MODULI OF CURVES WITH LEVEL STRUCTURE: ADMISSIBLE COVERS AND THEIR GRAPH-THEORETIC DATA

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ABSTRACT. Let $G = (\mathbb{Z}/p)^{2g}$ with a fixed symplectic form. We seek combinatorial data that stratify the boundary of $\mathcal{M}_g(p)$, viewed as the moduli space of connected symplectic G-covers. We restrict to admissible covers whose target curves have only genus-zero components.

For a weight-zero target dual graph Δ, the associated harmonic G-cover is encoded by a dilation class

$$\eta \in H_1(\Delta, G)$$

and a lifting class

$$\bar{\lambda} \in H^1(\Delta, G_D),$$

which record local monodromy and global gluing, respectively. We describe realizability, connectedness, and the symplectic condition in terms of these classes. We then construct a gluing morphism from a product of local genus-zero Hurwitz spaces to the boundary stratum associated to $(\Delta, \eta, \bar{\lambda})$. This reduces the study of the weight-zero boundary to a combinatorial classification of graph covers and a collection of local Hurwitz-space component problems.

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1. Introduction

Let $g \geq 2$, let p be prime, and set

$$G = (\mathbb{Z}/p)^{2g}.$$

We equip G with its standard symplectic form ω_{std}. The moduli space $\mathcal{M}_g(p)$ parametrizes smooth genus-g curves with full symplectic level-p structure. Analytically, it may be written as

$$\mathcal{M}_g(p) = \mathcal{T}_g / \operatorname{Mod}_g(p),$$

where

$$\operatorname{Mod}_g(p) = \ker\Big(\operatorname{Mod}_g \longrightarrow \operatorname{Sp}_{2g}(\mathbb{Z}/p)\Big).$$

For $p \geq 3$, the group $\operatorname{Mod}_g(p)$ is torsion-free, so $\mathcal{M}_g(p)$ is a fine moduli space rather than merely an orbifold.

A point of $\mathcal{M}_g(p)$ may equivalently be described by a smooth genus-g curve X together with a symplectic isomorphism

$$\phi\colon \big(H_1(X, \mathbb{Z}/p), \iota_X\big) \xrightarrow{\sim} \big(G, \omega_{\text{std}}\big),$$

where ι_X denotes the algebraic intersection pairing. By the covering-space correspondence, the same isomorphism determines a connected principal G-cover of X. Thus $\mathcal{M}_g(p)$ can be viewed as the symplectic component of the moduli space of connected principal $(\mathbb{Z}/p)^{2g}$-covers. This covering-space interpretation is compatible with the monodromy description of Hurwitz spaces [BR11; Sou25].

The purpose of this project is to find combinatorial data that stratify the boundary of a compactification of $\mathcal{M}_g(p)$. Admissible covers provide a natural compactification of the covering-space moduli problem by allowing both the source and target to become nodal while imposing a balanced local action at the nodes [BR11; SZ20]. We concentrate on the locus of connected admissible G-covers

$$P \longrightarrow C$$

for which every irreducible component of the target C has genus zero. Equivalently, the weighted dual graph

$$\Delta = \Gamma(C)$$

has weight zero at every vertex. Since C has arithmetic genus g, this means that

$$b_1(\Delta) = g.$$

This weight-zero locus contains the maximally degenerate boundary strata and is particularly well suited to a graph-theoretic description: all of the genus of the target is carried by cycles in its dual graph. The use of weighted dual graphs to index boundary strata and their tropicalizations is standard in the geometry of moduli spaces of curves [ACP15].

Passing from an admissible cover to dual graphs gives a harmonic G-cover

$$\Gamma \longrightarrow \Delta.$$

The graph Δ records the combinatorial type of the target curve, while the cover $\Gamma \to \Delta$ records the distribution of components, nodes, stabilizers, and gluing data upstairs. This passage from admissible covers to harmonic or tropical covers is developed in [BR11; LUZ19; LUZ24; El +24; Sou25]. The principal problem is to determine which graph covers arise from limits of points of $\mathcal{M}_g(p)$, which of them have connected source, and how the resulting combinatorial types determine strata in the boundary.

For a weight-zero target graph, the covering data separate into two complementary classes. The first is a dilation class

$$\eta \in H_1(\Delta, G),$$

which records local monodromy around the nodes and therefore determines the edge and vertex stabilizers. The second is a lifting or gluing class

$$\bar{\lambda} \in H^1(\Delta, G_D),$$

where G_D denotes the coefficient system determined by the dilation datum. This class records how the local fibers over the vertices are identified across the edges. The classification of abelian harmonic covers by dilation and lifting data is formulated in terms of dilated cohomology in [LUZ19; LUZ24].

For calculations, one may choose a lift

$$\lambda \in H^1(\Delta, G)$$

of $\bar{\lambda}$. Different lifts can determine the same graph cover, but the lift retains additional homological information relevant to the symplectic level structure. The distinction between the graph cover and the additional monodromy information carried by a chosen lift is also reflected in the decomposition of Hurwitz spaces into monodromy components [BR11; Sou25].

The connectedness condition is expressed by the subgroup

$$K(\eta, \lambda) = \left\langle \eta(\alpha), \lambda(\sigma) \;\middle|\; \alpha \in H^1(\Delta, \mathbb{F}_p),\ \sigma \in H_1(\Delta, \mathbb{F}_p) \right\rangle \subseteq G.$$

The associated graph cover is connected precisely when

$$K(\eta, \lambda) = G.$$

This criterion is the homological form of the usual monodromy criterion for connectedness and agrees with the spanning-tree description of connected abelian tropical covers [LUZ19; LUZ24; El +24; Sou25].

Because

$$\dim_{\mathbb{F}_p} H^1(\Delta, \mathbb{F}_p) = \dim_{\mathbb{F}_p} H_1(\Delta, \mathbb{F}_p) = g,$$

a pair arising from a full level structure gives an isomorphism

$$H^1(\Delta, \mathbb{F}_p) \oplus H_1(\Delta, \mathbb{F}_p) \xrightarrow{\sim} G, \qquad (\alpha, \sigma) \longmapsto \eta(\alpha) + \lambda(\sigma).$$

The evaluation pairing between graph cohomology and graph homology then induces an alternating form $\omega_{\eta,\lambda}$ on G. The pair (η, λ) belongs to the symplectic component corresponding to $\mathcal{M}_g(p)$ exactly when

$$\omega_{\eta,\lambda} = \omega_{\mathrm{std}}.$$

This recovers, in graph-theoretic form, the symplectic monodromy condition that distinguishes $\mathcal{M}_g(p)$ from the other components of the moduli space of principal G-covers [BR11; Sou25].

The resulting combinatorial datum for the boundary therefore consists of

$$(\Delta, \eta, \bar{\lambda}),$$

together with a lift λ when the symplectic structure must be remembered. The datum is required to satisfy four conditions:

(1) Δ is a connected stable graph of genus g whose vertices all have weight zero;
(2) η defines realizable local monodromy and dilation data;
(3) the associated G-cover $\Gamma \to \Delta$ is connected; and
(4) a lift λ of $\bar{\lambda}$ induces the prescribed symplectic form ω_{std} on G.

The data are considered up to the natural action of $\mathrm{Aut}(\Delta)$. The realizability condition in the second item is the abelian tropical-cover realizability criterion of [LUZ24]; in particular, each edge stabilizer must be cyclic.

The graph data determine the combinatorial type of a boundary stratum, but they do not by themselves specify the curves parametrized by that stratum. For this, one must attach a local Hurwitz space to each vertex of Δ and glue the corresponding local covers according to the graph cover $\Gamma \to \Delta$. Such vertexwise gluing morphisms are standard in admissible-cover compactifications [BR11; SZ20]. The resulting gluing map is the bridge between the combinatorial classification and the geometry of the boundary. In particular, it reduces the problem of finding connected components of a boundary stratum to the corresponding component problems for genus-zero local Hurwitz spaces, together with the finite identifications imposed by graph automorphisms and the gluing datum. The relevant genus-zero component calculations are closely related to those appearing in [BCK24].

Our main objectives are:

(i) to review the quotient, homological, and covering-space descriptions of $\mathcal{M}_g(p)$;
(ii) to relate $\mathcal{M}_g(p)$ to moduli spaces of connected admissible G-covers;
(iii) to formulate admissible covers in terms of harmonic covers of weighted dual graphs;
(iv) to classify weight-zero graph covers by dilation and lifting data;
(v) to characterize realizability and connectedness in terms of (η, λ);
(vi) to recover the symplectic form associated to a full level structure;
(vii) to construct the gluing morphism associated to a fixed combinatorial datum $(\Delta, \eta, \bar{\lambda})$; and
(viii) to use these data to organize the weight-zero boundary of $\mathcal{M}_g(p)$ into combinatorially defined strata.

Throughout, graph homology and cohomology are simplicial. Whenever coefficients are omitted, they are integral. Unless otherwise stated, Δ denotes a connected stable graph of genus g whose vertices all have weight zero, and all admissible covers are required to have connected source.

2. Moduli spaces and level structures

2.1. Teichmüller space and the mapping class group

Fix a closed oriented topological surface Σ_g of genus g.

Definition 2.1. The mapping class group of Σ_g is

$$\mathrm{Mod}_g = \pi_0\big(\mathrm{Homeo}^+(\Sigma_g)\big) = \mathrm{Homeo}^+(\Sigma_g)/\,\mathrm{Homeo}_0^+(\Sigma_g).$$

Thus Mod_g is the group of isotopy classes of orientation-preserving homeomorphisms of Σ_g.

