_{\mathrm{Born}_K}(V, W^b)$ are both subsets of the linear maps $V \to W$. We will show they coincide. Naturality in V and W is then immediate, since every comparison map is the identity on underlying linear maps.

We show the inclusion $\subseteq$ first. Suppose $u\colon V^t \to W$ is a continuous linear map. Let $B \in \mathcal{B}$ and let $U \subseteq W$ be an absolutely convex neighborhood of 0. Then $u^{-1}(U)$ is a neighborhood of 0 of V^t. So, it contains a bornivorous disk A. Since A absorbs B, we have $B \subseteq cA$ for some $c \in K^\times$. Thus, $u(B) \subseteq cu(A) \subseteq cU$. This implies that $u(B)$ is absorbed by every 0-neighborhood of W, i.e. $u(B)$ is bounded. So, $u\colon V \to W^b$ is bounded.

We finish with the inclusion $\supseteq$. Suppose $u\colon V \to W^b$ is bounded and let $U \subseteq W$ be an absolutely convex neighborhood of 0. Then, $u^{-1}(U)$ is a disk, and it is bornivorous. Indeed, for $B \in \mathcal{B}$, the set $u(B)$ is bounded in W, so $u(B) \subseteq cU$ for some $c \in K^\times$. Hence,

$B \subseteq c\,u^{-1}(U)$. This implies that $u^{-1}(U)$ is a neighborhood of 0 in V^t. Since as the absolutely convex neighborhoods of 0 in W form a base, we get that $u\colon V^t \to W$ is continuous.

Hence $\mathrm{Hom}_{\mathrm{LCVS}_K}(V^t, W) = \mathrm{Hom}_{\mathrm{Born}_K}(V, W^b)$ naturally, exhibiting $(-)^t$ as left adjoint to $(-)^b$. □

Earlier, we laid out some abelian properties of bornological spaces. It's clear that Born_K remains additive by being a subcategory of $\mathrm{Born}(\mathrm{VS}_K)$. However, the other properties descend to Born_K:

Proposition 5.20. Born_K *is closed under (co)kernels and (co)products.*

Though the final condition for a category to be abelian is usually expressed in terms of binormality – every monomorphism (resp. epimorphism) is a kernel (resp. cokernel) – it can be equivalently expressed as an abstraction of the first isomorphism theorem. Indeed, if $\mathcal{A}$ is an additive category with kernels and cokernels, then for a morphism $f\colon A \to B$ in $\mathcal{A}$, we can form the image $\mathrm{im}(f)$ as the kernel of the quotient $B \to \mathrm{coker}(f)$. This exactly comports with our intuition that the cokernel of f is "$B/\mathrm{im}(f)$". Dually, we can form the coimage $\mathrm{coim}(f)$ as the cokernel of the inclusion $\ker(f) \to A$, i.e. "the quotient $A/\ker(f)$". The universal property of cokernel induces a map $u\colon \mathrm{coim}(f) \to \mathrm{im}(f)$, where the image is the categorical image of f. If the map u is an isomorphism, we say that f is **strict**. If $\mathcal{A}$ is abelian, this identifies $\mathrm{coim}(f) \cong \mathrm{im}(f)$.

Sadly, this shows that CBorn_K (and by extension, $\mathrm{Ind}(\mathrm{Ban}_K)$) is not abelian. In these categories, for a morphism $f\colon V \to W$, the coimage and image are defined as they usually are on vector spaces, equipped with the pushforward bornology. Hence, the map $u\colon \mathrm{coim}(f) \to \mathrm{im}(f)$ is bijective. However, the bornologies on the two may not agree with each other, meaning that u is not always an isomorphism.

Furthermore, one can have morphisms with dense image that are not surjective, which are epimorphisms. We'd like to have some way of distinguishing these from strict epimorphisms, but this is not practical on the level of CBorn_K itself. Here is the classical example:

Example 5.21. Let K be a p-adic field, and let $K\langle x\rangle$ be the Tate algebra in one variable, i.e. the ring of analytic functions on the closed unit disk $\mathbb{D}$. The de Rham cohomology of D can be computed as the cohomology of the complex

$$0 \longrightarrow K\langle x\rangle \longrightarrow K\langle x\rangle \mathrm{d}x \longrightarrow 0,$$

where the middle arrow is $d(f) = \frac{df}{dx}\mathrm{d}x$. This is a valid complex in CBorn_K, since all of the terms are naturally Banach spaces. Furthermore, the image of d is dense since polynomials are dense in $K\langle x\rangle \mathrm{d}x$. However, d is not surjective.

Taking the algebraic cokernel of d would give that $H^1_{\mathrm{dR}}(D)$ is infinite-dimensional and equipped with the indiscrete topology. This is a very poor property for an affinoid domain to have. However, on the other hand, since the image of d is dense, taking the Banach cokernel of d would be the quotient $K\langle x\rangle \mathrm{d}x/\overline{\mathrm{im}(d)} = 0$. This does nothing to distinguish d from the identity on $K\langle x\rangle \mathrm{d}x$.

Despite these issues, there is still a residue of abelian structure on the two categories:

Proposition 5.22. Born_K *and* $\mathrm{Ind}(\mathrm{Ban}_K)$ *are* ***quasi-abelian****, i.e. they are additive, have all kernels and cokernels, and furthermore:*

1. *Strict epimorphisms are stable under pullback; if* $f\colon V \to W$ *is a strict epimorphism and* $g\colon W' \to W$ *is bounded, then given the pullback diagram*

$$\begin{array}{ccc} V' & \xrightarrow{f'} & W' \\ \downarrow & & \downarrow{\scriptstyle g} \\ V & \xrightarrow[f]{} & W \end{array}$$

   f' must also be a strict epimorphism.

2. *Strict monomorphisms are stable under pushout; if* $f\colon V' \to W'$ *is a strict monomorphism and* $g\colon V' \to V$ *is bounded, then given the pushout diagram*

$$\begin{array}{ccc} V' & \xrightarrow{f'} & W' \\ \downarrow{\scriptstyle g} & & \downarrow \\ V & \xrightarrow[f]{} & W \end{array}$$

   f must also be a strict monomorphism.

To prove this theorem, we will need a few preliminary results. First is the following proposition:

Proposition 5.23. *If* $\mathcal{C}$ *is quasi-abelian, then* $\mathrm{Ind}(\mathcal{C})$ *is quasi-abelian.*

We'll need a few preliminary lemmas and some setup. Recall that:

1. For a finite category $\mathcal{J}$, any functor $\mathcal{J} \to \mathrm{Ind}(\mathcal{C})$ is isomorphic to "$\mathrm{colim}_{i\in\mathcal{I}}$" D_i for some small filtered category $\mathcal{I}$ and some diagram $D\colon \mathcal{I} \to \mathcal{C}^{\mathcal{J}}$, where the quotation marks stand for the formal colimit of presheaves within $\mathrm{PreSh}(\mathcal{C})$, where here our presheaves are additive.

2. For D as above we can identify

$$\lim_{\mathcal{J}} \text{“}\mathrm{colim}_{\mathcal{I}}\text{”}\, D \cong \text{“}\mathrm{colim}_{\mathcal{I}}\text{”} \lim_{\mathcal{J}} D,$$

   i.e. the functor "$\mathrm{colim}_{\mathcal{I}}$"" is exact. This follows by the fact that these (co)limits are computed in the additive presheaf category, and hence pointwise in Ab where the analogous result holds.

More details on the above can be found in Chapter 6 of [KS06].

Lemma 5.24. *Let $\mathcal{C}$ be a quasi-abelian category, and let $\mathcal{I}$ be a filtered category. Let $f\colon F \to G$ be a morphism in $\mathcal{C}^{\mathcal{I}}$ with $f_i\colon F(i) \to G(i)$ a strict epimorphism for all i. Then, "$\operatorname{colim}_{\mathcal{I}}$"" f_i is a strict epimorphism in* $\mathrm{PreSh}(\mathcal{C})$.

Proof. Kernels and cokernels are computed pointwise in $\mathrm{PreSh}(\mathcal{C})$. Hence, since each f_i is a strict epimorphism and hence $\operatorname{coker}(\ker(f)) \to G_i$ is an isomorphism, we get that $\operatorname{coker}(\ker(f)) \to G$ is an isomorphism in $\mathcal{C}^{\mathcal{I}}$. This then follows by the fact that "$\operatorname{colim}_{\mathcal{I}}$"" is exact. □

Lemma 5.25. *Let $f\colon F \to G$ be a strict epimorphism in $\mathcal{C}^{\mathcal{I}}$, and let $g\colon G' \to G$ be an arbitrary morphism in $\mathcal{C}^{\mathcal{I}}$. Then, the cospan*

$$F \to G \leftarrow G'$$

is isomorphic to "$\operatorname{colim}_{i\in\mathcal{I}}$"$(F_i \to G_i \leftarrow G'_i)$, *where each map $f_i\colon F_i \to G_i$ is a strict epimorphism.*

Proof. By 1) from earlier, we can choose a representation "colim_I"$\left(F_i \xrightarrow{a_i} G_i \xleftarrow{g_i} G'_i\right)$ of the cospan, so $f =$ "colim_I"a_i. Factor each a_i through its coimage,

$$a_i = \left(C_i \xrightarrow{c_i} G_i\right) \circ \left(F_i \xrightarrow{e_i} C_i\right), \qquad C_i := \operatorname{coker}(\ker a_i),$$

with e_i a strict epimorphism. Since "colim_I" is exact, we have "colim_I"$C_i = \operatorname{coker}(\ker f) = \operatorname{coim}(f)$, while "$\operatorname{colim}_I$"$e_i$ is the canonical strict epimorphism $F \to \operatorname{coim}(f)$ and "colim_I"c_i is the canonical map $\operatorname{coim}(f) \to G$. Because f is a strict epimorphism, the latter is an isomorphism $\theta\colon$ "colim_I"$C_i \xrightarrow{\sim} G$.

Transporting the cospan along θ, it becomes

$$F \xrightarrow{\theta^{-1}f} \text{"}\operatorname{colim}_I\text{"}C_i \xleftarrow{\theta^{-1}g} G'.$$

We note that $\theta^{-1}f$ can be rewritten as "colim_I"e_i. Using

$$\operatorname{Hom}_{\mathrm{Ind}(\mathcal{C})}\Big(G',\ \text{"}\operatorname{colim}_I\text{"}C\Big) = \varprojlim_i\ \text{"}\operatorname{colim}_{i'}\text{"}\operatorname{Hom}_{\mathcal{C}}\left(G'_i, C_{i'}\right)$$

and the filteredness of I, after a cofinal reindexing, the morphism $\theta^{-1}g$ is represented levelwise by maps $g_i\colon G'_i \to C_i$; pulling the system $C_\bullet$ back along the cofinal functor does not change the strictness of the e_i. Relabelling C_i as G_i and e_i as f_i gives the claim. □

We note that both of these lemmas have dual statements.

Proof of Proposition 5.23. By the two recalled facts, $\mathrm{Ind}(\mathcal{C})$ is additive and has kernels and cokernels (computed pointwise in Ab), hence is pre-abelian; it remains to verify the two stability axioms.

1. Let
$$\begin{array}{ccc} X \times_Y Y' & \xrightarrow{f'} & Y' \\ \downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Y. \end{array}$$
be a pullback square in $\mathrm{Ind}(\mathcal{C})$, with f a strict epimorphism. By Lemma 5.25, we can represent the cospan as "colim_I"$\left(X_i \xrightarrow{f_i} Y_i \xleftarrow{g_i} Y_i'\right)$ with each f_i a strict epimorphism. Form the pullbacks $P_i := X_i \times_{Y_i} Y_i'$ in C, with projections $f_i' \colon P_i \to Y_i'$. Since C is quasi-abelian, each f_i' is a strict epimorphism. A pullback is a finite limit, so it commutes with "colim_I": thus $X \times_Y Y' =$ "colim_I" P_i and $f' =$ "colim_I" f_i'. By Lemma 5.24, f' is a strict epimorphism.

2. This is exactly dual. Represent the span via the dual half of Lemma 5.25 with each leg a strict monomorphism, form the pushouts levelwise (a finite colimit, again commuting with "colim_I"), apply the $\mathcal{C}$-level quasi-abelian axiom to each component, and conclude with the dual half of Lemma 5.24.

□

Remark 5.26. This is, to our knowledge, the first full proof of Proposition 5.23. The literature makes references to [Sch87] and the appendices of [AM06], but neither contain full proofs.

Next is a study of strict morphisms of complete bornological spaces:

Proposition 5.27. *A bounded morphism of bornological spaces $f \colon V \to W$ over K is strict if and only if for any bounded subset $B' \subseteq W$, there exists a bounded subset $B \subseteq V$ such that*
$$B' \cap f(V) = f(B)$$

Proof. Since the category of K-vector spaces is abelian, we already know that the induced map $u \colon \mathrm{coim}(f) \to \mathrm{im}(f)$ is bijective. So, we just need to check that the two bornologies agree. However, this follows by definition of the pushforward bornology on both constructions. □

Proof of Proposition 5.22. By Proposition 5.23, we just need to show that Ban_K is quasi-abelian to show that $\mathrm{Ind}(\mathrm{Ban}_K)$ is. In Ban_K, the strict epimorphisms are bounded surjections, and the strict monomorphisms are closed embeddings.

1. Suppose that we have a pullback square in Ban_K
$$\begin{array}{ccc} V' & \xrightarrow{f'} & W' \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle g} \\ V & \xrightarrow[f]{} & W \end{array}$$

where f is a surjective bounded map. Let $w \in W'$. Then, since f is surjective, there exists some $v \in V$ such that $f(v) = g(w)$. By definition of the fiber product, this means that $(v, w) \in V'$, with $f'(v, w) = w$. So, f' is surjective.

2. Suppose we have a pushout square in Ban_K

$$\begin{array}{ccc} V' & \xrightarrow{f'} & W' \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle g} \\ V & \xrightarrow[f]{} & W \end{array}$$

where f' is a closed embedding. Recall that we can describe the map $f\colon V \to W$ as the composition

$$V \hookrightarrow V \oplus W' \to W$$

where $V \oplus W'$ is the projection to $W \cong V \oplus W'/\langle (g(v), 0) - (0, f'(w))\rangle$. That f is injective follows by a dual set theoretic argument to the above, so closedness is the meat of the argument.

We'd like to show that $f(V)$ is closed in W. Let $[(a_n, 0)] \to a$ be a convergent sequence in $f(T)$. Since the subspace $\langle (h(v), 0) - (0, f'(v)\rangle \subseteq V \oplus W'$ is closed, we can lift $\{a_n\}_{n=1}^{\infty}$ to a convergent sequence $(a_n + x, f'(v_n)) \to (a + x, w)$ in $V \oplus W'$. However, since f' is a closed embedding, $w = f'(v)$ for some $v \in V'$. This implies that the projection of $(a + x, w)$, which is the limit of the sequence $[(a_n, 0)]$, is equal to $[(a_n + v', 0)]$ for some $v' \in V'$. Hence, the limit is in $f(V)$, and we are done.

That CBorn_K is additive and has all cokernels follows from the fact that Born_K is additive and has all cokernels. Explicitly, kernels and biproduct remain unchanged from how they are computed in Born_K, while cokernels are completed. Now for the pullback and pushout criteria:

1. Suppose that we have a pullback square in CBorn_K

$$\begin{array}{ccc} V' & \xrightarrow{f'} & W' \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle g} \\ V & \xrightarrow[f]{} & W \end{array}$$

where f is a strict surjective bounded map. f' is surjective for set theoretic reasons as outlined above, so we just need to check compatibility of the bornologies on its image and coimage. Let B be a bounded subset of W'. Since $g(B)$ is a bounded subset of W and since f is strict, there is a bounded subset B' of V such that $g(B) = f(B')$. Let p_1 and p_2 denote the projections $V \oplus W'$ to V and W' respectively. Then, $p_1^{-1}(B') \cap p_2^{-1}(B)$ is a bounded subset of $V \oplus W'$. Furthermore, since $V' \subseteq V \oplus W'$, this implies that $V' \cap p_1^{-1}(B') \cap p_2^{-1}(B)$ is a bounded subset of V'. This implies that $v(V' \cap p_1^{-1}(B') \cap p_2^{-1}(B)) \supseteq B$, and so we are done by Proposition 5.27.

2. Suppose that we have a pushout square in CBorn_K

$$\begin{array}{ccc} V' & \xrightarrow{f'} & W' \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle g} \\ V & \xrightarrow[f]{} & W \end{array}$$

where f' is a strict injective bounded map. f is injective for set theoretic reasons as outlined above, so we just need to check compatibility of the bornologies on its image and coimage. Let π denote the canonical projection $V \oplus W' \to W$. Let B be a bounded subset of W, and let B' be a bounded subset of $V \oplus W'$ such that $\pi(B') = B$. We just need to show that $f^{-1}(\pi(B))$ is bounded in V. Since B' is a bounded subset of $V \oplus W'$, there exists some absolutely convex bounded subsets C and C' of V and W' respectively such that $B' \subseteq C \oplus C'$. Since f' is a strict monomorphism, $f'^{-1}(C')$ is an absolutely convex subset of V'. This implies that

$$f^{-1}(\pi(B')) \subseteq C + h(f'^{-1}(C'))$$

and we are done by Proposition 5.27.

□

5.1.3 Bornological Tensor Products

Finally, we discuss the closed symmetric monoidal structure on CBorn_K:

Definition 5.28. Let V and W be bornological spaces of convex type over K. The **projective tensor product** of V and W is given by the tensor product $V \otimes_K W$ with bornology generated by the absolutely convex hulls of the sets

$$A \otimes B = \{x \otimes y \mid x \in A, y \in B\}.$$

Here, A and B are absolutely convex sets in the bases of the bornologies of V and W respectively. We denote the projective tensor product by $V \otimes_b W$. In the case that V and W are complete, the tensor product is taken to be the completion, denoted $V \widehat{\otimes}_b W$.

Since $\mathrm{Hom}_{\mathrm{Born}_K}(V, W)$ is already a K-vector space, we just need to describe a bornology on it to get an internal hom. This is given by:

Definition 5.29. Let V and W be bornological spaces of convex type over K. The **equiboundedness bornology** on $\mathrm{Hom}_{\mathrm{Born}_K}(V, W)$ is the bornology whose bounded sets are subsets $H \subseteq \mathrm{Hom}_{\mathrm{Born}_K}(V, W)$ such that for all bounded subsets $B \subseteq V$, the set

$$H(B) = \bigcup_{f \in H} f(B)$$

is bounded in W. We denote $\mathrm{Hom}_{\mathrm{Born}_K}(V, W)$ equipped with the equiboundedness bornology by $\underline{\mathrm{Hom}}(V, W)$.

In order to show that $\underline{\mathrm{Hom}}(V,W)$ gives an internal hom in CBorn_K, we first note the following:

Lemma 5.30. *If W is a convex type (resp. separated/complete) bornological space, then $\underline{\mathrm{Hom}}(V,W)$ is convex type (resp. separated/complete).*

The internal Hom in Born_K allows us to show a bornological analogue of the universal property of tensor product:

Proposition 5.31. *Let $\underline{\mathrm{Bi}}_{\mathrm{Born}_K}(V,W;X)$ be the space of bilinear maps $V \times W \to X$ with an analogous equibounded bornology to the above. Then, there is an isomorphism*

$$\underline{\mathrm{Bi}}_{\mathrm{Born}_K}(V,W;X) \cong \underline{\mathrm{Hom}}(V \otimes_b W, X).$$

An analogous statement also holds in CBorn_K.

Proof. That the analogous statement also holds in CBorn_K follows by the fact that completion is a left adjoint. If X is complete, then we have

$$\underline{\mathrm{Hom}}(V \otimes_b W, X) \cong \underline{\mathrm{Hom}}(V \widehat{\otimes}_b W, X).$$

□

We can finally show:

Theorem 5.32. *$- \otimes_b -$ and $\underline{\mathrm{Hom}}(-,-)$ provide Born_K with the structure of a closed symmetric monoidal category. This descends to such a structure on CBorn_K via considering the completed tensor product.*

Proof. The unit of the tensor product is obviously K in its von Neumann bornology. Checking the diagrams is fairly tedious and easy, so we omit it.

For the identification

$$\mathrm{Hom}_{\mathrm{Born}_K}(V \otimes_b W, X) \cong \mathrm{Hom}_{\mathrm{Born}_K}(V, \underline{\mathrm{Hom}}(W,X))$$

we show that $\underline{\mathrm{Bi}}(V,W;X) \cong \mathrm{Hom}_{\mathrm{Born}_K}(V, \underline{\mathrm{Hom}}(W,X))$. The result then follows by Proposition 5.31.

We know that we have an isomorphism on the level of vector spaces by linear algebra, taking the left polar of a bilinear map $V \times W \to X$. So we just need to show that the bornologies agree. Let $H \subseteq \underline{\mathrm{Bi}}(V,W;X)$ be equibounded. Let $H_b \subseteq \underline{\mathrm{Hom}}(V, \underline{\mathrm{Hom}}(W,X))$ denote the set of left polars of H. Then, if $A \subseteq V$ is bounded, we have

$$H_b(A) = \bigcup_{\phi_b \in H_b} \phi_b(A).$$

Hence, we are done if we show that $\phi_b(A) \subseteq \underline{\mathrm{Hom}}(W,X)$ is equibounded. If $B \subseteq W$ is bounded, then

$$\phi_b(A)(B) = \bigcup_{\psi_b \in \phi_b(A)} \psi_b(B) = b_\psi(A \times B)$$

where b_ψ is the bilinear form corresponding to ψ_b. However, b_ψ is an equibounded bilinear map, so $b_\psi(A \times B)$ is bounded. Hence, $\phi_b(A)$ is equibounded, thus $H_b(A)$ is equibounded.

That the result descends to completed tensor product follows by the fact that completion is a left adjoint, similarly to the proof of Proposition 5.31. □

To summarize this entire subsection, we have shown:

Theorem 5.33. CBorn_K *is a closed (co)complete symmetric monoidal quasi-abelian category.*

5.2 Categories of Complexes and Derived Categories of Quasi-Abelian Categories

We've shown that complete bornological spaces form a quasi-abelian category. Such categories carry a well-behaved derived category, due to Schneiders [Sch99] in the triangulated setting and Kelly [Kel24] ∞-categorically. The construction is similar to the abelian case: it is the homotopy category of a model structure on chain complexes. The main subtlety is that, because of non-strict morphisms, the correct weak equivalences are the *strict* quasi-isomorphisms, and the resulting cohomology objects live not in $\mathcal{C}$ but in an enveloping abelian category (the left or right heart). This allows us to detect non-strict quasi-isomorphisms as nontrivial, which was the main goal in the first place.

Throughout, $\mathcal{C}$ is a quasi-abelian category. For the existence statements, we also assume that $\mathcal{C}$ is locally presentable and has enough projectives, both of which hold for CBorn_K by our work throughout this section.

5.2.1 Complexes and the Projective Model Structure

Let $\mathcal{C}$ be a quasi-abelian category. Complexes in $\mathcal{C}$ are defined analogously to how they are in abelian categories:

Definition 5.34. A **chain complex** in $\mathcal{C}$ is a sequence $A_\bullet$ of objects with differentials $d_n \colon A_n \to A_{n-1}$ satisfying $d_{n-1} \circ d_n = 0$. Chain maps and chain homotopies are defined as usual, giving an additive category $\mathrm{Ch}(\mathcal{C})$ and its homotopy category $K(\mathcal{C})$ obtained by quotienting by chain homotopy. We write $X[1]$ for the shift, $X[1]_n := X_{n-1}$ with differential $-d_X$. For a chain map $f \colon X \to Y$, we write $\mathrm{Cone}(f)$ for the mapping cone $\mathrm{Cone}(f)_n = Y_n \oplus X_{n-1}$, with differential

$$\begin{bmatrix} d_Y & f \\ 0 & -d_X \end{bmatrix},$$

fitting into a distinguished triangle

$$X \longrightarrow Y \longrightarrow \mathrm{Cone}(f)$$

in $K(\mathcal{C})$.

The main difference is the notion of exactness, which we have to be careful about due to the existence of non-strict morphisms.

Definition 5.35. A sequence

$$A \xrightarrow{f} B \xrightarrow{g} C$$

in $\mathcal{C}$ is **strictly exact** at B if f is strict and $\ker(g) = \mathrm{im}(f)$. A complex $X_\bullet$ is *strictly acyclic* if it is strictly exact at every degree, i.e. each d_n is strict and $\ker(d_n) = \mathrm{im}(d_{n+1})$.

Definition 5.36. A chain map $f\colon X \to Y$ is a **strict quasi-isomorphism** if $\mathrm{Cone}(f)$ is strictly acyclic. We write W for the class of strict quasi-isomorphisms.

The naive cohomology $\ker(d_n)/\mathrm{im}(d_{n+1})$ taken in $\mathcal{C}$ is poorly behaved, so defining a quasi-isomorphism in terms of it is ill-fit. The main content of [Sch99] is to show that there are well-behaved cohomology functors valued in an abelian category, recovered through the t-structures we construct later, and that $f \in W$ if and only if f induces isomorphisms on them. At the level of the homotopy category, this is given in the following way:

Proposition 5.37 ([Sch99])**.** *The strictly acyclic complexes form a thick triangulated Serre subcategory $\mathcal{N} \subseteq K(\mathcal{C})$, and W is the class of maps whose cone lies in $\mathcal{N}$. Hence, the Verdier quotient*

$$D(\mathcal{C}) := K(\mathcal{C})/\mathcal{N} \cong K(\mathcal{C})[W^{-1}]$$

*is a triangulated category which we call the **derived category** of $\mathcal{C}$.*

Similarly to the classical derived category of an abelian category, we can realize $D(\mathcal{C})$ as the localization of a model structure on $\mathrm{Ch}(\mathcal{C})$. The exact structure of $\mathcal{C}$ supplies a projective class: its projectives are the objects P for which $\mathrm{Hom}_{\mathcal{C}}(P, -)$ carries strict epimorphisms to surjections, and "enough projectives" means every object admits a strict epimorphism from a projective. For $\mathcal{C} = \mathrm{CBorn}_K$, this holds with projectives generated by the $\ell^1(S)$ by 5.50. We will need the following definitions:

Definition 5.38. For a projective object $P \in \mathcal{C}$ and $n \in \mathbb{Z}$, the **sphere** $S^n(P)$ is P concentrated in degree n, and the **disk** $D^n(P)$ is P in degrees n and $n-1$ with identity differential. Let $\mathcal{P}$ be a set of projective generators and set

$$I = \{S^{n-1}(P) \hookrightarrow D^n(P) \mid P \in \mathcal{P},\ n \in \mathbb{Z}\}, \qquad J = \{0 \hookrightarrow D^n(P) : P \in \mathcal{P},\ n \in \mathbb{Z}\}.$$

Each $D^n(P)$ is strictly acyclic and contractible, so the maps in J are strict quasi-isomorphisms.

