D-Modules and Beilinson–Bernstein Localization

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Introduction

The theory of D-modules provides an algebraic framework for studying differential equations. Given a commutative k-algebra R one can define a ring of differential operators $D_{R/k}$. Similarly given a variety X there exists a sheaf of differential operators $\mathcal{D}_X$. A D-module is a module over such a ring or sheaf. They can be thought of as R-modules or $\mathcal{O}_X$-modules that also carry a compatible action of derivations. In this paper we introduce the necessary background to explain two important applications of D-modules: Bernstein-Sato theory, and Beilinson-Bernstein localization.

Given nonzero $f \in k[x_1, \ldots, x_n]$, the Bernstein-Sato polynomial of f, or b-function, is a nonzero monic polynomial $b_f(s) \in k[s]$ of minimal degree such that there exists a differential operator $P(s)$ satisfying

$$P(s) \cdot f^{s+1} = b_f(s) \cdot f^s.$$

In Part I, we use the theory of D-modules, particularly over the Weyl algebra, to show the existence of the Bernstein-Sato polynomial over a field k of characteristic zero. The proof works by showing a certain module over the Weyl algebra is holonomic, and therefore of finite length. This forces a descending chain of submodules to stabilize, which implies the existence of the b-function. One consequence of this existence is that the localization $k[x_1, \ldots, x_n, f^{-1}]$ is finitely generated as a module over A_n.

The idea behind this proof is not confined to the polynomial ring $k[x_1, \ldots, x_n]$. Mebkhout and Narváez-Macarro have shown the existence of the b-function for commutative, Noetherian, regular k-algebras A that satisfy three further conditions detailed in Section 2.1. Their result also generalizes the b-functions to exist for any module of minimal dimension over the differential operators $D_{A/k}$. For K a non-archimedean field of characteristic zero, their result applies in particular to the Tate algebra $K\langle x_1, \ldots, x_n\rangle$, and so to rigid analytic geometry.

This leads us to a result of Bitoun and Bode that applies the b-functions which we highlight in Section 3. Let $X = \mathrm{Sp}(A)$ be a smooth affinoid space with free tangent sheaf and $Z = \{f = 0\}$ a hypersurface, with $U = X - Z$ the complement. There exists a completed sheaf of differential operators $\widehat{\mathcal{D}}_X$ on X. Given a meromorphic connection on X with singularities along Z there are two natural completions, one over $\widehat{\mathcal{D}}_X$ and the other over $\widehat{\mathcal{D}}_U$, which is the $\widehat{\mathcal{D}}$-module pushforward of the meromorphic connection along $U \hookrightarrow X$. There is a comparison map θ between the two completions, and the result of Bitoun and Bode [BB21] proves that if the roots of the relevant b-functions satisfy an arithmetic condition known as positive type, then θ is an isomorphism. This isomorphism in particular gives a criterion for the $\widehat{\mathcal{D}}$-pushforward of the meromorphic connection to be coadmissible as $\widehat{\mathcal{D}}_X$-module. An example from [BB21] is also shown where the positive type condition is indeed necessary.

In Part II we turn to another application of D-modules, Beilinson-Bernstein localization, one of the foundational results in geometric representation theory. Let G be a complex semisimple algebraic group with Lie algebra $\mathfrak{g}$. The representation theory of $\mathfrak{g}$ is an important question studied by many mathematicians. The theorem of Beilinson and Bernstein gives a correspondence between the representations of $\mathfrak{g}$ and $\mathcal{D}_X$-modules, where $X = G/B$ is the flag variety. This allows the use of geometry to generate and study the representations.

We treat in this paper the untwisted case, where the correspondence is between representations of $\mathfrak{g}$ with trivial infinitesimal character and quasicoherent modules over $\mathcal{D}_X$. The proof can be split into two steps. We show the first step, which is an isomorphism

$$U(\mathfrak{g})/(Z(\mathfrak{g})_+) \xrightarrow{\sim} \Gamma(X, \mathcal{D}_X).$$

The ideal $(Z(\mathfrak{g})_+)$ is that generated by the elements of the center with no constant term. The map above arises by differentiating the action of G on the structure sheaf $\mathcal{O}_X$. The proof of this step involves studying the associated graded map, which can be shown to equal the map induced on functions by the Springer resolution. The second step, which we do not show in this paper, proves the flag variety is $\mathcal{D}$-affine, in the sense that taking global sections gives an equivalence between quasicoherent $\mathcal{D}_X$-modules and $\Gamma(X, \mathcal{D}_X)$-modules. Together, these two results imply the theorem of Beilinson and Bernstein.

The two parts of the paper intersect at the end in Section 6.6. We present a result of Ardakov and Wadsley that establishes a Beilinson-Bernstein correspondence when G is an algebraic group over a non-archimedean field. The sheaf of differential operators is then considered over the rigid analytic flag variety. We finish by applying the results of Bitoun and Bode to the example of $G = \mathrm{SL}_2$.

This paper is expository. Section 1 follows closely the treatment of [Cou95] on the subject. The rest of Part I is based on the papers of Mebkhout and Narváez-Macarro [MN91] and Bitoun and Bode [BB21]. Part II draws on a variety of sources for the general theory of Lie algebras as well as the proof of Beilinson-Bernstein localization. All of the algebraic group and Lie algebra theory needed can be found in [Bor91], [Hum72], and [FH91]. Many of the proofs and facts building to Beilinson-Bernstein localization come from [HTT08] and [CG10].

Section 1 assumes only a basic knowledge of algebra. Section 2 uses some elementary knowledge of homological algebra, and familiarity with some rigid analytic geometry is useful for Section 3. Part II assumes some experience with algebraic geometry and sheaves. Knowledge of algebraic groups and Lie algebras will help with motivation, but we introduce the necessary theory to understand the proofs, citing some results.

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Part I. Bernstein-Sato Polynomials and $\widehat{\mathcal{D}}$-modules

1. Bernstein-Sato Theory of Polynomial Rings

1.1. **The Weyl algebra.** Fix a field k of characteristic 0, and consider the polynomial ring $k[x_1, \ldots, x_n]$. Let x_i also denote the element of $\mathrm{End}_k(k[x_1, \ldots, x_n])$ that is multiplication by x_i. Let $\partial_i \in \mathrm{End}_k(k[x_1, \ldots, x_n])$ be differentiation with respect to x_i, i.e. $\partial_i(f) = \frac{\partial f}{\partial x_i}$.

Definition 1.1. The Weyl algebra A_n or $A_n(k)$, is the subalgebra of $\mathrm{End}_k(k[x_1, \ldots, x_n])$ generated by the x_i and ∂_i for $i = 1, 2, \ldots, n$.

Let $[\cdot, \cdot]$ denote the commutator bracket, i.e. for P, Q in any ring, $[P, Q] = PQ - QP$. From the commuting of partial derivatives and of polynomial multiplication we have that for all $1 \leq i, j \leq n$ that

$$[x_i, x_j] = [\partial_i, \partial_j] = 0.$$

If $i \neq j$, then x_j commutes with ∂_i so $[\partial_i, x_j] = 0$. The last key relation of the Weyl algebra comes from the product rule. Observe that if $f \in k[x_1, \ldots, x_n]$, then

$$\partial_i x_i(f) = f + x_i \frac{\partial f}{\partial x_i},$$

thus

$$[\partial_i, x_i] = \partial_i x_i - x_i \partial_i = 1.$$

So for any $1 \leq i, j \leq n$, $[\partial_i, x_j] = \delta_{ij}$.

These relations in fact determine the Weyl algebra. To be precise, let $k\{x_1, \ldots, x_n, y_1, \ldots, y_n\}$ be the free algebra over k in $2n$ variables. Let I be the two-sided ideal generated by $[x_i, x_j], [y_i, y_j]$ and $[y_i, x_j] - \delta_{ij}$ for all $1 \leq i, j \leq n$. Then the k-algebra morphism $\varphi : k\{x_1, \ldots, x_n, y_1, \ldots, y_n\} \to A_n(k)$ defined by $x_i \mapsto x_i$ and $y_j \mapsto \partial_j$ is surjective and has kernel I. In particular,

$$k\{x_1, \ldots, x_n, y_1, \ldots, y_n\}/I \cong A_n(k).$$

1.2. **Standard basis.** The commutator relations of the Weyl algebra lead to a nice basis. Given some word in terms of the generators of A_n we can use the relation that $[\partial_i, x_j] = \delta_{ij}$ to shift all of the ∂'s to the right side of the x's, and the resulting expression will be a linear combination of terms of the form $x_1^{\alpha_1} \cdots x_n^{\alpha_n} \partial_1^{\beta_1} \cdots \partial_n^{\beta_n}$. To represent this basis we will use multi-index notation. If $\alpha \in \mathbb{N}^n$ then $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, and the similar definition for ∂^α.

Proposition 1.2. *The set $B = \{x^\alpha \partial^\beta \mid \alpha, \beta \in \mathbb{N}^n\}$ is a basis for A_n.*

Proof. From the above discussion it is clear that B spans A_n. Now we will show independence.

Consider some linear combination $\xi = \sum_{\alpha,\beta \in \mathbb{N}^n} c_{\alpha\beta} x^\alpha \partial^\beta$ such that at least one coefficient is nonzero. We will show $\xi \neq 0$.

Pick a multi-index $\beta' \in \mathbb{N}^n$ such that there exists α' so that $c_{\alpha'\beta'} \neq 0$, but for all $|\beta| < |\beta'|$ and α, $c_{\alpha\beta} = 0$. Then we have

$$\xi(x^{\beta'}) = \sum_{\alpha,\beta \in \mathbb{N}^n} c_{\alpha\beta} x^\alpha \partial^\beta (x^{\beta'})$$

If $|\beta| > |\beta'|$, then we differentiate too many times and $\partial^\beta(x^{\beta'}) = 0$. If $|\beta| < |\beta'|$, then we know already $c_{\alpha\beta} = 0$ for all α. If $|\beta| = |\beta'|$, then

$$\partial^\beta (x^{\beta'}) = \partial_1^{\beta_1}(x_1^{\beta_1'}) \cdots \partial_n^{\beta_n}(x_n^{\beta_n'})$$

is nonzero if and only if $\beta_i \leq \beta_i'$ for each i, which is only possible if $\beta = \beta'$. In this case then $\partial^{\beta'}(x^{\beta'}) = \beta'! = \beta_1'! \cdots \beta_n'!$. Thus

$$\xi(x^{\beta'}) = \beta'! \sum_{\alpha \in \mathbb{N}^n} c_{\alpha\beta'} x^\alpha.$$

By the definition of β' we know at least one coefficient above is nonzero, so $\xi(x^{\beta'}) \neq 0$, proving $\xi \neq 0$. Hence B is linearly independent and therefore a basis. □

Now we will see the Weyl algebra as a specific instance of what is known as the ring of differential operators associated to a commutative k-algebra R.