The group Mod_g is generated by Dehn twists. Its action on $H_1(\Sigma_g, \mathbb{Z})$ preserves the algebraic intersection pairing and therefore induces a homomorphism

$$\mathrm{Mod}_g \longrightarrow \mathrm{Sp}_{2g}(\mathbb{Z}).$$

This map is surjective. Its kernel is the Torelli group, denoted $\mathcal{I}_g$, so there is an exact sequence

$$1 \longrightarrow \mathcal{I}_g \longrightarrow \mathrm{Mod}_g \longrightarrow \mathrm{Sp}_{2g}(\mathbb{Z}) \longrightarrow 1.$$

Definition 2.2. The Teichmüller space $\mathcal{T}_g$ is the set of equivalence classes of pairs (X, f), where:

(i) X is a compact Riemann surface of genus g; and

(ii) $f : \Sigma_g \to X$ is an orientation-preserving homeomorphism, called a marking.

Two pairs (X, f) and (Y, h) are equivalent if there is a biholomorphism $u : X \to Y$ such that $u \circ f$ is isotopic to h.

The mapping class group acts on the right by

$$[(X, f)] \cdot [\phi] = [(X, f \circ \phi)].$$

Proposition 2.3. *There is a natural bijection between isomorphism classes of compact Riemann surfaces of genus g and points of $\mathcal{T}_g / \mathrm{Mod}_g$. Consequently,*

$$\mathcal{M}_g(\mathbb{C}) \cong \mathcal{T}_g / \mathrm{Mod}_g .$$

Proof. The map

$$[(X, f)] \longmapsto [X]$$

is well-defined because equivalent marked surfaces have biholomorphic underlying curves. It is surjective because every compact oriented surface of genus g is homeomorphic to Σ_g.

Suppose that (X, f) and (Y, h) have biholomorphic underlying surfaces. After using a biholomorphism to identify X and Y, the two markings differ by the mapping class

$$[f^{-1} \circ h] \in \mathrm{Mod}_g .$$

Conversely, acting by a mapping class changes only the marking and not the isomorphism class of the underlying curve. Thus the fibers are precisely the Mod_g-orbits. □

It is helpful to separate two reasons why points of $\mathcal{T}_g$ can be different. First, the underlying curves may have different complex structures. Second, even if the underlying complex structures agree, the markings may differ by a nontrivial mapping class.

For $g \geq 2$, the stabilizer of a point $[(X, f)] \in \mathcal{T}_g$ is naturally isomorphic to $\mathrm{Aut}(X)$. Since $\mathrm{Aut}(X)$ is finite, the quotient is an orbifold. This gives the familiar explanation that $\mathcal{M}_g$ is naturally a stack rather than a scheme: some curves have nontrivial automorphism groups.

2.2. The principal level subgroup

Reduction modulo p gives

$$\mathrm{Sp}_{2g}(\mathbb{Z}) \longrightarrow \mathrm{Sp}_{2g}(\mathbb{Z}/p).$$

Definition 2.4. The principal level-p subgroup is

$$\mathrm{Mod}_g(p) = \ker\Big(\mathrm{Mod}_g \longrightarrow \mathrm{Sp}_{2g}(\mathbb{Z}/p)\Big) .$$

The moduli space with full level-p structure is

$$\mathcal{M}_g(p) := \mathcal{T}_g / \mathrm{Mod}_g(p).$$

We have

$$\mathcal{I}_g \subseteq \mathrm{Mod}_g(p) \subseteq \mathrm{Mod}_g$$

and

$$\mathrm{Mod}_g \,/\, \mathrm{Mod}_g(p) \cong \mathrm{Sp}_{2g}(\mathbb{Z}/p).$$

Accordingly,

$$\mathcal{M}_g(p) \longrightarrow \mathcal{M}_g$$

is a finite Galois cover in the orbifold sense, with Galois group $\mathrm{Sp}_{2g}(\mathbb{Z}/p)$.

Theorem 2.5. *If $g \geq 2$ and $p \geq 3$, then $\mathrm{Mod}_g(p)$ is torsion-free. Consequently, it acts freely on $\mathcal{T}_g$, and $\mathcal{M}_g(p)$ is a fine moduli space rather than merely an orbifold.*

Proof sketch. The Torelli group $\mathcal{I}_g$ is torsion-free. Suppose $\phi \in \mathrm{Mod}_g(p)$ has finite order. Its image in $\mathrm{Sp}_{2g}(\mathbb{Z})$ lies in

$$\Gamma(p) = \ker\Big(\mathrm{Sp}_{2g}(\mathbb{Z}) \longrightarrow \mathrm{Sp}_{2g}(\mathbb{Z}/p)\Big).$$

For $p \geq 3$, the principal congruence subgroup $\Gamma(p)$ is torsion-free. Hence the image of ϕ in $\mathrm{Sp}_{2g}(\mathbb{Z})$ is trivial, so $\phi \in \mathcal{I}_g$. Since $\mathcal{I}_g$ is torsion-free, we conclude that $\phi = \mathrm{id}$. □

To see why this implies that the action is free, recall that the stabilizer of a marked Riemann surface is a finite subgroup of Mod_g. Its intersection with the torsion-free group $\mathrm{Mod}_g(p)$ must therefore be trivial.

Remark 2.6. The compactification $\overline{\mathcal{M}}_g(p)$ may still carry stack-like behavior along its boundary, depending on the chosen compactification. The torsion-free argument applies directly to the smooth level structure moduli problem.

2.3. Three descriptions of $\mathcal{M}_g(p)$

Let

$$G = (\mathbb{Z}/p)^{2g}$$

and equip G with the standard symplectic form ω_{std}.

Definition 2.7. A full symplectic level-p structure on a smooth genus-g curve X is a symplectic isomorphism

$$\varphi : (G, \omega_{\mathrm{std}}) \xrightarrow{\sim} (H_1(X, \mathbb{Z}/p), \iota_X),$$

where ι_X is the algebraic intersection pairing.

Equivalently, a level structure is an ordered symplectic basis

$$B = (a_1, \ldots, a_g, b_1, \ldots, b_g)$$

of $H_1(X, \mathbb{Z}/p)$, satisfying

$$\iota_X(a_i, a_j) = 0, \qquad \iota_X(b_i, b_j) = 0, \qquad \iota_X(a_i, b_j) = \delta_{ij}.$$

There is also a covering-space interpretation. Connected principal G-covers of X correspond to surjective monodromy homomorphisms

$$\pi_1(X, x_0) \twoheadrightarrow G$$

up to conjugation in G. Since G is abelian, conjugation is trivial. Every homomorphism to G factors through abelianization and reduction modulo p, giving

$$\begin{aligned} \{\text{connected principal } G\text{-covers of } X\} &\cong \mathrm{Epi}(\pi_1(X, x_0), G) \\ &\cong \mathrm{Epi}(H_1(X, \mathbb{Z}), G) \\ &\cong \mathrm{Epi}(H_1(X, \mathbb{Z}/p), G). \end{aligned}$$

Because both $H_1(X, \mathbb{Z}/p)$ and G have dimension $2g$ over $\mathbb{F}_p$, every surjective map between them is an isomorphism. Thus

$$\mathrm{Epi}(H_1(X, \mathbb{Z}/p), G) = \mathrm{Isom}(H_1(X, \mathbb{Z}/p), G).$$

Proposition 2.8. *The following data are equivalent:*

(i) a point of $\mathcal{T}_g / \mathrm{Mod}_g(p)$;

(ii) a smooth genus-g curve X with a full symplectic level-p structure;

(iii) a connected principal G-cover of X whose monodromy isomorphism

$$H_1(X, \mathbb{Z}/p) \xrightarrow{\sim} G$$

preserves the symplectic form.

Proof. A marking $f : \Sigma_g \to X$ identifies $H_1(X, \mathbb{Z}/p)$ with $H_1(\Sigma_g, \mathbb{Z}/p)$. Two markings induce the same level structure precisely when they differ by an element of $\mathrm{Mod}_g(p)$. This proves the equivalence of (i) and (ii).

A full level structure gives an isomorphism

$$H_1(X, \mathbb{Z}/p) \xrightarrow{\sim} G,$$

and therefore a connected principal G-cover. Conversely, the monodromy of a connected principal G-cover is an isomorphism because the source and target have equal finite dimension. Requiring this isomorphism to be symplectic gives exactly a full symplectic level structure. □

The interpretation by principal bundles agrees with the smooth, unramified locus of the relevant Hurwitz space [BR11]. Tropical analogues of these spaces are studied in [El +24; Sou25].

2.4. The components of the space of all principal G-covers

If one drops the symplectic condition, the moduli space of connected principal G-covers is generally disconnected. For a fixed curve X, an isomorphism

$$\phi : H_1(X, \mathbb{F}_p) \xrightarrow{\sim} G$$

transports the intersection form on $H_1(X, \mathbb{F}_p)$ to a symplectic form on G. Two isomorphisms lie in the same symplectic component when they differ by an element of $\mathrm{Sp}_{2g}(\mathbb{F}_p)$.