Theorem 5.39 ([Sch99; Kel24; CH02])**.** *Let $\mathcal{C}$ be a locally presentable quasi-abelian category with enough projectives. Then* $\mathrm{Ch}(\mathcal{C})$ *admits a cofibrantly generated and combinatorial model structure, the* projective model structure*, with generating cofibrations I, generating acyclic cofibrations J, and:*

1. *Weak equivalences given by strict quasi-isomorphisms W.*

2. *Fibrations given by degreewise strict epimorphisms. In particular every object is fibrant.*

3. *Cofibrant objects given by degreewise-projective complexes that are additionally K-projective (homotopically projective). In particular, every bounded-below complex of projectives is cofibrant.*

Proof Sketch. $\mathrm{Ch}(\mathcal{C})$ is locally presentable, so by [Lur09, Proposition A.2.6.13], it suffices to identify W, check that it is an accessible class closed under retracts and the two-out-of-three property, and verify the lifting/smallness conditions for I and J. Closure of W is immediate from Proposition 5.37. The domains and codomains of I, J are small since they are built from the compact generators $\mathcal{P}$. So, the small object argument applies.

A map $f\colon X \to Y$ has the right lifting property against J if and only if for every $P \in \mathcal{P}$ and n, $\mathrm{Hom}(P, X_n) \to \mathrm{Hom}(P, Y_n)$ is surjective. Since $\mathcal{P}$ are projective generators, this means that f is a degreewise strict epimorphism, giving 2). The right lifting property against I characterizes the acyclic fibrations as degreewise strict epimorphisms that lie in W, and the small-object factorizations then produce the cofibration–acyclic-fibration and acyclic-cofibration–fibration factorizations. Identifying the cofibrant objects with the K-projective complexes of projectives is given by constructing iterated pushouts along elements of I, with "acyclic" meant as "strictly acyclic". We leave the rest of the details to [Kel24]. □

Remark 5.40. The hypotheses are used to guarantee enough projectives and the smallness needed for the small object argument, as both are part of the setup we will invoke with CBorn_K. Kelly's work [Kel24] further upgrades this to a symmetric monoidal model structure for the completed tensor product in CBorn_K that we defined in the last section, which is what we will use to make $D(\mathrm{CBorn}_K)$ symmetric monoidal in Section 5.3.

5.2.2 The Derived ∞-Category, Stability and t-Structure

Now that we've discussed the model structure on complexes in a quasi-abelian category, we can move on to defining the derived ∞-category. The construction is analogous to the usual setting in abelian categories, and the derived category has the same homotopical properties as $D(\mathcal{A})$ for an abelian category $\mathcal{A}$ (it is stable, presentable, symmetric monoidal if the original category is, etc.). The main differences are some subtleties in choices of t-structure stemming from various abelian envelopes of a quasi-abelian category $\mathcal{C}$ that we will discuss momentarily.

Definition 5.41. The **derived ∞-category** $\mathcal{D}(\mathcal{C})$ is the underlying ∞-category of the projective model structure of Theorem 5.39. In other words, it is the Dwyer–Kan localization $\mathrm{Ch}(\mathcal{C})[W^{-1}]$. Its homotopy category is the triangulated category $D(\mathcal{C})$ given by Proposition 5.37.

Since the model structure is combinatorial, $\mathcal{D}(\mathcal{C})$ is a presentable ∞-category (this is by [Lur17], Proposition 1.3.4.22 and [Lur09], Section A.3.7). In particular, it admits all small limits and colimits, and is pointed by the zero complex. This is needed to show that it is stable.

The last two things we will need for stability is to give explicit descriptions of loops and suspensions. This is given by the following two lemmas:

Lemma 5.42. *The shift functor* $[1]\colon \mathrm{Ch}(\mathcal{C}) \to \mathrm{Ch}(\mathcal{C})$ *is an equivalence preserving and reflecting* W*, hence induces an equivalence* $[1]\colon \mathcal{D}(\mathcal{C}) \xrightarrow{\sim} \mathcal{D}(\mathcal{C})$ *with inverse* $[-1]$*.*

Proof. $[1]$ is an automorphism of the additive category $\mathrm{Ch}(\mathcal{C})$ with inverse $[-1]$, and $\mathrm{Cone}(f[1]) \cong \mathrm{Cone}(f)[1]$, so it preserves and reflects strict acyclicity. Hence, it preserves and reflects W. An equivalence of categories preserving the localizing class descends to an equivalence of localizations, so we are done. □

Lemma 5.43. *In* $\mathcal{D}(\mathcal{C})$*, the suspension functor is* $\Sigma \simeq [1]$*, and the loop functor is* $\Omega \simeq [-1]$*.*

Proof. To show this, we observe that cofiber sequences in $\mathcal{D}(\mathcal{C})$ are computed by mapping cones. Indeed, for $f\colon X \to Y$, the triangle

$$X \to Y \to \mathrm{Cone}(f)$$

is a homotopy cofiber sequence. We see this by replacing f by a cofibration with cofibrant domain and taking the quotient.

The suspension is the cofiber of $X \to 0$. Hence, by the cone formula

$$\mathrm{Cone}(X \to 0)_n = 0 \oplus X_{n-1} = X[1]_n,$$

so $\Sigma X \simeq X[1]$. Dually, $\Omega \simeq [-1]$. □

We can now prove:

Theorem 5.44. *For* $\mathcal{C}$ *a locally presentable quasi-abelian category with enough projectives,* $\mathcal{D}(\mathcal{C})$ *is a stable presentable* ∞*-category.*

Proof. By the discussion above, $\mathcal{D}(\mathcal{C})$ is presentable. In particular, it is pointed and admits finite limits and colimits. By Lemma 5.43, the loop functor is $\Omega \simeq [-1]$. This is an equivalence by Lemma 5.42. A pointed ∞-category admitting finite limits and colimits is stable when its loop functor is an equivalence ([Lur17], Corollary 1.4.2.27). Hence, $\mathcal{D}(\mathcal{C})$ is stable. □

The last, and arguably most subtle thing we need to do, is to write down the t-structure on $D(\mathcal{C})$. Because there are non-strict morphisms in $\mathcal{C}$, two viable t-structures appear on $D(\mathcal{C})$: the left and right t-structure, equipped with **left** and **right** hearts $LH(\mathcal{C})$ and $RH(\mathcal{C})$ respectively. In the case that $\mathcal{C}$ is abelian, these t-structures collapse into the usual one. Due to this, one can think of the left and right hearts as two different ways of constructing an abelian envelope of $\mathcal{C}$: one by formally forcing all monomorphisms to be strict (the left heart), and one by formally forcing all epimorphisms to be strict (the right heart). It turns out that one forces the other, though the resulting abelian categories are not equivalent. We will explicitly describe the left t-structure, since it is what we will use throughout the rest of this document. Keep in mind that the following definition is more of a definition/theorem, since the left heart can be defined purely category-theoretically.

Definition 5.45. Let $K(\mathcal{C})$ be the homotopy category of $\mathcal{C}$. The **left heart** of $\mathcal{C}$, $LH(\mathcal{C})$, is the localization of the full subcategory of $K(\mathcal{C})$ consisting of complexes

$$A_\bullet = (0 \longrightarrow A_1 \xrightarrow{f_A} A_0 \longrightarrow 0)$$

where f_A is a monomorphism, by the multiplicative system consisting of morphisms $u\colon A_\bullet \to B_\bullet$ such that

$$\begin{array}{ccc} A_1 & \xrightarrow{f_A} & A_0 \\ \downarrow{\scriptstyle u_1} & & \downarrow{\scriptstyle u_0} \\ B_1 & \xrightarrow{f_B} & B_0 \end{array}$$

is a (co)cartesian square in $K(\mathcal{C})$.

Theorem 5.46. *$LH(\mathcal{C})$ is an abelian category, with a strongly exact (i.e. finite (co)limit preserving) embedding $I_\ell\colon \mathcal{C} \to LH(\mathcal{C})$ given by sending $A \in \mathcal{C}$ to the class of the complex*

$$0 \longrightarrow 0 \longrightarrow A \longrightarrow 0.$$

I_ℓ identifies $\mathcal{C}$ with the full subcategory of $LH(\mathcal{C})$ given by the equivalence classes of those objects such that f_A is strict, and admits a left adjoint $C\colon LH(\mathcal{C}) \to \mathcal{C}$. Furthermore, a sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is strictly exact in $\mathcal{C}$ if and only if it is exact in $LH(\mathcal{C})$. By the universal property of localization, I_ℓ then induces an exact functor $D(\mathcal{C}) \to D(LH(\mathcal{C}))$, which is an equivalence of categories identifying the left t-structure on $D(\mathcal{C})$ with the standard t-structure on $D(LH(\mathcal{C}))$.

Proof Sketch. We will sketch the proof that $LH(\mathcal{C})$ is abelian as well as the description of the left adjoint $C\colon LH(\mathcal{C}) \to \mathcal{C}$, leaving the rest to [Sch99]. Let

$$A_\bullet = (0 \longrightarrow A_1 \xrightarrow{f_A} A_0 \longrightarrow 0), \qquad B_\bullet = (0 \longrightarrow B_1 \xrightarrow{f_B} B_0 \longrightarrow 0)$$

be objects of $LH(\mathcal{C})$, and let $g_\bullet\colon A_\bullet \to B_\bullet$. Then, by diagram chase, it is clear that

$$\ker(g_\bullet) = (0 \longrightarrow \ker(g_1) \longrightarrow \ker(g_0) \longrightarrow 0), \qquad \operatorname{coker}(g_\bullet) = (0 \longrightarrow \operatorname{coker}(g_1) \longrightarrow \operatorname{coker}(g_0) \longrightarrow 0),$$

and that direct sum is defined analogously to how it is for chain complexes as well. The only thing left to show is that every morphism is strict. Similarly to the above, we can define $\operatorname{coim}(g_\bullet)$ and $\operatorname{im}(g_\bullet)$ analogously to how they are defined for complexes, giving a map $u_\bullet\colon \operatorname{coim}(g_\bullet) \to \operatorname{im}(g_\bullet)$. However, the square

$$\begin{array}{ccc} \operatorname{coim}(g_1) & \longrightarrow & \operatorname{coim}(g_0) \\ \downarrow{\scriptstyle u_1} & & \downarrow{\scriptstyle u_0} \\ \operatorname{im}(g_1) & \longrightarrow & \operatorname{im}(g_0) \end{array}$$

is (co)cartesian, meaning that $u_\bullet$ is an isomorphism by definition of $LH(\mathcal{C})$ being a localization at such morphisms.

We form the left adjoint $C\colon LH(\mathcal{C}) \to \mathcal{C}$ by sending the class of a complex

$$(0 \longrightarrow A_1 \xrightarrow{f_A} A_0 \longrightarrow 0)$$

to $\operatorname{coker}(f_A)$. □

Given that $LH(\mathcal{C})$ is abelian, we can now give a well-defined notion of (co)homology for complexes valued in $\mathcal{C}$:

Definition 5.47. Let $(M_\bullet, d_\bullet)$ be a complex in $\mathcal{C}$. We define the n**th homology** of $M_\bullet$, $H_n(M_\bullet)$ to be the nth homology of $I_\ell(M_\bullet) \in \mathrm{Ch}(LH(\mathcal{C}))$. Explicitly, $H_n(M_\bullet)$ is the class of the 2-term complex

$$(0 \longrightarrow \operatorname{coim}(d_{n+1}) \longrightarrow \ker(d_n) \longrightarrow 0)$$

Remark 5.48.

1. The homology of a complex is zero when it is strictly exact at a given degree. Indeed, if $M_\bullet$ is strictly exact at M_n, then we have an identification $\operatorname{coim}(d_{n+1}) \cong \ker(d_n)$ since $\operatorname{coim}(d_{n+1}) \cong \operatorname{im}(d_{n+1}) = \ker(d_n)$. This implies that strictly acyclic complexes are exact when viewed in $LH(\mathcal{C})$. Hence, if $f\colon X \to Y$ is a strict quasi-isomorphism, then $\operatorname{Cone}(f)$ is acyclic in $LH(\mathcal{C})$, implying that f is quasi-isomorphism when viewed in $LH(\mathcal{C})$. This is how one gets the functor $D(\mathcal{C}) \to D(LH(\mathcal{C}))$ described in the statement of Theorem 5.46.

2. This allows us to fully resolve the problems with the de Rham complex of the rigid unit disk

   $$0 \longrightarrow K\langle x\rangle \longrightarrow K\langle x\rangle \mathrm{d}x \longrightarrow 0$$

   that we mentioned earlier. Taking cohomology in our setup, we get that

   $$H^1_{\mathrm{dR}}(D) = [K\langle x\rangle / K \xrightarrow{[d]} \operatorname{im}(d)] \in LH(\mathrm{CBorn}_K),$$

   which is a nonzero object, since $[d]$ is a bounded bijection but not an isomorphism. This shows that d fails to be an epimorphism in $LH(\mathrm{CBorn}_K)$, precisely remembering its failure to be strict. However, we still get that $C(H^1_{\mathrm{dR}}(D)) = 0$, since it is the usual Banach cokernel.

Although the left heart is described in terms of complexes of monomorphisms, we can show that it is determined by the projective objects of $\mathcal{C}$, analogously to the usual situation with abelian categories:

Theorem 5.49.

1. *An object P of $\mathcal{C}$ is projective if and only if $I_\ell(P)$ is projective in $LH(\mathcal{C})$.*

2. *$\mathcal{C}$ has enough projectives if and only if $LH(\mathcal{C})$ has enough projectives. Moreover, in this case, any projective object of $LH(\mathcal{C})$ is isomorphic to $I_\ell(P)$ for some projective object $P \in \mathcal{C}$.*

Hence, if $\mathcal{C}$ has enough projectives, then $D(\mathcal{C})$ is determined by the projective objects of $\mathcal{C}$. In other words, denoting $\mathcal{P}(\mathcal{C})$ (resp. $\mathcal{P}(LH(\mathcal{C}))$ the full subcategory of projective objects of $\mathcal{C}$ (resp. $LH(\mathcal{C})$) and $K^-(-)$ the homotopy category of bounded below complexes, we have $K^-(\mathcal{P}) \cong D^-(\mathcal{C})$. More generally, if $\mathcal{C}$ is elementary quasi-abelian, we can identify $LH(\mathcal{C}) \cong \mathrm{Add}(\mathcal{P}^{op}, \mathrm{Ab})$.

Proof.

1. Let P be a projective object of $\mathcal{C}$. Consider an epimorphism $u\colon A_\bullet \to B_\bullet$ in $LH(\mathcal{C})$ and a morphism $f\colon I_\ell(P) \to B_\bullet$. We need to show that f factors through u. Since we can replace $A_\bullet$ and $B_\bullet$ by isomorphic objects, we can assume that A and B are represented respectively by the complexes of monomorphisms

$$(0 \longrightarrow A_1 \xrightarrow{g_A} A_0 \longrightarrow 0), \qquad (0 \longrightarrow B_1 \xrightarrow{g_B} B_0 \longrightarrow 0),$$

that u comes from the morphism of complexes

$$\begin{array}{ccc} A_1 & \xrightarrow{g_A} & A_0 \\ \downarrow{\scriptstyle u_1} & & \downarrow{\scriptstyle u_0} \\ B_1 & \xrightarrow{g_B} & B_0, \end{array}$$

and that f is represented by the morphism of complexes

$$\begin{array}{ccc} 0 & \longrightarrow & P \\ \downarrow & & \downarrow{\scriptstyle \phi} \\ B_1 & \xrightarrow{f_B} & B_0. \end{array}$$

Since u is an epimorphism, the map

$$A_0 \oplus B_1 \xrightarrow{(u_0, f_B)} B_0$$

is a strict epimorphism. Furthermore, since P is projective in $\mathcal{C}$, there are two morphisms $\phi'\colon P \to A_0, \phi''\colon P \to B_1$ such that $\phi = u_0 \circ \phi' + f_B \circ \phi''$. Hence, the morphism of complexes

$$\begin{array}{ccc} 0 & \longrightarrow & P \\ \downarrow & & \downarrow{\scriptstyle \phi'} \\ A_1 & \xrightarrow{f_A} & A_0 \end{array}$$

induces a map $f' \colon I_\ell(P) \to A$ such that $u \circ f' = f$. Hence, $I_\ell(P)$ is projective.

To show the converse, since I_ℓ is exact, the adjunction $C \dashv I_\ell$ implies that C sends a projective object of $LH(\mathcal{C})$ to a projective object of $\mathcal{C}$.

2. Suppose that $\mathcal{C}$ has enough projectives, and let $A \in LH(\mathcal{C})$. We know that there is an epimorphism
$$I_\ell(E) \to A$$
where E is an object of $\mathcal{C}$ by the adjunction. Now, since $\mathcal{C}$ has enough projectives, we can choose a strict epimorphism $P \to E$ in $\mathcal{C}$ where P is projective. Hence,
$$I_\ell(P) \to I_\ell(E)$$
is an epimorphism in $LH(\mathcal{C})$ and $I_\ell(P)$ is projective by our work in the last part. Hence, $LH(\mathcal{C})$ has enough projectives.

Conversely, suppose that $LH(\mathcal{C})$ has enough projectives and let $E \in \mathcal{C}$. Then, there is an epimorphism
$$P \to I_\ell(E)$$
in $LH(\mathcal{C})$ where P is projective in $LH(\mathcal{C})$. Since C is a left adjoint, it is cokernel preserving, and thus sends epimorphisms in $LH(\mathcal{C})$ to strict epimorphisms in $\mathcal{C}$. Hence,
$$C(P) \to E$$
is a strict epimorphism. Since $C(P)$ is projective, $\mathcal{C}$ has enough projectives.

Now, since any projective object $Q \in LH(\mathcal{C})$ is a quotient of $I_\ell(P)$ where P is a projective object of $\mathcal{C}$, it is a direct summand of such an object. This implies that it is isomorphic to $I_\ell(P')$ for some projective object P' of $\mathcal{C}$.

For the last statement, by Theorem 5.46, we have an equivalence $D(\mathcal{C}) \cong D(LH(\mathcal{C}))$. By the classical theory of abelian categories, we have an equivalence $D(LH(\mathcal{C})) \cong K^-(\mathcal{P}(LH(\mathcal{C})))$. Then, by our work in this proposition, we have an identification $K^-(\mathcal{P}(LH(\mathcal{C}))) \cong K^-(\mathcal{P}(\mathcal{C}))$. □

5.3 Animated Bornological Algebras

With derived categories of quasi-abelian categories in hand, we now have two goals:

1. To show that $D(\mathrm{CBorn}_K) \cong D(\mathrm{Ind}(\mathrm{Ban}_K))$.
2. To show that $D(\mathrm{CBorn}_K)$ realizes a derived algebraic context.

The first requires us to show that CBorn_K and $\mathrm{Ind}(\mathrm{Ban}_K)$ have the same class of projective objects, which is the following result:

Proposition 5.50. *For an archimedean field K, the compact projective objects in* CBorn_K *and* $\mathrm{Ind}(\mathrm{Ban}_K)$ *are the retracts of spaces of summable sequences*

$$\ell^1(S) = \left\{ f \colon S \to K \;\middle|\; \sum_{s \in S} |f(s)| < \infty \right\}$$

equipped with the summation norm for all sets S. In particular, this implies that $D(\mathrm{CBorn}_K) \cong D(\mathrm{Ind}(\mathrm{Ban}_K))$.

To prove this, we will need the following result:

Proposition 5.51. *Let $\mathcal{C}$ be a locally presentable category generated by a class of projective objects $\mathcal{P}$ under finite colimits. Then,* $\mathrm{Ind}(\mathcal{C})$ *and* $\mathrm{Ind}^m(\mathcal{C})$ *are generated by the compact projective objects of $\mathcal{C}$.*

Proof of Proposition 5.50. We just need to show that Ban_K is generated under finite colimits by $\ell^1(S)$ for $S \in \mathrm{Sets}$ and that they are compact projectives. The result will follow by Proposition 5.51 and 5.16. Let $V \in \mathrm{Ban}_K$, and let B_V denote the unit ball in V. The space $\ell^1(B_V)$ is generated by f_v for $v \in B_V$, where $f_v \colon B_V \to K$ is given by $f_v(v) = 1$, $f_v(s) = 0$ otherwise. This immediately shows that the map

$$g \colon \ell^1(B_V) \to V, \qquad f_v \mapsto v$$

is surjective, since for $w \in \ell^1(B_V)$, $||w|| g(w/||w||) = w$. By covering $\ker(g)$ by $\ell^1(B_{\ker(g)})$ in the same way, we get a presentation of V as the cokernel

$$\ell^1(B_{\ker(g)}) \longrightarrow \ell^1(B_V) \longrightarrow V \longrightarrow 0.$$

Now we show that $\ell^1(S)$ is projective. Let $\pi \colon W \to V$ be surjective bounded linear map, and let $g \colon \ell^1(S) \to V$ be bounded. Since $\ell^1(S)$ is generated by the sequences f_s for $s \in S$ as above, by the axiom of choice, we can describe a map

$$\ell^1(S) \to W, \qquad f_s \mapsto w$$

where w is a choice of element in $\pi^{-1}(g(f_s))$ scaled to lie in the unit ball.

The above, along with 2.2, implies that

$$D(\mathrm{CBorn}_K)^{cn} \cong \mathcal{P}_\Sigma(\ell^1) \cong D(\mathrm{Ind}(\mathrm{Ban}_K))^{cn}.$$

Furthermore, it is a general fact that for a presentable symmetric monoidal stable ∞-category with right complete t-structure $(\mathcal{C}, \mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0}, \mathcal{C}^\heartsuit)$, we can identify $\mathcal{C} \cong \mathcal{C}_{\geq 0} \otimes \mathrm{Spt}$, where the tensor product is taken in $\mathrm{Pr}^{L,st}$. This implies that

$$D(\mathrm{CBorn}_K) \cong D(\mathrm{CBorn}_K)^{cn} \otimes \mathrm{Spt} \cong D(\mathrm{Ind}(\mathrm{Ban}_K))^{cn} \otimes \mathrm{Spt} \cong D(\mathrm{Ind}(\mathrm{Ban}_K)).$$

□

We will denote the collection of $\ell^1(S)$ for all sets S by ℓ^1.

Remark 5.52. We note that there is an analogue for non-archimedean fields with the space of zero sequences

$$c_0(S) = \left\{ f \colon S \to K \mid \forall \epsilon > 0, \exists \text{ finitely many } s \in S \text{ s.t.} |f(s)| < \epsilon \right\}.$$

To show that $D(\mathrm{CBorn}_K)$ realizes a derived algebraic context, the last thing we need to show is that ℓ^1 is closed under symmetric powers. This follows by:

Lemma 5.53. *For sets S and T, we have*

1. $\ell^1(S \sqcup T) \cong \ell^1(S) \oplus \ell^1(T)$.
2. $\ell^1(S \times T) \cong \ell^1(S) \widehat{\otimes}_{\mathbb{C}} \ell^1(T)$.
3. $\ell^1(S^{\times n}/S_n) \cong \mathrm{Sym}^n(\ell^1(S))$.

Proof.

1. For the map $\ell^1(S) \oplus \ell^1(T) \to \ell^1(S \sqcup T)$, given maps $f \colon S \to \mathbb{C}, g \colon T \to \mathbb{C}$ we have a diagram

$$S \longrightarrow \mathbb{C} \longleftarrow T$$

   Hence, the natural choice is sending the pair (f, g) to the unique map from $S \sqcup T$ by universal property of coproduct. Explicitly, this is

$$(f \sqcup g) \colon S \sqcup T \to \mathbb{C}, \qquad (s, S) \to f(s), (t, T) \to g(t).$$

   It is clearly summable and clearly an isomorphism.

2. We construct a map $\ell^1(S) \widehat{\otimes}_{\mathbb{C}} \ell^1(T) \to \ell^1(S \times T)$ via sending $f \otimes g$ to the map

$$(f \times g) \colon S \times T \to \mathbb{C}, \qquad (s, t) \to f(s)g(t).$$

3. There is an inclusion $\ell^1(S^{\times n}/S_n) \hookrightarrow \ell^1(S^{\times n})$ by sending $f \colon S^{\times n}/S_n \to \mathbb{C}$ to the corresponding map $\hat{f} \colon S^{\times n} \to \mathbb{C}$ sending all elements in the same orbit of S_n to the same number. This gives a diagram

$$\begin{array}{ccc} \ell^1(S)^{\widehat{\otimes} n} & \xleftarrow{\sim} & \ell^1(S^{\times n}) \\ \downarrow & & \uparrow \\ \mathrm{Sym}^n(\ell^1(S)) & \xleftarrow[\phi]{} & \ell^1(S^{\times n}/S_n) \end{array}$$

and our map $\ell^1(S^{\times n}/S_n) \to \mathrm{Sym}^n(\ell^1(S))$ is the induced dashed arrow on the bottom. It's clearly surjective, so we need to show that it is injective. Suppose that $\phi(f) = 0$. Then, thinking of f as a map $S^{\times n} \to \mathbb{C}$ where f sends elements of the same orbit of S_n to the same number, f's image in $\ell^1(S)^{\widehat{\otimes} n}$ lies in the submodule generated by $g - \sigma(g)$ for $g \in \ell^1(S)^{\widehat{\otimes} n}$ and $\sigma \in S_n$. Since f sends elements in the same orbit of S_n to the same complex number, this means that f's image in $\ell^1(S)^{\widehat{\otimes} n}$ is zero. This implies that f is zero in $\ell^1(S^{\times n})$, hence it is zero in $\ell^1(S^{\times n}/S_n)$.

□

All of our work finally gives:

Corollary 5.54. *The tuple*

$$(D(\mathrm{CBorn}_K), D(\mathrm{CBorn}_K)_{\geq 0}, D(\mathrm{CBorn}_K)_{\leq 0}, \ell^1) \cong (D(\mathrm{Ind}(\mathrm{Ban}_K)), D(\mathrm{Ind}(\mathrm{Ban}_K))_{\geq 0}, D(\mathrm{Ind}(\mathrm{Ban}_K))_{\leq 0}, \ell^1)$$

is a derived algebraic context.

Proof. We equip $D(\mathrm{CBorn}_K)$ with the right-complete left t-structure and designate ℓ^1 as $D(\mathrm{CBorn}_K)^0$. That ℓ^1 consists of compact projective generators follows from Proposition 5.50, and that it is closed under $D(\mathrm{CBorn}_K)^{\heartsuit}$-symmetric powers follows from Lemma 5.53. □

We've already seen that our work identifies $D(\mathrm{CBorn}_{\mathbb{C}})^{cn}$ with $\mathcal{P}_\Sigma(\ell^1)$. Furthermore, since $\mathbb{C}$ contains $\mathbb{Q}$, the same proof as the proof of Theorem 2.12 will identify $\mathcal{C}\mathrm{Alg}(\mathrm{CBorn}_{\mathbb{C}})^{cn}$ with the animation of symmetric algebras over ℓ^1. This will be crucial in our theory of derived complex analytic spaces. Hence, combining this with Theorem 4.14 and the general facts about descendable morphisms from Section 3.2, our work in these first five sections culminates in:

Theorem 5.55. *There is a strong relative geometry tuple* $(\mathrm{Aff}_{D(\mathrm{CBorn}_K)}, \tau_{\mathrm{desc}}, \mathrm{descendable}, \mathrm{all})$. *Furthermore, associated to it is a sheafy presentable six-functor formalism*

$$\mathrm{QCoh}\colon \mathrm{Corr}(\mathrm{Shv}(\mathrm{Aff}_{D(\mathrm{CBorn}_K)}, \tau_{\mathrm{desc}}), E') \to \mathrm{Pr}^L$$

which is $*$-local on the target, $!$-local, and tame.