1.3. **Ring of differential operators.** Given a field k and commutative k-algebra R we can define the ring of differential operators of R as a sub-algebra $D_{R/k}$ of $\mathrm{End}_k(R)$. The definition is inductive, first define $D^0_{R/k} = R \subseteq \mathrm{End}_k(R)$, where we view $r \in R$ as an endomorphism by the map $a \mapsto ar$. Now for $n \geq 1$, define

$$D^n_{R/k} = \{D \in \mathrm{End}_k(R) \mid \forall f \in R, [D, f] \in D^{n-1}_{R/k}\}. \tag{1.1}$$

It is not hard to check inductively that $D^0_{R/k} \subseteq D^1_{R/k} \subseteq \dots$. We then define $D_{R/k} = \bigcup_{n\geq 0} D^n_{R/k}$. We say that $P \in D_{R/k}$ has order m if $P \in D^m_{R/k}$ but $P \notin D^\ell_{R/k}$ for any $\ell < m$. Thus $D^m_{R/k}$ is the set of operators of order $\leq m$.

Our first goal is to show that $D_{R/k}$ is a k-algebra, although generally noncommutative. The commutator is k-linear in both entries, thus we see inductively immediately that $D^n_{R/k}$ is a k-vector space. In fact if $D \in D^n_{R/k}$ and $a \in R$, then for all $b \in R$,

$$[aD, b] = aDb - baD = a[D, b],$$

and $[D, b] \in D^{n-1}_{R/k}$ so inductively we see as well $aD \in D^n_{R/k}$, so $D^n_{R/k}$ is a R-module.

Note $D_{R/k}$ is closed under addition as if $D \in D^n_{R/k}$ and $D' \in D^m_{R/k}$ where $m > n$, then it also holds that $D \in D^m_{R/k}$, then $D + D' \in D^m_{R/k} \subset D_{R/k}$.

The following shows that $D_{R/k}$ is in fact a filtered ring, and one such that its grading is commutative. For the unaware reader we will make these definitions precise in Section 1.5.

Theorem 1.3.

(1) $D^m_{R/k} \cdot D^n_{R/k} \subseteq D^{m+n}_{R/k}$
(2) $[D^m_{R/k}, D^n_{R/k}] \subseteq D^{m+n-1}_{R/k}$.

Proof. (1) We'll induct on $m + n$ for both. When $m + n = 0$, the statement of (1) is just $R \cdot R \subseteq R$. For (2) we know that $[D^0_{R/k}, D^0_{R/k}] = \{0\}$, and our convention is that $D^m_{R/k} = \{0\}$ whenever $m < 0$. Now assume that both results hold whenever $m + n < \ell$ for some $\ell \in \mathbb{N}$, and consider some $D \in D^m_{R/k}, D' \in D^n_{R/k}$ where $m + n = \ell$. Then for any $f \in R$,

$$[DD', f] = D[D', f] + [D, f]D'.$$

But $[D, f] \in D^{m-1}_{R/k}, [D', f] \in D^{n-1}_{R/k}$, hence by the induction we have that $D[D', f], [D, f]D' \in D^{m+n-1}_{R/k}$, so $[DD', f] \in D^{m+n-1}_{R/k}$. Thus $DD' \in D^{m+n}_{R/k}$, proving (1).

Jacobi's Identity which holds for the commutator bracket in any ring implies that

$$[[D, D'], f] = [[D, f], D'] - [[D', f], D].$$

But $[D, f] \in D^{m-1}_{R/k}$ and $[D', f] \in D^{n-1}_{R/k}$, thus by the inductive hypothesis, both $[[D, f], D']$ and $[[D', f], D]$ are in $D^{m+n-2}_{R/k}$. Thus $[D, D'] \in D^{m+n-1}_{R/k}$. □

As a confirmation that this is the right definition, in the case of $R = k[x_1, \dots, x_n]$, it turns out that $D_{R/k}$ is in fact equal to A_n, the Weyl algebra. The proof of this fact uses induction and a number of lemmas and identities, all of which can be found in [Cou95, Chapter 3].

As an example, in the case of $R = k[x]$, consider the surjective homomorphism $\psi : k\{y_1, y_2\} \to D_{k[x]/k}$ defined by $y_1 \mapsto x$, and $y_2 \mapsto \partial \in D^1_{k[x]/k}$, the derivative operator. Then by the relations from the product rule, $[y_2, y_1] - 1$ is contained in the kernel of ψ, so there is an induced map

$$\overline{\psi} : k\{y_1, y_2\}/([y_2, y_1] - 1) = A_1(k) \to D_{k[x]/k}.$$

This map is an isomorphism.

1.4. **Degree of an operator.** From the definition of the ring of differential operators we see that given $D \in A_n$, there is a natural notion of the order of D defined as the largest integer m such that $D \in D^m_{k[x]/k}$. In the context of the Weyl algebra there is a second notion of the size of the operator that we will call the *degree* of the operator.

Definition 1.4. Let $D \in A_n$ be written in canonical form as $D = \sum_{\alpha,\beta \in \mathbb{N}^n} c_{\alpha\beta} x^\alpha \partial^\beta$. Then the degree of D is

$$\deg(D) = \max\{|\alpha| + |\beta| : c_{\alpha\beta} \neq 0\}.$$

If $D = 0$ define $\deg(D) = -\infty$.

For example the degree of $x_1^2\partial_2$ is three, and the degree of $\partial_1 x_1$ is two because we can rewrite it as $x_1\partial_1 + 1$. The degree satisfies the following nice properties.

Proposition 1.5. *Let $D, D' \in A_n$, then*

(1) $\deg(D + D') \leq \max\{\deg D, \deg D'\}$,
(2) $\deg(DD') = \deg(D) + \deg(D')$,
(3) $\deg([D, D']) \leq \deg D + \deg D' - 2$.

Proof. If we write D and D' in canonical form, then (1) is clear.

It suffices to prove (3) for standard monomials. If we have proven the result for monomials, then the bilinearity of the commutator bracket and part (1) implies the general result. First note if D or D' is zero, then the inequality holds since $\deg([D, D']) = -\infty$. If $\deg D$ or $\deg D' = 0$, then again $[D, D'] = 0$, so the inequality holds. Since nonzero scalars do not affect degree and the commutator is k-linear, we can now assume $D = x^\alpha\partial^\beta$ and $D' = x^\gamma\partial^\eta$ for nonzero multi-indices $(\alpha, \beta), (\gamma, \eta) \in \mathbb{N}^{2n}$. Whenever the commutation relation $[\partial_i, x_j] = \delta_{ij}$ is used to replace $\partial_i x_j$ with $x_j\partial_i$, the correction term δ_{ij} has degree two less than the corresponding degree two term. Thus applying these relations to make swaps, we see that

$$\partial^\beta x^\alpha = x^\alpha \partial^\beta + R_{\alpha,\beta},$$

where $\deg R_{\alpha,\beta} \leq |\alpha| + |\beta| - 2$. Then

$$\begin{aligned} DD' &= x^\alpha \partial^\beta x^\gamma \partial^\eta \\ &= x^\alpha (x^\gamma \partial^\beta + R_{\gamma,\beta})\partial^\eta \\ &= x^{\alpha+\gamma}\partial^{\beta+\eta} + x^\alpha R_{\gamma,\beta}\partial^\eta. \end{aligned}$$

Similarly then $D'D = x^{\alpha+\gamma}\partial^{\beta+\eta} + x^\gamma R_{\alpha,\eta}\partial^\beta$. Then

$$[D, D'] = x^\alpha R_{\gamma,\beta}\partial^\eta - x^\gamma R_{\alpha,\eta}\partial^\beta.$$

Writing $R_{\gamma,\beta}$ and $R_{\alpha,\eta}$ in canonical form, each of the summands above are in canonical form and have degree less than or equal to

$$|\alpha| + |\gamma| + |\beta| + |\eta| - 2 = \deg D + \deg D' - 2,$$

hence $\deg([D, D']) \leq \deg D + \deg D' - 2$.

To prove (2), consider arbitrary differential operators

$$D = \sum_{\alpha,\beta} c_{\alpha\beta} x^\alpha \partial^\beta, \quad D' = \sum_{\gamma,\eta} d_{\gamma\eta} x^\gamma \partial^\eta.$$

We can then define polynomials which capture the top degree part of D and D'. If $\deg D = m, \deg D' = r$, define

$$p_D(X,Y) = \sum_{|\alpha|+|\beta|=m} c_{\alpha\beta} X^\alpha Y^\beta, \quad p_{D'}(X,Y) = \sum_{|\gamma|+|\eta|=r} d_{\gamma\eta} X^\gamma Y^\eta,$$

as elements of $k[X_1, \ldots, X_n, Y_1, \ldots, Y_n]$. If we identify the monomials $X^\epsilon Y^\delta$ with $x^\epsilon \partial^\delta$, then the calculation that we made above for monomials shows that the degree $m+r$ part of DD' is precisely the product $p_D(X,Y)p_{D'}(X,Y)$. Since $k[X,Y]$ is a domain, the degree $m+r$ part of DD' is nonzero, so $\deg(DD') \geq m+r$. Since $\deg(DD') \leq m+r$ was already clear, we have equality. □

Now we can let $B_m = \{D \in A_n \mid \deg(D) \leq m\}$ be the k-vector space of operators of degree $\leq m$. Then $A_n = \bigcup_{i\geq 0} B_i$ and by Proposition 1.5 we have $B_i B_j \subseteq B_{i+j}$. This gives what is called a filtration on A_n. We will give the general definition shortly, but first we need the definition of a graded ring.

1.5. **Graded and filtered rings.**

Definition 1.6. A k-algebra R is graded if there exist k-vector spaces $\{R_i\}_{i\geq 0}$ such that $R = \bigoplus_{i\geq 0} R_i$, and $R_i R_j \subseteq R_{i+j}$ for all $i, j \geq 0$.

In the above setting we call the R_i the homogeneous components of R, and elements of R_i are called homogeneous of degree i.

Definition 1.7. If $S = \bigoplus_{i\geq 0} S_i$ is another graded k-algebra, then a k-algebra homomorphism $\phi : R \to S$ is graded if $\phi(R_i) \subseteq S_i$ for each $i \geq 0$.