Thus the connected components are indexed by

$$\mathrm{Sp}_{2g}(\mathbb{F}_p) \backslash \mathrm{GL}_{2g}(\mathbb{F}_p).$$

Their number is

$$\frac{|\mathrm{GL}_{2g}(\mathbb{F}_p)|}{|\mathrm{Sp}_{2g}(\mathbb{F}_p)|}.$$

Using

$$|\mathrm{GL}_n(\mathbb{F}_p)| = p^{n(n-1)/2} \prod_{i=1}^{n} (p^i - 1)$$

and

$$|\mathrm{Sp}_{2g}(\mathbb{F}_p)| = p^{g^2} \prod_{i=1}^{g} (p^{2i} - 1),$$

we obtain

$$\frac{|\mathrm{GL}_{2g}(\mathbb{F}_p)|}{|\mathrm{Sp}_{2g}(\mathbb{F}_p)|} = p^{g(g-1)} \prod_{i=1}^{g} (p^{2i-1} - 1).$$

This agrees with the monodromy description of components of Hurwitz spaces [BR11; Sou25].

Example 2.9. For $g = 2$ and $p = 3$,

$$\frac{|\mathrm{GL}_4(\mathbb{F}_3)|}{|\mathrm{Sp}_4(\mathbb{F}_3)|} = 3^2(3-1)(3^3-1) = 9 \cdot 2 \cdot 26 = 468.$$

Thus the space of connected principal $(\mathbb{Z}/3)^4$-covers has 468 components. The component corresponding to the standard symplectic form is $\mathcal{M}_2(3)$.

2.5. Relation with admissible covers

Admissible covers compactify the moduli problem by allowing nodal source and target curves. We use the balanced formulation appearing in the theory of Hurwitz stacks [BR11]. Related admissible-cover spaces occur in [BCK24; SZ20].

For a finite group G, an admissible G-cover consists, roughly, of a finite map

$$P \longrightarrow C$$

of nodal curves, together with a G-action on P, such that:

(i) the map is a principal G-bundle away from nodes and marked points;
(ii) the action is compatible with the map;
(iii) prescribed local monodromy conditions hold at marked points;
(iv) the map has compatible local equations over nodes; and
(v) the action at every node is balanced on the two branches.

Locally at a node, an admissible cover has equations of the form

$$xy = 0, \qquad uv = 0, \qquad x = u^r, \qquad y = v^r.$$

A generator of the stabilizer acts by

$$(u, v) \longmapsto (\zeta u, \zeta^{-1} v),$$

which is the balanced condition.

When source and target are smooth and unmarked, the nodal conditions are vacuous, so an admissible cover is simply a principal G-bundle. Let $\mathrm{Adm}_g^\circ(G)$ denote the smooth locus of the admissible-cover space, and let $\mathrm{CAdm}_g^\circ(G)$ denote the connected-source locus. Then

$$\mathcal{M}_g(p) \hookrightarrow \mathrm{CAdm}_g^\circ(G) \hookrightarrow \mathrm{Adm}_g^\circ(G).$$

The first inclusion selects the component whose monodromy is symplectic.

Different authors impose connectedness at different stages. In particular, the total admissible-cover stack may permit disconnected source curves, while the spaces in [Sou25] are typically formulated using connected curves. This distinction is important in comparisons among [BR11; BCK24; Sou25].

3. Weighted graphs and graph covers

3.1. Weighted dual graphs

Weighted dual graphs record the combinatorial types of stable curves. Their role in the tropicalization of moduli spaces is developed in [ACP15].

Definition 3.1. A weighted graph Γ consists of:

(i) a finite set $V(\Gamma)$ of vertices;
(ii) a finite set $H(\Gamma)$ of half-edges;
(iii) an involution
$$\tau : H(\Gamma) \longrightarrow H(\Gamma);$$
(iv) an incidence map
$$\partial : H(\Gamma) \longrightarrow V(\Gamma);$$
(v) a weight function
$$\mathrm{wt} : V(\Gamma) \longrightarrow \mathbb{Z}_{\geq 0}.$$

The two-element orbits of τ are the edges of Γ, and the fixed points are its legs.

For $v \in V(\Gamma)$, let

$$T_v\Gamma = \partial^{-1}(v)$$

be the tangent set at v, and define

$$\mathrm{val}(v) = |T_v\Gamma|.$$

Definition 3.2. A weighted graph is stable if

$$2\,\mathrm{wt}(v) - 2 + \mathrm{val}(v) > 0 \qquad \text{for every } v \in V(\Gamma).$$

The genus of a connected weighted graph is

$$g(\Gamma) = b_1(\Gamma) + \sum_{v \in V(\Gamma)} \mathrm{wt}(v).$$

Definition 3.3. Let C be a stable nodal curve. Its dual graph $\Gamma(C)$ is defined as follows:

(i) vertices correspond to irreducible components of C;
(ii) edges correspond to nodes of C;
(iii) the two half-edges of an edge correspond to the two branches of the normalization at the node;
(iv) legs correspond to marked points; and
(v) the weight of a vertex is the genus of the corresponding normalized irreducible component.

If G acts on C, then G acts on $\Gamma(C)$. Under the standard admissibility hypotheses,

$$\Gamma(C/G) \cong \Gamma(C)/G.$$

This compatibility is part of the general relationship between admissible covers and their tropicalizations [BR11; LUZ24; El +24].

3.2. Harmonic morphisms

We follow the conventions for harmonic morphisms and tropical covers used in [LUZ19; LUZ24]; see also [El +24; Sou25].

Set

$$X(\Gamma) = V(\Gamma) \sqcup H(\Gamma).$$

Definition 3.4. A harmonic morphism of graphs is a graph morphism

$$f : \Gamma' \longrightarrow \Gamma$$

together with a degree function

$$d_f : X(\Gamma') \longrightarrow \mathbb{Z}_{\geq 0}$$

such that:

(i) if $e' = \{h'_1, h'_2\}$, then

$$d_f(h'_1) = d_f(h'_2);$$

(ii) for every $v' \in V(\Gamma')$ and every $h \in T_{f(v')}\Gamma$,

$$d_f(v') = \sum_{\substack{h' \in T_{v'}\Gamma' \\ f(h') = h}} d_f(h').$$

The second condition says that the total expansion in every tangent direction downstairs is independent of the chosen tangent direction. The degree is therefore independent of the chosen vertex or edge:

$$\deg(f) = \sum_{v' \in f^{-1}(v)} d_f(v') = \sum_{e' \in f^{-1}(e)} d_f(e').$$

Definition 3.5. A harmonic G-cover is a harmonic morphism

$$f : \Gamma' \longrightarrow \Gamma$$

with a G-action on Γ' such that:

(i) f and d_f are G-invariant;
(ii) G acts transitively on every fiber; and
(iii) $\deg(f) = |G|$.

For $x' \in X(\Gamma')$, the local degree is the order of the stabilizer:

$$d_f(x') = |\operatorname{Stab}_G(x')|.$$

Indeed, the orbit of x' has size

$$\frac{|G|}{|\operatorname{Stab}_G(x')|},$$

and the total degree over the corresponding point downstairs is $|G|$.

3.3. Metric and weighted graph covers

A metric graph is a graph Γ together with a positive edge length function

$$\ell : E(\Gamma) \longrightarrow \mathbb{R}_{>0}.$$

A harmonic morphism of metric graphs is required to satisfy

$$\ell(f(e')) = d_f(e')\ell(e')$$

for every edge $e' \in E(\Gamma')$.

For weighted graphs, define the local Euler characteristic

$$\chi(v) = 2 - 2\,\mathrm{wt}(v) - \mathrm{val}(v).$$

The local ramification contribution of a harmonic morphism is

$$\mathrm{Ram}_f(v') = d_f(v')\chi(f(v')) - \chi(v').$$

Definition 3.6. A harmonic morphism of weighted graphs is unramified if

$$\mathrm{Ram}_f(v') = 0 \qquad \text{for every } v' \in V(\Gamma').$$

Equivalently,

$$2\,\mathrm{wt}(v') - 2 + \mathrm{val}(v') = d_f(v')\left(2\,\mathrm{wt}(f(v')) - 2 + \mathrm{val}(f(v'))\right).$$

This is the local graph-theoretic Riemann–Hurwitz formula. It should not be confused with the assertion that the corresponding algebraic cover is unramified at every point; dilation along graph edges can arise from admissible ramification at nodes [LUZ24; El +24].

4. Dilation data and realizability

Fix a finite abelian group G, usually

$$G = (\mathbb{Z}/p)^{2g},$$

and a stable weighted target graph Δ. The terminology of dilation data and the realizability criterion used below follow [LUZ19; LUZ24]; related formulations occur in [El +24; Sou25].

4.1. Stabilizer data

Definition 4.1. For a harmonic G-cover

$$f : \Gamma \longrightarrow \Delta,$$

the G-dilation datum is the collection of subgroups

$$D(v) \leq G, \qquad D(h) \leq G,$$

where:

(i) $D(v)$ is the stabilizer of any vertex over $v \in V(\Delta)$;
(ii) $D(h)$ is the stabilizer of any half-edge over $h \in H(\Delta)$.

Since G is abelian, the stabilizers of two points in the same orbit are equal rather than merely conjugate. Therefore $D(v)$ and $D(h)$ are independent of the chosen lift.