6 Derived Complex Analytic and Dagger Spaces

The goal of this section is to define a notion of derived complex analytic space, as well as the setting in which we will work with them. There will be two main actors in this setting:

1. A derived notion of Stein algebra and Stein space.
2. A derived notion of dagger affinoid algebra and dagger affinoid space.

Classically, complex analytic geometry is the study of spaces that are locally Stein. Hence, our notion of derived complex analytic space will roughly be spaces that are locally derived Stein. However, the Grothendieck topology one is forced to put on derived Stein spaces is poorly behaved; it is not finitary, i.e. the affine objects can fail to be "quasi-compact", causing headaches down the road. Hence, we will show that we can embed our derived complex analytic spaces into a category of derived dagger spaces equipped with the better behaved rational topology (similarly to rigid geometry).

6.1 Classical Dagger Affinoid Spaces, Stein Spaces, and Dagger Stein Spaces

Before launching into the derived setting, we explain the relationship between classical dagger affinoid and Stein spaces, along with their bornological structures. This will be crucial for our derived setting, as these will serve as the π_0 of our derived objects, and many arguments will reduce to them.

We begin with the fundamental objects of dagger geometry. Let $K = \mathbb{C}$ throughout. Recall that for a polyradius $\rho = (\rho_1, \dots, \rho_n) \in \mathbb{R}^n_{>0}$, the Banach algebra of convergent power series on the closed polydisk $\overline{D}_\rho(0) = \{z \in \mathbb{C}^n \mid |z_i| \leq \rho_i\}$ is

$$\mathcal{O}(\overline{D}_\rho(0)) = \left\{ f(z) = \sum_{\alpha \in \mathbb{N}^n} a_\alpha z^\alpha \;\middle|\; \|f\|_\rho := \sum_\alpha |a_\alpha| \rho^\alpha < \infty \right\},$$

a Banach algebra under $\| \cdot \|_\rho$.

Definition 6.1. The **ring of overconvergent power series** in n variables is

$$\mathbb{C}\langle z_1, \dots, z_n \rangle^\dagger := \bigcup_{\rho' > 1} \mathcal{O}(\overline{D}_{\rho'}(0)),$$

where the union is taken over polyradii $\rho' = (\rho'_1, \dots, \rho'_n)$ with each $\rho'_i > 1$, ordered by $\rho' \leq \rho''$ if $\rho'_i \leq \rho''_i$ for all i. We endow $\mathbb{C}\langle z \rangle^\dagger$ with the bornology making it the colimit

$$\mathbb{C}\langle z_1, \dots, z_n \rangle^\dagger \cong \mathrm{colim}_{\rho' > 1} \mathcal{O}(\overline{D}_{\rho'}(0))$$

in $\mathrm{CBorn}_\mathbb{C}$, with transition maps given by restriction. In particular, $\mathbb{C}\langle z \rangle^\dagger \in \mathrm{Ind}(\mathrm{Ban}_\mathbb{C})$ via Theorem 5.16, with the Ind-Banach presentation given by the above colimit.

Remark 6.2. Concretely, $f = \sum_\alpha a_\alpha z^\alpha$ lies in $\mathbb{C}\langle z \rangle^\dagger$ if and only if it converges on some open polydisk strictly containing $\overline{D}_1(0)$, i.e. there exists $\rho' > 1$ componentwise such that $\sum_\alpha |a_\alpha| (\rho')^\alpha < \infty$.

Definition 6.3. A **dagger affinoid algebra** over $\mathbb{C}$ is a quotient

$$A^\dagger = \mathbb{C}\langle z_1, \dots, z_n \rangle^\dagger / I$$

for some $n \geq 0$ and some finitely generated ideal I. We endow $A^\dagger$ with the pushforward bornology, making it an object of $\mathrm{CBorn}_\mathbb{C}$. We denote the category of dagger affinoid algebras over $\mathbb{C}$, with morphisms given by bounded $\mathbb{C}$-algebra maps, by $\mathrm{AfndAlg}^\dagger_\mathbb{C}$.

The **dagger affinoid space** associated to $A^\dagger$ is $\mathrm{MaxSpec}(A^\dagger)$, the set of maximal ideals of $A^\dagger$ equipped with the Grothendieck topology generated by **rational subdomains**: for $f_0, \dots, f_k \in A^\dagger$ with $A^\dagger = (f_0, \dots, f_k)$, the rational subdomain is

$$\mathrm{MaxSpec}(A^\dagger)\Big(\tfrac{f_1,\dots,f_k}{f_0}\Big) := \{\, x \in \mathrm{MaxSpec}(A^\dagger) \mid |f_i(x)| \le |f_0(x)|,\ 1 \le i \le k \,\},$$

with ring of functions $A^\dagger\langle f_1/f_0, \dots, f_k/f_0\rangle^\dagger := A^\dagger\langle T_1, \dots, T_k\rangle^\dagger/(f_0T_i - f_i)$.

Next, we recall Stein algebras. Recall that a complex space X is **Stein** if it is holomorphically convex and holomorphically separable. Equivalently, by Grauert's Theorem/Cartan's Theorem B, $H^q(X, \mathcal{F}) = 0$ for all coherent sheaves $\mathcal{F}$ and $q \ge 1$.

Definition 6.4. A **Stein algebra** over $\mathbb{C}$ is a $\mathbb{C}$-algebra of the form $\mathcal{O}(X)$ for a Stein space X, where $\mathcal{O}(X)$ denotes the ring of global holomorphic functions. We denote the category of Stein algebras with bounded $\mathbb{C}$-algebra morphisms by $\mathrm{SteinAlg}_{\mathbb{C}}$.

The topology of uniform convergence on compact subsets of X gives $\mathcal{O}(X)$ the structure of a Frechet space. We can use this to describe a Pro-Banach structure on $\mathcal{O}(X)$ in the usual way, which will be important for understanding its relationship with dagger affinoid algebras:

Proposition 6.5. *Let X be a Stein space. Write $X = \bigcup_{n\ge 0} K_n$ as a union of compact holomorphically convex subsets with $K_n \subseteq \mathrm{int}(K_{n+1})$. Then there are natural restriction maps $\mathcal{O}(K_{n+1}) \to \mathcal{O}(K_n)$ between Banach algebras of holomorphic functions on the K_n's, and the natural map*

$$\mathcal{O}(X) \xrightarrow{\sim} \varprojlim_n \mathcal{O}(K_n)$$

is an isomorphism of Fréchet algebras. This realizes $\mathcal{O}(X)$ as a Pro-Banach algebra, i.e. an object of $\mathrm{Pro}(\mathrm{Ban}_{\mathbb{C}})$, via the system $\{\mathcal{O}(K_n)\}_{n\ge 0}$.

Proof. The Pro-Banach presentation follows because the defining seminorms of the Fréchet topology are the norms $\|f\|_{K_n} := \sup_{x\in K_n} |f(x)|$. □

Remark 6.6. The distinction between the Ind-Banach structure on dagger affinoid algebras and the Pro-Banach structure on Stein algebras is crucial. Dagger affinoid algebras are inductive limits of Banach algebras (algebras of functions on growing polydisks), while Stein algebras are projective limits (functions on a Stein domain, constrained to shrinking compact subsets). This asymmetry is what makes the relationship between the two non-trivial, and it will be important in our derived setting.

We now make this relationship precise. The key observation is the following:

Proposition 6.7. *Let $A^\dagger = \mathbb{C}\langle z_1, \dots, z_n\rangle^\dagger/I$ be a dagger affinoid algebra over $\mathbb{C}$, and let $X = \mathrm{MaxSpec}(A^\dagger) \subseteq \bar{D}_1(0)$ be the associated dagger affinoid space. Then:*

1. *X is a Stein compactum, i.e. it is compact and admits a fundamental system of open Stein neighborhoods $\{U\}$ in $\bigcup_{\rho'>1} D_{\rho'}(0)$, each of which is a closed analytic subspace of a (Stein) open polydisk.*

2. *There is a natural isomorphism of complete bornological algebras*

$$A^\dagger \cong \varinjlim_{U \supseteq X} \mathcal{O}(U),$$

the colimit over Stein neighborhoods of X ordered by reverse inclusion, exhibiting $A^\dagger$ as the ***germ algebra*** *of X.*

Proof Sketch. Since $A^\dagger$ is a finitely generated quotient of $\mathbb{C}\langle z\rangle^\dagger$, the zero set $X = V(I)$ is a compact analytic subset of $\bar{D}_1(0)$. For each $\rho' > 1$ the open polydisk $D_{\rho'}(0)$ is a domain of holomorphy, hence Stein, and the polydisks $\{D_{\rho'}(0)\}_{\rho'>1}$ form a neighborhood basis of $\overline{D}_1(0)$. Intersecting with the ambient analytic subset gives a fundamental system of Stein neighborhoods of X, proving 1. For 2, an overconvergent function on X is, by definition, the restriction of a holomorphic function on some neighborhood, and two agree on X if and only if they agree on a smaller one, which is the colimit description. The transition maps are restrictions of Banach algebras $\mathcal{O}(U') \to \mathcal{O}(U)$ for $U \subseteq U'$, so the colimit is taken in $\mathrm{Ind}(\mathrm{Ban}_{\mathbb{C}})$. □

The relationship in the other direction requires the following:

Definition 6.8. A Stein algebra $\mathcal{O}(X)$ is **Pro-dagger affinoid** if X admits an exhaustion by dagger affinoid domains $X_0 \subseteq X_1 \subseteq \cdots$ with $X = \bigcup_{n\geq 0} X_n$, such that the restriction maps $\mathcal{O}(X_{n+1}) \to \mathcal{O}(X_n)$ form a cofiltered system of Banach algebras with

$$\mathcal{O}(X) \cong \varprojlim_n \mathcal{O}(X_n)$$

in the category of bornological algebras, with the Pro-Banach structure of Proposition 6.5.

Theorem 6.9. *Every Stein algebra over $\mathbb{C}$ is Pro-dagger affinoid. Concretely speaking, for any Stein space X, there exists a cofiltered system of dagger affinoid domains $\{X_n\}_{n\geq 0}$ with $X = \bigcap_n X_n$ and a natural isomorphism*

$$\mathcal{O}(X) \cong \varprojlim_n \mathcal{O}(X_n)$$

of Pro-Banach algebras in $\mathrm{Born}_{\mathbb{C}}$.

We now upgrade these algebra statements to spaces. Recall the classical duality between Stein spaces and their algebras of holomorphic functions:

Theorem 6.10 (Grauert–Remmert [GR12], Forster [For67])**.** *The functor*

$$\mathcal{O}\colon \mathrm{Stein}_{\mathbb{C}}^{\mathrm{op}} \to \mathrm{SteinAlg}_{\mathbb{C}}$$

sending a Stein space X to its algebra of global sections $\mathcal{O}(X)$ is an equivalence of categories.

Using this equivalence, we pass to the analogous statement for dagger affinoid spaces:

Corollary 6.11. *The functor* $\mathrm{MaxSpec}\colon (\mathrm{AfndAlg}_{\mathbb{C}}^{\dagger})^{op} \to \mathrm{Afnd}_{\mathbb{C}}^{\dagger}$ *is an equivalence, with inverse* $\mathcal{O}$*. Combining the Pro-dagger affinoid presentation of Theorem 6.9 with the Grauert–Remmert equivalence* $\mathcal{O}\colon \mathrm{Stein}_{\mathbb{C}}^{\mathrm{op}} \cong \mathrm{SteinAlg}_{\mathbb{C}}$ *yields inclusions*

$$\mathrm{SteinAlg}_{\mathbb{C}} \hookrightarrow \mathrm{Pro}(\mathrm{AfndAlg}_{\mathbb{C}}^{\dagger}), \qquad \mathrm{SteinSp}_{\mathbb{C}} \hookrightarrow \mathrm{Ind}(\mathrm{Afnd}_{\mathbb{C}}^{\dagger}).$$

Inspired by this result, we can restrict a more specific class of Stein algebras and Stein spaces, as results on them scale to the derived setting more easily:

Definition 6.12 ([BBK18])**.** A **dagger Stein algebra** is a complete bornological $\mathbb{C}$-algebra A isomorphic to an inverse limit of $\mathbb{C}$-dagger affinoid algebras

$$A \cong \varprojlim_n A_n$$

such that each map $A_{n+1} \to A_n$ is a Weierstrass localization and $\mathrm{MaxSpec}(A_n)$ is contained in the interior of $\mathrm{MaxSpec}(A_{n+1})$ (considered in the Zariski topology) for each n. We denote the category of dagger Stein algebras by $\mathrm{SteinAlg}^{\dagger}$.

Remark 6.13. We note that the usual definition of dagger Stein algebra involves the bornological spectrum, which we did not define in Section 5. This definition is equivalent. Furthermore, dagger Stein spaces can be defined analogously on the opposite category.

Later, we'll need a refinement of Theorem 6.9 for dagger Stein spaces, which follows from [BBK18]. Recall that a morphism of rings $A \to B$ is called a **homotopy epimorphism** if the map $B \otimes_A^{\mathbb{L}} B \to B$ is an isomorphism in the derived category. We extend this notation for bornological algebras, trading $\otimes^{\mathbb{L}}$ for $\widehat{\otimes}^{\mathbb{L}}$.

Proposition 6.14 ([BBK18])**.** *Let* X_0 *be a dagger Stein space over* $\mathbb{C}$ *with dagger affinoid exhaustion* $V_1 \subseteq V_2 \subseteq \cdots \cup_m V_m = X_0$*. Suppose that for each* m*, we have a finite rational cover* $\{V_m^{\ell}\}_{\ell}$ *of the dagger affinoid* V_m*. Then, there is a family* $\{W_\alpha \hookrightarrow X_0\}_\alpha$ *of open Stein subsets of* X_0 *such that:*

1. $\bigcup_\alpha W_\alpha = X_0$.
2. *Each inclusion* $\mathcal{O}(X_0) \to \mathcal{O}(W_\alpha)$ *is a homotopy epimorphism.*
3. *Each* W_α *satisfies* $W_\alpha \subseteq V_m^{\ell}$ *for some* m *and* ℓ*. In particular, the restriction* $\mathcal{O}(V_m^{\ell}) \to \mathcal{O}(W_\alpha)$ *is defined and is a homotopy epimorphism.*

6.2 Derived Dagger Affinoid and Stein Spaces

Definition 6.15. A **derived dagger affinoid algebra** is an object $A \in \mathcal{C}\mathrm{Alg}(D(\mathrm{CBorn}_{\mathbb{C}}))^{cn}$ such that:

1. $\pi_0(A)$ is a classical derived dagger affinoid algebra.

2. $\pi_n(A)$ is a finitely generated $\pi_0(A)$-module.

We denote the category of derived dagger affinoid algebras $\mathrm{dAfndAlg}^\dagger$.

We let $\mathrm{dAfnd}^\dagger$ denote $(\mathrm{dAfndAlg}^\dagger)^{op}$, the category of **dagger affinoid spaces**.

Definition 6.16. A **derived dagger Stein algebra** is an object $A \in \mathcal{C}\mathrm{Alg}(D(\mathrm{CBorn}_{\mathbb{C}}))^{cn}$ such that:

1. $\pi_0(A)$ is a classical dagger Stein algebra.
2. $\pi_n(A)$ is a co-admissible $\pi_0(A)$-module. In other words, if X is the Stein space that $\pi_0(A)$ corresponds to, then $\pi_n(A)$ is the global sections of a coherent sheaf $\mathcal{F}_n$ on X.

We denote the category of derived dagger Stein algebras as $\mathrm{dSteinAlg}^\dagger$.

A **derived dagger Stein space** is $\mathrm{Spec}(A) \in \mathrm{Aff}_{D(\mathrm{CBorn}_{\mathbb{C}})}$ for a derived dagger Stein algebra A. We denote the category of derived dagger Stein spaces by $\mathrm{dStein}^\dagger$.

Finally, we let Aff denote $\mathcal{C}\mathrm{Alg}(D(\mathrm{CBorn}_C))^{op}$, and let ι and j denote the inclusion functors $\mathrm{dAfnd}^\dagger \to \mathrm{Aff}$ and $\mathrm{dStein}^\dagger \to \mathrm{Aff}$ respectively. We note that we work with non-connective objects, as they will allow us to have a good analogue of "affine morphism". Furthermore, we also note that for any derived dagger affinoid space $X = \mathrm{Spec}(A) \in \mathrm{dAfnd}^\dagger$ (resp. dagger Stein space in $\mathrm{dStein}^\dagger$), we will use X_0 as a short hand for the dagger affinoid space (resp. dagger Stein space) associated to $\pi_0(A)$. Concretely, this is the Berkovich spectrum $\mathcal{M}(\pi_0(A))$.

We now define a number of Grothendieck topologies on each of these objects. The goal will be to give an analogue of the analytic topology for Stein spaces and the rational topology for dagger affinoids, applying the rational topology along to Aff. We will need to be quite careful with the rational topology for reasons that will be explained below.

Definition 6.17. We define a topology τ' on $\mathrm{dAfnd}^\dagger$ as follows: the covering sieves are generated by finite families $\{X_i \to X\} = \{\mathrm{Spec}(A_i) \to \mathrm{Spec}(A)\}$ where each X_i is such that:

1. The restriction maps $A \to A_i$ are derived strong homotopy epimorphisms.
2. The classical dagger affinoid spaces $X_{i,0}$ form a cover of X_0.
3. The cover of X_0 by $X_{i,0}$'s can be refined by an open cover: there exists a collection $\{U_j\}_{j\in J}$ cover X_0 such that for all j, there exists an i with $U_j \subseteq X_{i,0}$.

Definition 6.18. 1. A map $A \to A_i$ in $\mathrm{dAfnd}^\dagger$ is called a **rational localization** if it is a derived strong homotopy epimorphism, and $\pi_0(A) \to \pi_0(A_i)$ is a rational localization of classical dagger affinoids. We say that the corresponding object $\mathrm{Spec}(A_i)$ is a **rational subdomain** of $\mathrm{Spec}(A)$.

2. An **iterated rational subdomain** is a morphism $\mathrm{Spec}(A_n) \to \mathrm{Spec}(A_0)$ that can be factored through finitely many rational subdomains, i.e. there exist $A_1, \cdots, A_{n-1}$ with $A_i \to A_{i+1}$ a rational localization for $0 \leq i \leq n-1$ such that the map $\mathrm{Spec}(A_n) \to \mathrm{Spec}(A_0)$ factors as

$$\mathrm{Spec}(A_n) \to \mathrm{Spec}(A_{n-1}) \to \cdots \to \mathrm{Spec}(A_1) \to \mathrm{Spec}(A_0).$$

Definition 6.19. We define the **roomy iterated rational topology** τ_{rat} to have covering sieves generated by finite families $\{X_i \to X\}_i = \{\mathrm{Spec}(A_i) \to \mathrm{Spec}(A)\}_i$ such that:

1. $\bigcup_i X_{i,0} = X_0$.
2. Each map $\mathrm{Spec}(A_i) \to \mathrm{Spec}(A)$ realizes $\mathrm{Spec}(A_i)$ as an iterated rational subdomain of $\mathrm{Spec}(A)$.
3. The cover can be refined by an open cover. In particular, there exist open subsets $\{U_j\}_j$ in X_0 such that $\bigcup_j U_j = X_0$, and for all j, there exists some i such that $U_j \subseteq X_{i,0}$.

That τ_{rat} is a topology follows by the fact that rational localizations (and thus iterated rational localizations) are closed under tensor product, and that iterated rational localizations are the smallest class of morphisms containing all rational localizations that is closed under composition by definition. The "roominess" (statement 3) is what does the heavy lifting in the following, as it is what allows the roomy iterated rational topology to be both descendable and subcanonical (the starting point for a good topology, as we saw in sections 2 and 3).

We will need to show that τ_{rat} is subcanonical, as well as that quasi-coherent sheaves satisfy descent in τ_{rat}, to build to later global results. We will accomplish this by showing them for τ', which contains τ_{rat}.

Proposition 6.20 (Tate Acyclicity in τ')**.** *Let $\{X_i \to X\}_i = \{\mathrm{Spec}(A_i) \to \mathrm{Spec}(A)\}_i$ be a covering in τ'. Then, the map $A \to \prod_i A_i$ is descendable, and* QCoh *satisfies Čech descent along the family. In particular, $\tau' \subseteq \tau_{desc}^{\mathrm{dAfnd}^\dagger}$, where $\tau_{desc}^{\mathrm{dAfnd}^\dagger}$ is the descendable topology on* $\mathrm{dAfnd}^\dagger$.

Proof. We recommend that the reader read Section 7.2 on locales and their relationship to analytic geometry before proceeding, as our proof is fundamentally reliant on them.

Since our covers are finite and idempotent, it suffices to show that the joint pullback functor $\mathrm{QCoh}(\mathrm{Spec}(A)) \to \prod \mathrm{QCoh}(\mathrm{Spec}(A_i))$ is conservative. By [CS26, Theorem 11.8 1)], we know that there is map of locales

$$\pi_A \colon \mathcal{S}(A) \to X_0,$$

where by X_0 we mean the locale of closed subsets on the topological space X_0. Furthermore, on the locale $\mathcal{S}(A)$, we know that the assignment $U \mapsto \mathrm{C}(U)$ (see section 7.2 for a definition) for an open subset U is a sheaf; this means that for an open cover $\{U_k\}$ of $\mathcal{S}(A)$, the functor $\prod_k j_{U_k}^* \colon \mathrm{QCoh}(A) \to \prod_k \mathrm{C}(U_k)$ is conservative. Finally, letting $\pi_A^* \colon \mathcal{O}(X_0) \to \mathcal{O}(\mathcal{S}(A))$ be

the corresponding map of frames, their work also implies that $\pi_A^*(X_{i,0})$ corresponds to the idempotent algebra A_i.

Since $\{\mathrm{Spec}(A_i) \to \mathrm{Spec}(A)\}_i$ is a covering in τ', we have that the $X_{i,0}$'s cover X_0. This induces a closed cover of X_0 refined by the open cover $\{U_j\}_{j\in J}$. However, on the locale, we have $U_j \cap X_{i,0}^c = \varnothing$.

I claim that this implies $\pi_A^*(U_j) \cap \pi_A^*(X_{i,0})^c = \varnothing$. Since maps of frames preserve the minimum element and meets, we already have $\pi_A^*(U_j)\cap\pi_A^*(X_{i,0}^c) = \pi_A^*(U_j\cap X_{i,0}^c) = \varnothing$. All that remains is to verify $\pi_A^*(X_{i,0}^c) = \pi_A^*(X_{i,0})^c$. This requires a minor verification as complements of locales are formal. Indeed, when we write $X_{i,0}^c$, we do not mean the complement of $X_{i,0}$ within the topological space X_0; we mean the the same element of the poset corresponding to the locale on X_0, viewed in the opposite ordering. However, since π_A^* is a map of posets, it is a map of the underlying sets. This means that whether we take the complement before or applying π_A^* does not change the element of the underlying set, as changing the ordering of posets does not affect the underlying map of sets. In other words, $\pi_A^*(X_{i,0}^c) = \pi_A^*(X_{i,0})^c$, and we have $\pi_A^*(U_j) \cap \pi_A^*(X_{i,0})^c = \pi_A^*(U_j \cap X_{i,0}^c) = \varnothing$. By [CS26, Lemma 6.4.3], this implies that $j_{U_j}^*$ factors through the pullback functor $\mathrm{QCoh}(\mathrm{Spec}(A)) \to \mathrm{QCoh}(\mathrm{Spec}(A_i))$. Hence, the joint map $\prod_j j_{U_j}^* \colon \mathrm{QCoh}(\mathrm{Spec}(A)) \to \prod_j \mathrm{C}(U_j)$ factors through the joint pullback functor $\mathrm{QCoh}(\mathrm{Spec}(A)) \to \prod_i \mathrm{QCoh}(\mathrm{Spec}(A_i))$. However, since $j_{U_j}^*$ is conservative, this implies that the joint pullback functor must be conservative. □

Remark 6.21. It is worth flagging that denoting the map of locales of closed and open subsets (in dual orderings) both as π_A^* is an abuse of notation. They are the same maps of sets but in different orderings, so maps of different posets.

Since the descendable topology is subcanonical, we immediately obtain the following corollary:

Corollary 6.22. *τ' is a subcanonical Grothendieck topology on* $\mathrm{dAfnd}^\dagger$.

Furthermore, since roomy iterated rational covers are particular examples of dagger affinoid covers refinable by opens, we get two more important corollaries:

Corollary 6.23. *τ_{rat} is a subcanonical Grothendieck topology on* $\mathrm{dAfnd}^\dagger$*, and satisfies Tate acyclicity.*

Remark 6.24. Due to the analogy between dagger affinoid spaces and rigid geometry, one may be tempted to define the topology on $\mathrm{dAfnd}^\dagger$ purely terms of rational subdomains. Indeed, much of the work we do in this section has already been established for derived enhancements of classical nonarchimedean affinoids in the rational topology (see [Soo25]). However, during the process of writing this work, we were sent counterexamples which showed that in the archimedean case, a) rational subdomains are not closed under composition, and b) there exist rational subdomains of dagger affinoid spaces that lack interiors. This leads to a counterexample to the famous (and quite useful) Gerritzen–Grauert theorem that one would assume to remain true over $\mathbb{C}$: there exist dagger affinoid spaces that cannot be written as a finite union of (iterated) rational subdomains. A proof of "archimedean Gerritzen–Grauert" was written by Bambozzi [Bam14], but contains several gaps that lead to the proof not being

sound (in particular, he assumes the existence of rational subdomains that have interiors in a way that contradicts the counterexample we were given). Much of our adjustments in this section and beyond come out of an avoidance of these counterexamples and Gerritzen–Grauert, as an earlier approach crucially relied on them.

Definition 6.25. The **rational Stein topology** τ_{rat}^{St} on $\mathrm{dStein}^\dagger$ is the topology whose covers are given by (not necessarily finite) families $\{X_i \to X\} := \{\mathrm{Spec}(A_i) \to \mathrm{Spec}(A)\}$ such that each morphism $A \to A_i$ is a derived strong homotopy epimorphism and the family of classical dagger Stein spaces $X_{i,0}$ cover X_0. We will call a leg of a rational Stein cover an **open derived Stein subspace** of X.

That τ_{rat}^{St} is subcanonical is shown in [BKK24].

This allows us to define our notions of analytic spaces in both topologies as the "locally representable" objects:

Definition 6.26.

1. A **derived Stein space** is an object $X = \mathrm{Spec}(A)$ in the image of the Yoneda embedding $\mathrm{dStein}^\dagger \hookrightarrow \mathrm{Shv}(\mathrm{dStein}^\dagger, \tau_{rat}^{St})$.
2. Let $X = \mathrm{Spec}(A)$ be a derived Stein space. An **analytic subspace** of X is a subsheaf $U \hookrightarrow X$ such that there exists a small collection $\{U_i \hookrightarrow U\}_i$ of Stein subspaces of U such that $\coprod_i U_i \to U$ is an effective epimorphism.
3. Let $X \in \mathrm{Shv}(\mathrm{dStein}^\dagger, \tau_{rat}^{St})$. An **analytic subspace** of X is a subsheaf $U \hookrightarrow X$ such that for every derived Stein space $X' \to X$ mapping to X, the pullback $U' \hookrightarrow X'$ is an analytic subspace in the sense of the above.
4. A **derived complex analytic space** is an object $X \in \mathrm{Shv}(\mathrm{dStein}^\dagger, \tau_{rat}^{St})$ such that there exists a small family $\{U_i \hookrightarrow X\}$ of derived Stein analytic subspaces such that $\coprod_i U_i \to X$ is an effective epimorphism. We denote the category of derived complex analytic spaces by $\mathrm{dAn}_{\mathbb{C}}$.
5. The **analytic topology** on $\mathrm{dAn}_{\mathbb{C}}$ is defined such that the covering sieves are generated by small families $\{U_i \hookrightarrow X\}$ of analytic subspaces such that $\coprod_i U_i \to X$ is an effective epimorphism.