The filtration $\{B_i\}_{i\geq 0}$ that we mentioned above is an example of the following.

Definition 1.8. A filtration of a k-algebra R is a family of k-vector spaces $\mathcal{F} = \{F_i\}_{i\geq 0}$ such that

(1) $F_0 \subseteq F_1 \subseteq \cdots \subseteq R$,
(2) $R = \bigcup_{i\geq 0} F_i$,
(3) $F_i \cdot F_j \subseteq F_{i+j}$.

Then we say R is a filtered algebra, and for convention let $F_j = \{0\}$ for $j < 0$.

The filtration $\mathcal{B} = \{B_i\}_{i\geq 0}$ on A_n is known as the Bernstein filtration, and will be key to studying A_n-modules.

The power of filtrations come from the fact that given a filtered algebra R, we can construct an associated graded algebra denoted $\mathrm{gr}(R)$. The associated graded algebra is often easier to study, and the hope is that properties can then be passed back to prove results about the original filtered algebra. Now we will show how $\mathrm{gr}(R)$ is constructed.

Let R be a k-algebra with a filtration $\mathcal{F} = \{F_i\}_{i\geq 0}$. For each $i \geq 1$ let σ_i be the projection $\sigma_i : F_i \to F_i/F_{i-1}$. Then as a vector space we define

$$\mathrm{gr}^{\mathcal{F}}(R) := \bigoplus_{i\geq 0} F_i/F_{i-1}.$$

Given two homogeneous elements $\sigma_i(a), \sigma_j(b)$, define multiplication by

$$\sigma_i(a)\sigma_j(b) = \sigma_{i+j}(ab).$$

To see that this is well-defined, let $a' \in F_i, b' \in F_j$ be such that $\sigma_i(a) = \sigma_i(a')$ and $\sigma_j(b) = \sigma_j(b')$. Then

$$ab - a'b' = a(b - b') + (a - a')b',$$

and $b - b' \in F_{j-1}$, $a - a' \in F_{i-1}$, so $ab - a'b' \in F_{i+j-1}$, proving $\sigma_{i+j}(ab) = \sigma_{i+j}(a'b')$. This defines multiplication on homogeneous elements, then extending linearly defines multiplication on all of $\mathrm{gr}^{\mathcal{F}}(R)$, turning it into a graded algebra.

As an example and as a critical result, let us calculate the associated graded ring to A_n with respect to the Bernstein filtration $\mathcal{B}$.

Theorem 1.9. *The graded algebra $\mathrm{gr}^{\mathcal{B}}(A_n)$ is isomorphic to $k[y_1, \ldots, y_{2n}]$, the polynomial ring in $2n$ variables.*

Proof. We first claim $\mathrm{gr}^{\mathcal{B}}(A_n)$ is commutative. Consider $\sigma_i(d)$ and $\sigma_j(d')$ homogeneous elements of degree i and j, with $d \in B_i, d' \in B_j$. Then by (3) of Proposition 1.5, $[d, d'] \in B_{i+j-2}$. Applying σ_{i+j} we have $\sigma_{i+j}(dd') = \sigma_{i+j}(d'd)$, so $\sigma_i(d)\sigma_j(d') = \sigma_j(d')\sigma_i(d)$. Thus multiplication by homogeneous elements is commutative, and every element is a finite sum of homogeneous elements, thus $\mathrm{gr}^{\mathcal{B}}(A_n)$ is commutative.

Let $\sigma_1(x_i) = z_i$ and $\sigma_1(\partial_i) = z_{n+i}$. We will show the z_i generate the algebra. Any homogeneous element in $\mathrm{gr}^{\mathcal{B}}(A_n)$ is of the form $\sigma_m(d)$ where $d \in B_m$. Then d is a linear combination of monomials of the form $x^\alpha \partial^\beta$ where $|\alpha| + |\beta| \leq m$. Since σ_m kills terms in B_{m-1}, $\sigma_m(d)$ is a linear combination of terms of the form $\sigma_m(x^\alpha \partial^\beta)$ where $|\alpha| + |\beta| = m$. Then

$$\sigma_m(x^\alpha \partial^\beta) = \sigma_1(x_1)^{\alpha_1} \cdots \sigma_1(x_n)^{\alpha_n} \sigma_1(\partial_1)^{\beta_1} \cdots \sigma_1(\partial_n)^{\beta_n} = z_1^{\alpha_1} \cdots z_n^{\alpha_n} z_{n+1}^{\beta_1} \cdots z_{2n}^{\beta_n}.$$

Thus $\sigma_m(d)$ is a polynomial in the z_i. Now every element of $\mathrm{gr}^{\mathcal{B}}(A_n)$ is a finite sum of homogeneous terms thus in fact $z_1, \ldots, z_{2n}$ generate $\mathrm{gr}^{\mathcal{B}}(A_n)$.

Now since $\mathrm{gr}^{\mathcal{B}}(A_n)$ is commutative, there exists a surjective algebra homomorphism

$$\phi : k[y_1, \ldots, y_{2n}] \to \mathrm{gr}^{\mathcal{B}}(A_n)$$

defined by $y_i \mapsto z_i$. This is also a graded k-algebra morphism since the y_i which are degree 1 map to the z_i which are degree 1.

Lastly we show ϕ is injective. Consider a polynomial $f \in k[y_1, \ldots, y_{2n}]$ such that $\phi(f) = 0$. If we consider a homogeneous decomposition $f = f_0 + \cdots + f_\ell$, then $\phi(f_0), \ldots, \phi(f_\ell)$ are all of different degree in $\mathrm{gr}^{\mathcal{B}}(A_n)$, thus $\phi(f) = 0$ if and only if each $\phi(f_i) = 0$. Thus we can assume f is homogeneous, so write

$$f(y_1, \ldots, y_{2n}) = \sum_{|\alpha|+|\beta|=m} c_{\alpha\beta} y_1^{\alpha_1} \cdots y_n^{\alpha_n} y_{n+1}^{\beta_1} \cdots y_{2n}^{\beta_n}.$$

Let $d = \sum_{|\alpha|+|\beta|=m} c_{\alpha\beta} x^\alpha \partial^\beta \in A_n$, then $\phi(f) = \sigma_m(d)$, so $\phi(f) = 0$ if and only if $d \in B_{m-1}$. But d is a linear combination of monomials $x^\alpha \partial^\beta$ such that $|\alpha| + |\beta| = m$, so the only way $d \in B_{m-1}$ is if all coefficients $c_{\alpha\beta}$ are zero, in which case $f = 0$. Hence ϕ is injective and thus an isomorphism. $\square$

Remark 1.10. As said before, we can also consider the order filtration on A_n, filtered by $A_n = \bigcup_{i \geq 0} D^i_{k[x_1,\ldots,x_n]/k}$. Letting $R = k[x_1, \ldots, x_n]$ we have

$$D^i_{R/k} = \sum_{|\beta| \leq i} R\partial^\beta.$$

With a proof similar to that above, one can prove in this case that the associate graded ring $\mathrm{gr}(A_n)$ is isomorphic to $(k[x_1,\dots,x_n])[\xi_1,\dots,\xi_n]$, the polynomial ring in n variables over $k[x_1,\dots,x_n]$. As a ring this is isomorphic to the polynomial ring in $2n$ variables as is the case with the Bernstein filtration. The difference is that under this grading the x_i are degree 0 and the ξ_i have degree 1.

Definition 1.11. Given two filtered algebras $R = \bigcup_{i\geq 0} F_iR$ and $S = \bigcup_{i\geq 0} F_iS$, an algebra homomorphism $\phi : R \to S$ is filtered if it preserves the filtration, i.e. $\phi(F_iR) \subseteq F_iS$.

If we have a filtered map $\phi : R \to S$ then we get induced maps

$$\phi_i : F_iR/F_{i-1}R \to F_iS/F_{i-1}S.$$

Taking the infinite direct sum we obtain a graded map

$$\mathrm{gr}\,\phi : \mathrm{gr}\,R \to \mathrm{gr}\,S.$$

Proposition 1.12. *If the graded map* $\mathrm{gr}\,\phi : \mathrm{gr}\,R \to \mathrm{gr}\,S$ *is an isomorphism then* $\phi : R \to S$ *is an isomorphism.*

Proof. Take nonzero $r \in R$, and let $n \geq 0$ be the smallest integer such that $r \in F_nR$. Then $\sigma_n(r) \in F_nR/F_{n-1}R$ is nonzero, so $(\mathrm{gr}\,\phi)(\sigma_n(r)) \in F_nS/F_{n-1}S$ is nonzero. By definition though, $(\mathrm{gr}\,\phi)(\sigma_n(r)) = \sigma_n(\phi(r))$, so $\phi(r) \neq 0$, proving ϕ is injective.

For surjectivity we induct to show $F_nS = \phi(F_nR)$. For $n = 0$, the map $\phi|_{F_0R} : F_0R \to F_0S$ is an isomorphism, proving the claim. Assume now $F_{n-1}S = \phi(F_{n-1}R)$. Since $F_nR/F_{n-1}R \to F_nS/F_{n-1}S$ is an isomorphism, for any $s \in F_nS$, there exists $r \in F_nR$ such that

$$\phi(r) \equiv s \pmod{F_{n-1}S}.$$

Take $s' \in F_{n-1}(S)$ such that $\phi(r) - s = s'$. By the inductive hypothesis there exists $r' \in F_{n-1}R$ such that $\phi(r') = s'$. Then

$$\phi(r - r') = \phi(r) - \phi(r') = s,$$

so $s \in \phi(F_nR)$, completing the induction. □

1.6. **Graded and filtered modules.** When studying modules over a filtered or graded ring R we can consider filtrations and grading structures on the module that behave well with the structure of R. Formally we have the following definitions.

Definition 1.13. Let R be a graded k-algebra with $R = \bigoplus_{i\geq 0} R_i$. A left R-module M is a graded module if there exists k-vector space $\{M_i\}_{i\geq 0}$ such that $M = \bigoplus_{i\geq 0} M_i$, and $R_iM_j \subseteq M_{i+j}$ for all $i, j \geq 0$.

The M_i are called the homogeneous components of degree i. Analogously we can define filtered modules.

Definition 1.14. Let R be a k-algebra with filtration $\mathcal{F} = \{F_i\}_{i\geq 0}$. A left R-module M is a filtered module with respect to $\mathcal{F}$ if there exists a family of k-vector spaces $\Gamma = \{\Gamma_i\}_{i\geq 0}$ such that

(1) $\Gamma_0 \subseteq \Gamma_1 \subseteq \cdots \subseteq M$,
(2) $\bigcup_{i\geq 0} \Gamma_i = M$,
(3) $F_i\Gamma_j \subseteq \Gamma_{i+j}$ for all $i, j \geq 0$.