The fiber over a vertex is the transitive G-set

$$f^{-1}(v) \cong G/D(v),$$

and the fiber over a half-edge is

$$f^{-1}(h) \cong G/D(h).$$

Moreover,

$$d_f(v') = |D(f(v'))|, \qquad d_f(h') = |D(f(h'))|.$$

4.2. Extended homology and monodromy

The monodromy datum of a weighted graph can be recorded in an extended homology group

$$H_1^{\mathrm{ext}}(\Delta, G) = H_1(\Delta, G) \oplus \bigoplus_{v \in V(\Delta)} G^{2\,\mathrm{wt}(v)}.$$

A class (η, ξ) consists of:

(i) an element $\eta(h) \in G$ for every half-edge $h \in H(\Delta)$, satisfying

$$\eta(h) + \eta(\bar{h}) = 0$$

and

$$\sum_{h \in T_v\Delta} \eta(h) = 0 \qquad \text{for every } v \in V(\Delta);$$

(ii) an element

$$\xi(v) \in G^{2\,\mathrm{wt}(v)}$$

for every vertex v, recording monodromy around a symplectic basis of the positive-genus component corresponding to v.

The associated dilation datum is

$$D(h) = \langle \eta(h) \rangle$$

and

$$D(v) = \langle \eta(h) \mid h \in T_v\Delta \rangle + \langle \text{entries of } \xi(v) \rangle .$$

The balancing relation

$$\sum_{h \in T_v\Delta} \eta(h) = 0$$

is the graph-theoretic analogue of the relation saying that the product of local monodromies around punctures on a component is the identity.

Theorem 4.2 (Realizability criterion)**.** *A harmonic G-cover of weighted graphs is realizable as the dual graph of an admissible cover of curves if and only if its dilation datum is induced by a class in*

$$H_1^{\mathrm{ext}}(\Delta, G).$$

This is the realizability criterion for abelian tropical covers [LUZ24].

Corollary 4.3. *If the stabilizer of an edge is not cyclic, then the graph cover is not realizable.*

Proof. In a realizable admissible cover, the stabilizer of a branch over a node is generated by its local monodromy. In the homological description,

$$D(h) = \langle \eta(h) \rangle,$$

which is cyclic. Therefore a noncyclic edge stabilizer cannot occur in a realizable cover. This obstruction is also explicit in [LUZ24; El +24; Sou25]. □

For

$$G = (\mathbb{Z}/p)^{2g},$$

every cyclic subgroup has order 1 or p. Hence a realizable edge stabilizer is either trivial or a one-dimensional subspace of the $\mathbb{F}_p$-vector space G.

4.3. The weight-zero case

Assume every vertex of Δ has weight zero. After orienting the edges, a half-edge assignment is equivalent to a function

$$\eta : E(\Delta) \longrightarrow G.$$

The local balancing condition becomes

$$\sum_{t(e)=v} \eta(e) - \sum_{s(e)=v} \eta(e) = 0 \qquad \text{for every } v.$$

Lemma 4.4. *For a weight-zero graph Δ, admissible dilation data are precisely the elements of $H_1(\Delta, G)$.*

Proof. The simplicial chain groups are

$$C_1(\Delta, G) = G^{E(\Delta)}, \qquad C_0(\Delta, G) = G^{V(\Delta)}.$$

The boundary map is determined by

$$\partial_1(e) = t(e) - s(e).$$

For $\eta \in C_1(\Delta, G)$, the v-component of $\partial_1(\eta)$ is

$$\sum_{t(e)=v} \eta(e) - \sum_{s(e)=v} \eta(e).$$

Thus the balancing condition is precisely

$$\partial_1(\eta) = 0.$$

Since a graph has no 2-simplices,

$$H_1(\Delta, G) = \ker(\partial_1).$$

□

The orientation of the edges is auxiliary. Reversing the orientation of an edge replaces its label by its negative, leaving the corresponding homology class unchanged.

5. Lifting data and classification of graph covers

Dilation data determine the fibers and local stabilizers but do not determine how these fibers are glued across edges. The remaining information is called lifting or gluing data. We use the dilated cohomology formalism of [LUZ19; LUZ24].

5.1. Dilated cohomology

Fix a dilation datum D. For a vertex v, set

$$C(v) = D(v).$$

For an edge e with endpoints v and w, set

$$C(e) = D(v) + D(w).$$

Consider the two-term cochain complex

$$\bigoplus_{v \in V(\Delta)} G/C(v) \longrightarrow \bigoplus_{e \in E(\Delta)} G/C(e),$$

whose differential is induced by

$$(g_v)_{v \in V(\Delta)} \longmapsto (g_{t(e)} - g_{s(e)})_{e \in E(\Delta)}.$$

Definition 5.1. The first dilated cohomology group $H^1(\Delta, G_D)$ is the first cohomology of this complex.

Concretely, an element is represented by a tuple

$$(g^e)_{e \in E(\Delta)}, \qquad g^e \in G/C(e),$$

and two tuples are equivalent,

$$(g^e) \sim (\widetilde{g}^e),$$

if there are elements $g^v \in G/C(v)$ such that

$$g^e - \widetilde{g}^e = g^{t(e)} - g^{s(e)} \quad \text{in } G/C(e).$$

Changing g^v amounts to choosing a different distinguished lift of the vertex v. Thus, taking the quotient by coboundaries exactly captures the equivalence that arises from choosing different distinguished lifts at the vertices.

There is a natural morphism of cochain complexes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & G^{V(\Delta)} & \xrightarrow{\delta^0} & G^{E(\Delta)} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \displaystyle\bigoplus_{v \in V(\Delta)} G/C(v) & \longrightarrow & \displaystyle\bigoplus_{e \in E(\Delta)} G/C(e) & \longrightarrow & 0. \end{array}$$

It induces a surjection

$$H^1(\Delta, G) \twoheadrightarrow H^1(\Delta, G_D).$$

Indeed, if $C^\bullet$ denotes the kernel complex, then the associated long exact sequence includes

$$H^1(\Delta, G) \longrightarrow H^1(\Delta, G_D) \longrightarrow H^2(C^\bullet).$$

Since the complexes are concentrated in degrees 0 and 1,

$$H^2(C^\bullet) = 0.$$

5.2. Classification by dilation and lifting data

Theorem 5.2 (Classification by dilated cohomology)**.** *For a fixed target graph Δ, admissible G-covers with dilation datum D are classified by classes*

$$\bar{\lambda} \in H^1(\Delta, G_D),$$

up to the action of $\mathrm{Aut}(\Delta)$*. Equivalently, their isomorphism classes are parametrized by*

$$\left\{(\eta, \bar{\lambda}) : \eta \in H_1^{\mathrm{ext}}(\Delta, G),\ \bar{\lambda} \in H^1(\Delta, G_D)\right\} / \mathrm{Aut}(\Delta).$$

This is the dilated-cohomology classification of abelian tropical covers [LUZ19; LUZ24].

For computations, it is often convenient to choose a lift

$$\lambda \in H^1(\Delta, G)$$

of $\bar{\lambda}$. The class $\bar{\lambda}$ determines the graph cover, but distinct lifts may encode different symplectic information, that is, different covers of curves,

Remark 5.3. Before quotienting by $\mathrm{Aut}(\Delta)$, the target graph is regarded as labelled. An automorphism of Δ moves both the dilation and lifting data. Thus two labelled pairs may determine isomorphic unlabelled G-covers.

6. Connectedness

The connectedness criteria in this section reformulate the monodromy and lifting descriptions used for abelian tropical covers [LUZ19; LUZ24]. Similar criteria occur in the study of tropical unramified p-covers [El +24; Sou25].

6.1. A spanning-tree criterion

Choose a spanning tree

$$T \subseteq \Delta$$

and let

$$F = E(\Delta) \setminus E(T)$$

be the set of non-tree edges. Choose one lift $\widetilde{T}$ of T to the covering graph. For each $f \in F$, choose a lift $\widetilde{f}$. After fixing one endpoint in $\widetilde{T}$, the other endpoint lies in a translate $g_f\widetilde{T}$ for some $g_f \in G$. The element g_f is defined up to the corresponding vertex stabilizers.

Theorem 6.1 (Spanning-tree connectedness criterion)**.** *The covering graph is connected if and only if*

$$G = \left\langle D(v),\, g_f \;\middle|\; v \in V(\Delta),\ f \in F \right\rangle.$$

More generally, the number of connected components is the index of this subgroup in G.

Proof. Let

$$K = \left\langle D(v), g_f \;\middle|\; v \in V(\Delta),\ f \in F \right\rangle.$$

All translates of $\widetilde{T}$ are indexed by elements of G. If $a \in D(v)$, then $\widetilde{T}$ and $a\widetilde{T}$ meet at the vertex over v, so they lie in the same connected component. If $f \in F$, then the chosen lift $\widetilde{f}$ joins $\widetilde{T}$ to $g_f\widetilde{T}$. Hence $\widetilde{T}$ and $k\widetilde{T}$ lie in the same component for every $k \in K$.

Conversely, any path between two translates of $\widetilde{T}$ can be decomposed into moves within lifted copies of the tree, moves through vertex stabilizers, and crossings of non-tree edges. Each such move changes the translation label by an element of $D(v)$ or by one of the g_f. Therefore two translates lie in the same connected component only if their labels differ by an element of K.

Thus connected components are indexed by G/K, and the graph is connected exactly when $K = G$. □

6.2. A two-vertex example

Let Δ consist of two vertices v, w joined by one edge. Let

$$A = D(v), \qquad B = D(w).$$

There are no non-tree edges.