The same definitions define **dagger affinoid spaces**, **analytic subspaces** of dagger affinoid spaces, **analytic subspaces** of $X \in \mathrm{Shv}_{\tau_{rat}}(\mathrm{dAfnd}^\dagger)$, **derived dagger spaces** $\mathrm{dAn}_{\mathbb{C}}^\dagger$, and the **analytic topology** on them, respectively, replacing $\mathrm{dStein}^\dagger$ with $\mathrm{dAfnd}^\dagger$ and τ_{rat}^{St} with τ_{rat} when needed.

We prove a result that will be crucial in the next section:

Proposition 6.27. *Let $X = \mathrm{Spec}(A) \in \mathrm{dStein}^\dagger$ and let $W_0 \hookrightarrow X_0$ be an open Stein subset such that $\pi_0(A) \to \mathcal{O}(W_0)$ is a homotopy epimorphism. Then, there is a derived dagger algebra B_W and a morphism $A \to B_W$ such that:*

1. *$\pi_0(B_W) = \mathcal{O}(W_0)$ is a dagger Stein algebra, and $\pi_n(B_W) = \pi_n(A) \otimes_{\pi_0(A)} \mathcal{O}(W_0)$ is a coadmissible $\pi_0(B_W)$ module. Hence,* $\mathrm{Spec}(B_W) \in \mathrm{dStein}^\dagger$.

2. *The morphism $A \to B_W$ is a derived strong homotopy epimorphism.*

Moreover, B_W is functorial in W_0: a homotopy-epimorphism restriction $\mathcal{O}(W_0) \to \mathcal{O}(W_0')$ induces a canonical map $B_W \to B_{W'}$ under A. The same statement holds with $\mathrm{dStein}^\dagger$ *replaced by* $\mathrm{dAfnd}^\dagger$ *when W_0 is a dagger affinoid subdomain and we replace "coadmissible" with "finitely generated" and "Stein" with "iterated rational subdomain".*

Essentially, this statement (and its follow up, Lemma 6.31) will allow us to automatically lift any arguments done on π_0 to derived ones, comporting with the intuition in derived geometry that "all geometry is done on π_0". Later on in our proofs, we may use "lifting by Proposition 6.27" as short hand for this. We will make heavy use of the theory of cotangent complexes and Postnikov towers in the proof. Instead of writing a separate section, we will remind the reader of a few important results on this theory. Here, $\tau_{\leq n}$ denotes the nth level truncation of a connective algebra, and $\mathbb{L}_{B/A}$ denotes the cotangent complex of a morphism of connective algebras $A \to B$.

Lemma 6.28 ([Lur17], Corollary 7.4.1.28)**.** *For $A \in \mathcal{C}\mathrm{Alg}(D(\mathrm{CBorn})_{\mathbb{C}})^{cn}$, and for all $n \geq 1$, there is a canonical square-zero extension $\tau_{\leq n}A \to \tau_{\leq n-1}A$ with fiber $\pi_n(A)[n]$. Furthermore, $A \simeq \varprojlim_n \tau_{\leq n}A$.*

We will also need the following easy lemma:

Lemma 6.29. *If $f\colon A \to B$ is a homotopy epimorphism, then f is formally étale (i.e. $\mathbb{L}_{B/A} = 0$).*

Proof. Base changing the cotangent complex of f along f itself yields $\mathbb{L}_{(B\widehat{\otimes}^{\mathbb{L}}_A B/B)} \simeq \mathbb{L}_{B/A}\widehat{\otimes}_B(B\widehat{\otimes}^{\mathbb{L}}_A B)$. Since f is a homotopy epimorphism, the multiplication map $B\widehat{\otimes}^{\mathbb{L}}_A B \to B$ is an equivalence, which composes with the unit $B \to B\widehat{\otimes}^{\mathbb{L}}_A B$ to the identity. Hence, the left hand side of the above equivalence is $\mathbb{L}_{B/B} = 0$. Hence, $\mathbb{L}_{B/A} \otimes_B B = \mathbb{L}_{B/A} = 0$. □

Remark 6.30. We also make the following observation: let $\widetilde{A} \to A$ be a square-zero extension of connective algebras in $D(\mathrm{CBorn}_{\mathbb{C}})$, classified by a morphism $\eta\colon \mathbb{L}_A \to M[k]$ with $M \in \mathrm{Mod}_A$ for some k, and let $A \to B$ be such that $\mathbb{L}_{B/A} = 0$. The cofiber sequence $\mathbb{L}_A\widehat{\otimes}^{\mathbb{L}}_A B \to \mathbb{L}_B \to \mathbb{L}_{B/A}$ then collapses to an equivalence $\mathbb{L}_A \otimes_A B \simeq \mathbb{L}_B$. Hence, η can be identified with a canonical class

$$\eta_S\colon \mathbb{L}_B \simeq \mathbb{L}_A\widehat{\otimes}^{\mathbb{L}}_A B \xrightarrow{\eta\otimes 1} (M\widehat{\otimes}^{\mathbb{L}}_A B)[k].$$

The corresponding square-zero extension $\widetilde{B}$ of B by $M \otimes_A B$ is the essentially unique such extension receiving a map from $\widetilde{A}$ over $A \to B$. Its defining property is the pushout $\widetilde{B}\otimes_{\widetilde{A}} A \simeq B$. This gives $\mathbb{L}_{\widetilde{B}/\widetilde{A}}\widehat{\otimes}^{\mathbb{L}}_{\widetilde{B}} B \simeq \mathbb{L}_{B/A} = 0$. However, since $\widetilde{B} \to B$ is a nilpotent thickening, base change along B is conservative, so we have $\mathbb{L}_{\widetilde{B}/\widetilde{A}} \simeq 0$. Hence, $\widetilde{A} \to \widetilde{B}$ is formally étale.

Proof of Proposition 6.27. For notational convenience, let R denote $\pi_0(A)$, B_0 denote $\mathcal{O}(W_0)$, and $r\colon R \to B_0$ the given homotopy epimorphism. We build a tower $\{B_n\}_{n\geq 0}$ in $D(\mathrm{CBorn}_{\mathbb{C}})$ with compatible maps $\tau_{\leq n}A \to B_n$ such that each $B_n \to B_{n-1}$ is a square-zero extension by $(\pi_n(A) \otimes_R B_0)[n]$ and each $\tau_{\leq n}A \to B_n$ is formally étale.

The base case is $n = 0$, where we let $B_0 = \mathcal{O}(W_0)$. The map $\tau_0 A = R \to B_0$ is r, which is formally étale by Lemma 6.29. For the inductive step, suppose that B_{n-1} and a formally étale map $\tau_{\leq n-1}A \to B_{n-1}$ are given. By Lemma 6.28, $\tau_{\leq n}A$ is the square zero extension of $\tau_{\leq n-1}A$ classified by $\eta_n\colon \mathbb{L}_{\tau_{\leq n-1}A} \to \pi_n(A)[n+1]$. Applying Remark 6.30 with the A replaced by $\tau_{\leq n-1}A$, $\widetilde{A}$ replaced by $\tau_{\leq n}A$, B replaced by B_{n-1}, M replaced by $\pi_n(A)[n]$, and $\mathbb{L}_{B_{n-1}/\tau_{\leq n-1}A} = 0$ yields the canonical class

$$(\eta_n)_{B_{n-1}} : \mathbb{L}_{B_{n-1}} \simeq \mathbb{L}_{\tau_{\leq n-1}A} \widehat{\otimes}^{\mathbb{L}}_{\tau_{\leq n-1}A} B_{n-1} \xrightarrow{\eta_n \otimes 1} \left(\pi_n(A) \widehat{\otimes}_R B_0\right)[n+1],$$

where $\pi_n(A)\widehat{\otimes}^{\mathbb{L}}_{\tau_{\leq n-1}A}B_{n-1} \simeq \pi_n(A) \otimes_R B_0$, since the $\tau_{\leq n-1}A$-action on $\pi_n(A)$ factors through R and $\pi_0(B_{n-1}) = B_0$. Let B_n be the corresponding square-zero extension of B_{n-1}. Then, $\tau_{\leq n}A \to B_n$ exists and is again formally étale by Remark 6.30. Let $B_W := \varprojlim_n B_n$, with $A \simeq \varprojlim_n \tau_{\leq n}A \to \varprojlim_n B_n = B_W$ the induced map.

The next step is to compute the homotopy groups of B_W. The fiber of $B_n \to B_{n-1}$ is $(\pi_n(A)\widehat{\otimes}^{\mathbb{L}}_R B_0)[n]$, so the long exact sequence of homotopy groups gives $\pi_j(B_n) = \pi_j(B_{n-1})$ for $j < n$, $\pi_n(B_n) = \pi_n(A)\widehat{\otimes}_R B_0$, and $\pi_j(B_n) = 0$ for $j > n$. Hence, $B_n = \tau_{\leq n}B_W$, the tower is the Postnikov tower of B_W, and

$$\pi_0(B_W) = \mathcal{O}(W_0), \qquad \pi_n(B_W) \simeq \pi_n(A)\widehat{\otimes}_R \mathcal{O}(W_0) \text{ for } n \geq 1.$$

This tensor product is underived; the higher Tor groups $\mathrm{Tor}_i^R(\pi_n(A), \mathcal{O}(W_0))$ vanish for $i > 0$ by Remark 6.32.

Now, by hypothesis, $\pi_0(B_W) = \mathcal{O}(W_0)$ is a dagger Stein algebra. For $n \geq 1$, let $\{V_m\}$ be a dagger affinoid exhaustion of X_0. Since $\pi_n(A)$ is coadmissible, we can write $\pi_n(A) = \varprojlim_m (\pi_n(A) \otimes_{\pi_0(A)} \mathcal{O}(V_m))$ as a compatible system of finitely generated modules over the affinoid pieces. Since the restriction maps have vanishing higher Tor as argued above, restricting to W_0 sends it to a coadmissible $\mathcal{O}(W_0)$-module. Hence, $\mathrm{Spec}(B_W) \in \mathrm{dStein}^{\dagger}$. Furthermore, by our earlier work, the canonical map $\pi_*(A)\widehat{\otimes}_{\pi_0(A)}\pi_0(B_W) \to \pi_*(B_W)$ is the isomorphism $\pi_*(A)\widehat{\otimes}_R\mathcal{O}(W_0) \xrightarrow{\sim} \pi_*(B_W)$. Hence, the map $A \to B_W$ is derived strong.

We now show that the map $A \to B_W$ is a homotopy epimorphism. For the homotopy epimorphism condition, we have a convergent base change spectral sequence

$$E_2^{p,q} = \mathrm{Tor}_p^{\pi_*(A)}(\pi_*(B_W), \pi_*(B_W))_q \implies \pi_{p-q}(B_W \widehat{\otimes}^{\mathbb{L}}_A B_W).$$

where the subscript q is the qth graded piece. Since $\pi_*(B_W) = \pi_*(A)\widehat{\otimes}_R\mathcal{O}(W_0)$ and base change is transitive, we have

$$\pi_*(B_W)\widehat{\otimes}^{\mathbb{L}}_{\pi_*(A)}\pi_*(B_W) \simeq \pi_*(A) \otimes_{\pi_0(A)} \left(\mathcal{O}(W_0)\widehat{\otimes}^{\mathbb{L}}_{\pi_0(A)}\mathcal{O}(W_0)\right) \simeq \pi_*(A) \otimes_{\pi_0(A)} \mathcal{O}(W_0) = \pi_*(B_W).$$

So, the page E_2 is concentrated in degree 0 and thus equal to $\pi_*(B_W)$, and the multiplication map $B_W \widehat{\otimes}^{\mathbb{L}}_A B_W \to B_W$ is an equivalence on homotopy groups, thus an equivalence. Hence, the map $A \to B_W$ is a homotopy epimorphism.

Finally, we show functoriality. A homotopy epimorphism restriction $\mathcal{O}(W_0) \to \mathcal{O}(W_0')$ makes $R \to \mathcal{O}(W_0')$ a homotopy epimorphism. Hence, it is formally étale. Running the first construction for W_0' and using the uniqueness statement of Remark 6.30 produces a canonical compatible family $B_n \to B_n'$ under $\tau_{\leq n} A$. Hence, this yields a canonical map $B_W \to B_W'$ under A. The uniqueness makes this functorial.

Finally, the affinoid variant follows since rational localizations $\pi_0(A) \to \mathcal{O}(W_0)$ have all relevant vanishing higher Tor-groups by Remark 6.32. Again, we replace "coadmissible" with "finitely generated everywhere", giving $\mathrm{Spec}(B_W) \in \mathrm{dAfnd}^\dagger$. □

Finally, we record a technical general lemma about derived bornological algebras that will be important later:

Lemma 6.31. *Let $A \to A_i$ be a derived strong morphism of connective algebras in $\mathcal{C}\mathrm{Alg}(D(\mathrm{CBorn}_{\mathbb{C}})$. Let $A \to C$ be any connective A-algebra such that $\pi_0(A) \to \pi_0(A_i)$ is transversal to $\pi_m(C)$ over $\pi_0(A)$ for all $m \geq 0$. Then, the space of A-algebra factorizations $A_i \to C$ is empty or contractible. It is nonempty if and only if the classical map $\pi_0(A) \to \pi_0(C)$ factors through $\pi_0(A_i)$.*

Proof. Since $A \to A_i$ is a homotopy epimorphism, we have that $\mathrm{Map}_{\mathcal{C}\mathrm{Alg}(A)}(A_i, C)$ is (-1)-truncated. A factorization exists if and only if the map $C \to C \widehat{\otimes}^{\mathbb{L}}_A A_i$ is an equivalence.

Since $A \to A_i$ is transversal to C, the base change spectral sequence degenerates. Hence, we compute

$$\pi_*(C \widehat{\otimes}^{\mathbb{L}}_A A_i) \cong \pi_*(C) \otimes_{\pi_*(A)} \pi_*(A_i) \cong \pi_*(C) \otimes_{\pi_0(A)} \pi_0(A_i).$$

Now, suppose that $\pi_0(A) \to \pi_0(C)$ factors through $\pi_0(A_i)$. Then, $\pi_*(C)$ is a $\pi_0(A_i)$ module. Since $\pi_0(A) \to \pi_0(A_i)$ is a flat homotopy epimorphism, we have

$$\pi_*(C) \otimes_{\pi_0(A)} \pi_0(A_i) \cong \pi_*(C) \otimes_{\pi_0(A_i)} (\pi_0(A_i) \otimes_{\pi_0(A)} \pi_0(A_i)) \cong \pi_*(C).$$

So, $C \to C \widehat{\otimes}^{\mathbb{L}}_A A_i$ is an equivalence and the factorization exists. Conversely, the factorization forces the statement on π_0 by applying π_0. □

Remark 6.32. We note that all relevant morphisms in our settings (morphisms of derived Stein spaces and morphisms of derived dagger affinoid spaces) can be verified to have vanishing relevant Tor groups. Indeed, in the Stein setting, this is [EP96, Corollary 4.2.5], where it is shown that the global sections of a coherent sheaf on a Stein space have all relevant Tor groups vanishing. In the affinoid setting, we are considering Tor groups of finitely generated modules over dagger affinoid algebras. In this case, the algebraic and completed tensor products agree, and so this is handled by [BK17, Remark 5.8]. Hence, Lemma 6.31 can be applied in all of our relevant settings, and the vanishing Tor-group arguments go through in the proof of Proposition 6.27.

Remark 6.33. Before concluding, we record a crucial detail on the relationship between derived dagger Stein spaces and derived dagger affinoid spaces. In [BKK24], it is shown that a derived dagger Stein algebra $A \in \mathrm{dStein}^\dagger$ admits a **geometric presentation**, i.e. a tower

$$\cdots \longrightarrow A_{n+1} \longrightarrow \overline{A}_{n+1} \longrightarrow A_n \longrightarrow \overline{A}_n \longrightarrow \cdots$$

where each $\overline{A}_n$ is a derived dagger affinoid algebra and each A_n is a derived dagger Stein algebra, such that the transition maps $A_n \to \overline{A}_n$, $\overline{A}_n \to A_{n-1}$ and projections $A \to \overline{A}_n$, $A \to A_n$ are homotopy epimorphisms, and

$$A \simeq \varprojlim_n A_i \simeq \varprojlim_n \overline{A}_i.$$

Furthermore, the $\mathrm{Spec}(A_i)$'s form a cover of $\mathrm{Spec}(A)$ in τ_{rat}^{St} by [BBK18], Theorem 5.7. Important to note in this construction is that $\mathcal{M}(\pi_0(\overline{A}_i))$ will have an interior, and that each map $\overline{A}_n \to \overline{A}_{n-1}$ is a rational localization. Indeed, since $\pi_0(A)$ is a classical dagger Stein algebra, one can take the $\overline{A}_i$'s to be the derived base change (via Proposition 6.27) of the dagger affinoid presentation guaranteed by the definition of classical dagger Stein algebra. Additionally, each restriction map $\pi_0(A) \to B_i$ will be a flat homotopy epimorphism by [BBK18]. By definition, the B_i's are all Weierstrass localizations of each other (in the classical sense), and $\mathcal{M}(B_i)$ embeds into the interior of $\mathcal{M}(B_{i+1})$. This gives a series of iterated rational subdomains with interior $\mathrm{Spec}(\overline{A}_i) \subseteq \mathrm{Spec}(\overline{A}_{i+1})$, which will be *crucial* for constructing roomy iterated rational covers of particular dagger affinoids in the next section.

6.3 Nonsense with Grothendieck Topologies

Our goal is to prove the following theorem:

Theorem 6.34. *There is a fully faithful geometric morphism of ∞-topoi* $\mathrm{Shv}(\mathrm{dStein}^\dagger, \tau_{rat}^{St}) \hookrightarrow \mathrm{Shv}(\mathrm{dAfnd}^\dagger, \tau_{rat})$. *In particular, this restricts to a fully faithful functor* $\mathrm{dAn}_{\mathbb{C}} \to \mathrm{dAn}_{\mathbb{C}}^\dagger$.

Here, we prove a few results on functors of ∞-sites and their relationship to their Kan extensions to sheaves. In particular, we will show that for a functor of sites $f\colon (\mathcal{C}, \tau) \to (\mathcal{D}, \tau')$ that is continuous and cocontinuous, there are fully faithful extensions f_* and $f_!$ from $\mathrm{Shv}(\mathcal{C}, \tau) \to \mathrm{Shv}(\mathcal{D}, \tau')$. This yields our goal for proving Theorem 6.34: find a topology τ on Aff such that we get a diagram

$$\begin{array}{ccc} (\mathrm{dStein}^\dagger, \tau_{rat}^{St}) & \xrightarrow{\ j\ } & (\mathrm{Aff}, \tau) \\ & & \uparrow{\scriptstyle \iota} \\ & & (\mathrm{dAfnd}^\dagger, \tau_{rat}) \end{array}$$

with j and ι continuous and cocontinuous. The topology we construct will be engineered for ι to be continuous and cocontinuous, so the difficulty will be in showing that j is as well. The result will follow from general nonsense with the adjoints of the induced functors on sheaves.

Since we have already used them earlier, we assume that the reader is familiar with the definitions of ∞-sites through covering sieves. For the rest of this section, fix ∞-sites $(\mathcal{C}, \tau)$ and $(\mathcal{D}, \tau')$.

If $f\colon \mathcal{C} \to \mathcal{D}$ is any functor, then by precomposition we get a functor

$$f'^*\colon \mathrm{PreSh}(\mathcal{D}) \to \mathrm{PreSh}(\mathcal{C}).$$

Left and right Kan extension induce left and right adjoints $f'_!$ and f'_* respectively. Letting $j_{\mathcal{C}}$ denote the functor $\mathrm{Shv}(\mathcal{C}, \tau) \to \mathrm{PreSh}(\mathcal{C})$ and $(-)^+$ denote sheafification, we will use $f_!$ (resp. f_*) to denote the composite

$$\mathrm{Shv}(\mathcal{C}, \tau) \xrightarrow{j_{\mathcal{C}}} \mathrm{PreSh}(\mathcal{C}) \xrightarrow{f'_!} \mathrm{PreSh}(\mathcal{D}) \xrightarrow{(-)^+} \mathrm{Shv}(\mathcal{D}, \tau').$$

(resp. replace $f'_!$ with f'_*). We do so similarly for f^* with the composite

$$\mathrm{Shv}(\mathcal{D}, \tau') \xrightarrow{j_{\mathcal{C}}} \mathrm{PreSh}(\mathcal{D}) \xrightarrow{f'^*} \mathrm{PreSh}(\mathcal{C}) \xrightarrow{(-)^+} \mathrm{Shv}(\mathcal{C}, \tau).$$

We recall the following definitions:

Definition 6.35. Suppose that $\mathcal{C}$ is closed under fiber products.

1. f is **continuous** if the composite

   $$\mathrm{Shv}(\mathcal{D}, \tau') \xrightarrow{j_{\mathcal{D}}} \mathrm{PreSh}(\mathcal{D}) \xrightarrow{f^*} \mathrm{PreSh}(\mathcal{C})$$

   factors through the full subcategory $\mathrm{Shv}(\mathcal{C}, \tau) \subseteq \mathrm{PreSh}(\mathcal{C})$. Equivalently ([Soo26], Lemma 2.12), if $\mathcal{C}^0_{/X} := \{U_i \to X\}$ is a τ-covering sieve on $\mathcal{C}$, then the sieve generated by $f(\mathcal{C}^0_{/X})$ is a τ'-covering sieve on $\mathcal{D}$, and if $T \to X$ is an arbitrary morphism in $\mathcal{C}$, then the canonical map

   $$f(T \times_X U_i) \to f(T) \times_{f(X)} f(U_i)$$

   is an equivalence.

2. Dually, f is **cocontinuous** if the composite

   $$\mathrm{Shv}(\mathcal{C}, \tau) \xrightarrow{j_{\mathcal{C}}} \mathrm{PreSh}(\mathcal{C}) \xrightarrow{f'_*} \mathrm{PreSh}(\mathcal{D})$$

   factors through the full subcategory $\mathrm{Shv}(\mathcal{D}, \tau') \subseteq \mathrm{PreSh}(\mathcal{D})$. Equivalently ([Soo26], Lemma 2.14), for every $X \in \mathcal{C}$ and τ'-covering sieve $\mathcal{D}^0_{/f(X)} \subseteq \mathcal{D}_{f(X)}$, the subcategory

   $$f^{-1}(\mathcal{D}^0_{/f(X)})_{/X} := \mathcal{C}_{/X} \times_{\mathcal{D}_{/f(X)}} \mathcal{D}^0_{/f(X)} \subseteq \mathcal{C}_{/X}$$

   is a τ-covering sieve.

We recall a property of the functors $f_!$, f_*, and f^* in the case that f is continuous:

Proposition 6.36 ([Soo26], Lemma 2.8, Lemma 2.10)**.** *Let $f\colon (\mathcal{C},\tau)\to(\mathcal{D},\tau')$ be a continuous and cocontinuous functor of ∞-sites. Then, there are adjunctions*

$$f_! \dashv f^* \dashv f_*$$

as functors between $\mathrm{Shv}(\mathcal{C},\tau)$ *and* $\mathrm{Shv}(\mathcal{D},\tau')$*. Furthermore, if $\mathcal{C}$ and $\mathcal{D}$ have finite limits and f is left exact, then so are $f_!$ and f_*.*

We will need one corollary of this in Section 7, which shows how these adjoints respect images under the Yoneda embedding:

Corollary 6.37 ([Soo26])**.** *Let $(-)^+$ denote τ'-sheafification on* $\mathrm{PreSh}(\mathcal{D})$*. Let $X\in\mathcal{C}$, and denote its image under the Yoneda embedding by X as well. If $f\colon(\mathcal{C},\tau)\to(\mathcal{D},\tau')$ is a continuous functor, then there is a canonical equivalence $f_!(X^+)\simeq f(X)^+$.*

Given, $(\mathcal{C},\tau)$ and a functor $f\colon\mathcal{C}\to\mathcal{D}$ we now define a topology on $\mathcal{D}$ which upgrades f to being cocontinuous:

Definition 6.38. We define the Grothendieck topology $f_*\tau$ on $\mathcal{D}$ in the following way: For each $Y\in\mathcal{D}$, a full subcategory $\mathcal{D}^0_{/Y}\subseteq\mathcal{D}_{/Y}$ is a $f_*\tau$-covering sieve if and only if for every $X\in\mathcal{C}$ and every morphism $f(X)\to Y$ in $\mathcal{D}$, the subcategory

$$f^{-1}(\mathcal{D}^0_{/Y})_{/X} := \mathcal{C}_{/X}\times_{\mathcal{D}_{/Y}}\mathcal{D}^0_{/Y}\subseteq\mathcal{C}_{/X}$$

is a τ-covering sieve.