Although it is not part of the standard definition, when $R = A_n$ and $\mathcal{F} = \mathcal{B}$ is the Bernstein filtration, we require

(4) Γ_j is a finite dimensional k-vector space for each $j \geq 0$.

Example 1.15. Consider A_n with the Bernstein filtration $\mathcal{B}$. Then $\mathcal{B}$ gives a filtration of A_n as a left module over A_n. Conditions (1), (2) and (3) are immediate, and to see condition (4) remember that we have a finite basis for B_j given by $\{x^\alpha \partial^\beta \mid \alpha, \beta \in \mathbb{N}^n, |\alpha| + |\beta| \leq j\}$.

Example 1.16. Consider the polynomial ring $k[x_1, \ldots, x_n]$ as a left A_n-module. Then $C = \{C_i\}_{i \geq 0}$ is a filtration of $k[x_1, \ldots, x_n]$ with respect to $\mathcal{B}$, where C_i is the k-vector space of polynomials of degree $\leq i$. Conditions (1), (2) and (4) are clear. To see condition (3), we can consider just monomial operators. If $|\alpha| + |\beta| \leq i$ and $f \in k[x_1, \ldots, x_n]$ is of degree j, then

$$\deg(x^\alpha \partial^\beta \cdot f) \leq |\alpha| + \deg(\partial^\beta \cdot f) \leq |\alpha| + \deg(f) \leq i + j.$$

Given a filtered algebra R we constructed an associated graded algebra, and we can make a similar construction for a filtered module M. Assume the notation from Definition 1.14, we will construct a $\mathrm{gr}^{\mathcal{F}} R$-module denoted $\mathrm{gr}^{\Gamma} M$. As a k-vector space define

$$\mathrm{gr}^{\Gamma} M = \bigoplus_{i \geq 0} \Gamma_i / \Gamma_{i-1}.$$

Denote the projection maps $\mu_i : \Gamma_i \to \Gamma_i/\Gamma_{i-1}$. If $a \in F_j$ and $m \in \Gamma_i$, define the action of $\mathrm{gr}^{\mathcal{F}} R$ on $\mathrm{gr}^{\Gamma} M$ by

$$\sigma_j(a) \cdot \mu_i(m) = \mu_{i+j}(am).$$

A simple check similar to the one we did earlier will show that this operation is well defined. This defines the scalar action on homogeneous elements, which we can then extend linearly turning $\mathrm{gr}^{\Gamma} M$ into a $\mathrm{gr}^{\mathcal{F}} R$-module. If $m \in \Gamma_i \backslash \Gamma_{i-1}$, then the formal symbol of m means $\mu_i(m) \in \mathrm{gr}^{\Gamma} M$.

1.7. **Induced filtrations.** Let M be a left A_n-module with a filtration Γ with respect to $\mathcal{B}$. If $N \subseteq M$ is a submodule, there turn out to be nice induced filtrations on both N and M/N.

To define the induced filtration on N, let $\Gamma' = \{\Gamma'_i\}_{i \geq 0}$ where $\Gamma'_i = \Gamma_i \cap N$. It is clear this is a filtration on N. Furthermore if we consider the composition of maps

$$\Gamma_i \cap N \hookrightarrow \Gamma_i \twoheadrightarrow \Gamma_i/\Gamma_{i-1},$$

the kernel of this map is $\Gamma_{i-1} \cap N$. Hence we have naturally induced k-linear maps

$$\nu_i : \Gamma_i \cap N / \Gamma_{i-1} \cap N \to \Gamma_i / \Gamma_{i-1}.$$

Combining these gives an injective k-linear map $\nu : \mathrm{gr}^{\Gamma'} N \to \mathrm{gr}^{\Gamma} M$. One can check this map is in fact a homomorphism of $\mathrm{gr}^{\mathcal{B}} A_n$-modules by confirming this for homogeneous elements.

For M/N, let $\Gamma''_i = \Gamma_i/\Gamma_i \cap N$. Then there are natural inclusions $\Gamma''_i \hookrightarrow M/N$, and in these identifications $\Gamma''_0 \subseteq \Gamma''_1 \subseteq \cdots \subseteq M/N$. It is not hard to see that the other properties are met for Γ'' to be a filtration on M/N.

Then

$$\Gamma''_i/\Gamma''_{i-1} = (\Gamma_i/\Gamma_i \cap N)/(\Gamma_{i-1}/\Gamma_{i-1} \cap N) \cong \Gamma_i/(\Gamma_i \cap N + \Gamma_{i-1}).$$

Then let

$$\pi_i : \Gamma_i/\Gamma_{i-1} \to \Gamma''_i/\Gamma''_{i-1}$$

be the projection map. Combining all of these maps gives a surjective k-linear map

$$\pi : \mathrm{gr}^{\Gamma} M \to \mathrm{gr}^{\Gamma''}(M/N).$$

Once again it is not hard to check π is in fact a $\mathrm{gr}^{\mathcal{B}} A_n$-module homomorphism. The fact that we have short exact sequences

$$0 \longrightarrow \Gamma'_i/\Gamma'_{i-1} \xrightarrow{\nu_i} \Gamma_i/\Gamma_{i-1} \xrightarrow{\pi_i} \Gamma''_i/\Gamma''_{i-1} \longrightarrow 0$$

for all i, implies that ν and π form a short exact sequence

$$0 \longrightarrow \mathrm{gr}^{\Gamma'} N \xrightarrow{\nu} \mathrm{gr}^{\Gamma} M \xrightarrow{\pi} \mathrm{gr}^{\Gamma''}(M/N) \longrightarrow 0.$$

1.8. **Noetherian rings and modules.**

Definition 1.17. Let R be a ring. An R-module M is Noetherian if every increasing chain of submodules

$$N_1 \subseteq N_2 \subseteq \cdots$$

eventually stabilizes, that is there exists some integer m such that $N_m = N_{m+1} = \cdots$.

Theorem 1.18 (Properties of Noetherian Modules)**.** *Let M be a left R-module and N and N' are submodules.*

(1) *M is Noetherian if and only if every submodule of M is finitely generated.*
(2) *M is Noetherian if and only if N and M/N are Noetherian.*
(3) *If $M = N + N'$ and N and N' are Noetherian, then M is Noetherian.*

Proof. These are all standard results, and proofs can be found at [Cou95, 8.1]. □

Definition 1.19. A ring R is left Noetherian if R is Noetherian as a left R-module over itself.

Theorem 1.20 (Hilbert's Basis Theorem)**.** *If R is a commutative Noetherian ring, then $R[x]$ is Noetherian.*

Proof. Once again a proof is given at [Cou95, Theorem 8.2.1]. □

Inductively this shows that $R[x_1, \ldots, x_n]$ is Noetherian. In particular, polynomial rings of finite variables over k are Noetherian.

This next result justifies the study of the associated graded module of a filtered module, showing Noetherianity descends through grading. This will allow us in particular to show A_n is Noetherian.

Theorem 1.21. *Let M be a left A_n-module with a filtration Γ with respect to the Bernstein filtration $\mathcal{B}$ on A_n. If $\mathrm{gr}^{\Gamma} M$ is Noetherian as a $\mathrm{gr}^{\mathcal{B}} A_n$ module, then M is a Noetherian A_n-module.*

Proof. Let N be any submodule of M, let Γ' be the induced filtration on N defined as $\Gamma' = \{\Gamma_i \cap N\}_{i \geq 0}$. Then from section 1.7, $\mathrm{gr}^{\Gamma'} N \subseteq \mathrm{gr}^{\Gamma} M$, so $\mathrm{gr}^{\Gamma'} N$ is finitely generated. We can always find a homogeneous generating set, so let $n_1, \ldots, n_t \in N$ be such that $n_i \in \Gamma_{s_i}$ and $\mu_{s_1}(n_1), \ldots, \mu_{s_t}(n_t)$ generate $\mathrm{gr}^{\Gamma'} N$. Let $s = \max(s_1, \ldots, s_t)$, and we will show N is generated by Γ'_s.

Suppose Γ'_s doesn't generate N. Let ℓ be the smallest integer such that there exists $u \in \Gamma'_\ell$ such that u is not a A_n-linear combination of elements in Γ'_s. Clearly $\ell > s$. Since the $\mu_{s_i}(n_i)$ generate $\mathrm{gr}^{\Gamma'} N$, there exists $r_i \in B_{\ell - s_i}$ such that

$$\mu_\ell(u) = \sum_{i=1}^{t} \sigma_{\ell - s_i}(r_i)\mu_{s_i}(n_i) = \mu_\ell\left(\sum_{i=1}^{t} r_i n_i\right).$$

Thus $u - \sum_{i=1}^{t} r_i n_i \in \Gamma'_{\ell-1}$. Then by the minimality of ℓ $u - \sum_{i=1}^{t} r_i n_i$ is an A_n- linear combination of elements of Γ'_s. But also each n_i is in Γ'_s, so u must be an A_n-linear combination of elements of Γ'_s, a contradiction.

Thus Γ'_s generates N. By condition (4) in Definition 1.14, Γ'_s has finite dimension as a vector space over k, and choosing a finite basis gives a generating set for the A_n-module N. Hence M is Noetherian. □

Corollary 1.22. *A_n is left Noetherian.*

Proof. Consider A_n as a left A_n-module with the Bernstein filtration. Then the associated graded module is just $\mathrm{gr}^{\mathcal{B}} A_n$ viewed as a left module over itself. By Theorem 1.9 this is the polynomial ring in $2n$ variables over k, which is Noetherian. Thus applying Theorem 1.21 gives that A_n is left Noetherian. □

The following result tells us that Noetherian modules over a Noetherian ring R are just modules that are finitely generated.

Proposition 1.23. *Let R be a left Noetherian ring. Then a left R-module M is Noetherian if and only if it is finitely generated.*

Proof. The direction Noetherian $\implies$ finitely generated always holds by (1) of Theorem 1.18. For the other direction, let M be a finitely generated module. If M has a generating list of length m, then there exists a surjective homomorphism $R^m \to M$. Thus we can write M as a quotient of R^m. By (2) and (3) of Theorem 1.18, R^m is Noetherian and quotients of Noetherian modules are Noetherian hence M is Noetherian. □

1.9. **Good filtrations.** Let us drop the generality now and return to our setting of considering filtered left A_n-modules M with filtration Γ with respect to the Bernstein filtration $\mathcal{B}$ on A_n.