Lemma 6.2. *The covering graph is connected if and only if*

$$A + B = G.$$

Proof. The spanning-tree criterion gives the result immediately. It can also be seen directly. The vertex set upstairs is

$$G/A \sqcup G/B.$$

The edge indexed by $g \in G$ joins gA to gB. The vertex A is connected to gB if and only if one can write

$$g = a + b \qquad (a \in A,\ b \in B).$$

Thus all vertices lie in one component precisely when $A + B = G$. □

This graph is a bipartite coset graph, closely related to the Bass–Serre graph associated to the subgroups A and B.

6.3. An invariant homological criterion

Now assume all vertices of Δ have weight zero. Let

$$\eta \in H_1(\Delta, G)$$

be the dilation class, and let

$$\lambda \in H^1(\Delta, G)$$

be a lift of the gluing class. The universal coefficient theorem gives

$$H_1(\Delta, G) \cong \mathrm{Hom}(H^1(\Delta, \mathbb{F}_p), G)$$

and

$$H^1(\Delta, G) \cong \mathrm{Hom}(H_1(\Delta, \mathbb{F}_p), G).$$

Define

$$K(\eta, \lambda) = \left\langle \eta(\alpha), \lambda(\sigma) \;\middle|\; \alpha \in H^1(\Delta, \mathbb{F}_p),\ \sigma \in H_1(\Delta, \mathbb{F}_p) \right\rangle.$$

Proposition 6.3. *The subgroup $K(\eta, \lambda)$ is independent of the choice of lift λ of $\bar{\lambda}$. Moreover, the number of connected components of the covering graph is*

$$[G : K(\eta, \lambda)].$$

In particular, the graph cover is connected if and only if

$$K(\eta, \lambda) = G.$$

Proof. Let λ' be another lift of $\bar{\lambda}$. Then

$$\lambda' - \lambda$$

lies in the kernel of

$$H^1(\Delta, G) \longrightarrow H^1(\Delta, G_D).$$

This kernel is generated by cochains whose edge values lie in the subgroups determined by the dilation datum. Those subgroups are contained in the subgroup generated by the image of η. Therefore replacing λ by λ' does not change $K(\eta, \lambda)$.

After choosing a spanning tree and dual bases of homology and cohomology, the image of η generates the stabilizer data, while the image of λ generates the gluing translations associated to non-tree edges. Thus $K(\eta, \lambda)$ is exactly the subgroup appearing in the spanning-tree criterion. □

7. The symplectic condition

Let

$$G = (\mathbb{Z}/p)^{2g},$$

and suppose Δ is a connected weight-zero graph of genus g. Then

$$\dim_{\mathbb{F}_p} H_1(\Delta, \mathbb{F}_p) = \dim_{\mathbb{F}_p} H^1(\Delta, \mathbb{F}_p) = g.$$

Suppose (η, λ) defines a connected cover. The images of η and λ together span G.

Definition 7.1. Define an alternating form $\omega_{\eta,\lambda}$ on G by requiring

$$\omega_{\eta,\lambda}\big(\eta(\alpha), \eta(\beta)\big) = 0$$

for all $\alpha, \beta \in H^1(\Delta, \mathbb{F}_p)$,

$$\omega_{\eta,\lambda}\big(\lambda(\sigma), \lambda(\tau)\big) = 0$$

for all $\sigma, \tau \in H_1(\Delta, \mathbb{F}_p)$, and

$$\omega_{\eta,\lambda}\big(\eta(\alpha), \lambda(\sigma)\big) = \alpha(\sigma).$$

Skew-symmetry then gives

$$\omega_{\eta,\lambda}\big(\lambda(\sigma), \eta(\alpha)\big) = -\alpha(\sigma).$$

Proposition 7.2. *If*

$$H^1(\Delta, \mathbb{F}_p) \oplus H_1(\Delta, \mathbb{F}_p) \longrightarrow G, \qquad (\alpha, \sigma) \longmapsto \eta(\alpha) + \lambda(\sigma)$$

is an isomorphism, then $\omega_{\eta,\lambda}$ is a nondegenerate symplectic form on G.

Proof. Choose dual bases

$$\alpha_1, \dots, \alpha_g \quad \text{and} \quad \sigma_1, \dots, \sigma_g$$

satisfying

$$\alpha_i(\sigma_j) = \delta_{ij}.$$

With respect to the basis

$$\eta(\alpha_1), \dots, \eta(\alpha_g), \lambda(\sigma_1), \dots, \lambda(\sigma_g)$$

of G, the matrix of $\omega_{\eta,\lambda}$ is

$$\begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix}.$$

This matrix is alternating and invertible. □

Definition 7.3. After fixing a standard symplectic form ω_{std} on G, the data (η, λ) are called symplectic if

$$\omega_{\eta,\lambda} = \omega_{\text{std}}.$$

An automorphism of Δ acts simultaneously on homology and cohomology and preserves the evaluation pairing. Hence $\omega_{\eta,\lambda}$ depends only on the $\mathrm{Aut}(\Delta)$-orbit of (η, λ).

Remark 7.4. The graph cover depends only on

$$\bar{\lambda} \in H^1(\Delta, G_D),$$

but the symplectic form may depend on the chosen lift

$$\lambda \in H^1(\Delta, G).$$

Thus a single graph cover may occur in the boundary of more than one component of the moduli space of principal G-covers. This agrees with the decomposition of Hurwitz spaces by monodromy and symplectic data [BR11; Sou25].

8. From algebraic level structures to graph data

Let X be a smooth genus-g surface, and let

$$\phi : H_1(X, \mathbb{F}_p) \xrightarrow{\sim} G$$

be a symplectic isomorphism. Suppose X degenerates to a stable curve C whose dual graph Δ has only weight-zero vertices.

8.1. The homological splitting

We construct a (choice-free) map that takes the data of a symplectic map $\phi : H_1(X, \mathbb{F}_p) \to (\mathbb{F}_p)^{2g}$ along with a choice of Δ and produces the data of an admissible symplectic graph cover of Δ. More specifically, we produce a pair (η, λ) that satisfies the connectedness and symplecticness conditions.

Let us begin with a symplectic map $\phi : H_1(X, \mathbb{F}_p) \to G$ where X is a compact genus g surface. Let C be a stable nodal curve obtained from X and let Δ be its corresponding dual graph. There is a natural splitting $H_1(X, \mathbb{F}_p) = A \oplus B$ into the part generated by the meridian cycles and the part generated by the longitudinal cycles. Let us denote these two subspaces A and B respectively. When we degenerate to C, we only contract curves in B.

We note that there is a surjective map $H_1(X, \mathbb{F}_p) \to H_1(C, \mathbb{F}_p) \cong H_1(\Delta, \mathbb{F}_p)$ whose kernel is contained in B. By a rank argument, we see that the kernel is exactly B and so $A \cong H_1(\Delta, \mathbb{F}_p)$. We also note that $B \hookrightarrow \mathrm{Hom}(H_1(\Delta, \mathbb{F}_p), \mathbb{F}_p)$ where $b \mapsto ([a] \mapsto i(b, a))$. Considering ranks shows equality again. Since these are all free modules, we can write a natural isomorphism

$$H_1(X, \mathbb{F}_p) \cong H^1(\Delta, \mathbb{F}_p) \oplus H_1(\Delta, \mathbb{F}_p).$$

Under this splitting, the algebraic intersection pairing becomes the evaluation pairing. In particular,

$$\langle \alpha, \beta \rangle = 0 \quad \text{for } \alpha, \beta \in H^1(\Delta, \mathbb{F}_p),$$
$$\langle \sigma, \tau \rangle = 0 \quad \text{for } \sigma, \tau \in H_1(\Delta, \mathbb{F}_p),$$

and

$$\langle \alpha, \sigma \rangle = \alpha(\sigma).$$

This decomposition is compatible with the graph-of-groups description of a smoothing [BR11].

This also gives us a natural morphism

$$\mathrm{Hom}(H_1(X, \mathbb{F}_p), G) \cong H_1(\Delta, G) \oplus H^1(\Delta, G).$$

Thus, $\phi \in \mathrm{Hom}(H_1(X, \mathbb{F}_p), G)$ corresponds to a unique element $(\eta, \lambda) \in H_1(\Delta, G) \oplus H^1(\Delta, G)$. Applying ϕ to the two summands produces

$$\eta : H^1(\Delta, \mathbb{F}_p) \longrightarrow G$$

and

$$\lambda : H_1(\Delta, \mathbb{F}_p) \longrightarrow G.$$

Theorem 8.1. *There is a natural isomorphism*

$$\mathrm{Hom}(H_1(X, \mathbb{F}_p), G) \cong H_1(\Delta, G) \oplus H^1(\Delta, G).$$

Under this isomorphism, a symplectic level structure determines a pair (η, λ) satisfying:

(i) the associated graph cover is connected;
(ii) η defines realizable dilation data; and
(iii)

$$\omega_{\eta,\lambda} = \omega_{\mathrm{std}}.$$

Proof. Using the splitting,

$$\begin{aligned}\operatorname{Hom}(H_1(X,\mathbb{F}_p),G) &\cong \operatorname{Hom}(H^1(\Delta,\mathbb{F}_p),G)\oplus \operatorname{Hom}(H_1(\Delta,\mathbb{F}_p),G)\\ &\cong H_1(\Delta,G)\oplus H^1(\Delta,G).\end{aligned}$$

Since ϕ is an isomorphism, the images of η and λ span G. By the connectedness criterion, the associated cover is connected.