Proposition 6.39 ([Soo26])**.**

1. *If $f\colon\mathcal{C}\to\mathcal{D}$ is a functor, with $(\mathcal{C},\tau)$ an ∞-site, then $f\colon(\mathcal{C},\tau)\to(\mathcal{D},f_*\tau)$ is cocontinuous.*
2. *Furthermore, if $\mathcal{C}$ and $\mathcal{D}$ have finite limits, and $f\colon(\mathcal{C},\tau)\to(\mathcal{D},f_*\tau)$ is a fully faithful and left exact, then f is also continuous.*

Proof. 1 is essentially by definition. For 2, let $X\in\mathcal{C}$ and let $\mathcal{C}^0_{/X}\subseteq\mathcal{C}_{/X}$ be a τ-covering sieve. Let $\mathcal{D}^1_{/f(X)}\subseteq\mathcal{D}_{/f(X)}$ be the sieve generated by the image of $\mathcal{C}^0_{/X}$. By [Soo26, Lemma 2.12] (this is where left exactness is used), it suffices to show that the sieve $\mathcal{D}^1_{/f(X)}$ generated by the image of $\mathcal{C}^0_{/X}\to\mathcal{D}_{/f(X)}$ is a $f_*\tau$-covering sieve. Fix a morphism $f(X')\to f(X)$. Since f is fully faithful, this induces a morphism $X'\to X$. We need to check that $\mathcal{C}_{/X'}\times_{\mathcal{D}_{/f(X)}}\mathcal{D}^1_{/f(X)}$ is a τ-covering sieve. By construction, this sieve consists of morphisms $Y\to X'$ such that $f(Y)\to f(X)$ factors over some $f(Y')\to f(X)$ with $Y'\to X\in\mathcal{C}^0_{/X}$. Since f is fully faithful, this is equivalent to saying that $Y\to X$ factors over $Y'\to X$. Hence,

$$\mathcal{C}_{/X'}\times_{\mathcal{C}_{/X}}\mathcal{C}^0_{/X} = \mathcal{C}_{/X'}\times_{\mathcal{D}_{/f(X)}}\mathcal{D}^1_{/f(X)}$$

as full subcategories of $\mathcal{C}_{/X'}$, so $\mathcal{D}^1_{/f(X)}$ is a $f_*\tau$-covering sieve by definition. □

Lemma 6.40 ([Soo26])**.** *In the setting of Statement 2 of the above proposition, $f_!$ and f_* are fully faithful.*

Proof. We just need to show that the counit morphism $f^*f_* \implies \mathrm{id}$ is an equivalence, as the equivalence $\mathrm{id} \simeq f_*f_!$ follows by passing to left adjoints. Since f is both continuous and cocontinuous, the functors f'^* and f'_* preserve sheaves on the level of presheaves. Hence, the natural transformation $f^*f_* \implies \mathrm{id}$ is identified with $f'^*f'_* \implies \mathrm{id}$ restricted to sheaves. But, the latter is an equivalence; since f is fully faithful, f'_* is fully faithful. □

We will need two more small technical results:

6.4 Compatibility of Sheaves

We now apply the buildup from the last few sections to prove Theorem 6.34. As mentioned in the last subsection, the brunt of the proof relies on showing that j is continuous and cocontinuous:

Proposition 6.41. *The inclusion functor $j\colon \mathrm{dStein}^\dagger \to \mathrm{Aff}$ is cocontinuous with respect to τ_{rat}^{St} on the source and $\iota_*\tau_{rat}$ on the target.*

Proof. We need to show that for $X = \mathrm{Spec}(A) \in \mathrm{dStein}^\dagger$ and for any $\iota_*\tau_{rat}$-covering sieve $S_{/j(X)}$ on $j(X)$, $j^{-1}(S_{/j(X)})_X$ is a τ_{rat}^{St}-covering sieve. Furthermore, by definition of the pushforward topology $\iota_*\tau'$, for every derived dagger affinoid $U \in \mathrm{dAfnd}^\dagger$ and every map $\iota(U) \to j(X)$, the pullback sieve

$$\iota^{-1}(S_{/j(X)})_{/U} := \mathrm{dAfnd}^\dagger_{/U} \times_{\mathrm{Aff}_{/j(X)}} S_{/j(X)}$$

is a τ'-covering sieve on U.

The result will follow fairly quickly from constructing a derived dagger affinoid exhaustion of X analogous to 6.9. Let X_0 have dagger exhaustion

$$V_1 \subseteq V_2 \subseteq \cdots \subseteq \bigcup_m V_m = X,$$

and thus $\pi_0(A) = \varprojlim_m \mathcal{O}(V_m)$. By the affinoid case of Proposition 6.27, the derived restriction B_m is such that $\pi_0(B_m) = \mathcal{O}(V_m)$ and $\pi_n(B_m) = \pi_n(A) \otimes_{\pi_0(A)} \mathcal{O}(V_m)$ is finitely generated over $\pi_0(B_m)$. Furthermore, since A is connective, B_m is as well. Hence,

$$\widetilde{V}_m := \mathrm{Spec}(B_m) \in \mathrm{dAfnd}^\dagger$$

and $A \to B_m$ is derived strong and a homotopy epimorphism. Thus, $\widetilde{V}_m \to X$ is a morphism in Aff with image on π_0 the subdomain inclusion $V_m \hookrightarrow X_0$.

By the defining property of $\iota_*\tau_{rat}$ along $\iota(\widetilde{V}_m) \to j(X)$, the sieve

$$\iota^{-1}(S_{/j(X)})_{/\widetilde{V}_m} := \mathrm{dAfnd}^\dagger_{/\widetilde{V}_m} \times_{\mathrm{Aff}_{/j(X)}} S_{/j(X)}$$

is a τ_{rat}-covering of $\widetilde{V}_m$. Hence, it contains a finite cover by affinoid subdomains

$$\{\widetilde{V}_m^\ell \to \widetilde{V}_m\}_\ell$$

such that each composite $\widetilde{V}_m^\ell \to \widetilde{V}_m \to j(X)$ is in $S_{/j(X)}$.

Let $\{V_m^\ell\}_\ell$ denote the finite cover associate to π_0 on the dagger affinoid V_m. By Proposition 6.14, we obtain a family $\{W_\alpha \hookrightarrow X_0\}_\alpha$ of open Stein rational subdomains with $\bigcup_\alpha W_\alpha = X_0$, each W_α relatively compact in some $V_{m(\alpha)}^{\ell(\alpha)}$, and each inclusion $\mathcal{O}(X_0) \to \mathcal{O}(W_0)$ a homotopy epimorphism.

Now, fix α and let $W, \widetilde{V}^\ell$ denote W_α and $\widetilde{V}_{m(\alpha)}^{\ell(\alpha)}$ respectively. By Proposition 6.27, the derived restriction B_W gives $\mathrm{Spec}(B_W) \in \mathrm{dStein}^\dagger$, with $A \to B_W$ a derived strong homotopy epimorphism. Furthermore, since $W \subseteq V_{m(\alpha)}^\ell$, the restriction on the level of π_0 $\mathcal{O}(V_{m(\alpha)}^\ell) \to \mathcal{O}(W)$ is a homotopy epimorphism. Hence, by applying Proposition 6.27, we get a factorization

$$\mathrm{Spec}(B_W) \longrightarrow \widetilde{V}^\ell \longrightarrow \widetilde{V}_{m(\alpha)} \longrightarrow j(X).$$

By our earlier work, the composite $\widetilde{V}^\ell \to j(X)$ is in $S_{/j(X)}$, and $\mathrm{Spec}(B_W) \in \mathrm{dStein}^\dagger$. Hence, the morphism $\mathrm{Spec}(B_W) \to X$ is in $j^{-1}(S_{/j(X)})_{/\widetilde{V}_m}$. By Propositions 6.14 and 6.27, this implies that the family

$$\{\mathrm{Spec}(B_{W_\alpha}) \to X\}_\alpha$$

is a cover of X in τ_{rat}^{St} with each element in $j^{-1}(S_{/j(X)})_X$. Hence, the sieve generated by it is a covering sieve in τ_{rat}^{St} within $j^{-1}(S_{/j(X)})_X$. By upward closure of sieves, we conclude that $j^{-1}(S_{/j(X)})_X$ is a covering sieve. □

We now prove continuity:

Proposition 6.42. *The functor j is continuous with τ_{rat}^{St} on the source and ι' on the target.*

Proof. let $X = \mathrm{Spec}(A) \in \mathrm{dStein}^\dagger$. Fix a τ_{rat}^{St}-covering sieve $\{\mathrm{Spec}(A_i) \to X\}_i$. We need to show that:

1. $\{j(\mathrm{Spec}(A_i)) \to j(X)\}$ is a $\iota_*\tau_{rat}$-covering sieve.
2. For any morphism $\mathrm{Spec}(B) \to X$, the map

$$j(\mathrm{Spec}(B) \times_X \mathrm{Spec}(A_i)) \to j(\mathrm{Spec}(B)) \times_{j(X)} j(\mathrm{Spec}(A_i))$$

   is an equivalence.

2 follows by the fact that $\mathrm{dStein}^\dagger$ is a full subcategory of Aff (hence j is left exact), and that $\mathrm{dSteinAlg}^\dagger$ is closed under tensor products (see [BBK18] and [BKK24]). For 1, let $T_{/j(X)}$ be the sieve that the covering generates on $j(X)$, and let $\{W_{i,0} \hookrightarrow X_0\}$ be the covering on the level of π_0. Let $U = \mathrm{Spec}(B) \in \mathrm{dAfnd}^\dagger$, and let $\iota(U) \to j(X)$ be a morphism in Aff. We need to show that

$$\iota^{-1}(T_{/j(X)})_{/U} := \mathrm{dAfnd}^\dagger_{/U} \times_{\mathrm{Aff}_{j(X)}} T_{/j(X)}$$

is a τ_{rat}-covering on U. Let V denote the dagger affinoid corresponding to π_0 on U. A morphism $\mathrm{Spec}(C) \to U$ is in $\iota^{-1}(T_{/j(X)})_{/U}$ if and only if the composite $\mathrm{Spec}(C) \to X$ factors in Aff through some $\mathrm{Spec}(A_i)$. Since each $A \to A_i$ is transversal to $\pi_m(C)$ for all $m \geq 0$ by Cartan's Theorem B, Lemma 6.31 shows that the map factorizes only when $\pi_0(A) \to \pi_0(C)$ factors through $\pi_0(A_i)$. Hence, it suffices to produce a finite τ_{rat}-cover of U legs realize a rational refinement of $f^{-1}(X_{i,0}) \hookrightarrow V$, where $f\colon V \to X_0$ is the map induced by $A \to B$ on π_0.

Since each $W_{i,0}$ is Stein, Remark 6.33 equips it with a dagger affinoid exhaustion $W_{i,0} = \bigcup_m V_{i,0}^m$ by iterated rational subdomains satisfying $V_{i,0}^m \subseteq \mathrm{int}(V_{i,0}^{m+1})$, where the interior is taken in the Stein space $W_{i,0}$. Pulling back along f, let $Z_i^m := f^{-1}(V_{i,0}^m)$. This fiber product is the set-theoretic preimage with its naive structure (by Lemma 6.31). Furthermore, since rational subdomains are stable under pullback, each Z_i^m is an iterated rational subdomain of V. This doesn't involve any claims about the interior of Z_i^m in an ambient polydisk; indeed, as stressed before, V could have empty interior. However, the roominess of the cover is supplied intrinsically by pulling opens back from X_0 in the following way. The sets $\mathrm{int}(V_{i,0}^m)$ are open in X_0 and satisfy $\bigcup_{i,m} \mathrm{int}(V_{i,0}^m) = \bigcup_i W_{i,0} = X_0$, so their preimages $f^{-1}(\mathrm{int}(V_{i,0}^m))$ are open in V, cover V, and satisfy $f^{-1}(\mathrm{int}(V_{i,0}^m)) \subseteq Z_i^m$. Since V is compact, finitely many of these opens already cover V; let $\{Z_j\}_j$ denote the corresponding finite family of legs. Then $\{Z_j \to V\}_j$ is a finite family of iterated rational subdomains covering V, and the opens $f^{-1}(\mathrm{int}(V_{i,0}^m))$ satisfy condition 3 in the definition of the roomy iterated rational topology. Hence $\{Z_j \to V\}_j$ is a finite roomy iterated rational cover refining $\{W_{i,0} \cap V \to V\}$.

Lifting each $Z_j \to V$ to derived dagger affinoids C_k in U by the affinoid case of Proposition 6.27 gives $\mathrm{Spec}(C_j) \to U$ with $B \to C_j$ an iterated rational localization with nonempty interior on the level of π_0, hence a finite τ_{rat} cover. Since $R_j \subseteq X_{i(j),0}$ on π_0, Lemma 6.31 gives a derived factorization $A \to A_{i(j)} \to C_k$, so each $\mathrm{Spec}(C_j) \to X$ is in $T_{/j(X)}$. Hence, $\iota^{-1}(T_{/j(X)})_{/U}$ contains a τ_{rat}-covering sieve, so it is τ_{rat}-covering by upward closure. □

Proof of Theorem 6.34. Our situation is captured by the following diagram:

$$\begin{array}{ccc} (\mathrm{dStein}^\dagger, \tau_{rat}^{St}) & \xrightarrow{j} & (\mathrm{Aff}, \iota_* \tau_{rat}) \\ & & \uparrow{\scriptstyle \iota} \\ & & (\mathrm{dAfnd}^\dagger, \tau_{rat}) \end{array}$$

ι is continuous and cocontinuous by Proposition 6.39, and thus $\iota_!$ is left-exact and fully faithful. so all that is left to show that the essential image of $j_!$ is contained in the essential image of $\iota_!$. This, along with Propositions 6.41 and 6.42, is enough to show that the functor $\iota^* j_!\colon \mathrm{Shv}(\mathrm{dStein}^\dagger, \tau_{rat}^{St}) \to \mathrm{Shv}(\mathrm{dAfnd}^\dagger, \tau_{rat})$ is fully faithful.

To do this, it suffices to show that the natural map

$$\iota_! \iota^* j_!(X) \to j_!(X)$$

is an equivalence for any $X \in \mathrm{Shv}(\mathrm{dStein}^\dagger, \tau_{rat}^{St})$. Since $\iota_!$, ι^*, and $j_!$ all commute with colimits, it suffices to show this in the case $X = \mathrm{Spec}(A)$. We first show that $\mathrm{Spec}(A) \simeq$

$\lim_n \mathrm{Spec}(A_n)$ on the level of sheaves. This is true in $\mathrm{dStein}^\dagger$ by the existence of the geometric presentation, but the Yoneda embedding does not necessarily preserve this colimit; it is a nontrivial statement.

By the covariant Yoneda lemma, it suffices to show that for any sheaf $Y \in \mathrm{Shv}(\mathrm{dStein}^\dagger, \tau_{rat}^{St})$, we have

$$\mathrm{Map}_{\mathrm{Shv}}(\mathrm{Spec}(A), Y) \simeq \mathrm{Map}_{\mathrm{Shv}}\left(\varinjlim_n \mathrm{Spec}(A_n), Y\right).$$

However, since the A_n's form a cover of $\mathrm{Spec}(A)$ in τ_{rat}^{St}, we have

$$\begin{aligned}\mathrm{Map}_{\mathrm{Shv}}\left(\varinjlim_n \mathrm{Spec}(A_n), X\right) &\simeq \varprojlim_n \mathrm{Map}_{\mathrm{Shv}}(\mathrm{Spec}(A_n), X) \\ &\simeq \varprojlim_n X(\mathrm{Spec}(A_n)) \simeq X(\mathrm{Spec}(A)) \simeq \mathrm{Map}_{\mathrm{Shv}}(\mathrm{Spec}(A), X).\end{aligned}$$

where the second and fourth equivalences are the contravariant Yoneda lemma and the third is by definition of sheaf on the cover $\{\mathrm{Spec}(A_i) \to \mathrm{Spec}(A)\}$. By the geometric presentation of $\mathrm{Spec}(A)$, continuity of $j_!$, and cofinality of the $\overline{A}_n$'s, we have

$$j_!\left(\varinjlim_n \mathrm{Spec}(A_n)\right) \simeq \varinjlim_n j_!(\mathrm{Spec}(A_n)) \simeq \varinjlim_n \iota_!(\mathrm{Spec}(\overline{A}_n)).$$

This implies that

$$\begin{aligned}\iota_!\iota^* j_!(\mathrm{Spec}(A)) &\simeq \iota_!\iota^* \varinjlim_n j_!(\mathrm{Spec}(A_n)) \simeq \varinjlim_n \iota_!\iota^*\iota_!(\mathrm{Spec}(\overline{A}_n)) \\ &\simeq \varinjlim_n \iota_!(\mathrm{Spec}(\overline{A}_n)) \simeq j_!(\mathrm{Spec}(A)).\end{aligned}$$

We conclude by showing that this restricts to a fully faithful functor $\mathrm{dAn}_{\mathbb{C}} \hookrightarrow \mathrm{dAn}_{\mathbb{C}}^\dagger$. Let $X \in \mathrm{dAn}_{\mathbb{C}}$ have a Stein analytic cover by $U_i = \mathrm{Spec}(A_i)$'s such that $\coprod_i U_i \to X$ is an effective epimorphism. Since $j_!$ and ι^* are both colimit-preserving and left exact by Proposition 6.36, they preserve effective epimorphisms. Hence, $\coprod_i \iota^* j_!(U_i) \to \iota^* j_!(X)$ is an effective epimorphism.

Fix i and write $A_n := A_{n,i}$ for the dagger affinoid terms of the geometric presentation of A_i. By full faithfulness of $\iota_!$ and our earlier arguments,

$$\iota^* j_!(U_i) \simeq \iota^* \varinjlim_n \iota_!(\mathrm{Spec}(A_n)) \simeq \varinjlim_n \mathrm{Spec}(A_n)$$

in $\mathrm{Shv}(\mathrm{dAfnd}^\dagger, \tau_{rat})$. The transition maps $A_{n+1} \to A_n$ are homotopy epimorphisms, so the maps $\mathrm{Spec}(A_n) \to \iota^* j_!(U_i)$ are monomorphisms. Since the diagram, is filtered, this implies that $\coprod_n \mathrm{Spec}(A_n) \to \iota^* j_!(U_i)$ is an effective epimorphism (this is [Lur09], Proposition 6.2.3.7). Composing with the effective morphism $\coprod_i \iota^* j_!(U_i) \to \iota^* j_!(X)$ in the previous paragraph, the total map $\coprod_i \coprod_n \mathrm{Spec}(A_{n,i}) \to \iota^* j_!(X)$ is an effective epimorphism.

All that is left is to show that each $\mathrm{Spec}(A_{n,i})$ is a derived dagger affinoid analytic subspace of $\iota^* j_!(X)$. It suffices to check this after pulling back along an arbitrary dagger

affinoid $\mathrm{Spec}(B) \to \iota^* j_!(X)$. Let V be the classical dagger affinoid associated to $\pi_0(B)$. Since V is compact and $\{U_i\}$ covers x, after refining by a finite τ_{rat}-cover of $\mathrm{Spec}(B)$, we can assume the map factors through a single $\iota^* j_!(U_i)$ (this is completely analogous to the proof of Proposition 6.27). The composite $\mathrm{Spec}(A_n) \hookrightarrow \iota^* j_!(U_i)$ is a monomorphism cut out on π_0 by the dagger affinoid subdomain $\mathrm{Spec}(A_n)_0 \subseteq X_{i,0}$. Since $A_i \to A_n$ is a flat homotopy epimorphism, Lemma 6.31 with the affinoid case of Proposition 6.27 identifies the derived pullback $\mathrm{Spec}(A_n) \times_{\iota^* j_!(U_i)} \mathrm{Spec}(B)$ with a derived dagger affinoid whose classical space associated to π_0 is the intersection $\mathrm{Spec}(A_n)_0 \cap V$ and whose structure map to $\mathrm{Spec}(B)$ is derived strong.

However, similarly to the case of the proof of Proposition 6.42, $\mathrm{Spec}(A_n)_0$ is covered by the higher $\mathrm{Spec}(A_i)_0$'s in such a way that they are rational subdomains and have nonempty interior. Since $\mathrm{Spec}(A_n)_0$ is compact, this has a finite subcover which we will denote $\{R_k\}_k$. When intersected with V, this yields a finite roomy iterated rational cover $\{R_k \cap V\}_k$.

Hence, lifting each $R_k \cap V$ by Proposition 6.27 yields a finite derived rational cover $\{\mathrm{Spec}(C_k) \to \mathrm{Spec}(B)\}_k$ of the pullback. So, the pullback is an analytic subspace of $\mathrm{Spec}(B)$, so $\mathrm{Spec}(A_{n,i})$ is a derived dagger affinoid analytic subspace of $\iota^* j_!(X)$.

Therefore, $\iota^* j_!(X)$ admits a cover by derived dagger affinoid analytic subspaces whose coproduct maps to it by an effective epimorphism, i.e. $\iota^* j_!(X) \in \mathrm{dAn}^\dagger_{\mathbb{C}}$. □

Remark 6.43. It seems that the geometric presentation and derived dagger exhaustion we constructed in the proof of Proposition 6.41 are related. The former lacks detail in its construction, so they may in fact be the same. Furthermore, our result generalizes the classical result that the category of complex analytic spaces embeds fully faithfully into the category of dagger complex analytic spaces (see [Bam14], Section 6.4.2).

7 Analytification

Now that we have a good theory of derived complex analytic and dagger spaces, we move toward reproducing and strengthening some of the foundational theorems in complex analytic geometry. The first of these is the analytification process and GAGA. In classical analytic/algebraic geometry, this is a theorem which compares coherent sheaves : If X is a finite type $\mathbb{C}$-scheme, then it admits a covering by $\mathrm{Spec}(R_i)$ where R_i are finite type $\mathbb{C}$-algebras. The underlying $\mathbb{C}$-points of the $\mathrm{Spec}(R_i)$'s equipped with the analytic topology and sheaf of holomorphic functions are Stein spaces, giving rise to a complex analytic space X^{an}. If X is projective, one can show the following:

Theorem 7.1 (GAGA (Serre, [Ser56]))**.** *There is an equivalence of abelian categories*

$$\mathrm{Coh}(X) \cong \mathrm{Coh}(X^{an}).$$

GAGA has an immense number of important corollaries. Perhaps the most significant of these are Chow's Theorem and the algebraization of analytic maps:

Theorem 7.2 (Chow's Theorem)**.** *Any analytic subspace of $\mathbb{P}^n_{\mathbb{C}}$ admits the structure of a projective variety.*

Theorem 7.3 (Algebraization of Analytic Maps)**.** *Let X and Y be proper $\mathbb{C}$-schemes. Then, there is a bijection*

$$\mathrm{Hom}_{\mathrm{Sch}_{\mathbb{C}}}(X, Y) \cong \mathrm{Hom}_{\mathrm{An}_{\mathbb{C}}}(X^{an}, Y^{an}).$$

In this section, we will construct a similar analytification process for derived schemes, constructing associated derived complex analytic and dagger analytic spaces. We will then use it to show a derived enhancement of GAGA and Chow's theorem for complex analytic spaces (which implies the classical statement), as well as a nonabelian dagger analytic GAGA.

7.1 Analytification as a Geometric Morphism

Analytification for $X \in \mathrm{Shv}(\mathrm{AniAlg}, \tau_{zar})$ is fairly straightforward. Before getting into its construction, there is one quirk worth pointing out. In the end, we'd like for quasi-coherent sheaves to glue locally on our analytic spaces. This requires passing them through Aff. However, in the last section, we only showed that $\mathrm{Shv}(\mathrm{dStein}^{\dagger}, \tau_{rat}^{St})$ and $\mathrm{Shv}(\mathrm{dAfnd}^{\dagger}, \tau_{rat})$ embed into $\mathrm{Shv}(\mathrm{Aff}, \iota_*\tau_{rat})$, not necessarily accounting for descendability (where quasi-coherent sheaves actually glue).

Because of this, we need to make the following definition:

Definition 7.4. The **descendable and roomy iterated rationally induced topology** on Aff is defined to be $\tau_{dri} := \tau_{desc} \cap \iota_*\tau_{rat}$.

The important result we will need is:

Proposition 7.5. *The inclusion functor $\iota\colon \mathrm{dAfnd}^{\dagger} \to \mathrm{Aff}$ is continuous and cocontinuous for τ_{rat} on the source and τ_{dri} on the target.*

Proof. It is a general fact that for an ∞-category $\mathcal{C}$ with two topologies τ_1 and τ_2 with $\tau_1 \supseteq \tau_2$, the identity functor $\mathcal{C} \to \mathcal{C}$ is cocontinuous with τ_1 on the source and τ_2 on the target. Since $\iota_*\tau_{rat} \supseteq \tau_{dri}$, ι is cocontinuous.

All that is left is to show continuity, which boils down to showing that any roomy iterated rational cover is descendable. This was done in Proposition 6.20. □

This gives us another adjoint triple $\iota'_! \dashv \iota'^* \dashv \iota'_*$, where $\iota'_!$ and ι'_* are fully faithful, and all three functors are left exact. By Theorem 6.34, we know that $\iota^* j_!$ is fully faithful. So, composing with $\iota'_!$ gives us a fully faithful left exact functor

$$h_!\colon \mathrm{Shv}(\mathrm{dStein}^{\dagger}, \tau_{rat}^{St}) \hookrightarrow \mathrm{Shv}(\mathrm{Aff}, \tau_{dri}).$$

with right adjoint

$$h^* := j^*\iota_*\iota'^*\colon \mathrm{Shv}(\mathrm{Aff}, \tau_{dri}) \to \mathrm{Shv}(\mathrm{dStein}^{\dagger}, \tau_{rat}^{\mathrm{St}})$$

Remark 7.6. Several important subtleties about our setup must be stressed, as they will inform much of the rest of our work in this section and beyond.

1. Though we have a fully faithful functor $h_!\colon \mathrm{Shv}(\mathrm{dStein}^\dagger, \tau_{rat}^{\mathrm{St}}) \hookrightarrow \mathrm{Shv}(\mathrm{Aff}, \tau_{dri})$, there is no (co)continuous functor on the level of sites $(\mathrm{dStein}^\dagger, \tau_{rat}^{\mathrm{St}}) \to (\mathrm{Aff}, \tau_{dri})$. This is essentially because descendable covering sieves must contain finite covers, and there exist noncompact Stein spaces. One can think of the functor $h_!$ as "working with the associated dagger space" associated to a complex analytic space in the test category $\mathrm{Shv}(\mathrm{Aff}, \tau_{dri})$.

2. Due to this extra wrinkle, one may wonder why we do not just work with $\mathrm{Shv}(\mathrm{Aff}, \iota_*\tau_{rat})$ as we did in the last section. This is because the six-functor formalism we will put on $\mathrm{dAn}_{\mathbb{C}}$ and $\mathrm{dAn}^\dagger$ is inherited from the six functor formalism we get on $\mathrm{Shv}(\mathrm{Aff}, \tau_{desc})$ via Theorem 5.55. Quasi-coherent sheaves do not glue in the same way purely in $\iota_*\tau_{rat}$, so we do not get a six functor formalism on complex analytic spaces by purely working with $j_!$.

3. We also note that since τ_{dri} is contained in $\iota_*\tau_{rat}$, there is an inclusion $\mathrm{Shv}(\mathrm{Aff}, \tau_{dri}) \hookrightarrow \mathrm{Shv}(\mathrm{Aff}, \iota_*\tau_{rat})$ which we denote k_*. This functor has a left adjoint given by τ_{dri}-sheafification, denoted k^*. It is fairly quick to see that $k^* \circ j_! \simeq h_!$ in the following way: the functor underlying $\iota_!$ and $\iota'_!$ on the level of sites are both ι. So, $\iota_! \simeq k_*\iota'_!$. Since sheafification is trivial on objects that are already sheaves, this gives that $k^*\iota_! \simeq \iota'_!$. However, in the proof of Theorem 6.34 we showed that $\iota_!\iota^* j_! \simeq j_!$. Hence, we have

$$h_! = \iota'_!\iota^* j_! \simeq k^*\iota_!\iota^* j_! \simeq k^* j_!.$$

   In other words, one can think of τ_{dri}-sheafification for complex analytic spaces as refining by roomy iterated rational dagger subdomains. By uniqueness of adjoints, we have $h^* \simeq j^* k_*$.

Constructing analytification amounts to constructing a functor $\mathrm{AniAlg}_{\mathbb{C}} \to D(\mathrm{CBorn}_{\mathbb{C}})^{cn}$ with good (co)continuity properties. This is induced by giving polynomial algebras over $\mathbb{C}$ their fine bornology, yielding a functor $\mathrm{Poly}_{\mathbb{C}} \to \mathrm{CBorn}_{\mathbb{C}}$, hence a functor $\mathrm{Poly}_{\mathbb{C}} \to D(\mathrm{CBorn}_{\mathbb{C}})^{cn}$. This is coproduct preserving, since $\mathbb{C}[x_1, \cdots, x_n] \otimes_{\mathbb{C}} \mathbb{C}[x_{n+1}, \cdots, x_m] \cong \mathbb{C}[x_1, \cdots, x_m]$ and the bornologies agree. Hence, by Theorem 2.3, we get a functor $\mathrm{AniAlg}_{\mathbb{C}} \to D(\mathrm{CBorn}_{\mathbb{C}})$ whose opposite we denote $a\colon \mathrm{dAff}_{\mathbb{C}} \to \mathrm{Aff}$.