Definition 1.24. We say a filtration Γ on M is a good filtration if $\mathrm{gr}^{\Gamma} M$ is finitely generated (equivalently Noetherian) over $\mathrm{gr}^{\mathcal{B}} A_n$.

Note that Theorem 1.21 implies that if M has a good filtration then M is finitely generated (equivalently Noetherian). In fact every finitely generated M admits a good filtration. If $m_1, \ldots, m_t$ are a generating set, then as one can confirm taking $\Gamma_i = B_i m_1 + \cdots + B_i m_t$ for $i \geq 0$ gives a good filtration on M as $\mathrm{gr}^{\Gamma} M$ is generated by $\mu_0(m_1), \ldots, \mu_0(m_t)$.

Good filtrations with respect to $\mathcal{B}$ can be characterized quite cleanly by the following property.

Proposition 1.25. *Let M be a filtered left A_n-module with filtration Γ. Then Γ is a good filtration if and only if there exists an integer t_0 such that for all $t \geq t_0$ and $i \geq 0$, $B_i \Gamma_t = \Gamma_{i+t}$.*

We need a simple lemma first.

Lemma 1.26. *Let $\mathcal{B} = \{B_i\}_{i \geq 0}$ be the Bernstein filtration, then $B_i B_j = B_{i+j}$ for all $i, j \geq 0$.*

Proof. The inclusion $B_i B_j \subseteq B_{i+j}$ has been proven already. The other direction just follows from the fact that B_{i+j} is spanned by monomials $x^{\alpha} \partial^{\beta}$ where $|\alpha| + |\beta| \leq i + j$, and any such monomial can be expressed as a product of two monomials, one of degree $\leq i$ and one of degree $\leq j$. □

Proof of Proposition 1.25. Suppose there does exist such a t_0. Each Γ_i is finite dimensional as a vector space over k, so for each $0 \leq j \leq t_0$ pick finitely many elements of Γ_j such that their images in the quotient Γ_j/Γ_{j-1} span the vector space. Then the left $\mathrm{gr}^{\mathcal{B}} A_n$-module generated by the formal symbols of these elements contain Γ_j/Γ_{j-1} for each $0 \leq j \leq t_0$. Then the fact that $B_i\Gamma_{t_0} = \Gamma_{i+t_0}$ for all $i \geq 0$ implies that the module they generate contains every summand Γ_j/Γ_{j-1}, so they in fact generate $\mathrm{gr}^{\Gamma} M$, showing it is finitely generated.

For the reverse direction assume $\mathrm{gr}^{\Gamma} M$ is finitely generated over $\mathrm{gr}^{\mathcal{B}} A_n$. We can always find a homogeneous generating set, so let $m_1, \ldots, m_\ell \in M$ and $t_1, \ldots, t_\ell \in \mathbb{N}$ be such that $\mu_{t_1}(m_1), \ldots, \mu_{t_\ell}(m_\ell)$ generate $\mathrm{gr}^{\Gamma} M$. Let $t_0 = \max(t_1, \ldots, t_\ell)$, we will show that t_0 satisfies the desired property.

Fix $t \geq t_0$. When $i = 0$ it is clear that $B_0\Gamma_t = \Gamma_t$ since $1 \in B_0$. Now assume that $B_{i-1}\Gamma_t = \Gamma_{i+t-1}$. Then consider $u \in \Gamma_{i+t} \backslash \Gamma_{i+t-1}$. By hypothesis then

$$\mu_{i+t}(u) \in \sum_{j=1}^{\ell} \sigma_{i+t-t_j}(B_{i+t-t_j})\mu_{t_j}(m_j).$$

Thus there exists $w \in \Gamma_{i+t-1}$ such that

$$u + w \in \sum_{j=1}^{\ell} B_{i+t-t_j} m_j.$$

By Lemma 1.26, $B_{i+t-t_j} = B_i B_{t-t_j}$, and $B_{t-t_j} m_j \subseteq \Gamma_t$, so $u + w \in B_i\Gamma_t$. But by the inductive hypothesis we have that $w \in \Gamma_{i+t-1} = B_{i-1}\Gamma_t \subseteq B_i\Gamma_t$, thus $u \in B_i\Gamma_t$. This proves $\Gamma_{i+t} \subseteq B_i\Gamma_t$, and the reverse direction is true by definition. □

This characterization of good filtrations allows us to closely relate any two good filtrations.

Proposition 1.27. *Let Γ and Ω be two filtrations of a left A_n-module M with respect to $\mathcal{B}$. If Γ is a good filtration, then there exists $k \in \mathbb{N}$ such that $\Gamma_j \subseteq \Omega_{j+k}$ for all $j \geq 0$.*

Proof. Let t_0 be as given in Proposition 1.25. Consider a finite basis of Γ_{t_0} as a k-vector space. Since $M = \bigcup_{i \geq 0} \Omega_i$, there exists some $k \in \mathbb{N}$ such that Ω_k contains each element of the basis of Γ_{t_0}. Then $\Gamma_{t_0} \subseteq \Omega_k$. For any $j \geq 0$,

$$\Gamma_j \subseteq B_j\Gamma_{t_0} = \Gamma_{j+t_0} \subseteq B_j\Omega_k \subseteq \Omega_{j+k}.$$

□

Corollary 1.28. *If Γ and Ω are two good filtrations on a left A_n-module M with respect to $\mathcal{B}$, then there exists $k \in \mathbb{N}$ such that for all $j \geq 0$, $\Omega_{j-k} \subseteq \Gamma_j \subseteq \Omega_{j+k}$.*

Proof. Applying Proposition 1.27 twice there exists k' and $k'' \in \mathbb{N}$ such that for all $j \geq 0$, $\Gamma_j \subseteq \Omega_{j+k'}$ and $\Omega_j \subseteq \Gamma_{j+k''}$. These inequalities can be extended to any $j \in \mathbb{Z}$ with the convention from before that $\Omega_j = \Gamma_j = \{0\}$ when $j < 0$. Let $k = \max(k', k'')$. Then for all $j \geq 0$,

$$\Omega_{j-k} \subseteq \Gamma_{j-k+k''} \subseteq \Gamma_j \subseteq \Omega_{j+k'} \subseteq \Omega_{j+k}.$$

□

Using this property that relates any two given good filtrations on M to each other we will be able to define in the following section the dimension and multiplicity associated to a finitely generated A_n-module M.

1.10. **Dimension and multiplicity.** Let M be a finitely generated left A_n-module with a good filtration Γ. We will define a dimension for M that captures how quickly the $\dim \Gamma_i$ grows. The following theorem will allow us to show that this growth is well behaved.

Theorem 1.29 (Existence of Hilbert Polynomial)**.** *Let $N = \bigoplus_{i\geq 0} N_i$ be a finitely generated graded module over the polynomial ring $k[x_1, \ldots, x_n]$. Then there exists a polynomial $H_N(t) \in \mathbb{Q}[t]$ and a positive integer s_0 such that for all $s \geq s_0$,*

$$\sum_{i=0}^{s} \dim_k(N_i) = H_N(s).$$

Proof. The existence relies on induction on n the number of variables and the fact that every polynomial $f \in \mathbb{Q}[t]$ such that $f(\mathbb{Z}) \subseteq \mathbb{Z}$ is a $\mathbb{Z}$-linear combination of binomial coefficients $\binom{t}{i}$. See [Cou95, Theorem 9.1.1] for the specifics of the proof. □

Then $H_N(t)$ is called the Hilbert polynomial of N. Note that this polynomial is unique as two polynomials in $\mathbb{Q}[t]$ that agree on infinitely many values must be equal. Now return to our setting of M being an A_n-module with a good filtration Γ. Then $\operatorname{gr}^\Gamma M$ is a finitely generated graded module over $\operatorname{gr}^{\mathcal{B}} A_n$ which is a polynomial ring in $2n$ variables over k. This implies that there exists a polynomial which we denote $H(t, \Gamma, M) \in \mathbb{Q}[t]$ such that for large enough s,

$$\sum_{i=0}^{s} \dim_k(\Gamma_i/\Gamma_{i-1}) = H(s, \Gamma, M).$$

Since $\dim_k(\Gamma_i/\Gamma_{i-1}) = \dim_k \Gamma_i - \dim_k \Gamma_{i-1}$ and $\dim_k \Gamma_{-1} = 0$, the left hand side in fact just equals $\dim_k \Gamma_s$.

Definition 1.30. The dimension $d(M)$ of M is the degree of the polynomial $H(t, \Gamma, M)$ above. The multiplicity $m(M)$ of M is $m(M) = d(M)! a_{d(M)}$, where $a_{d(M)}$ is the leading coefficient. See [Cou95, Chapter 9] for a proof of the fact that $m(M)$ is always an integer.

The definitions above appear to depend on the good filtration Γ. This is not the case though, as Corollary 1.28 allows us to closely relate the Hilbert polynomials for two different good filtrations.

Proposition 1.31. *The definitions of $d(M)$ and $m(M)$ are independent of the choice of good filtration on M.*

Proof. Let Γ and Ω be two good filtrations on M. By Corollary 1.28 there exists $k \in \mathbb{N}$ such that for all $j \geq 0$, $\Omega_{j-k} \subseteq \Gamma_j \subseteq \Omega_{j+k}$. In particular if we take $j \gg 0$, then

$$H(j-k, \Omega, M) \leq H(j, \Gamma, M) \leq H(j+k, \Omega, M).$$

If we divide through by $H(j, \Omega, M)$ and take the limit as $j \to \infty$, we get that $\lim_{j\to\infty} \frac{H(j,\Gamma,M)}{H(j,\Omega,M)} = 1$. Thus $H(t, \Gamma, M)$ and $H(t, \Omega, M)$ must have the same degree and leading coefficient so $d(M)$ and $m(M)$ do not depend on the chosen good filtration. □

Example 1.32. Consider A_n as an A_n-module with the Bernstein filtration. Then

$$B_i = \operatorname{span}_k\{x^\alpha \partial^\beta \mid \alpha, \beta \in \mathbb{N}^n, |\alpha| + |\beta| \leq i\},$$

so $\dim B_i = \binom{2n+i}{2n}$. As a polynomial in i this has a leading term $\frac{1}{(2n)!} i^{2n}$, so A_n has dimension $d(A_n) = 2n$, and multiplicity $m(A_n) = (2n)! \frac{1}{(2n)!} = 1$.