The dilation data arise from local monodromy around vanishing cycles and therefore satisfy the realizability criterion of [LUZ24]. Finally, ϕ preserves the intersection pairing. Under the splitting, this pairing is the evaluation pairing, so the form induced on G is precisely ω_{std}. □

8.2. A basis calculation

Choose a symplectic basis

$$a_1,\dots,a_g,b_1,\dots,b_g$$

of $H_1(X,\mathbb{F}_p)$. Suppose the b_i are contracted in the degeneration. Then:

(i) the b_i determine local monodromy and hence the dilation class;
(ii) the a_i determine cycles in the graph and hence the lifting class.

If $\alpha_i \in H^1(\Delta,\mathbb{F}_p)$ and $\sigma_i \in H_1(\Delta,\mathbb{F}_p)$ are dual bases, then

$$\eta(\alpha_i)=\phi(b_i),\qquad \lambda(\sigma_i)=\phi(a_i).$$

The symplectic condition becomes

$$\omega_{\mathrm{std}}\big(\phi(b_i),\phi(a_j)\big)=\delta_{ij}.$$

Example 8.2 (The theta graph)**.** Let $g=2$, and let Δ be the theta graph. Choose dual bases

$$\{\alpha_1,\alpha_2\}\subset H^1(\Delta,\mathbb{F}_p),\qquad \{\sigma_1,\sigma_2\}\subset H_1(\Delta,\mathbb{F}_p).$$

For a symplectic isomorphism

$$\phi : H_1(X,\mathbb{F}_p)\xrightarrow{\sim}(\mathbb{F}_p)^4,$$

the corresponding graph data are

$$\eta(\alpha_i)=\phi(b_i),\qquad \lambda(\sigma_i)=\phi(a_i).$$

The stabilizer of the node obtained by contracting b_i is

$$\langle\phi(b_i)\rangle,$$

while the gluing translation along the corresponding graph cycle is $\phi(a_i)$. The identity

$$\omega_{\mathrm{std}}\big(\eta(\alpha_i),\lambda(\sigma_j)\big)=\delta_{ij}$$

recovers the symplectic condition.

9. Graphs of groups

The preceding homological description has a parallel formulation in terms of graphs of groups and the topology of Hurwitz spaces. We follow the graph-of-groups perspective for degenerating covers in [BR11].

9.1. Fundamental groups of graphs of groups

Definition 9.1. A graph of groups consists of:

(i) a connected graph Δ;
(ii) a group G_v for every $v \in V(\Delta)$;
(iii) a group $G_h = G_{\bar{h}}$ for every edge;
(iv) injective homomorphisms

$$\partial_h : G_h \longrightarrow G_{\partial(h)}.$$

Choose a maximal tree $T \subseteq \Delta$. The fundamental group $\pi_1(\Delta, \mathcal{G}, T)$ is generated by the vertex groups and the oriented half-edges, subject to

$$h = 1 \qquad (h \in T),$$
$$h\bar{h} = 1,$$

and

$$h\, \partial_{\bar{h}}(x)\, h^{-1} = \partial_h(x) \qquad (x \in G_h).$$

Its isomorphism type is independent of the choice of T.

For a nodal curve with dual graph Δ, assign to each vertex v the fundamental group of the corresponding normalized component with its special points removed. Assign to each edge group an infinite cyclic group generated by a loop around the corresponding branch. The two edge maps record the boundary loops around the two branches of the node.

Theorem 9.2. *The fundamental group of this graph of groups is naturally isomorphic to the fundamental group of a nearby smooth fiber of a smoothing of the nodal curve.*

This is the graph-of-groups description used in the topology of Hurwitz spaces [BR11].

9.2. Abelianization

Because

$$G = (\mathbb{Z}/p)^{2g}$$

is abelian of exponent p, every homomorphism

$$\pi_1(\Delta, \mathcal{G}) \longrightarrow G$$

factors through the abelianization after quotienting by p.

Proposition 9.3. *If every vertex of Δ has weight zero, then*

$$\mathrm{Hom}(\pi_1(\Delta, \mathcal{G}), G) \cong H_1(\Delta, G) \oplus H^1(\Delta, G).$$

Proof. After abelianization, the stable letters corresponding to edges outside a maximal tree generate a free abelian group naturally identified with

$$H^1(\Delta, \mathbb{Z}).$$

The abelianizations of the punctured-sphere vertex groups, modulo the relations identifying the boundary loops along edges, give

$$H_1(\Delta, \mathbb{Z}).$$

Thus the relevant abelianization is

$$H_1(\Delta, \mathbb{Z}) \oplus H^1(\Delta, \mathbb{Z}).$$

Applying $\mathrm{Hom}(-, G)$ gives

$$\begin{aligned} \mathrm{Hom}\Big(H_1(\Delta, \mathbb{Z}) \oplus H^1(\Delta, \mathbb{Z}), G\Big) &\cong \mathrm{Hom}(H_1(\Delta, \mathbb{Z}), G) \oplus \mathrm{Hom}(H^1(\Delta, \mathbb{Z}), G) \\ &\cong H^1(\Delta, G) \oplus H_1(\Delta, G). \end{aligned}$$

Reordering the summands proves the claim. □

The $H_1(\Delta, G)$-summand records local monodromy, while the $H^1(\Delta, G)$-summand records stable-letter or gluing data. Thus the graph-of-groups and dilated-cohomology descriptions are two presentations of the same topological information.

10. Examples and computations

10.1. A separating edge

Let Δ consist of two vertices v, w joined by one edge. Suppose

$$A = D(v), \qquad B = D(w),$$

and the edge stabilizer is trivial. Then

$$V(\Gamma) = G/A \sqcup G/B.$$

For each $g \in G$, there is an edge joining gA to gB.

The graph is connected if and only if

$$A + B = G.$$

If

$$G = (\mathbb{F}_p)^4$$

and A, B are complementary two-dimensional subspaces, then

$$|G/A| = |G/B| = p^2,$$

and every vertex on one side is connected to every vertex on the other side. Thus

$$\Gamma \cong K_{p^2,p^2}.$$

For $p = 3$, this gives $K_{9,9}$. The number of two-dimensional subspaces of $\mathbb{F}_3^4$ is the Gaussian binomial coefficient

$$\binom{4}{2}_3 = \frac{(3^4-1)(3^3-1)}{(3^2-1)(3-1)} = 130.$$

For each A, the number of complementary two-dimensional subspaces B is

$$3^{2\cdot 2} = 81.$$

Hence there are

$$130 \cdot 81 = 10530$$

ordered complementary pairs (A, B), and

$$\frac{10530}{2} = 5265$$

unordered pairs.

If the pair is required to be symplectic, then A and B must be complementary nondegenerate symplectic planes. In the standard symplectic component, choosing A determines

$$B = A^{\perp,\omega}.$$

For $G = (\mathbb{F}_3)^4$, there are 90 symplectic planes, producing 45 unordered symplectic decompositions.

10.2. A single loop

Suppose Δ has one vertex v and one loop. Let

$$A = D(v),$$

and let

$$t \in G/A$$

be the gluing class. The covering graph has vertex set G/A. Starting at A, following the lifted loop moves to $t + A$; repeated traversal reaches

$$nt + A \qquad (n \in \mathbb{Z}).$$

Therefore the graph is connected if and only if

$$A + \langle t \rangle = G.$$

If $G = (\mathbb{F}_3)^4$, connectedness with one gluing element forces

$$\dim A \geq 3.$$

In the realizable genus-two case, one typically has

$$\dim A = 3$$

and a one-dimensional edge stabilizer

$$D(e) = \langle a \rangle \subseteq A.$$

Then

$$|G/A| = 3,$$

so there are three vertices upstairs. If the edge stabilizer has order 3, the fiber over the loop consists of

$$|G/D(e)| = 27$$

edges. The resulting graph is a triangle with nine parallel edges between each neighboring pair of vertices.

10.3. A rose

Let Δ be a rose with one vertex and r loops. Let

$$A = D(v),$$

and let

$$t_1, \ldots, t_r \in G/A$$

be the gluing classes. Then

$$V(\Gamma) = G/A,$$

and the cover is connected if and only if

$$A + \langle t_1, \ldots, t_r \rangle = G.$$

When $r = g$, $\dim_{\mathbb{F}_p} A = g$, and A is Lagrangian, a symplectic cover is obtained by choosing the t_i as a basis of a complementary Lagrangian subspace. The dilation and gluing subspaces are then dual under the symplectic pairing.

10.4. The theta graph

Let Δ consist of two vertices joined by three edges. Choose one edge in a spanning tree. The remaining edges determine two gluing elements t_1, t_2. If the vertex stabilizers are A, B, then connectedness is equivalent to

$$A + B + \langle t_1, t_2 \rangle = G.$$

If the three oriented edge labels are a, b, c, the local balancing condition gives

$$a + b + c = 0.$$

Thus two edge labels determine the third. Realizability requires

$$D(e_a) = \langle a \rangle, \qquad D(e_b) = \langle b \rangle, \qquad D(e_c) = \langle c \rangle$$

to be cyclic [LUZ24; El +24].