The next ingredient is the following:

Proposition 7.7. *a is continuous with respect to τ_{zar} on the source and τ_{dri} on the target. This gives an induced adjunction $a_! \dashv a^*$, where the functor*

$$a_!\colon \mathrm{Shv}(\mathrm{dAff}_{\mathbb{C}}, \tau_{zar}) \to \mathrm{Shv}(\mathrm{Aff}, \tau_{dri})$$

is left exact.

Proof. We need to show that a is continuous for τ_{desc} and $\iota_*\tau_{rat}$ on the target, respectively. For τ_{desc}, we know that Zariski localizations are descendable on $\mathrm{AniAlg}_{\mathbb{C}}$ by our work in Section 2. This gives that the functor $(\mathrm{dAff}_{\mathbb{C}}, \tau_{zar}) \to (\mathrm{dAff}_{\mathbb{C}}, \tau_{desc})$, which is the identity on objects, is continuous. So, this reduces to showing that the functor a is continuous with

respect to the descendable topology on the source and target, i.e. a descendable morphism of animated $\mathbb{C}$-algebras promotes to a descendable morphism in $\mathcal{C}\mathrm{Alg}(D(\mathrm{CBorn}_{\mathbb{C}}))$. However, this follows by the fact that $D(\mathrm{VS}_{\mathbb{C}}) \to D(\mathrm{CBorn}_{\mathbb{C}})$ is an exact functor of stable presentably symmetric monoidal ∞-categories.

For $\iota_*\tau_{rat}$, we follow the proof of [Soo26], Lemma 4.18. For the sake of brevity, let $\mathcal{C} = \mathrm{dAff}_{\mathbb{C}}$, let $\mathcal{D} = \mathrm{Aff}$, and let $\mathcal{E} = \mathrm{dAfnd}^{\dagger}$. Let $X = \mathrm{Spec}(A) \in \mathcal{C}$ and let $S_{/X}$ be a Zariski covering sieve on X, given explicitly by a family of maps $\{A \to A_i\}_i$ of animated algebras (and corresponding maps $\{\mathrm{Spec}(A_i) \to \mathrm{Spec}(A)\}_i = \{X_i \to X\}_i$). By passing to a refinement if necessary, we can assume that A_i are of the form $A_i = A[f_i^{-1}]$, where the maps $f_i \colon \mathbb{C} \to A$ are morphisms in $D(\mathrm{VS}_{\mathbb{C}})$ generating the unit ideal in $\pi_0(A)$. Let $\mathcal{D}^1_{/a(X)}$ denote the sieve generated by the image of $S_{/X} \to \mathcal{D}_{/a(X)}$. By [Soo26, Lemma 2.12], it suffices to show that $\mathcal{D}^1_{/a(X)}$ is a $\iota_*\tau_{rat}$-covering sieve.

Let $Y = \mathrm{Spec}(B) \in \mathrm{Afnd}$ and let $\iota(Y) \to a(X)$ be any morphism. We need to show that

$$\mathcal{E}^1_{/Y} := \mathcal{E}_{/Y} \times_{\mathcal{D}_{i(Y)}} \mathcal{D}^1_{/a(X)}$$

is a τ_{rat}-covering sieve. Let

$$Z_i := a(X_i) \times_{a(X)} \iota(Y).$$

Then, each $Z_i \to \iota(Y)$ is induced by some $B \to B[f_i^{-1}]$, where f_i is the composite morphism $\mathbb{C} \to A \to B$ in $D(\mathrm{CBorn}_{\mathbb{C}})$, with f_i generating the unit ideal in $\pi_0(B)$. Since the f_i generate the unit ideal in $\pi_0(B)$, the continuous functions $|f_i|$ have no common zero on the compact space $Y_0 = \mathcal{M}(\pi_0(B))$, so by compactness there is a uniform bound

$$\max_i |f_i(x)| \geq c > 0 \qquad \text{for all } x \in Y_0.$$

Fix any real $\lambda > 1$. We use the overlapping rational localizations

$$B_i^{\lambda} := B\left\langle \frac{f_1}{\lambda f_i}, \cdots, \widehat{\frac{f_i}{\lambda f_i}}, \cdots, \frac{f_n}{\lambda f_i} \right\rangle^{\dagger}, \qquad Y_i^{\lambda} := \mathrm{Spec}(B_i^{\lambda}),$$

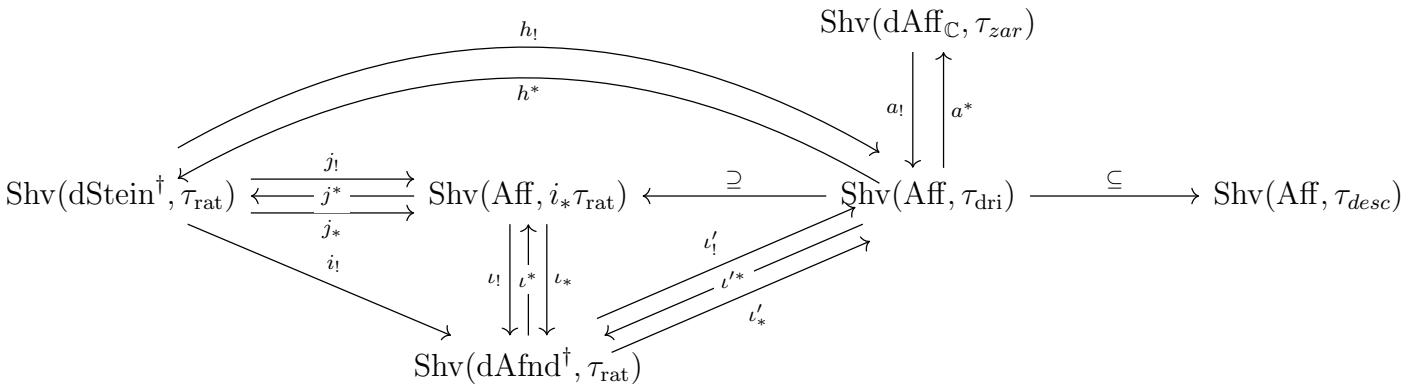
whose associated classical dagger affinoid subdomains are $Y_{i,0}^{\lambda} = \{x : |f_j(x)| \leq \lambda\,|f_i(x)|\ \forall j\}$. Since $\lambda > 1$, each $Y_{i,0}^{\lambda}$ contains the corresponding sharp Laurent piece, so the family $\{Y_{i,0}^{\lambda}\}_i$ covers Y_0. Furthermore, each $B \to B_i^{\lambda}$ is a rational localization, so each $Y_i^{\lambda} \to Y$ realizes Y_i^{λ} as a rational subdomain of Y. On the level of π_0, these are the classical rational localizations $\pi_0(B) \to \pi_0(B_i^{\lambda})$, promoted to derived strong homotopy epimorphisms by the affinoid case of Proposition 6.27.

We finish by verifying roominess. Consider the strict loci

$$U_i^{\lambda} := \{\, x \in Y_0 \mid |f_j(x)| < \lambda\,|f_i(x)|\ \forall j\},$$

which are open in Y_0 and satisfy $U_i^{\lambda} \subseteq Y_{i,0}^{\lambda}$. They cover Y_0: given $x \in Y_0$, choose i maximizing $|f_i(x)|$. Then $|f_i(x)| \geq c > 0$, and $|f_j(x)| \leq |f_i(x)| < \lambda\,|f_i(x)|$ for all j since $\lambda > 1$, so $x \in U_i^{\lambda}$. Thus, $\{U_i^{\lambda}\}_i$ is an open refinement of $\{Y_{i,0}^{\lambda}\}_i$ so $\{Y_i^{\lambda} \to Y\}_i$ is a roomy iterated rational cover. Since each $Y_i^{\lambda} \to Y$ factors as $\iota(Y_i^{\lambda}) \to Z_i \to \iota(Y)$, every leg lies in $\mathcal{E}^1_{/Y}$. Since $\{Y_i^{\lambda} \to Y\}_i$ is a roomy iterated rational cover, it generates a τ_{rat}-covering sieve on Y contained in $\mathcal{E}^1_{/Y}$. So, by upward closure of covering sieves, $\mathcal{E}^1_{/Y}$ is itself τ_{rat}-covering. Hence, $\mathcal{D}^1_{/a(X)}$ is an $\iota_*\tau_{\mathrm{rat}}$-covering sieve, and we are done. $\square$

To recap, we are working with the following diagram in full:



We are now ready to define analytification:

Definition 7.8. We define **complex analytification** to be the functor

$$(-)^{an} := h^* a_! \colon \mathrm{Shv}(\mathrm{AniAlg}, \tau_{zar}) \to \mathrm{Shv}(\mathrm{dStein}^{\dagger}, \tau_{rat}^{St})$$

and **dagger analytification** to be the functor

$$(-)^{an,\dagger} := \iota'^* a_! \colon \mathrm{Shv}(\mathrm{AniAlg}, \tau_{zar}) \to \mathrm{Shv}(\mathrm{dAfnd}^{\dagger}, \tau_{rat}).$$

We note that these are both left exact left adjoints, and thus geometric morphisms.

Remark 7.9. We note that the equivalence $j^*\iota_*\iota'^* = h^* \simeq j^* k_*$ derived in Remark 7.6 implies that $X^{an} \simeq j^*\iota_*(X^{an,\dagger})$, meaning that complex analytification can be gotten from dagger analytification. We conjecture that the essential images of $(-)^{an}$ and $(-)^{an,\dagger}$ are equivalent when restricted to proper objects.

Before proceeding, we need to prove that the analytifications of derived schemes are in fact derived analytic spaces in both the dagger and complex analytic case:

Proposition 7.10. *Let $X \in \mathrm{dSch}_{\mathbb{C}}$ be locally finitely presented; in other words, there exists a cover $\coprod_i \mathrm{Spec}(A_i) \to X$ where $\mathrm{Spec}(A_i)$ is a finitely presented animated $\mathbb{C}$-algebra. Then, $X^{an,\dagger} \in \mathrm{dAn}_{\mathbb{C}}$, and $X^{an} \in \mathrm{dAn}_{\mathbb{C}}$.*

Proof. Let X be a locally finitely presented derived $\mathbb{C}$-scheme, with an effective epimorphism $\coprod_i \mathrm{Spec}(A_i) \to X$ where $\mathrm{Spec}(A_i)$ is finitely presented. Since dagger analytification is a left exact left adjoint, the map

$$\coprod_i \mathrm{Spec}(A_i)^{an,\dagger} \to X^{an,\dagger}$$

is an effective epimorphism. To conclude, all we need to show is

1. Each $\mathrm{Spec}(A_i)^{an,\dagger} \in \mathrm{dAn}_{\mathbb{C}}^{\dagger}$.

2. Each $\mathrm{Spec}(A_i)^{an,\dagger} \hookrightarrow X^{an,\dagger}$ is an analytic subspace.

For 1, we begin with affine space $\mathbb{A}^n$. I claim that $\mathbb{A}^{n,an,\dagger}$ can be written as $\varinjlim_\rho \mathrm{Spec}(\mathbb{C}\langle X/\rho\rangle^\dagger)$, with transition maps the Weierstrass localizations $\mathbb{C}\langle X/\rho\rangle^\dagger \to \mathbb{C}\langle X/\rho'\rangle^\dagger$ for $\rho' \leq \rho$. Here, the $\mathbb{C}\langle X/\rho\rangle^\dagger$ are the algebras of overconvergent functions on the polydisk of radius ρ. Indeed, for $\mathrm{Spec}(B) \in \mathrm{dAfnd}^\dagger$, by adjunction we have

$$\mathrm{Map}_{\mathrm{Shv}(\mathrm{dAfnd}^\dagger,\tau_{rat})}(\mathrm{Spec}(B), \mathbb{A}^{n,an,\dagger}) = \mathrm{Map}_{\mathrm{Shv}(\mathrm{Aff},\tau_{dri})}(\iota_!'\mathrm{Spec}(B), a_!(\mathrm{Spec}(\mathbb{C}[X]))) \simeq \mathrm{Map}_{\mathcal{C}\mathrm{Alg}}(\mathbb{C}[X], B)$$

where we have used subcanonicity of τ_{rat} and Corollary 6.37. Pick the system of polyradii $\rho_m = \{2^m, \cdots, 2^m\}$. Since the roomy iterated rational topology is finitary, we have

$$\varinjlim_{m\in\mathbb{N}} \mathrm{Map}_{\mathrm{Shv}(\mathrm{Aff},\tau_{dri})}(\mathrm{Spec}(B), \mathrm{Spec}(\mathbb{C}\langle X/\rho_m\rangle^\dagger) \simeq \mathrm{Map}_{\mathrm{Shv}(\mathrm{Aff},\tau_{dri})}(\mathrm{Spec}(B), \varinjlim_{m\in\mathbb{N}} \mathrm{Spec}(\mathbb{C}\langle X/\rho_m\rangle^\dagger)).$$

Dropping the subscripts on mapping spaces for notational convenience, I claim that the canonical map

$$\varinjlim_{m\in\mathbb{N}} \mathrm{Map}(\mathbb{C}\langle X/\rho_m\rangle^\dagger, B) \to \mathrm{Map}(\mathbb{C}[X], B)$$

is an equivalence. Since each $\mathbb{C}[X] \to \mathbb{C}\langle X/\rho_m\rangle^\dagger$ is a homotopy epimorphism, it is clear that it is a monomorphism. Furthermore, each dagger algebra is a colimit of Wiener algebras $\mathbb{C}\langle X/\rho_m\rangle^\dagger \simeq \mathrm{colim}_{r>\rho_m}$, the Wiener algebras $\mathbb{C}\langle X/\rho_m\rangle$ will be cofinal in the colimit, so it suffices to show that the map is an epimorphism on them.

It suffices to show that the map is surjective on connected components. If B is a classical dagger affinoid algebra, we first write it as an Ind-Banach algebra $B \simeq \varinjlim_\alpha B_\alpha$. Since the algebras $\mathbb{C}\langle X/\rho_m\rangle$ are ℓ^1-spaces, they are compact projective. Hence, we can write

$$\varinjlim_{m\in\mathbb{N}} \mathrm{Map}(\mathbb{C}\langle X/\rho_m\rangle, B) \simeq \lim_{m\in\mathbb{N}} \varinjlim_\alpha \mathrm{Map}(\mathbb{C}\langle X/\rho_m\rangle, B_\alpha) \simeq \varinjlim_\alpha \lim_{m\in\mathbb{N}} \mathrm{Map}(\mathbb{C}\langle X/\rho_m\rangle, B_\alpha).$$

Since B_α and $\mathbb{C}\langle X/\rho_m\rangle$ are Banach algebras, the set $\mathrm{Map}(\mathbb{C}\langle X/\rho_n\rangle, B_\alpha)$ is identified with

$$\left\{(b_1, \cdots, b_n) \in B_\alpha \;\middle|\; \sum_{(k_i)_{i=1}^n \in \mathbb{N}^n} c_{(k_i)_{i=1}^n} \prod_{i=1}^n (b_i/2^m)^{k_i} \text{ converges } \forall\, (c_{(k_i)_{i=1}^n}) \text{ s.t. } \sum_{(k_1,\cdots,k_n)\in\mathbb{N}^n} |c_{(k_i)_{i=1}^n}| < \infty\right\}.$$

The limit of such mapping spaces is B_α^n itself, which can be identified with $\mathrm{Map}(\mathbb{C}[X], B_\alpha)$. However, n is finite and finite products commute with filtered colimits, so we get $\varinjlim_n B_\alpha^n \simeq (\varinjlim_n B_\alpha)^n \simeq B^n$, i.e.

$$\varinjlim_\alpha \mathrm{Map}(\mathbb{C}[X], B_\alpha) \simeq \varinjlim_\alpha \mathrm{Map}(\mathbb{C}[X], B_\alpha)$$

completing the proof of the claim.

In general, take a morphism $\mathbb{C}\langle X/\rho_n\rangle^\dagger \to \pi_0(B)$ for some n. By restriction, this gives a map $\mathbb{C}[X] \to \pi_0(B)$. This lifts to $\mathbb{C}[X] \to B$ since $\mathbb{C}[X]$ is projective. But, since the map $\mathbb{C}[X] \to \mathbb{C}\langle X/\rho_n\rangle$ is a homotopy epimorphism, [BKK24, Corollary 2.1.36] guarantees that $\mathbb{C}\langle X/\rho_n\rangle \to \pi_0(B)$ lifts to a morphism $\mathbb{C}\langle X/\rho_n\rangle \to B$ under $\mathbb{C}[X]$. This implies that the left arrow in the commutative diagram

$$\begin{array}{ccc} \varinjlim_{n\in\mathbb{N}} \mathrm{Map}_{\mathcal{C}\mathrm{Alg}}(\mathbb{C}\langle X/\rho_n\rangle, B) & \longrightarrow & \mathrm{Map}_{\mathcal{C}\mathrm{Alg}}(\mathbb{C}[X], B) \\ \downarrow & & \downarrow \\ \varinjlim_{n\in\mathbb{N}} \mathrm{Map}_{\mathcal{C}\mathrm{Alg}}(\mathbb{C}\langle X/\rho_n\rangle, \pi_0(B)) & \longrightarrow & \mathrm{Map}_{\mathcal{C}\mathrm{Alg}}(\mathbb{C}[X], \pi_0(B)) \end{array}$$

is surjective on π_0. The bottom arrow is an isomorphism, and the right arrow is an isomorphism on π_0 by projectivity of $\mathbb{C}[X]$. Hence, the top arrow is surjective on π_0.

Now, a general finitely presented $\mathbb{C}$-algebra A fits in a pullback diagram

$$\begin{array}{ccc} \mathrm{Spec}(A) & \longrightarrow & \mathbb{A}^n \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\mathbb{C}) & \longrightarrow & \mathbb{A}^c \end{array}$$

for some n and c. In other words, we can write

$$A \simeq \mathbb{C}[X_1, \cdots, X_n] \otimes^{\mathbb{L}}_{\mathbb{C}[Y_1,\cdots,Y_c]} \mathbb{C}$$

where $\mathbb{C}[Y_1, \cdots, Y_c] \to \mathbb{C}[X_1, \cdots, X_n]$ sends Y_j to the relations $f_j(X_1, \cdots, X_n)$ and $\mathbb{C}[Y_1, \cdots, Y_c] \to \mathbb{C}$ is the augmentation. Since analytification is left exact, it preserves this finite limit, yielding

$$\mathrm{Spec}(A)^{an,\dagger} \simeq \mathbb{A}^{n,an,\dagger} \times_{\mathbb{A}^{c,an,\dagger}} \mathrm{Spec}(\mathbb{C}).$$

Base change the polydisk cover we just constructed along the map $\mathrm{Spec}(A)^{an,\dagger} \to \mathbb{A}^{n,an,\dagger}$. For each ρ, the f_j are bounded on the compact polydisk $\overline{D}^n_\rho$. Hence, the map $\overline{D}^n_\rho \to \mathbb{A}^{c,an,\dagger}$ factors through some $\overline{D}^c_\sigma$. Hence, the corresponding piece is

$$\overline{D}^n_\rho \times_{\mathbb{A}^{c,an,\dagger}} \mathrm{Spec}(\mathbb{C}) = \mathrm{Spec}\left(\mathbb{C}\langle X/\rho\rangle^\dagger \widehat{\otimes}^{\mathbb{L}}_{\mathbb{C}\langle Y/\sigma\rangle^\dagger} \mathbb{C}\right).$$

On the level of π_0, this is $\mathbb{C}\langle X/\rho\rangle^\dagger/(f_1, \cdots, f_c)$, which is a dagger affinoid algebra by definition. Furthermore, the higher homotopy groups are finitely presented over π_0, since base change along the finitely presented map $\mathbb{C}\langle Y/\sigma\rangle^\dagger \to \mathbb{C}$ stays finitely presented and π_0 is Noetherian. Since effective epimorphisms are stable under base change, we get that

$$\coprod_\rho \left(\overline{D}^n_\rho \times_{\mathbb{A}^{n,an,\dagger}} \mathrm{Spec}(A)^{an,\dagger}\right) \to \mathrm{Spec}(A)^{an,\dagger}$$

is an effective epimorphism by dagger affinoids. Hence, $\mathrm{Spec}(A)^{an,\dagger} \in \mathrm{dAn}^\dagger_{\mathbb{C}}$.

For 2, the $\mathrm{Spec}(A_i) \to X$ are Zariski open immersions, i.e. monomorphisms. Since dagger analytification is left exact, the analytified maps remain monomorphisms. So, all we need to check is pulling back along an arbitrary dagger affinoid $\mathrm{Spec}(B) \to X^{an,\dagger}$ and producing a finite roomy rational cover of the pullback in $\mathrm{Spec}(B)$. This is analogous to the computation produced at the end of the proof of Theorem 6.34. Hence, $X^{an,\dagger} \in \mathrm{dAn}^\dagger_{\mathbb{C}}$.

By Remark 7.6, we know that $X^{an} = j^*\iota_*(X^{an,\dagger})$. However, $j^*\iota_*$ is the right adjoint of $\iota^* j_!$. By Theorem 6.34, we know that $\iota^* j_!$ restricts to a functor $\mathrm{dAn}_{\mathbb{C}} \to \mathrm{dAn}^\dagger_{\mathbb{C}}$. Hence, $j^*\iota_*$ will restrict to a functor $\mathrm{dAn}^\dagger_{\mathbb{C}} \to \mathrm{dAn}_{\mathbb{C}}$. So, since $X^{an,\dagger} \in \mathrm{dAn}^\dagger_{\mathbb{C}}$, $j^*\iota_*(X^{an,\dagger}) = X^{an} \in \mathrm{dAn}_{\mathbb{C}}$. $\square$

We conclude this section by identifying the dagger and complex analytifications of a locally finitely presented derived scheme. This will allow for all results proved on dagger analytic quasicoherent sheaves to carry over to complex analytic quasicoherent sheaves.

Proposition 7.11. *For $X \in \mathrm{dSch}_{\mathbb{C}}^{\mathrm{lfp}}$, there is an equivalence $i_!(X^{an}) \simeq X^{an,\dagger}$, where $i_!$ is the embedding $\mathrm{dAn}_{\mathbb{C}} \hookrightarrow \mathrm{dAn}_{\mathbb{C}}^{\dagger}$ constructed in Theorem 6.34*

Proof. By similar gluing procedures to proofs later in this section (using that all involved functors are geometric morphisms), it suffices to show the result on $\mathbb{A}^n$. However, this follows by the geometric presentation of $\mathbb{A}^{n,an}$ being equivalent to $\mathrm{colim}_n \mathrm{Spec}(\mathbb{C}\langle x/\rho_n\rangle^{\dagger})$, and the fact that $\mathbb{A}^{n,an,\dagger}$ is equivalent to the same colimit as shown in the proof of the last proposition. □

7.2 GAGA, à la Clausen-Scholze

In [CS26], GAGA is reformulated in terms of sheaves of ∞-categories on locales associated to complex analytic spaces. The category of quasi-coherent sheaves on the analytification is redressed as a localization with respect to sheaves where "coordinate variables behave analytically". At the outset, this seems like a bit of an esoteric generalization of GAGA, but it earns its keep for several reasons. The first, and most critical one, is that it allows for a recovery of the classical statement of GAGA for coherent sheaves on classical spaces (which we do not cover here, see Clausen–Scholze for an explanation). The second is that it allows for an abstract formalism for GAGA that can be applied to any suitable setting of analytic geometry. The third is that we can use it to produce a more classical looking statement of GAGA. In this section, we will apply the second point to the $D(\mathrm{CBorn}_{\mathbb{C}})$-setting, following the notes of Clausen–Scholze as well as a recent preprint of Yamada [Yam26]. The next section will be concerned with producing a quasi-coherent GAGA statement in terms of the analytification functors constructed in the last section, fulfilling the third point. We will blackbox most proofs in this background section.

Our abstract GAGA will ultimately rely on a theory of "pointless" topology, built on the following objects:

Definition 7.12. A **locale** is a poset whose elements abstract the open sets of a topological space. Concretely speaking, it is a poset $(L, \leq)$ such that

1. Every subset $A \subseteq L$ has an infimum (or **meet**), denoted $\bigwedge A$, and a supremum (or **join**), denoted $\bigvee A$. These bootstrap to the notion of the join and meet of a collection of subsets $\{A_i\}_{i\in I}$.
2. The general distributive law is satisfied: for subsets $\{A_i\}_{i\in I}$ and B of L, we have
$$B \wedge \left(\bigvee_{i\in I} A_i \right) = \bigvee_{i\in I} (B \wedge A_i).$$

A morphism of locales $X \to Y$ is given by a map of posets $Y \to X$ respecting all joins and finite meets.

Given a locale, there is an obvious notion of complement, giving an obvious definition of closed subsets. One can axiomatize the properties of the closed sets, leading to an equivalent definition of locale in terms of them.

We invoke locales due to them providing a common ground for both spaces and certain ∞-categories in the following way:

Example 7.13 ([BF11])**.** Let $\mathcal{C}$ be a closed symmetric monoidal stable ∞-category with all colimits. Then, we can define a locale, $\mathcal{S}(\mathcal{C})$ as follows:

The idempotent algebras in $\mathcal{C}\mathrm{Alg}(\mathcal{C})$ are the closed subsets of the locale. The order and operations are defined in the following way for idempotent algebras A and A', with associated elements $Z, Z' \in \mathcal{S}(\mathcal{C})$:

1. $A \leq A'$ if and only if there is a map $A' \to A$ which commutes with the unit map.
2. $Z \cap Z'$ corresponds to $A \otimes A'$.
3. $Z \subseteq Z'$ if and only if $A \otimes A' \simeq A$.
4. $Z \cup Z' = Z \sqcup_{Z \cap Z'} Z'$ corresponds to the fiber of $A \oplus A' \to A \otimes A'$.
5. $\bigcap_i Z_i$ corresponds to $\mathrm{colim}_i A_i$.

The next step is to put a sheaf of ∞-categories on $\mathcal{S}(\mathcal{C})$ for an arbitrary $\mathcal{C}$ satisfying the assumptions of Example 7.13. Sheaves on locales are quite similar to sheaves on a topological space, but can be interpreted in terms of sites. We recall the definition here for completeness (see [Joh02], C.1, for more, as well as [Aok23] for a deeper discussion in our setting):

Definition 7.14. Let X be a locale.

1. A **presheaf** on X valued in a (possibly ∞-) category $\mathcal{C}$ is a contravariant functor $\mathcal{F} \colon X^{op} \to \mathcal{C}$, where we consider X as the associated category. As usual, for each open subset $U \subseteq X$, we call $\mathcal{F}(U)$ the **sections** of $\mathcal{F}$ on U. If $V \leq U$, then we have a **restriction map** $\rho_{U,V} \colon \mathcal{F}(U) \to \mathcal{F}(V)$.
2. A presheaf $\mathcal{F}$ on X is a **sheaf** if for every family of sections $\{s_i \in F(U_i) \mid i \in I\}$ such that $\rho_{U_i, U_i \cap U_j}(s_i) = \rho_{U_j, U_i \cap U_j}(s_j)$ for all j, there exists a unique $s \in \mathcal{F}(\bigcup_i U_i)$ such that $\rho_{U, U_i}(s) = s_i$.