Example 1.33. Consider $k[x_1, \dots, x_n]$ as an A_n-module with the standard filtration by polynomial degree. This is a good filtration as one can easily verify the condition in Proposition 1.25. If C_i is the set of polynomials of degree $\leq i$, then

$$\dim C_i = |\{\alpha \in \mathbb{N}^n \mid |\alpha| \leq i\}| = \binom{n+i}{n}.$$

As a polynomial in i the leading term of this is $\frac{1}{n!}i^n$, thus $d(k[x_1, \dots, x_n]) = n$ and $m(k[x_1, \dots, x_n]) = 1$.

Given a good filtration Γ on A_n-module M and a submodule $N \subseteq M$, let Γ' and Γ'' be the induced filtrations on N and M/N as in Section 1.7. Then $\mathrm{gr}^{\Gamma'} N \subseteq \mathrm{gr}^{\Gamma} M$ and $\mathrm{gr}^{\Gamma''}(M/N)$ is a quotient of $\mathrm{gr}^{\Gamma} M$, so by Theorem 1.18, Γ' and Γ'' are also good filtrations. Then the short exact sequence

$$0 \longrightarrow \Gamma'_i/\Gamma'_{i-1} \longrightarrow \Gamma_i/\Gamma_{i-1} \longrightarrow \Gamma''_i/\Gamma''_{i-1} \longrightarrow 0$$

implies $\dim(\Gamma'_i/\Gamma'_{i-1}) + \dim(\Gamma''_i/\Gamma''_{i-1}) = \dim(\Gamma_i/\Gamma_{i-1})$. Taking $s \gg 0$ and summing over this identity from $i = 0, \dots, s$ forces the following equality of polynomials,

$$H(t, \Gamma, M) = H(t, \Gamma', N) + H(t, \Gamma'', M/N). \tag{1.2}$$

This leads to the following results.

Theorem 1.34. *Let M be a finitely generated A_n-module, and $N \subseteq M$ a submodule. Then*

1. *$d(M) = \max(d(N), d(M/N))$.*
2. *If $d(N) = d(M/N)$, then $m(M) = m(N) + m(M/N)$.*

Proof. If $d(N) \neq d(M/N)$, then

$$\max(d(N), d(M/N)) = \deg(H(t, \Gamma', N) + H(t, \Gamma'', M/N)) = d(M)$$

by (1.2). If $d(N) = d(M/N)$, then since the Hilbert polynomials must be increasing functions, the leading coefficients of both $H(t, \Gamma', N)$ and $H(t, \Gamma'', M/N)$ are positive so they do not cancel out, implying that $d(M) = d(N) = d(M/N)$. Furthermore by (1.2) the leading coefficient of $H(t, \Gamma, M)$ is the sum of those for the other two Hilbert polynomials, so

$$m(M) = m(N) + m(M/N).$$

□

Corollary 1.35. *For any positive integer ℓ, $d(A_n^\ell) = 2n$.*

Proof. We proved the result already for $\ell = 1$. Assume that for some $\ell \geq 1$ that $d(A_n^{\ell-1}) = 2n$. Then consider the submodule $A_n^{\ell-1} \subseteq A_n^\ell$ by looking at just the first $\ell - 1$ factors. Then $A_n^\ell / A_n^{\ell-1} \cong A_n$, so by (1) in Theorem 1.34,

$$d(A_n^\ell) = \max(d(A_n^{\ell-1}), d(A_n)) = 2n.$$

□

This immediately leads to an upper bound on dimension.

Proposition 1.36. *Let M be a finitely generated A_n-module, then $d(M) \leq 2n$.*

Proof. Suppose M has a generating list of size $\ell \in \mathbb{N}$, so there exists a surjective homomorphism $\phi : A_n^\ell \to M$. Then $A_n^\ell / \ker \phi \cong M$, so by (1) of Theorem 1.34 and the preceding corollary we have

$$2n = d(A_n^\ell) = \max(d(\ker \phi), d(M)).$$

Hence $d(M) \leq 2n$. □

1.11. **Bernstein's inequality.** After obtaining this upper bound, we'll now show that n is a lower bound on dimension. This important result is known as *Bernstein's Inequality.*

Theorem 1.37 (Bernstein's Inequality)**.** *Let M be a finitely generated nonzero A_n-module. Then $d(M) \geq n$.*

The proof relies on the following lemma.

Lemma 1.38. *Let M be a finitely generated A_n-module with filtration Γ. Suppose $\Gamma_0 \neq \{0\}$. The linear map that sends $b \in B_i$ to the multiplication by b map*

$$B_i \to \mathrm{Hom}_k(\Gamma_i, \Gamma_{2i}),$$
$$b \mapsto (m \mapsto bm)$$

is injective for any $i \geq 0$.

Proof. This is equivalent to proving that $b\Gamma_i \neq \{0\}$ for any $0 \neq b \in B_i$. We induct on i. When $i = 0$, the statement is trivial as $b \in k$.

Now assume the result holds for some $i \geq 0$. Consider $0 \neq d \in B_{i+1}$ such that $d\Gamma_{i+1} = 0$. Then clearly $d \notin k$, so some x_j or ∂_j appears in the canonical form for d. If x_j appears, then a straightforward calculation in canonical form shows that $[d, \partial_j]$ is nonzero. Then $[d, \partial_j] \in B_i$, and for all $m \in \Gamma_i$ we have

$$[d, \partial_j] \cdot m = d \cdot (\partial_j m) - \partial_j (d \cdot m) = 0,$$

because $\partial_j m \in \Gamma_{i+1}$ and $m \in \Gamma_i \subseteq \Gamma_{i+1}$. This implies that $[d, \partial_j]\Gamma_i = 0$, a contradiction. If we conversely started with the assumption ∂_j appears in the canonical form of d, swapping the roles of x_j and ∂_j would yield the contradiction. □

Proof of Bernstein's Inequality. Let M have a good filtration Γ with $\Gamma_0 \neq 0$. Then pick s_0 large enough so that for $s \geq s_0$, $\dim \Gamma_s = H(s, \Gamma, M)$. Then by Lemma 1.38, for any $s \geq s_0$ we have

$$\dim_k B_s \leq \dim_k \mathrm{Hom}_k(\Gamma_s, \Gamma_{2s}) = \dim_k \Gamma_s \cdot \dim_k \Gamma_{2s}.$$

Re-expressing both sides this inequality becomes

$$\binom{2n+s}{2n} \leq H(s, \Gamma, M) \cdot H(2s, \Gamma, M).$$

This is a polynomial inequality which holds for all large enough s, hence the degree of the LHS must be $\leq$ to the degree of the RHS. The degree of the LHS is $2n$, and the degree of the RHS is $2d(M)$, thus $2n \leq 2d(M)$, so $d(M) \geq n$. □

1.12. **Holonomic modules.** The modules M of minimal dimension are very important and will lead to our proof of the existence of the Bernstein-Sato polynomial.

Definition 1.39. A finitely generated A_n-module is *holonomic* if it is zero or has minimal dimension, that is dimension n.

We saw earlier that $d(k[x_1, \ldots, x_n]) = n$, so $k[x_1, \ldots, x_n]$ is our first example of a holonomic module. As a non-example, recall $d(A_n^\ell) = 2n$ for any positive integer ℓ. The minimality of the dimension for holonomic modules leads to the following nice results.

Proposition 1.40. *Submodules and quotients of holonomic modules are holonomic.*

Proof. Let M be a holonomic module and N a submodule. Then by (1) of Theorem 1.34,

$$n = d(M) = \max(d(N), d(M/N)).$$

Hence $d(N), d(M/N) \leq n$. Then Bernstein's inequality implies in fact $d(N) = d(M/N) = n$. □

The power that holonomic modules will play in proving the existence of the Bernstein-Sato polynomial comes from the fact that holonomic modules are Artinian. While a Noetherian module is one such that every ascending chain of submodules stabilizes, an Artinian module is one where every descending chain of submodules stabilizes.

Definition 1.41. An R-module M is Artinian if every descending chain of submodules $N_1 \supseteq N_2 \supseteq \cdots$ stabilizes, i.e. there exists a positive integer ℓ such that $N_\ell = N_{\ell+1} = \cdots$.

Proposition 1.42. *Let M be left R-module, and N a submodule. Then M is Artinian if and only if N and M/N are Artinian.*

Proof. The proof is similar to the Noetherian case which can be found in [Cou95]. □

Proposition 1.43. *Holonomic modules are Artinian.*

Proof. Let M be a holonomic module. Consider a strict descending chain of submodules

$$M = N_0 \supsetneq N_1 \supsetneq N_2 \supsetneq \cdots \supsetneq N_\ell.$$

Then each N_i is also holonomic, so $d(N_i) = n$ for $0 \leq i \leq \ell$. By (2) in Theorem 1.34 we have

$$m(M) = m(N_1) + m(N_0/N_1).$$

Then applying this relation again to N_1 and N_2 and then repeating, we obtain that

$$m(M) = m(N_\ell) + \sum_{i=0}^{\ell-1} m(N_i/N_{i+1}).$$

Thus $m(M) \geq \ell$, as for each i, $m(N_i/N_{i+1}) \geq 1$. This places an upper bound on the possible length of a descending chain, hence M is Artinian. □

The following result will be useful later on.

Lemma 1.44. *Let M be a left A_n-module with filtration Γ with respect to the Bernstein filtration. If for $i \gg 0$,*

$$\dim_k \Gamma_i \leq \frac{c}{n!} i^n$$

for some constant $c \geq 0$, then M is holonomic and $m(M) \leq c$.

Proof. Let $N \subseteq M$ be any finitely generated submodule. We claim that N is then holonomic.

Let Ω be a good filtration on N. Since $\Gamma' = \{\Gamma_i \cap N\}_{i\geq 0}$ is a filtration on N (not necessarily good), Proposition 1.27 implies that there exists $k \in \mathbb{N}$ such that for all $j \geq 0$,

$$\Omega_j \subseteq \Gamma_{j+k} \cap N.$$

Then for $j \gg 0$,

$$H(j, \Omega, N) = \dim_k \Omega_j \leq \dim_k(\Gamma_{j+k} \cap N) \leq \frac{c}{n!}(j+k)^n.$$

Taking $j \to \infty$ we have that $\deg H(t, \Omega, N) \leq n$, and by comparing leading coefficients $m(N) \leq c$. So $d(N) \leq n$, thus by Bernstein's inequality $d(N) = n$.