10.5. The dumbbell graph

Let Δ consist of two looped vertices joined by a bridge. Since the bridge is separating, its homology label is zero. Let

$$A = D(v), \qquad B = D(w),$$

and let

$$t_1 \in G/A, \qquad t_2 \in G/B$$

be the gluing data for the two loops. Then the connectedness criterion is

$$A + B + \langle t_1, t_2 \rangle = G.$$

In a realizable elementary abelian cover, the stabilizers of the loop edges are either trivial or one-dimensional. The bridge does not contribute independent monodromy, but it determines how the two local covering graphs are joined.

11. Gluing Map

The pair $(\eta, \bar{\lambda})$ determines the dual graph of an admissible G-cover, but a boundary stratum contains additional geometric data. For each vertex of the target graph, one must choose a cover of the corresponding normalized irreducible component and then identify the marked points lying over the branches of the nodes. The gluing morphism assembles these local covers into a nodal admissible cover with the prescribed dual graph. This is the usua construction for admissible covers, expressed here in terms of dilation and lifting data; see [BR11; SZ20].

Fix a connected harmonic G-cover

$$f \colon \Gamma \longrightarrow \Delta$$

associated to a realizable pair

$$(\eta, \bar{\lambda}), \qquad \eta \in H_1(\Delta, G), \qquad \bar{\lambda} \in H^1(\Delta, G_D).$$

We assume that every vertex of Δ has weight zero although the map can be constructed in general.

For a vertex $v \in V(\Delta)$, let

$$T_v\Delta = \{h \in H(\Delta) \mid \partial(h) = v\}$$

be the set of half-edges incident to v. The restriction of the dilation class to v is the tuple

$$\eta(v) = \big(\eta(h)\big)_{h \in T_v\Delta}.$$

Since η is a cycle, these local monodromies satisfy the balancing relation

$$\sum_{h \in T_v\Delta} \eta(h) = 0.$$

This is the additive form of the relation asserting that the product of the local monodromies around the punctures of a rational component is the identity. It is therefore the necessary local relation for a cover of the punctured component corresponding to v; see [BR11; LUZ24].

The vertex stabilizer is

$$D(v) = \langle \eta(h) \mid h \in T_v\Delta \rangle \subseteq G.$$

For an edge $e = \{h, \bar{h}\}$, its stabilizer is

$$D(e) = \langle \eta(h) \rangle = \langle \eta(\bar{h}) \rangle.$$

Indeed,

$$\eta(\bar{h}) = -\eta(h).$$

Thus the two local monodromies at the branches of the node corresponding to e are inverse. This is precisely the balanced condition required in the definition of an admissible cover [BR11; SZ20]. In particular, the edge stabilizer is cyclic, in agreement with the realizability criterion for abelian tropical covers [LUZ24].

For each $v \in V(\Delta)$, let

$$\overline{\mathcal{H}}_v = \overline{\mathcal{H}}_{0,D(v),\eta(v)}$$

denote the local Hurwitz space parametrizing admissible $D(v)$-covers

$$P_v \longrightarrow C_v$$

with the following properties:

(1) C_v is a stable rational curve;
(2) the marked points of C_v are indexed by the half-edges $h \in T_v\Delta$; and
(3) the local monodromy at the marked point indexed by h is $\eta(h)$.

Such local Hurwitz spaces are the vertexwise factors that occur in the boundary descriptions of admissible-cover moduli spaces [BR11; SZ20]. Although the target C_v has genus zero, the genus of the source P_v is determined by the Riemann–Hurwitz formula and need not be zero.

The local cover

$$P_v \longrightarrow C_v$$

describes one distinguished irreducible component in the fiber over v. The full G-equivariant curve over C_v is obtained by induction from $D(v)$ to G:

$$\widetilde{P}_v = G \times^{D(v)} P_v.$$

As a curve, it is a disjoint union

$$\widetilde{P}_v \cong \bigsqcup_{[g] \in G/D(v)} gP_v.$$

Its components are therefore naturally indexed by

$$G/D(v) \cong f^{-1}(v),$$

which is the vertex fiber of the harmonic graph cover. This agrees with the coset description of the fibers determined by dilation data in [LUZ19; LUZ24].

Changing the distinguished component

$$P_v \subseteq \widetilde{P}_v$$

amounts to translating it by an element of G. Such a change alters the chosen labels of the components but not the isomorphism class of the induced G-curve. The same ambiguity appeared in the definition of dilated cohomology: changing the distinguished lift of a vertex modifies the edge translations by a coboundary [LUZ19; LUZ24].

We next explain how the lifting class determines the identifications between the local pieces. Choose a lift

$$\lambda \in H^1(\Delta, G)$$

of

$$\bar{\lambda} \in H^1(\Delta, G_D).$$

After choosing distinguished lifts of the vertices of Δ, the class λ may be represented by edge translations

$$\lambda(e) \in G.$$

These translations specify which components in the fibers over the two endpoints of e are joined by the lifted edges of Γ.

More explicitly, suppose e is oriented from $s(e)$ to $t(e)$. A chosen lift of the initial vertex determines a component indexed by a coset

$$g + D(s(e)).$$

The lift of e joins this component to the component over $t(e)$ indexed by

$$g + \lambda(e) + D(t(e)).$$

The value of $\lambda(e)$ is defined only modulo the subgroups contributed by the two endpoints. This is why the graph cover is naturally classified by the dilated cohomology class

$$\bar{\lambda} \in H^1(\Delta, G_D)$$

rather than by an ordinary G-valued cochain; compare [LUZ19; LUZ24].

If the distinguished lift over a vertex v is translated by $g_v \in G$, then the edge representatives change according to

$$\lambda(e) \longmapsto \lambda(e) + g_{t(e)} - g_{s(e)}.$$

Thus a different choice of distinguished vertex lifts changes λ by a coboundary. The resulting graph cover depends only on $\bar{\lambda}$, as predicted by the classification by dilated cohomology [LUZ19; LUZ24].

We now perform the gluing. Each half-edge

$$\widetilde{h} \in H(\Gamma)$$

determines a marked point

$$x_{\widetilde{h}}$$

on the component of $\widetilde{P}_v$ indexed by the vertex $\partial(\widetilde{h})$. If $\overline{\widetilde{h}}$ is the opposite half-edge, then $\widetilde{h}$ and $\overline{\widetilde{h}}$ form an edge of Γ. We identify the corresponding marked points:

$$x_{\widetilde{h}} \sim x_{\overline{\widetilde{h}}}.$$

Performing these identifications for every edge of Γ produces a nodal curve

$$P = \left(\bigsqcup_{v \in V(\Delta)} \widetilde{P}_v \right) \Big/ \sim .$$

The corresponding construction downstairs identifies the marked point $q_h \in C_{\partial(h)}$ with the marked point $q_{\bar{h}} \in C_{\partial(\bar{h})}$. It produces the stable nodal curve

$$C = \left(\bigsqcup_{v \in V(\Delta)} C_v \right) \Big/ \sim .$$

By construction,

$$\Gamma(P) \cong \Gamma, \qquad \Gamma(C) \cong \Delta.$$

Because the local monodromies at the two branches of every node are $\eta(h)$ and $-\eta(h)$, the action on the two tangent directions is balanced. The local covers therefore glue to an admissible G-cover

$$P \longrightarrow C;$$

this is the standard local gluing condition for admissible covers [BR11; SZ20].

The source curve P is connected because its dual graph Γ is connected and each local source component used in the construction is connected. Consequently, the condition

$$K(\eta, \lambda) = G$$

from the connectedness criterion is exactly the condition needed for the gluing construction to land in the connected-source locus. In this way, the homological connectedness criterion agrees with the geometric connectedness condition imposed on admissible covers.

Let

$$\overline{\mathcal{H}}(\Delta, \eta, \bar{\lambda})$$

denote the locally closed boundary stratum parametrizing connected admissible G-covers whose target dual graph is Δ, whose dilation class is η, and whose lifting class is $\bar{\lambda}$. The preceding construction defines a gluing morphism

$$\xi_{\Delta,\eta,\bar{\lambda}} \colon \prod_{v \in V(\Delta)} \overline{\mathcal{H}}_{0,D(v),\eta(v)} \longrightarrow \overline{\mathcal{H}}(\Delta, \eta, \bar{\lambda}).$$

This is the vertexwise form of the gluing morphisms used to construct boundary strata in moduli spaces of admissible covers [BR11; SZ20].

Proposition 11.1. *The gluing morphism*

$$\xi_{\Delta,\eta,\bar{\lambda}}$$

is surjective. Up to canonical isomorphism, it is independent of the choices of distinguished components over the vertices and of the representative

$$\lambda \in H^1(\Delta, G)$$

of $\bar{\lambda}$.

Proof. Every admissible cover in $\overline{\mathcal{H}}(\Delta, \eta, \bar{\lambda})$ may be normalized at its nodes. Grouping the resulting components according to the vertices of Δ gives local covers

$$P_v \longrightarrow C_v$$

with monodromy tuple $\eta(v)$. Recovering the original cover amounts to identifying the marked branches according to the edges of Γ. Therefore every point of the stratum lies in the image of $\xi_{\Delta,\eta,\bar{\lambda}}$.