We can use this to put a sheaf of ∞-categories on $\mathcal{S}(\mathcal{C})$ for an ∞-category $\mathcal{C}$ satisfying the assumptions of example 7.13 in the following way:

Example 7.15. For a closed subset $Z \subseteq \mathcal{C}$ corresponding to an idempotent algebra A, we define

$$\mathrm{C}(Z) = \mathrm{Mod}_A(\mathcal{C}).$$

Since A is idempotent, we can identify $\mathrm{Mod}_A(\mathcal{C})$ with

$$\mathrm{Mod}_A(\mathcal{C}) = \{X \in \mathcal{C} \mid X \simeq X \otimes A\} \subseteq \mathcal{C}.$$

This allows us to define $\mathrm{C}(U)$ for $U = \mathcal{S}(\mathcal{C}) \setminus Z$ by taking the Verdier quotient

$$\mathrm{C}(U) = \mathcal{C}/\mathrm{Mod}_A(\mathcal{C}) = \mathcal{C}/\mathrm{C}(Z).$$

This is equipped with a localization functor $\mathcal{C} \to \mathrm{C}(U)$. That this is a sheaf on $\mathcal{S}(\mathcal{C})$ is proven in [CS26], Proposition 5.5.

From here on out, we will consider $\mathcal{C} = \mathrm{Mod}_{\mathbb{C}[T]_{fin}}(D(\mathrm{CBorn}_{\mathbb{C}}))$. It turns out that we can divvy up $\mathcal{S}(\mathcal{C})$ in terms of the algebras of overconvergent functions discussed in Section 6.1, giving $\mathcal{S}(\mathcal{C})$ the structure of a "categorical analytified affine line". In this affine line, we can control the behavior of T as an "analytic variable". Indeed, it can be shown:

Proposition 7.16. *Let $\mathbb{C}\{|T| \leq 1\}$ denote the ring of overconvergent functions on the closed unit disk. Concretely, this is the ring*

$$\bigcup_{r>1} \left\{ \sum_{n=0}^{\infty} a_n T^n \in \mathbb{C}[|T|] \;\middle|\; \sum_{n=0}^{\infty} |a_n| r^n < \infty \right\}.$$

Dually, let $\mathbb{C}\{|T| \geq 1\}$ denote the ring

$$\bigcup_{r<1} \bigcup_{m \in \mathbb{Z}_{\geq 0}} \left\{ \sum_{n=-\infty}^{m} a_n T^n \in \mathbb{C}((T^{-1})) \;\middle|\; |a_n| r^n \to 0 \right\}.$$

Then, $\mathbb{C}\{|T| \leq 1\}$ and $\mathbb{C}\{|T| \geq 1\}$ are idempotent algebras in $\mathcal{C}$, and their localizations (e.g. the object $\mathbb{R}\mathrm{Hom}_{\mathbb{C}[T]}(\mathrm{fib}(\mathbb{C}[T] \to \mathbb{C}\{|T| \geq 1\}), \mathbb{C}[T])$ in the case of $\mathbb{C}\{|T| \geq 1\}$) correspond to $\mathbb{C}\{|T| > 1\}$ and $\mathbb{C}\{|T| < 1\}$ respectively.

Proof. This is [Yam26], Lemmas 4.3 and 4.5. □

We can use this "analytified affine line" setup to control the behavior of arbitrary polynomials.

Proposition 7.17 ([Yam26], Proposition 4.7.)**.** *For any $r > 0$, let $\{|T| < r\}$ denote the open subset determined by $\mathbb{C}\{|T/r| < 1\}$ of $\mathcal{S}(\mathcal{C})$, and define $\{|T| \leq r\}$, $\{|T| > r\}$, and $\{|T| \geq r\}$ similarly.*

Let R be any discrete commutative $\mathbb{C}$-algebra, let $f \in R$ and let $r > 0$. We let $\{|f| < r\}$ denote the preimage of $\{|T| < r\}$ induced by the map

$$\mathbb{C}[T] \xrightarrow{T \mapsto f} R,$$

and define $\{|f| \leq r\}, \{|f| \geq r\}$, and $\{|f| > r\}$ similarly. For any elements $f, g \in C[T]$, $\alpha \in \mathbb{C}$, and $r, s > 0$, we have the following:

1. $\{|f| \leq 1\} = \bigcap_{r>1} \{|f| \leq r\}$ *and* $\{|f| \geq 1\} = \bigcap_{0<r<1} \{|f| \geq r\}$.
2. $\{|f| \leq 1\} \cup \{|f| \geq 1\} = \mathcal{S}(\mathbb{C}[T])$.

3. *If* $r < 1$, *then* $\{|f| \leq r\} \cap \{|f| \geq 1\} = \varnothing$.

4. $\{|f| \leq 1\} \cap \{|g| \leq 1\} \subseteq \{|fg| \leq 1\}$, *and* $\{|f| \geq 1\} \cap \{|g| \geq 1\} \subseteq \{|fg| \geq 1\}$.

5. *If* $|\alpha|_{\mathbb{C}} \leq 1$, *then* $\{|\alpha| \leq 1\} = \mathcal{S}(\mathbb{C}[T])$. *If* $|\alpha|_{\mathbb{C}} \geq 1$, *then* $\{|\alpha| \geq 1\} = \mathcal{S}(\mathbb{C}[T])$.

6. $\{|f| \leq r\} \cap \{|g| \leq s\} \subseteq \{|f + g| \leq r + s\}$.

Remark 7.18. This shows that the sets $\{|f| \leq r\}$ seem to behave like closed subsets of the Berkovich spectrum of $\mathbb{C}[T]$. It does turn out that the locale associated to the Berkovich spectrum $\mathcal{M}(\mathbb{C}[T])$ admits a morphism from $\mathcal{S}(\mathcal{C})$. This shows that $\mathcal{S}(\mathcal{C})$ is a "categorified analytified affine line", since $\mathcal{M}(\mathbb{C}[T])$ is simply $\mathrm{Spec}(\mathbb{C}[T]) = \mathbb{A}^1$ in its analytic topology. This will also remain true for more general affine objects we will consider shortly. Thus, we call the locale associated to $\mathcal{M}(A)$ for an Ind-Banach $\mathbb{C}$-algebra A a **categorified locale**.

We can now set up GAGA. Let $\mathcal{S}(R)$ denote $\mathcal{S}(\mathrm{Mod}_R(D(\mathrm{CBorn}_{\mathbb{C}})))$ for any discrete commutative $\mathbb{C}$-algebra (considered in its fine bornology) R. We can define the open subset

$$\mathcal{S}(R, f) = \bigcup_{r>0} \{|f| < r\} \subseteq \mathcal{S}(R),$$

the open subset of $\mathcal{S}(R)$ in which f "behaves like an analytic variable". By multiplicativity and the triangle inequality, we have

$$\mathcal{S}(R, fg), \mathcal{S}(R, f + g) \supseteq \mathcal{S}(R, f) \cap \mathcal{S}(R, g).$$

Hence, if R is finite type and generated by $X_1, \cdots, X_n$, then the subset

$$S(R, R) := \bigcap_{f \in R} \mathcal{S}(R, f)$$

is an open subset of $\mathcal{S}(R)$, and is equal to the finite intersection

$$\bigcap_{i=1}^{n} \mathcal{S}(R, X_i).$$

This is our analytification, and we define

$$\mathrm{C}(R, R) = \mathrm{C}(\mathcal{S}(R, R))$$

in the sense of 7.13. For a general finite type $\mathbb{C}$-scheme X, defining the sheaf C and the open sets glue, so we can define $\mathcal{S}(X, X)$ and $\mathrm{C}(X, X)$ analogously by Kan extension. Rigorously, if X has a cover by finite type affine $\mathbb{C}$-schemes $\{\mathrm{Spec}(R_i)\}_{i \in I}$, we can define

$$\mathrm{C}(X, X) = \lim_i \mathrm{C}(R_i, R_i).$$

Using 7.17, we can show GAGA for proper schemes.

Theorem 7.19 (GAGA for Schemes, à la Clausen-Scholze ([Yam26], Theorem 4.8)). *For any proper $\mathbb{C}$-scheme X, we have*

$$\mathrm{C}(X, X) \simeq \mathrm{QCoh}(X).$$

For a derived $\mathbb{C}$-algebra R, one can define $\mathcal{S}(R, f)$ similarly to $\mathcal{S}(A, g)$ for a classical $\mathbb{C}$-algebra R; indeed, we treat the formal element f as a map from $\mathbb{C}[T]$ included $\mathrm{AniAlg}_{\mathbb{C}}$, and all the previous constructions go through. Hence, if R is a finite type derived $\mathbb{C}$-algebra, one can define $\mathrm{S}(R, R)$ analogously. Indeed, we obtain the coordinate variables X_i in R via the composite

$$\mathbb{C}[X_i] \to \mathbb{C}[X_1] \otimes^{\mathbb{L}} \cdots \otimes^{\mathbb{L}} \mathbb{C}[X_n] \simeq \mathbb{C}[X_1, \cdots, X_n] \to R$$

where the last projection is guaranteed by virtue of R being finite type.

It is similarly possible to prove a derived enhancement of 7.17, giving us an abstract setup for GAGA. Again, by Kan extension, this allows us to define $\mathrm{C}(X, X)$, yielding the following:

Theorem 7.20 (GAGA for Derived Schemes, á la Clausen-Scholze). *For any proper derived $\mathbb{C}$-scheme X, we have*

$$\mathrm{C}(X, X) \simeq \mathrm{QCoh}(X).$$

7.3 The Six-Functor Formalism on $\mathrm{dAn}^{\dagger}_{\mathbb{C}}$ and New Morphisms

Before proving derived GAGA and the algebraization of analytic maps in full, we will need to fully introduce the six-functor formalism on $\mathrm{dAn}^{\dagger}_{\mathbb{C}}$ as well as some new definitions of morphisms and their properties. We will also need to prove a few preliminary results on !-covers. They are as follows:

Definition 7.21. The six-functor formalism on $\mathrm{Shv}(\mathrm{dAfnd}^{\dagger}, \tau_{rat})$ is defined to be the restriction of the six-functor formalism $\mathrm{QCoh}\colon \mathrm{Corr}(\mathrm{Shv}(\mathrm{Aff}, \tau_{dri}), \widetilde{E}) \to \mathcal{C}\mathrm{Alg}(\mathrm{Pr}^{L})$ restricted to the essential image of $\mathrm{Shv}(\mathrm{dAfnd}^{\dagger}, \tau_{rat})$ in $\mathrm{Shv}(\mathrm{Aff}, \tau_{dri})$ under $\iota'_!$.

Definition 7.22. Let $X, Y, \in \mathrm{dAn}^{\dagger}_{\mathbb{C}}$ (resp. $\mathrm{dAn}_{\mathbb{C}}$), let $\mathrm{Spec}(A), \mathrm{Spec}(B) \in \mathrm{dAfnd}^{\dagger}$ (resp. $\mathrm{dStein}^{\dagger}$), and let $f\colon Y \to X, g\colon \mathrm{Spec}(A) \to \mathrm{Spec}(B)$ be morphisms.

1. We say that f is **affine** if for every map $\mathrm{Spec}(C) \to X$ with $\mathrm{Spec}(C) \in \mathrm{dAfnd}^{\dagger}$ (resp. $\mathrm{dStein}^{\dagger}$), the pullback $Y \times_X \mathrm{Spec}(C)$ is representable in $\mathrm{dAfnd}^{\dagger}$ (resp. $\mathrm{dStein}^{\dagger}$).

2. We say that X is **quasi-compact** if every cover of X in the analytic topology admits a finite subcover.

3. We say that f is **quasi-compact** if for every quasi-compact Z with a morphism $Z \to X$, the pullback $Y \times_X Z$ is quasi-compact.

4. We say that f is **quasi-separated** if the diagonal $Y \to Y \times_X Y$ is quasi-compact.

5. We say that g is a **closed immersion** if it is derived strong and the underlying map $B \to A$ is a strict epimorphism on π_0.

6. We say that f is a **closed immersion** if f is affine, and for every map $\mathrm{Spec}(C) \to X$ with $\mathrm{Spec}(C) \in \mathrm{dAfnd}^\dagger$, the map $\mathrm{Spec}(C) \times_X Y \to \mathrm{Spec}(C)$ is a closed immersion in the sense of the above.

7. We say that f is **separated** if the diagonal $Y \to Y \times_X Y$ is a closed immersion.

We abbreviate quasi-compact by qc, separated by s, and quasi-separated by qs. So, a quasi-compact quasi-separated morphism is denoted qcqs.

We prove a few important properties of these morphisms here.

Lemma 7.23. *Quasi-compact morphisms are stable under base change and composition.*

Proof. Let $f\colon Y \to X$ be a qc morphism, let $X' \to X$, and let $f'\colon Y \times_X X' \to X'$.

Let $\mathrm{Spec}(B) \to X'$ be an affinoid subspace, and compose to $\mathrm{Spec}(B) \to X' \to X$. By transitivity of fiber products, we have

$$(Y \times_X X') \times_{X'} \mathrm{Spec}(B) = Y \times_X \mathrm{Spec}(B).$$

Since f is qc, the right hand side is qc. Hence, the left hand side is qc as well.

For composition, let $h\colon X \to Z$ be qc, and let $\mathrm{Spec}(C) \to X''$ be an affinoid subspace. Let $X_B = X \times_{X''} \mathrm{Spec}(B)$. Similarly to above, we have

$$Y \times_{X''} \mathrm{Spec}(B) = Y \times_X X_B.$$

We know that X_B is qc. Choose a finite cover $\{U_i = \mathrm{Spec}(A_i)\}$ of X_B. Base changing this cover along $Y \times_X X_B \to X_B$ gives a finite cover of $Y \times_X X_B$. Since each U_i is a dagger affinoid mapping to X and f is qc, each $Y \times_X \mathrm{Spec}(A_i)$ is qc. Hence, it is a finite union of affinoid subspaces, and is thus qc. So, $h \circ f$ is qc. □

Lemma 7.24. *Let $f\colon Y \to X$ and $g\colon X \to Z$ with $g \circ f$ qc and g qs. Then, f is qc.*

Proof. We factor f through its graph $\Gamma_f\colon Y \to Y \times_Z X$ as $f = p_2 \circ \Gamma_f$, where $p_2\colon X \times_Z Y \to Y$ is projection onto the second factor. This sits in the pullback square

$$\begin{array}{ccc} Y & \xrightarrow{\Gamma_f} & Y \times_Z X \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f \times \mathrm{id}_X} \\ X & \xrightarrow{\Delta_g} & X \times_Z X \end{array}$$

Since g is qs, Δ_g is qc. Hence, Γ_f is qc by Lemma 7.23. The projection p_2 sits in the pullback square

$$\begin{array}{ccc} Y \times_Z X & \xrightarrow{p_2} & X \\ \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle g} \\ Y & \xrightarrow{g\circ f} & Z \end{array}$$

Hence, p_2 is a base change of $g \circ f$, so it is qc by Lemma 7.23. Since f is the composite of qc morphisms, f is qc by Lemma 7.23. □

Lemma 7.25. *Let $f\colon Y \to X$ and $g\colon X \to Z$ with $g \circ f$ qs. Then, f is qs.*

Proof. The diagonal $\Delta_f\colon Y \to Y \times_X Y$ is the base change of the diagonal $\Delta_{g\circ f}\colon Y \to Y \times_Z Y$ along the monomorphism $q\colon Y \times_X Y \hookrightarrow Y \times_Z Y$. Since $g \circ f$ is qs, $\Delta_{g\circ f}$ is qc. Hence, by base change, Δ_f is qc, and f is qs. □

The important takeaway from this is the following:

Proposition 7.26. *Let X, Y be qcqs analytic spaces (in $\mathrm{dAn}^\dagger_{\mathbb{C}}$ or $\mathrm{dAn}_{\mathbb{C}}$, and let $f\colon X \to Y$ be any morphism. Then, f is qcqs.*

Proof. Let $g\colon X \to *$ be the structure map. Then, since X is qcqs, g is qcqs. Then, the structure map $Y \to *$ can be written as $g \circ f$. Since Y is qcqs, $g \circ f$ is qcqs. Hence, by Lemmas 7.24 and 7.25, f is qcqs. □

With this built up, we also introduce some new notation, inspired by [Soo26], Section 3.1.6. Let $X \in \mathrm{dAn}^\dagger_{\mathbb{C}}$ and let $S \subseteq X_0$ be a closed subset of the underlying topological space of X. Here, the underlying topological space is the colimit of Berkovich spectra taken over the cover of X by analytic affinoid subspaces. We use the same notation X_0 as for the affinoid case. Let $V := X_0 \setminus S$, which we can identify with an analytic subspace $i\colon V \hookrightarrow X$. We define full subcategories $\Gamma_S\mathrm{QCoh}(X) \subseteq \mathrm{QCoh}(X)$ and $L_S\mathrm{QCoh}(X) \subseteq \mathrm{QCoh}(X)$ as the full subcategories spanned by objects M such that $i^*M \simeq 0$ and $i^!M \simeq 0$ respectively. If we assume that i is qc as well, then we have $i_* = i_!$ (see Remark 7.45), so base change implies that the inclusions of these full subcategories admit adjoints given by $\Gamma_S := \mathrm{Fib}(\mathrm{id} \to i_*i^*)$ and $L_S := \mathrm{Cofib}(i_!i^! \to \mathrm{id})$.

Our first lemma is the following:

Lemma 7.27. *Let $X \in \mathrm{dAn}^\dagger_{\mathbb{C}}$. Suppose we have sequences $V_1 \supseteq V_2 \supseteq V_3 \supseteq \cdots$ and $U_1 \subseteq U_2 \subseteq U_3 \subseteq \cdots$ of analytic subspaces of X such that for every $n \geq 1$, the morphisms $V_n \to X$ and $U_n \to X$ are quasi-compact, and $U_n \cup V_n = X$ and $U_n \cap V_{n+1} = \varnothing$.*

Write $S_n = X_0 \setminus V_n$ and $\Gamma_n\mathrm{QCoh}(X) := \Gamma_{S_n}\mathrm{QCoh}(X)$. Then, there is an equivalence of Pro-systems

$$\text{“}\lim_n\text{”}\mathrm{QCoh}^!(U_n) \simeq \text{“}\lim_n\text{”}\Gamma_n\mathrm{QCoh}(X)$$

compatible with the maps from $\mathrm{QCoh}(X)$.

Proof. Write $i_n \colon V_n \to X$ and $t_n \colon U_n \to X$ for the inclusions. We also write $\Gamma_n := \Gamma_{S_n}$ and write incl_n for the inclusion of $\Gamma_n \mathrm{QCoh}(X) \subseteq \mathrm{QCoh}(X)$. Consider the diagram

$$\cdots \to \mathrm{QCoh}(U_{n+1}) \xrightarrow{\Gamma_{n+1} t_{n+1}*} \Gamma_{n+1}\mathrm{QCoh}(X) \xrightarrow{t_n^! \mathrm{incl}_{n+1}} \mathrm{QCoh}(U_n) \xrightarrow{\Gamma_n t_n *} \Gamma_n \mathrm{QCoh(X)} \to \cdots .$$

We must show that there are natural equivalences

$$t_n^! \mathrm{incl}_{n+1} \Gamma_{n+1} t_{n+1*} \simeq t_{n,n+1}^!$$

and

$$\Gamma_n t_{n*} t_n^! \mathrm{incl}_{n+1} \simeq \Gamma_{n,n+1}.$$

To do this, we see that the left adjoint to $t_n^! \mathrm{incl}_{n+1}$ is given by t_{n*}. Indeed, this already factors through $\Gamma_{n+1}\mathrm{QCoh}(X)$, so by base change we have $i_{n+1}^* t_{n*} \simeq 0$ since $U_n \cap V_{n+1} = \varnothing$. So, by passing to adjoints, it suffices to show that

$$t_{n+1}^* \mathrm{incl}_{n+1} t_{n*} \simeq t_{n,n+1*}$$

and

$$t_{n*} t_n^* \mathrm{incl}_n \simeq \mathrm{incl}_{n,n+1}.$$

The first equivalence follows by base change. The second follows by descent applied to the covering $X = V_n \cup U_n$. □

The second lemma is the following:

Lemma 7.28. *Let $X \in \mathrm{dAn}_{\mathbb{C}}^{\dagger}$. Suppose that we have sequences $V_1 \supseteq V_2 \supseteq V_3 \supseteq \cdots$ and $U_1 \subseteq U_2 \subseteq U_3 \subseteq \cdots$ of analytic subspaces of X such that:*

1. *For every $n \geq 1$, the morphisms $V_n \to X$ and $U_n \to X$ are qc, and $U_n \cup V_n = X$ and $U_n \cap V_{n+1} = \varnothing$.*
2. *We have $\bigcup_{n \geq 1} U_n = X$.*

Then, the natural morphism $\mathrm{QCoh}^!(X) \to \lim_n \mathrm{QCoh}^!(U_n)$ is an equivalence.

Proof. We use the same notation as in the proof of the above lemma. I claim that the natural morphism

$$\mathrm{QCoh}(X) \to \lim_n \Gamma_n \mathrm{QCoh}(X), \qquad M \mapsto (\Gamma_n M)_n$$

is an equivalence. This functor is right adjoint to the functor sending $(M_n)_n \mapsto \mathrm{colim}_n M_n$.

Let $M \in \mathrm{QCoh}(X)$. First, we show that the counit morphism

$$\mathrm{colim}_n \mathrm{incl}_n \Gamma_n M \to M$$

is an equivalence. By descent, it suffices to check this after applying t_m^* for each $m \geq 0$. This follows by the fact that t_m^* commutes with colimits, $U_m \cap V_{m+1} = \varnothing$, and base change.

Now, we check the unit morphism. We need to show that for each Cartesian section $(M_n)_n$ belonging $\lim_n \Gamma_n \mathrm{QCoh}(X)$, and each $m \geq 0$, the morphism

$$M_m \to \Gamma_m(\mathrm{colim}_n \mathrm{incl}_n M_n)$$

is an equivalence. However, this follows by the fact that Γ_m commutes with colimits. So, the morphism we started with is an equivalence, and the result follows by 7.27. □

This gives an important result on !-coverings which we will use later.

Corollary 7.29. *With notations as above, the canonical morphism*

$$\mathrm{QCoh}^!(X) \to \lim_{[m]\in\Delta} \mathrm{QCoh}^!\left(\left(\coprod_{n\geq 1} U_n\right)^{m+1/X}\right)$$

is an equivalence. In particular, $\coprod_{n\geq 1} U_n \to X$ *is a* !*-cover.*

Proof. Since coproducts in $\mathrm{dAn}^\dagger_{\mathbb{C}}$ are universal, $\mathrm{Qcoh}^!$ commutes with products, this follows from Lemma 7.28 and the Bousfield-Kan formula for ∞-categories [Maz19]. □

7.4 Derived GAGA

In the last section, we showed that for a proper derived scheme X, we have $\mathrm{QCoh}(X) \simeq \mathcal{C}(X, X)$. Our goal is to relate $\mathcal{C}(X, X)$ to $\mathrm{QCoh}(a_!(X))$, recovering a more classical-looking quasi-coherent GAGA for our derived analytic geometry.

Theorem 7.30. *For a finite type derived* $\mathbb{C}$*-scheme* X*, there is an equivalence of* ∞*-categories*

$$\mathrm{C}(X, X) \simeq \mathrm{QCoh}(\iota'_!(X^{an,\dagger})).$$

To prove this result, we split into two key steps. The first is to show the result on $\mathbb{A}^n$:

Proposition 7.31. *We have*

$$\mathrm{C}(\mathbb{A}^n, \mathbb{A}^n) \simeq \mathrm{QCoh}(\iota'_!(\mathbb{A}^{n,an,\dagger})).$$

Proof of 7.31. We compute both sides in the case $X = \mathbb{A}_1$ and bootstrap from there. In this case, $X = \mathrm{Spec}(\mathbb{C}[X])$. The covering of X^{an} by closed disks of radius r $\overline{D}_r(0) = \mathrm{Spec}(\mathbb{C}\langle X/r\rangle^\dagger)$ is a $*$-covering. Hence, we can take

$$\mathrm{QCoh}(\iota'_!(X^{an,\dagger})) \simeq \varprojlim_{r>0} \mathrm{Mod}_{\mathbb{C}\langle X/r\rangle^\dagger} D(\mathrm{CBorn}_{\mathbb{C}})^{cn}$$

for the right-hand side, where the transition functors are base change. On the level of the locale, this is

$$\mathrm{QCoh}(\iota'_!(X^{an,\dagger})) \simeq \varprojlim_{r>0} \mathrm{C}(\{|X| < r\} \simeq \lim_{n\in\mathbb{N}} \mathrm{C}(|X| < 2^n).$$

On the left hand side, we have

$$\mathcal{S}(X,X) = \bigcup_{r>0} \{|X| < r\}.$$

Thus,

$$\mathrm{C}(X,X) = \mathrm{C}\bigg(\bigcup_{r>0}\{|X|<r\}\bigg) = \mathrm{C}\bigg(\bigcup_{n\in\mathbb{N}}\{|X|<2^n\}\bigg) \simeq \varprojlim_{n\in\mathbb{N}} \mathrm{C}(\{|X|<2^n\})$$

since C is a sheaf. However, on the level of the locale, we have the containments

$$\{|X| < 2^n\} \subseteq \{|X| \leq 2^n\} \subseteq \{|X| < 2^{n+1}\}.$$

Applying $\mathrm{C}(-)$ gives the system

$$\mathrm{C}(\{|X|<2^{n+1}\}) \longrightarrow \mathrm{C}(\{|X|\leq 2^n\}) \longrightarrow \mathrm{C}(\{|X|<2^n\}) \longrightarrow \cdots,$$

exhibiting both of the above systems as confinal subsystems of a larger set of restrictions. This implies that their limits are equivalent, i.e. $\mathrm{QCoh}(\iota_!'(X^{an,\dagger})) \simeq \mathrm{C}(X,X)$. The proof for $X = \mathbb{A}^n$ is analogous, replacing the radii with polyradii and iterating over the sets $\{|X_i| < r\}$ and $\{|X_i| \leq r\}$ for each side. □

The second step is to show abstractly that the quotient arguments we need for a general finite type derived $\mathbb{C}$-algebra go through. The result will then follow by the fact that QCoh and $\mathrm{C}(-)$ are sheaves. This takes the form of a relative Künneth formula for $\mathrm{C}(-,-)$:

Proposition 7.32 (Künneth for $\mathrm{C}(-,-)$)**.** *Let $\mathbb{A}^{n_1}$ and $\mathbb{A}^{n_2}$ be affine spaces, and let S be an affine space $\mathbb{A}^c$ (allowing $c=0$) with maps $S \to \mathbb{A}^{n_i}$. Then, we have a comparison*

$$\mathrm{C}(\mathbb{A}^{n_1},\mathbb{A}^{n_1}) \otimes_{\mathrm{C}(S,S)} \mathrm{C}(\mathbb{A}^{n_2},\mathbb{A}^{n_2}) \simeq \mathrm{C}(\mathbb{A}^{n_1}\times_S \mathbb{A}^{n_2}, \mathbb{A}^{n_1}\times_S \mathbb{A}^{n_2}), \qquad M\boxtimes N \mapsto p_1^*M \otimes p_2^*N$$

where the tensor product on the left hand side is the Lurie tensor product.