We want to apply this result to show that M is holonomic, thus we need to prove that M is finitely generated. Assume that M is not finitely generated. Then there exists an infinite ascending chain of finitely generated submodules of M, say

$$N_0 \subsetneq N_1 \subsetneq N_2 \subsetneq \cdots.$$

Then by part (2) of Theorem 1.34, for all $j \geq 1$ we have that

$$m(N_j) = m(N_j/N_{j-1}) + m(N_{j-1}) = m(N_0) + \sum_{i=1}^{j} m(N_i/N_{i-1}) \geq j.$$

But $m(N_j) \leq c$, so $j \leq c$. This gives a contradiction as the chain is infinite. Hence M is finitely generated, so applying the first part of the proof M is holonomic. □

1.13. **The existence of the Bernstein-Sato polynomial.** After having built a lot of basic theory surrounding A_n-modules, we will finally prove in this section the existence of the Bernstein-Sato polynomial for the case of differential operators over a polynomial ring, i.e. the Weyl algebra.

Let X denote the variable set $\{x_1, \ldots, x_n\}$, and $f \in k[X]$ be a polynomial. Introduce a new variable s, and let $k(s)$ be the field of rational functions in s. Then $A_n(k(s))$ is the Weyl algebra over the field $k(s)$ with a natural identification of

$$A_n(k(s)) \cong A_n(k) \otimes_k k(s).$$

We can extend the action of $A_n(k(s))$ on $k(s)[X]$ to $k(s)(X)$, the field of rational functions in X with coefficients in $k(s)$ via the normal quotient rule for differentiation. Then the localization $k(s)[X]_f \subset k(s)(X)$ is the subset of rational functions of the form $\frac{g}{f^m}$, where $g \in k(s)[X]$ and m is a nonnegative integer. If we apply ∂_i to $\frac{g}{f^m}$, then the resulting unsimplified fraction will have a denominator of f^{2m}, hence $k(s)[X]_f$ is a $A_n(k(s))$-submodule of $k(s)(X)$.

Let f^s be a formal symbol such that if $Q \in k(s)[X]_f$ then ∂_j acts by

$$\partial_j \cdot (Qf^s) = \left(\frac{\partial Q}{\partial x_j} + sQf^{-1}\frac{\partial f}{\partial x_j}\right) f^s.$$

This definition comes from how we would expect ∂_j to act if s was an integer. This turns $k(s)[X]_f f^s$ into an $A_n(k(s))$-module. This is the setting for the Bernstein-Sato Polynomial. The idea is that we can specialize s to integer values $\ell \in \mathbb{Z}$ as well. Specifically for each $\ell \in \mathbb{Z}$ there is a k-linear map $\Phi_\ell : k[s][X]_f f^s \to k[X]_f$ defined by $\Phi_\ell(Q(s,x)f^s) = Q(\ell, x)f^\ell$. Then if $P(s) \in A_n[s]$ and $v \in k[s][X]_f f^s$ then $\Phi_\ell(P(s)v) = P(\ell)\Phi_\ell(v)$.

Theorem 1.45 (Existence of Bernstein-Sato Polynomial)**.** *There exists a differential operator $P(s) \in A_n[s]$ and a nonzero polynomial $b(s) \in k[s]$ such that*

$$P(s) \cdot f^{s+1} = b(s) \cdot f^s,$$

where $f^{s+1} = ff^s$.

Above we are viewing $A_n[s] \cong A_n(k) \otimes_k k[s]$ as a subring of $A_n(k(s))$. The main part of the proof is that the module we've defined above turns out to be holonomic, as we can then leverage that holonomic modules are Artinian.

Proposition 1.46. *The $A_n(k(s))$-module $k(s)[X]_f f^s$ is holonomic.*

Proof. Let $M = k(s)[X]_f f^s$. First for clarity note that a general element of M can be written in the form $\frac{g}{f^m} \cdot f^s$ where $g \in k(s)[X]$ and m is a nonnegative integer. Let $\deg(f) = d$, and define

$$\Gamma_i = \left\{ \frac{g}{f^i} \cdot f^s \mid \deg(g) \leq (d+1)i \right\}.$$

We will show that $\Gamma = \{\Gamma_i\}_{i \geq 0}$ is a filtration of M with respect to the Bernstein filtration on $A_n(k(s))$. Then we will show that Γ satisfies the bounds from Lemma 1.44, as that then implies M is holonomic.

Consider some element $\frac{g}{f^i} \cdot f^s \in M$. Then

$$\frac{g}{f^i} \cdot f^s = \frac{g f^{\deg g}}{f^{i+\deg g}} \cdot f^s.$$

Then $\deg(gf^{\deg g}) = (d+1)\deg g \leq (d+1)(\deg g + i)$, so $\frac{g}{f^i} \cdot f^s \in \Gamma_{i+\deg g}$. This shows $M = \bigcup_{i \geq 0} \Gamma_i$.

If $\frac{g}{f^i} \cdot f^s \in \Gamma_i$, then $\deg(g) \leq (d+1)i$. But

$$\frac{g}{f^i} \cdot f^s = \frac{gf}{f^{i+1}} \cdot f^s,$$

and

$$\deg(gf) = \deg(g) + d \leq (d+1)i + d = (d+1)(i+1),$$

so $\frac{g}{f^i} \cdot f^s \in \Gamma_{i+1}$. This proves $\Gamma_0 \subseteq \Gamma_1 \subseteq \cdots$.

Now we show $B_i \Gamma_j \subseteq \Gamma_{i+j}$. Let $\frac{g}{f^j} \cdot f^s \in \Gamma_j$, so $\deg(g) \leq (d+1)j$. Then

$$x_i \left(\frac{g}{f^j} \cdot f^s \right) = \frac{x_i g f}{f^{j+1}} \cdot f^s,$$

and $\deg(x_i g f) = d + 1 + \deg(g) \leq (d+1)(j+1)$, so $x_i \left(\frac{g}{f^j} \cdot f^s \right) \in \Gamma_{j+1}$. Also

$$\partial_i \left(\frac{g}{f^j} \cdot f^s \right) = \left(\frac{\frac{\partial g}{\partial x_i} f - jg \frac{\partial f}{\partial x_i} + sg \frac{\partial f}{\partial x_i}}{f^{j+1}} \right) \cdot f^s.$$

The numerator above has degree $\leq \deg(g) + d - 1$, and

$$\deg(g) + d - 1 \leq (d+1)j + d - 1 < (d+1)(j+1).$$

Hence $\partial_i((g/f^j) \cdot f^s) \in \Gamma_{j+1}$. This proves that $B_1 \Gamma_j \subseteq \Gamma_{j+1}$. As a repeated application of Lemma 1.26, $B_i = B_1^i$, so the in fact $B_i \Gamma_j \subseteq \Gamma_{i+j}$.

Finally, there's a clear isomorphism of $k(s)$-vector spaces between Γ_j and the space of polynomials in $k(s)[X]$ of degree $\leq (d+1)j$,

$$\dim_{k(s)} \Gamma_j = \binom{(d+1)j+n}{n}.$$

The leading term of the polynomial in j that is this binomial coefficient is $\frac{(d+1)^n j^n}{n!}$. Thus there exists a positive integer t such that for all $j \geq t$,

$$\dim_{k(s)} \Gamma_j \leq \frac{2(d+1)^n j^n}{n!}.$$

Thus by Lemma 1.44 if we take $c = 2(d+1)^n$, M is holonomic. □

We need one final idea before we prove Theorem 1.45. Let φ be a map on $k(s)[X]_f f^s$ defined by $\varphi(Q(s,X) \cdot f^s) = Q(s+1,X)f \cdot f^s$. Then intuitively φ is the shift map defined by $s \mapsto s+1$. It is clear φ is a bijection, and it is not hard to check that φ is A_n-linear, i.e. it commutes with both x_i, ∂_i, and k-scalars. Also, φ is semilinear in the sense that for $P(s) \in A_n(k(s))$, $\varphi(P(s)v) = P(s+1)\varphi(v)$.

Proof of Theorem 1.45. Consider the cyclic module $A_n(k(s))f^s$, which is a submodule of $k(s)[X]_f f^s$. Since we just proved the latter module is holonomic, it follows that $A_n(k(s))f^s$ is also holonomic. In particular, by Proposition 1.43 $A_n(k(s))f^s$ is Artinian, so the chain

$$A_n(k(s))f^s \supseteq A_n(k(s))ff^s \supseteq A_n(k(s))f^2f^s \supseteq \cdots \tag{1.3}$$

stabilizes. Then there exists a nonnegative integer ℓ such that

$$f^\ell f^s \in A_n(k(s))f^{\ell+1}f^s.$$

Thus there exists an operator $D(s) \in A_n(k(s))$ such that $f^\ell f^s = D(s)f^{\ell+1}f^s$. Applying $\varphi^{-\ell}$ yields that

$$f^s = D(s-\ell)ff^s.$$

Then $D(s-\ell)$ is still in $A_n(k(s))$. Then take $b(s) \in k[s]$ to be the product of all the denominators of rational functions which show up in $D(s-\ell)$, so that if we let $P(s) = b(s)D(s-\ell)$, then $P(s) \in A_n[s]$. Multiplying through by $b(s)$ does yield

$$b(s) \cdot f^s = P(s) \cdot f^{s+1}. \tag{1.4}$$

□

In the proof above the polynomial $b(s)$ satisfying (1.4) is not unique. To define a unique monic Bernstein-Sato polynomial $b_f(s)$ of minimal degree, note that the set of polynomials $b(s) \in k[s]$ which satisfy (1.4) form an ideal. Then pick $b_f(s)$ to be the unique monic generator.

Example 1.47. As the most basic example, when $n = 1$ and $f = x$, we have

$$\partial \cdot x^{s+1} = (s+1)x^s,$$

so $b_f(s) = s+1$.

As a corollary of Theorem 1.45, we obtain the following finite generation result of localizations.

Corollary 1.48. *Let $f \in k[X]$ be nonzero. Then $k[X, f^{-1}]$ is cyclic as a left module over A_n.*

Proof. Since $b_f(s) \in k[s]$ has finitely many roots, we can consider $m \in \mathbb{Z}$ such that $b_f(\ell) \neq 0$ for all $\ell \in \mathbb{Z}$ and $\ell \leq m$. Then setting $s = m$ in Theorem 1.45 we have that

$$P(m)f^{m+1} = b_f(m)f^m.$$

Since $b_f(m) \in k^\times$ this implies $f^m \in A_n f^{m+1}$. We can now repeat this process. Since $b_f(m-1) \in k^\times$, the equation $P(m-1)f^m = b_f(m-1)f^{m-1}$ implies that $f^{m-1} \in A_n f^m \subseteq A_n f^{m+1}$. Repeating this process shows that $f^j \in A_n f^{m+1}$ for all $j \leq m$. Clearly just by multiplying by f we can also obtain all powers $f^j \in A_n f^{m+1}$ where $j \geq m+1$. Thus $A_n f^{m+1}$ contains all powers of f, so in fact $k[X, f^{-1}]$ is cyclic and equal to $A_n f^{m+1}$. □

The Bernstein-Sato polynomial has many properties and applications to a variety of fields. When f is non-constant, $b_f(s)$ always has a root of $s = -1$. In fact a result due to Kashiwara states that when $k = \mathbb{C}$, the roots of $b_f(s)$ are all negative rational numbers.