Changing the distinguished component over v translates $\widetilde{P}_v$ by an element $g_v \in G$. On the graph data, this changes the edge translations by

$$\lambda(e) \longmapsto \lambda(e) + g_{t(e)} - g_{s(e)}.$$

The difference is a coboundary, so it does not change the dilated cohomology class $\bar{\lambda}$. The corresponding gluings are related by the G-equivariant isomorphism acting by g_v on the induced curve over C_v. Hence they determine isomorphic admissible covers. This is the geometric counterpart of the coboundary equivalence in dilated cohomology [LUZ19; LUZ24]. □

There is a further finite ambiguity arising from automorphisms of the combinatorial datum. Define

$$\mathrm{Aut}(\Delta, \eta, \bar{\lambda}) = \left\{ \alpha \in \mathrm{Aut}(\Delta) \;\middle|\; \alpha_* \eta = \eta,\ \alpha^* \bar{\lambda} = \bar{\lambda} \right\}.$$

This group permutes the vertices, the local Hurwitz factors, and the marked points indexed by the incident half-edges. The gluing morphism is invariant under this action and therefore factors through

$$\left[\prod_{v\in V(\Delta)} \overline{\mathcal{H}}_{0,D(v),\eta(v)} \Big/ \operatorname{Aut}(\Delta,\eta,\bar{\lambda})\right] \longrightarrow \overline{\mathcal{H}}(\Delta,\eta,\bar{\lambda}).$$

Thus it is the automorphism orbit of

$$(\Delta,\eta,\bar{\lambda}),$$

rather than a particular labelling of Δ, that determines the corresponding boundary stratum. This is analogous to the quotient by graph automorphisms in the boundary stratification of $\overline{\mathcal{M}}_g$ and in admissible-cover compactifications [ACP15; BR11; SZ20].

The significance of the gluing construction is that it separates the component problem into a local geometric part and a finite combinatorial part. On connected components, the gluing morphism induces a surjection

$$\left(\prod_{v\in V(\Delta)} \pi_0\left(\overline{\mathcal{H}}_{0,D(v),\eta(v)}\right)\right) \Big/ \operatorname{Aut}(\Delta,\eta,\bar{\lambda}) \twoheadrightarrow \pi_0\left(\overline{\mathcal{H}}(\Delta,\eta,\bar{\lambda})\right).$$

Accordingly, once the graph datum

$$(\Delta,\eta,\bar{\lambda})$$

has been fixed, determining the connected components of the corresponding boundary stratum reduces to determining the connected components of the local genus-zero Hurwitz spaces and then accounting for the identifications induced by the gluing symmetries. In the present weight-zero setting, the local component problem may be approached using the genus-zero component calculations and related arguments of [BCK24].[1]

The gluing morphism is therefore the final step in passing from the classification of harmonic graph covers to a stratification of the boundary of $\mathcal{M}_g(p)$. The graph Δ records the combinatorial type of the target, the class η records the local monodromy and stabilizers, and the class $\bar{\lambda}$ records the global identifications of the local fibers. The local Hurwitz factors supply the remaining geometric moduli. Together, these data describe the strata of the weight-zero boundary and reduce their connected-component problem to a collection of genus-zero Hurwitz-space problems.

12. Conclusion

The objective of this project is to describe the weight-zero boundary of $\mathcal{M}_g(p)$ by explicit combinatorial data. A point of $\mathcal{M}_g(p)$ admits three compatible descriptions:

$$\mathcal{T}_g/\operatorname{Mod}_g(p) \iff \text{a symplectic basis of } H_1(X,\mathbb{Z}/p) \iff \text{a connected symplectic } G\text{-cover of } X,$$

where

$$G = (\mathbb{Z}/p)^{2g}.$$

The last description places $\mathcal{M}_g(p)$ inside the moduli space of principal G-covers as the component with prescribed symplectic monodromy [BR11; Sou25]. It is also the description that extends most naturally to the boundary through admissible covers [BR11; SZ20].

We have restricted attention to connected admissible G-covers whose target curves have only genus-zero irreducible components. Their target dual graphs are therefore connected stable weight-zero graphs Δ satisfying

$$b_1(\Delta) = g.$$

[1] Note that there is an error in the statement of Proposition 2.4 of [BCK24], so a modified version of 2.4 will be eventually needed for the description of the connected components.

Weighted dual graphs index the corresponding boundary types, in accordance with the general relationship between the boundary stratification of moduli spaces of curves and their tropicalizations [ACP15]. Passing to the dual graphs of admissible covers produces harmonic G-covers

$$\Gamma \longrightarrow \Delta,$$

as in the tropical and admissible-cover constructions of [BR11; LUZ19; LUZ24; El +24; Sou25].

For a fixed target graph, the dual graph of the admissible cover is encoded by a dilation class and a lifting class:

$$\eta \in H_1(\Delta, G), \qquad \bar{\lambda} \in H^1(\Delta, G_D).$$

The class η records local monodromy and stabilizers, while $\bar{\lambda}$ records how the fibers over the vertices are glued across the edges. These classes determine a harmonic G-cover

$$\Gamma \longrightarrow \Delta$$

up to the action of $\mathrm{Aut}(\Delta)$. This is the dilated-cohomology classification of abelian tropical covers developed in [LUZ19; LUZ24].

Realizability imposes an additional condition on the dilation class. In particular, each edge stabilizer must be generated by a single local monodromy element and is therefore cyclic. More generally, the local monodromies must satisfy the balancing relations at every vertex. These are precisely the graph-theoretic conditions arising from the balanced local model of an admissible cover [BR11; LUZ24].

A lift

$$\lambda \in H^1(\Delta, G)$$

of $\bar{\lambda}$ retains the information inherited from the full level structure. The cover is connected precisely when

$$K(\eta, \lambda) = G,$$

in agreement with the monodromy and spanning-tree connectedness criteria for abelian tropical covers [LUZ19; LUZ24; El +24; Sou25]. It belongs to the symplectic component corresponding to $\mathcal{M}_g(p)$ precisely when

$$\omega_{\eta,\lambda} = \omega_{\mathrm{std}}.$$

Thus the relevant combinatorial types are the $\mathrm{Aut}(\Delta)$-orbits of triples

$$(\Delta, \eta, \bar{\lambda})$$

that are realizable and connected and that admit a lift inducing the standard symplectic form.

The basic reason for the appearance of one homological class and one cohomological class is the splitting

$$H_1(X, \mathbb{F}_p) \cong H^1(\Delta, \mathbb{F}_p) \oplus H_1(\Delta, \mathbb{F}_p).$$

Under this splitting, the intersection pairing on the smooth fiber becomes the evaluation pairing

$$H^1(\Delta, \mathbb{F}_p) \times H_1(\Delta, \mathbb{F}_p) \longrightarrow \mathbb{F}_p.$$

This decomposition is compatible with the topological description of a degeneration by a graph of groups, which is also used in the study of Hurwitz spaces [BR11].

The same two types of data appear after abelianizing the fundamental group of the associated graph of groups:

$$\mathrm{Hom}\big(\pi_1(\Delta, \mathcal{G}), G\big) \cong H_1(\Delta, G) \oplus H^1(\Delta, G).$$

The $H_1(\Delta, G)$-summand records local monodromy, whereas the $H^1(\Delta, G)$-summand records the stable-letter or gluing data. The admissible-cover, tropical-cover, and graph-of-groups descriptions therefore encode the same monodromy information in different forms [BR11; LUZ19; LUZ24].

The gluing morphism turns this combinatorial classification into a boundary stratification. For each vertex v, the local monodromy tuple $\eta(v)$ determines a genus-zero Hurwitz space

$$\overline{\mathcal{H}}_{0,D(v),\eta(v)}.$$

The graph cover $\Gamma \to \Delta$ prescribes how the marked branches of these local covers are identified. Gluing them gives a surjective morphism

$$\prod_{v \in V(\Delta)} \overline{\mathcal{H}}_{0,D(v),\eta(v)} \longrightarrow \overline{\mathcal{H}}(\Delta, \eta, \bar{\lambda}).$$

This is the usual vertexwise clutching construction for admissible covers, reformulated in terms of dilation and lifting data [BR11; SZ20; LUZ24].

Consequently, the connected components of a boundary stratum are controlled by the connected components of the local genus-zero Hurwitz spaces, together with the finite identifications arising from automorphisms of the combinatorial datum. In the weight-zero setting considered here, the local component problem may be approached using the genus-zero results and related arguments of [BCK24]. The gluing map therefore separates the boundary component problem into a local geometric calculation and a finite combinatorial quotient.

The weight-zero boundary of $\mathcal{M}_g(p)$ is organized by the following hierarchy of data:

stable weight-zero target graph Δ
realizable dilation class η
lifting class $\bar{\lambda}$
connectedness condition $K(\eta, \lambda) = G$
symplectic condition $\omega_{\eta,\lambda} = \omega_{\text{std}}$
components of the associated local Hurwitz spaces.

The first line records the boundary type of the target [ACP15]; the next two lines classify the associated harmonic G-cover [LUZ19; LUZ24]; the fourth imposes connectedness [El +24; Sou25]; the fifth selects the symplectic component of the principal-cover moduli space [BR11; Sou25]; and the final line supplies the remaining geometric data through the admissible-cover gluing construction [BR11; SZ20; BCK24].

This framework connects the Hurwitz-stack perspective [BR11; SZ20] with the tropical-cover perspective [ACP15; LUZ19; LUZ24; El +24; Sou25]. It reduces the study of the weight-zero boundary of $\mathcal{M}_g(p)$ to a finite classification of realizable, connected, symplectic graph data followed by a collection of genus-zero Hurwitz-space calculations. In this way, harmonic graph covers provide the desired combinatorial data for stratifying the boundary of the moduli space of connected symplectic $(\mathbb{Z}/p)^{2g}$-covers.

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