Remark 7.33. It is likely that this holds in much more generality than just affine spaces (in fact, it should hold for all finite type derived schemes by expressing the local charts as fiber products of affines, similarly to how we do below), but we only prove it for them here as it is all we need. It is odd that such a proof is not present elsewhere in the literature.

We give a few lemmas for clarity before proving the full result. First, we recall a standard fact about the Lurie tensor product of module categories:

Lemma 7.34. *For $B_1, B_2, S \in \mathcal{C}\mathrm{Alg}(D(\mathrm{CBorn}_{\mathbb{C}}))$ with maps $S \to B_i$, we have*

$$\mathrm{Mod}_{B_i} \otimes_{\mathrm{Mod}_S} \mathrm{Mod}_{B_2} \simeq \mathrm{Mod}_{B_1 \widehat{\otimes}^{\mathbb{L}}_S B_2}, \qquad M\boxtimes N \mapsto p_1^*M \otimes p_2^*N.$$

Proof. This follows by the fact that $\mathrm{Mod}_{(-)}\colon \mathcal{C}\mathrm{Alg}(D(\mathrm{CBorn}_{\mathbb{C}})) \to \mathcal{C}\mathrm{Alg}(\mathrm{Pr}^L)$ is symmetric monoidal (see [Lur17], Theorem 4.8.5.16 and Section 4.8.4). □

The next is an important result on products of !-covers of affine space, which is the crux of our result:

Lemma 7.35. *Let $S \in \mathrm{dAfnd}^\dagger$ with maps $\mathbb{A}^{n_i} \to S$. Write $g_i \colon \coprod_\rho \overline{D}^{n_i}_\rho \to \mathbb{A}^{n_i}$ for the closed polydisk !-cover of $\mathbb{A}^{n_i}$ in the six-functor formalism on $\mathrm{Shv}(\mathrm{dAfnd}^\dagger, \tau_{rat})$ granted by 7.29. Then*

$$\left\{ \overline{D}^{n_1}_\rho \times_S \overline{D}^{n_2}_\psi \longrightarrow \mathbb{A}^{n_1} \times_S \mathbb{A}^{n_2} \right\}_{\rho,\psi}$$

is a !-cover, each leg lies in $\mathrm{dAfnd}^\dagger$, and every finite intersection of legs again lies in $\mathrm{dAfnd}^\dagger$.

Proof. We base change g_1 along $\mathbb{A}^{n_1} \times_S \mathbb{A}^{n_2} \to \mathbb{A}^{n_1}$. By !-locality, the resulting family $\{\overline{D}^{n_1}_\rho \times_S \mathbb{A}^{n_2} \to \mathbb{A}^{n_1} \times_S \mathbb{A}^{n_2}\}$ is a !-cover. We then base change g_2 along each $\overline{D}^{n_1}_\rho \times_S \mathbb{A}^{n_2} \to \mathbb{A}^{n_2}$. Again, by !-locality, these are !-covers, and a composite of !-covers is a !-cover. This gives the product family. Each leg is $\mathrm{Spec}(\mathbb{C}\langle X/\rho\rangle^\dagger \widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}(S)} \mathbb{C}\langle X'/\psi\rangle^\dagger)$, which is an object of $\mathrm{dAfnd}^\dagger$ an object of $\mathrm{dAfnd}^\dagger$. Finally, intersections of polydisk legs are computed radiuswise, so we get $\overline{D}^{n_i}_\rho \cap \overline{D}^{n_i}_{\rho'} = \overline{D}^{n_i}_{\min(\rho,\rho')}$. Hence, all finite intersections of legs are product polydisks, hence in $\mathrm{dAfnd}^\dagger$. □

Proof of Proposition 7.32. By the closed polydisk !-cover of $\mathbb{A}^{n_i}$ and descent for $\mathcal{C}$ in the six-functor formalism on $\mathrm{Shv}(\mathrm{dAfnd}^\dagger, \tau_{rat})$, we have

$$\mathrm{C}(\mathbb{A}^{n_i}, \mathbb{A}^{n_i}) \simeq \lim_{[m]\in\Delta} \prod \mathcal{C}\left(\left(\overline{D}^{n_i}_\rho \right)^{m+1/\mathbb{A}^{n_i}} \right),$$

a totalization all of whose terms are module categories over dagger affinoid algebras, since finite intersections of polydisk legs are polydisks. By Lemma 7.35, the product family is a !-cover of $\mathbb{A}^{n_1} \times_S \mathbb{A}^{n_2}$ with the same property, so

$$\mathcal{C}(\mathbb{A}^{n_1} \times_S \mathbb{A}^{n_2},\ \mathbb{A}^{n_1} \times_S \mathbb{A}^{n_2}) \simeq \lim_{[m]\in\Delta} \prod \mathcal{C}\left(\left(\overline{D}^{n_1}_\rho \times_S \overline{D}^{n_2}_\psi \right)^{m+1/\mathbb{A}^{n_1}\times_S\mathbb{A}^{n_2}} \right)$$

on each affinoid leg. Furthermore, Lemma 7.34 gives, levelwise and compatibly with the cosimplicial structure maps,

$$\mathrm{C}(\overline{D}^{n_1}_\rho) \otimes_{\mathcal{C}(S)} \mathrm{C}(\overline{D}^{n_2}_\psi) \simeq \mathrm{C}\big(\overline{D}^{n_1}_\rho \times_S \overline{D}^{n_2}_\psi\big).$$

It remains to pass the Lurie tensor product $\otimes_{\mathcal{C}(S)}$ through the two totalizations. The above limit computation is computed in Pr^R. By passing to Pr^L, we get the same description of $\mathrm{C}(\mathbb{A}^{n_i}, \mathbb{A}^{n_i})$ and $\mathrm{C}(\mathbb{A}^{n_1} \times_S \mathbb{A}^{n_2}, \mathbb{A}^{n_1} \times_S \mathbb{A}^{n_2})$, but instead as colimits over lower !-functors. Hence, applying Lurie tensor product will commute with these colimit presentations. Applying it to the levelwise equivalence yields an equivalence of cosimplicial objects, and taking totalizations gives

$$\mathcal{C}(\mathbb{A}^{n_1}, \mathbb{A}^{n_1}) \otimes_{\mathcal{C}(S,S)} \mathcal{C}(\mathbb{A}^{n_2}, \mathbb{A}^{n_2}) \simeq \mathcal{C}(\mathbb{A}^{n_1} \times_S \mathbb{A}^{n_2},\ \mathbb{A}^{n_1} \times_S \mathbb{A}^{n_2}).$$

The relative case over general S follows since base change to $\mathrm{C}(S)$ is symmetric monoidal. □

Proof of Theorem 7.30. For convenience's sake, we write $F := \mathrm{C}(-,-)$ and $G := \mathrm{QCoh}(\iota_!'((-)^{an,\dagger}))$. By Proposition 7.31, we know that F and G agree on $\mathbb{A}^n$. Now, every finitely presented derived $\mathbb{C}$-algebra R fits in a pullback diagram

$$\begin{array}{ccc} \mathrm{Spec}(R) & \longrightarrow & \mathbb{A}^n \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\mathbb{C}) & \longrightarrow & \mathbb{A}^c \end{array}$$

for some n and c. In other words, we can write

$$R \simeq \mathbb{C}[X_1, \cdots, X_n] \otimes^{\mathbb{L}}_{\mathbb{C}[Y_1,\cdots,Y_c]} \mathbb{C}$$

where $\mathbb{C}[Y_1, \cdots, Y_c] \to \mathbb{C}[X_1, \cdots, X_n]$ sends Y_j to the relations $f_j(X_1, \cdots, X_n)$ and $\mathbb{C}[Y_1, \cdots, Y_c] \to \mathbb{C}$ is the augmentation. Applying the relative Künneth formula of Proposition 7.32 over $S = \mathrm{C}(\mathbb{A}^c, \mathbb{A}^c)$ gives

$$F(R) \simeq F(\mathbb{A}^n) \otimes_{F(\mathbb{A}^c)} F(\mathrm{Spec}(\mathbb{C})).$$

The same base change holds for G since analytification is a left exact left adjoint and QCoh is a Künneth six functor formalism by our work in the next section. Since F and G agree on $\mathbb{A}^n$, $\mathbb{A}^c$, and $\mathrm{Spec}(\mathbb{C}) = \mathbb{A}^0$ by Proposition 7.31, they agree on R. The result then follows for general non-affine X by the fact that $\mathrm{C}(-)$ and QCoh are both sheaves in the relevant topology. □

By Theorem 7.30 and Theorem 7.20, we obtain a more classical looking derived GAGA:

Theorem 7.36 (Derived GAGA)**.** *For a proper derived $\mathbb{C}$-scheme X, there is an equivalence of ∞-categories*

$$\mathrm{QCoh}(X) \simeq \mathrm{QCoh}(\iota_!'(X^{an,\dagger})).$$

7.5 Tannaka Duality and Algebraization

Now that we've proven derived GAGA, we'd like to use it to prove derived enhancements of several classical theorems. The most important of these is the algebraization of analytic maps:

Theorem 7.37 (Algebraization of Analytic Maps)**.** *Let $X, Y \in \mathrm{dSch}$ be proper derived schemes. Then, there are equivalences of mapping spaces*

$$\mathrm{Map}_{\mathrm{dSch}}(X, Y) \simeq \mathrm{Map}_{\mathrm{dAn}^{\dagger}_{\mathbb{C}}}(X^{an,\dagger}, Y^{an,\dagger})$$

This theorem will essentially fall out of derived enhancement of Tannaka duality.

Theorem 7.38 (Tannaka Duality)**.** *For qcqs derived dagger analytic spaces $X, Y \in \mathrm{dAn}^{\dagger}_{\mathbb{C}}$ (resp. $\mathrm{dAn}_{\mathbb{C}}$) and any morphism $f\colon X \to Y$, pullback of quasi-coherent sheaves*

$$f \longmapsto f^* : \qquad \mathrm{Map}(X, Y) \longrightarrow \mathrm{Fun}^{\otimes, L}_{D(\mathrm{CBorn}_{\mathbb{C}})}(\mathrm{QCoh}(\iota_!'(Y)), \mathrm{QCoh}(\iota_!'(X)))$$

is fully faithful, with essential image the geometric functors.

We will take advantage of the notion of Tannakian and Künneth morphisms in a six-functor formalism first due to Kesting [Kes25], applying them to the six-functor formalisms on $\mathrm{dAn}_{\mathbb{C}}$ and $\mathrm{dAn}_{\mathbb{C}}^{\dagger}$.

Definition 7.39. Let $D\colon \mathrm{Corr}(\mathcal{C}, E) \to \mathrm{Pr}^L$ be a six-functor formalism. and let $f\colon X \to Z$ be a map in E.

1. We say that f is **Künneth** if for all morphisms $W \to Z$ in $\mathcal{C}$, the natural morphism
$$D(X) \otimes_{D(Z)} D(W) \to D(X \times_Z W), \qquad A \otimes B \mapsto p_1^*(A) \otimes p_2^*(B)$$
is an equivalence, with p_1 and p_2 the natural projections. We say that D is **Künneth** if any morphism in E is Künneth.

2. We say that f is **Tannakian** if the morphism induced by D
$$\mathrm{Hom}_{\mathcal{C}_{/S}}(Y, X) \to \mathrm{Fun}^{L,\otimes}_{D(S)}(D(X), D(Y))$$
is an equivalence for all $Y \in C_{/S}$. We say that D is **Tannakian** if any morphism in E is Tannakian.

Kesting's main theorem is the following:

Theorem 7.40 ([Kes25], Theorem 1.5)**.** *Let $(\mathcal{C}, \tau)$ be a subcanonical site, and let $D\colon \mathrm{Corr}(\mathcal{C}, E_0) \to \mathcal{C}\mathrm{Alg}(\mathrm{Pr}^L)$ be a Tannakian Künneth sheafy six-functor formalism. Consider its extension $D\colon \mathrm{Corr}(\mathrm{Shv}(\mathcal{C}, \tau), \widetilde{E}) \to \mathcal{C}\mathrm{Alg}(\mathrm{Pr}^L)$ given by Theorem 4.11. Let $f\colon X \to S$ be a morphism in $\widetilde{E}$, and assume that there is a !-cover $g\colon S' \to S$ with $S' \in \mathcal{C}$ and $g \in E_0$ (i.e. in the essential image of the Yoneda embedding $\mathcal{C} \to \mathrm{Shv}(\mathcal{C}, \tau)$). Then, f is Tannakian.*

This is quite close to our Tannaka duality. All that remains to show is that any morphism of qcqs derived analytic spaces is !-able, and that any qcqs derived complex analytic space admits a !-cover by an affine:

Proposition 7.41. *Let $f\colon Y \to X$ be a qcqs morphism of derived complex analytic (resp. complex dagger) spaces. Then f belongs to the class $\widetilde{E}$ of* !*-able morphisms, is* QCoh*-prim, and satisfies $f_! = f_*$. In particular, any morphism of qcqs derived complex analytic spaces is* !*-able and* QCoh*-prim.*

We'll need a few lemmas. The first is a reminder of Corollary 4.29:

Lemma 7.42. *Any morphism $f\colon \mathrm{Spec}(A) \to \mathrm{Spec}(B)$ in* Aff *is prim and satisfies $f_! = f_*$.*

Proof. We have a decomposition of the six-functor formalism into $I =$ isomorphisms and $P =$ all. Any morphism in P is prim in an abstract six-functor formalism, so f is prim. □

Next, we will need the following !-cover and !-ability results:

Lemma 7.43. *Let $f\colon Y \to X$ be an affine morphism in $\mathrm{dAn}_{\mathbb{C}}^{\dagger}$. Then, $f \in \widetilde{E}$.*

Proof. For any $W = \mathrm{Spec}(B) \in \mathrm{dAfnd}^\dagger$ with a map $W \to X$, the pullback of f to W is a map between two affine objects. By $*$-locality on the target, we are done. □

Lemma 7.44. *Let $X \in \mathrm{dAn}^\dagger_{\mathbb{C}}$ be a qcqs derived dagger analytic space. Let $U_i \hookrightarrow X$ be a finite affinoid cover of X. Then, $g\colon \coprod_i U_i \to X$ is a universal small !-cover of X.*

Proof of Proposition 7.41. The fact that the six-functor formalism $\mathrm{QCoh}\colon \mathrm{Corr}(\mathrm{Aff}, \mathrm{all}) \to \mathcal{C}\mathrm{Alg}(\mathrm{Pr}^L)$ is Künneth and Tannakian comes directly from the properties of module categories, see [Section 4.5][Lur17]. So, the result follows as long as we can show that any morphism of qcqs spaces is !-able, and that any qcqs space admits a !-cover by an affine.

We first show that f is !-able. Since $\widetilde{E}$ is $*$-local on the target, we can reduce to the case that $X = \mathrm{Spec}(B)$ is affinoid. Since we are now assuming that $f\colon Y \to X = \mathrm{Spec}(B)$ is a quasi-compact morphism to a derived affinoid space, Y has a finite cover by affinoid analytic subspaces. Denote this cover $\{U_i = \mathrm{Spec}(A_i) \to Y\}$ with covering map $g\colon \coprod_i U_i \to Y$. By Lemma 7.44, g is a universal !-cover. Each composite

$$\mathrm{Spec}(A_i) \hookrightarrow Y \to \mathrm{Spec}(B)$$

is a morphism between two objects of $\mathrm{dAfnd}^\dagger$, so is in $\widetilde{E}$ by definition of the six-functor formalism. Hence, by !-locality, $f \in \widetilde{E}$.

The proof that f is prim is an adaptation of the proof of [Cam26], Proposition 6.4.7. Credit goes to Juan Esteban Rodríguez Camargo for the approach. By [HM24, Lemma 4.5.7], it suffices to show that f is prim on a universal D^*-cover on X. So, without loss of generality, we can assume that X is affine. By [HM24, Lemma 4.5.8 ii), Lemma 4.7.4], it also suffices to show that f is prim after pre-composing with a descendable prim morphism $h\colon Y' \to Y$ (Here, descendable means that $h_*(\mathbf{1}_{Y'})$ is a descendable object of $\mathrm{QCoh}(Y)$).

We first show that f is prim in the case that it is affine, and then glue. In this case, if $X = \mathrm{Spec}(A)$ is affine, then so is Y. Hence, f is prim by Lemma 7.42. In the more general case of f qcqs, let $g\colon \coprod_i U_i \to Y$ be a finite cover by affine analytic subspaces. Then, the map g is affine and thus prim by the previous step. Furthermore, g is descendable and so g is prim and a descendable cover. Since the composite $\coprod_i U_i \to Y \to X$ is a coproduct of morphisms of affines, it is !-able and prim. Hence, f is as well.

For the qcqs case, there is a finite open cover $g\colon \coprod_i U_i \to Y$ such that g is quasi-compact and separated. Hence, g is prim by the last step, and it is descendable. Since the composite $\coprod_i U_i \to Y \to X$ is a finite coproduct of prim maps (since it is affine), f is prim. □

Remark 7.45. We note that more generally, this implies that any qc morphism f satisfies $f_* = f_!$.

Proof of Theorem 7.38. Since any morphism $f\colon X \to Y$ of qcqs derived complex analytic spaces is qcqs, it is !-able by Proposition 7.41. Furthermore, if $\{U_i\}$ is a finite affinoid cover of Y, it is a universal !-cover by 7.44. Hence, by Kesting's Theorem 7.40, f is Tannakian. □

Proof of Theorem 7.37. We have

$$\mathrm{Map}_{\mathrm{dSch}}(X, Y) \simeq \mathrm{Fun}^{\otimes, L}_{\mathrm{AniAlg}_{\mathbb{C}}}(\mathrm{QCoh}(Y), \mathrm{QCoh}(X)) \simeq \mathrm{Fun}^{\otimes, L}_{D(\mathrm{CBorn}(\mathbb{C}))}(\mathrm{QCoh}(\iota'_!(X^{an,\dagger})), \mathrm{QCoh}(\iota'_!(Y^{an,\dagger})))$$

$$\simeq \mathrm{Map}_{\mathrm{Shv}(\mathrm{dAff},\tau_{dri})}(\iota'_!(X^{an,\dagger}), \iota'_!(Y^{an,\dagger})) \simeq \mathrm{Map}_{\mathrm{dAn}^\dagger_{\mathbb{C}}}(X^{an,\dagger}, Y^{an,\dagger})$$

where the first equivalence is by Tannaka duality for derived schemes [Lur18] (this holds since proper objects are qcqs), the second equivalence is by derived GAGA, the third equivalence is by Theorem 7.38, and the final equivalence is by full faithfulness of $\iota_!$. □

References

[Aok23] Ko Aoki. "The sheaves–spectrum adjunction". In: *arXiv preprint arXiv:2302.04069* (2023).

[AM06] Michael Artin and Barry Mazur. *Etale homotopy*. Springer, 2006.

[BF11] Paul Balmer and Giordano Favi. "Generalized tensor idempotents and the telescope conjecture". In: *Proceedings of the London Mathematical Society* 102.6 (2011), pp. 1161–1185.

[Bam14] Federico Bambozzi. "On a generalization of affinoid varieties". In: *arXiv preprint arXiv:1401.5702* (2014).

[BB16] Federico Bambozzi and Oren Ben-Bassat. "Dagger geometry as Banach algebraic geometry". In: *Journal of Number Theory* 162 (2016), pp. 391–462.

[BBK18] Federico Bambozzi, Oren Ben-Bassat, and Kobi Kremnizer. "Stein domains in Banach algebraic geometry". In: *Journal of Functional Analysis* 274.7 (2018), pp. 1865–1927.

[BKK24] Oren Ben-Bassat, Jack Kelly, and Kobi Kremnizer. "A perspective on the foundations of derived analytic geometry". In: *arXiv preprint arXiv:2405.07936* (2024).

[BK17] Oren Ben-Bassat and Kobi Kremnizer. "Non-Archimedean analytic geometry as relative algebraic geometry". In: *Annales de la Faculté des sciences de Toulouse: Mathématiques*. Vol. 26. 1. 2017, pp. 49–126.

[BLR12] Siegfried Bosch, Werner Lütkebohmert, and Michel Raynaud. *Néron models*. Springer Science & Business Media, 2012.

[Bou55] N. Bourbaki. "Eléments de Mathématique. XV, Première Partie, Les Structures Fondamentales de L'Analyse. Livre V, Espaces Vectoriels Topologiques, Chaps. III, IV, & V. Actualités Scientifiques et Industrielles, 1189.(Hermann, Paris)". In: *The Mathematical Gazette* 39.329 (1955), pp. 254–255.

[Cam26] Juan Esteban Rodríguez Camargo. "Notes on solid geometry". In: *arXiv preprint arXiv:2603.03012* (2026).

[ČS24] Kestutis Česnavičius and Peter Scholze. "Purity for flat cohomology". In: *Annals of Mathematics* 199.1 (2024), pp. 51–180.

[CH02] J Daniel Christensen and Mark Hovey. "Quillen model structures for relative homological algebra". In: *Mathematical Proceedings of the Cambridge Philosophical Society*. Vol. 133. 2. Cambridge University Press. 2002, pp. 261–293.

[Cis19] Denis-Charles Cisinski. "Higher categories and homotopical algebra". In: *Cambridge studies in advanced mathematics* 180 (2019).

[CS26] Dustin Clausen and Peter Scholze. "Condensed mathematics and complex geometry". In: *arXiv preprint arXiv:2605.11731* (2026).

[Con06] Brian Conrad. "Relative ampleness in rigid geometry". In: *Annales de l'institut Fourier*. Vol. 56. 4. 2006, pp. 1049–1126.

[Del07] Pierre Deligne. "Catégories tannakiennes". In: *The Grothendieck Festschrift: A Collection of Articles Written in Honor of the 60th Birthday of Alexander Grothendieck*. Springer, 2007, pp. 111–195.

[EP96] Jörg Eschmeier and Mihai Putinar. *Spectral decompositions and analytic sheaves*. 10. Oxford University Press, 1996.

[For67] Otto Forster. "Zur Theorie der Steinschen Algebren und Moduln". In: *Mathematische Zeitschrift* 97.5 (1967), pp. 376–405.

[GR19] Dennis Gaitsgory and Nick Rozenblyum. *A study in derived algebraic geometry: Volume I: correspondences and duality*. Vol. 221. American Mathematical Society, 2019.

[GR12] Hans Grauert and Reinhold Remmert. *Coherent analytic sheaves*. Springer Science & Business Media, 2012.

[HM24] Claudius Heyer and Lucas Mann. "6-functor formalisms and smooth representations". In: *arXiv preprint arXiv:2410.13038* (2024).

[Hin13] Vladimir Hinich. "Dwyer-Kan localization revisited". In: *arXiv preprint arXiv:1311.4128* (2013).

[Hog66] Henri Hogbe-Nlend. "Complétion, tenseurs et nucléarité en bornologie". PhD thesis. Université de Bordeaux. Faculté des Sciences, 1966.

[Hou06] Christian Houzel. *Séminaire Banach*. Springer, 2006.

[Joh02] Peter T Johnstone. *Sketches of an Elephant: A Topos Theory Compendium*. Oxford University Press, 2002.

[KS06] Masaki Kashiwara and Pierre Schapira. *Categories and sheaves*. Springer, 2006.

[Kel24] Jack Kelly. *Homotopy in exact categories*. Vol. 298. 1490. American Mathematical Society, 2024.

[KKM22] Jack Kelly, Kobi Kremnizer, and Devarshi Mukherjee. "An analytic Hochschild-Kostant-Rosenberg theorem". In: *Advances in Mathematics* 410 (2022), p. 108694.

[KM25] Jack Kelly and Devarshi Mukherjee. "Localising invariants in derived bornological geometry". In: *arXiv preprint arXiv:2505.15750* (2025).

[Kes25] Youshua Kesting. "Categorical Künneth formulas for analytic stacks". In: *arXiv preprint arXiv:2507.08566* (2025).

[LZ12] Yifeng Liu and Weizhe Zheng. "Enhanced six operations and base change theorem for higher Artin stacks". In: *arXiv preprint arXiv:1211.5948* (2012).

[Lur09] Jacob Lurie. *Higher topos theory*. Princeton University Press, 2009.

[Lur17] Jacob Lurie. *Higher Algebra*. 2017. URL: https://www.math.ias.edu/~lurie/papers/HA.pdf.

[Lur18] Jacob Lurie. *Spectral Algebraic Geometry*. 2018. URL: https://www.math.ias.edu/~lurie/papers/SAG-rootfile.pdf.

[Mac45] George W Mackey. "On infinite-dimensional linear spaces". In: *Transactions of the American Mathematical Society* 57.2 (1945), pp. 155–207.

[Mac46] George W Mackey. "On convex topological linear spaces". In: *Transactions of the American Mathematical Society* 60.3 (1946), pp. 519–537.

[Man22] Lucas Mann. "A p-adic 6-functor formalism in rigid-analytic geometry". PhD thesis. Rheinische Friedrich-Wilhelms-Universitaet Bonn (Germany), 2022.

[Mat16] Akhil Mathew. "The Galois group of a stable homotopy theory". In: *Advances in Mathematics* 291 (2016), pp. 403–541.

[Maz19] Aaron Mazel-Gee. "On the Grothendieck construction for ∞-categories". In: *Journal of Pure and Applied Algebra* 223.11 (2019), pp. 4602–4651.

[NS18] Thomas Nikolaus and Peter Scholze. "On topological cyclic homology". In: (2018).

[Rak20] Arpon Raksit. "Hochschild homology and the derived de Rham complex revisited". In: *arXiv preprint arXiv:2007.02576* (2020).

[Sch87] Jean-Pierre Schneiders. "Cosheaves homology". In: *Bulletin de la Société Mathématique de Belgique. Série B* 39 (1987).

[Sch99] Jean-Pierre Schneiders. *Quasi-abelian categories and sheaves*. Société mathématique de France, 1999.

[Sch25] Peter Scholze. "Six-functor formalisms". In: *arXiv preprint arXiv:2510.26269* (2025).

[Sch26] Peter Scholze. "Lectures on condensed mathematics". In: *arXiv preprint arXiv:2605.03658* (2026).

[Ser56] Jean-Pierre Serre. "Géométrie algébrique et géométrie analytique". In: *Annales de l'Institut Fourier* 6 (Jan. 1956), pp. 1–42. DOI: 10.5802/aif.59.

[Soo25] Arun Soor. "Topics in derived analytic geometry". PhD thesis. PhD thesis, Mathematical Institute, University of Oxford, 2025.

[Soo26] Arun Soor. "Ind-Banach approach to Grothendieck duality in Rigid-analytic geometry". In: *arXiv preprint arXiv:2606.10057* (2026).

[TV05] Bertrand Toën and Gabriele Vezzosi. "Homotopical algebraic geometry I: Topos theory". In: *Advances in mathematics* 193.2 (2005), pp. 257–372.

[TV08] Bertrand Toën and Gabriele Vezzosi. *Homotopical Algebraic Geometry II: Geometric Stacks and Applications*. Vol. 2. American Mathematical Soc., 2008.

[Wae67] Lucien Waelbroeck. "Some theorems about bounded structures". In: *Journal of Functional Analysis* 1.4 (1967), pp. 392–408.

[Yam26] Yuto Yamada. “Analytification for Complex Geometry Revisited”. In: *arXiv preprint arXiv:2605.27961* (2026).