The polynomial $b_f(s)$ is also used to study singularities of the hypersurface $\{f = 0\}$. A key result of Kollár at [Kol97] states that the log canonical threshold for $\{f = 0\}$, a numerical invariant measuring the complexity of singularities, is equal to the negative of the largest root of $b_f(s)$.

The existence of the b-function and its applications show the power of the theory of holonomic A_n-modules as developed in Coutinho's book [Cou95]. In the proof above, the key fact is that holonomic modules are Artinian, forcing the descending chain in (1.3) to stabilize. This finite-length stabilization mechanism will remain important in general settings. However, the theory of A_n-modules as presented here also has limitations if we wish to generalize. The dimension theory relied heavily on the Bernstein filtration which is specific to the Weyl algebra. For more general rings of differential operators the order filtration is the natural filtration to consider. Furthermore, in this section we proved the existence of the b-function only for the specific A_n-module $k[x_1, \ldots, x_n]$. To obtain analogous b-functions for more general modules, one needs a framework ensuring that localization and base change to $k(s)$ preserve an appropriate finite-length condition. In the next section we will see how the results of Mebkhout and Narváez-Macarro resolve these issues using modules of minimum dimension over a class of filtered rings known as type Diff. This will recover the above Weyl algebra case as a special example, while also extending to broader settings such as Tate Algebras.

2. More General Bernstein-Sato Theory

We will start this section by laying out the setting and key parts of the proof by Mebkhout and Narváez-Macarro in [MN91] that establishes the existence of the Bernstein-Sato polynomial in much more general settings. In some of what follows ideas from homological algebra will be used. We will not introduce the needed definitions, yet they can all be found in a standard introductory book such as [Wei94]. We do not prove some of the results that lead to the main theorem, as such if one does not know homological algebra they can still get the main takeaway that is Theorem 2.16. Before we lay out the setting for the proof in [MN91], let us introduce some relevant algebraic definitions.

Definition 2.1. Let A be a commutative ring and $\mathfrak{p} \subseteq A$ a prime ideal. The height of $\mathfrak{p}$, denoted $\mathrm{ht}(\mathfrak{p})$ is the maximum value ℓ such that there exists a proper chain of prime ideals

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_\ell = \mathfrak{p}.$$

Definition 2.2. The Krull dimension of a commutative ring A, denoted $\dim(A)$ is the supremum of $\mathrm{ht}(\mathfrak{p})$ over all prime ideal $\mathfrak{p} \subseteq A$.

Definition 2.3. A commutative ring A is local if it has a unique maximal ideal.

If we have a commutative ring A with prime ideal $\mathfrak{p}$, then the localization $S^{-1}A = A_{\mathfrak{p}}$ where $S = A - \mathfrak{p}$ is an example of a local ring. To see this note that the ideal $S^{-1}\mathfrak{p}$ is exactly all of the non units in $A_{\mathfrak{p}}$. Thus if $I \subsetneq A_{\mathfrak{p}}$ is any proper ideal, then I does not contain any units so $I \subseteq S^{-1}\mathfrak{p}$.

Definition 2.4. Let A be a commutative noetherian ring. At each prime ideal $\mathfrak{p} \subseteq A$ consider the localization $A_{\mathfrak{p}}$, and denote the unique maximal ideal as $\mathfrak{m}_{\mathfrak{p}}$. If the minimal number of generators for $\mathfrak{m}_{\mathfrak{p}}$ is $\dim(A_{\mathfrak{p}})$ for all prime ideals $\mathfrak{p}$ then A is called a regular ring.

This next definition generalizes the standard differential operators that are a linear combination of ∂_i.

Definition 2.5. Let A be a commutative k-algebra. Define the set of k-derivations $\mathrm{Der}_k(A)$ as a vector subspace of $\mathrm{End}_k(A)$ by

$$\mathrm{Der}_k(A) = \{\varphi \in \mathrm{End}_k(A) \mid \varphi(xy) = x\varphi(y) + \varphi(x)y \text{ for all } x, y \in A\}.$$

2.1. **Setting.** Fix k a field of characteristic zero. Let A be a commutative Noetherian regular algebra over k. Furthermore A is required to satisfy the following conditions:

(1) Every maximal ideal of A has height n, this is said as A is equicodimensional of dimension n.
(2) The residue fields at every maximal ideal of A are algebraic over k.
(3) There exists $x_1, \ldots, x_n \in A$ and $\partial_1, \ldots, \partial_n \in \mathrm{Der}_k(A)$ such that $\partial_i(x_j) = \delta_{ij}$.

In this setting it turns out that the filtered pieces of the ring of differential operators $D^i_{A/k}$ is a free A-module with a basis $\{\partial^\alpha \mid \alpha \in \mathbb{N}^n, |\alpha| \leq i\}$. Furthermore, following a proof similar to that in Theorem 1.9 one can see that that the associated graded ring $\mathrm{gr}(D_{A/k})$ is isomorphic to the the polynomial ring $A[y_1, \ldots, y_n]$ by the association $\partial_i \mapsto y_i$. In particular $D_{A/k}$ is a ring of type Diff.

Definition 2.6. A filtered ring R is of type Diff if the associated graded ring is a commutative, noetherian, regular ring all of whose graded maximal ideals have the same height.

A ring R as above is then both left and right Noetherian, so the left and right global dimensions agree, and per the notation in [MN91] we will write this just as $\mathrm{dh}(R)$.

2.2. **Dimension of rings of type Diff.** Over a ring R of type Diff, we can define a module of minimal dimension, similar to the holonomic modules over A_n. Through this section let $R = \bigcup_{i \geq 0} R_i$ be a ring of type Diff.

Definition 2.7. Let M be a left R-module. A good filtration of M is a filtration $\Gamma = \{\Gamma_i\}_{i \geq 0}$ of M with respect to R such that each Γ_i is finitely generated as a module over R_0, and $\mathrm{gr}^\Gamma(M)$ is a finitely generated $\mathrm{gr}(R)$-module. By convention $\Gamma_i = 0$ for $i < 0$.

Given a good filtration Γ on M, define the dimension of M as $\dim(\mathrm{gr}(R)/\mathrm{ann}_{\mathrm{gr}(R)}(\mathrm{gr}^\Gamma(M)))$ where $\mathrm{ann}_{\mathrm{gr}(R)}(\mathrm{gr}^\Gamma(M))$ is the ideal $\{s \in \mathrm{gr}(R) \mid s \cdot \mathrm{gr}^\Gamma(M) = 0\}$. This dimension is independent of the choice of good filtration (see [Bjo79]), so we define it as $\dim_R(M)$. In the case of $R = A_n$, this is precisely the dimension we defined before in terms of the Hilbert polynomial. The next example illustrates this.

Example 2.8. Take $R = A_n$ with the order filtration, and $M = k[X]$ with the constant filtration, so $\Gamma_i = k[X]$ for $i \geq 0$. From the Hilbert polynomial definition, we showed the dimension of M was n. Under this constant filtration, the only nonzero graded piece of $\mathrm{gr}(M)$ is the zeroth piece which is $k[X]$. Since $\partial_i \in A_n$ is in the first graded piece of $\mathrm{gr}(R)$, it must annihilate $\mathrm{gr}(M)$. Thus if $\mathrm{gr}(R) \cong k[X][\xi_1, \ldots, \xi_n]$ where ξ_i is the image of ∂_i, then we have in fact that

$$\mathrm{ann}_{\mathrm{gr}(R)}(\mathrm{gr}(M)) = (\xi_1, \ldots, \xi_n).$$

Thus $\mathrm{gr}(R)/\mathrm{ann}_{\mathrm{gr}(R)}(\mathrm{gr}(M)) \cong k[X]$, and $\dim(k[X]) = n$.

With this definition of dimension we have the same result that if M has submodule N, then $\dim_R(M) = \max(\dim_R(N), \dim_R(M/N))$ (see [Bjo79]).

Now we introduce the role of homological algebra. The following results show its power, as the dimension $\dim_R(M)$ defined above is closely related to the vanishing of the Ext groups $\mathrm{Ext}^i_R(M, R)$ for $i < \mathrm{dh}(R)$.

Definition 2.9. Let M be a nonzero finitely generated R-module. Define

$$\mathrm{grade}_R(M) = \inf\{i \in \mathbb{N} \mid \mathrm{Ext}^i_R(M, R) \neq 0\}$$

Note that the $\mathrm{Ext}^i_R(-, R)$ is contravariant, so if M is a left module then $\mathrm{Ext}^i_R(M, R)$ will be a right module and vice versa. It also preserves finite generation, so we can consider for example the grade of the Ext group.

Theorem 2.10. [MN91, Theorem 1.2.2] *If M is a finitely generated R-module then*

(1) $\dim_R(M) = \dim \mathrm{gr}(R) - \mathrm{grade}_R(M)$,
(2) $\mathrm{grade}_R(\mathrm{Ext}^i_R(M, R)) \geq i$, *provided* $\mathrm{Ext}^i_R(M, R) \neq 0$.

Since $\mathrm{grade}_R(M) \leq \mathrm{dh}(R)$, this immediately leads to the inequality

$$\dim_R(M) \geq \dim \mathrm{gr}(R) - \mathrm{dh}(R). \tag{2.1}$$

Definition 2.11. An R-module M is of minimal dimension if we have an equality above, i.e. $\dim_R(M) = \dim \mathrm{gr}(R) - \mathrm{dh}(R)$, or if it is zero.

By Theorem 2.10 this is equivalent to $\mathrm{grade}_R(M) = \mathrm{dh}(R)$, so $\mathrm{Ext}^i_R(M, R) = 0$ whenever $i \neq \mathrm{dh}(R)$.

Example 2.12. When $R = A_n$, then the global dimension of A_n is n, and the dimension of $\mathrm{gr}(R) \cong k[X][\xi_1, \ldots, \xi_n]$ is $2n$. Thus an A_n-module M is of minimal dimension if $\dim_R(M) = 2n - n = n$. This shows that the definition of minimal dimension agrees with the definition of holonomic modules. Then the inequality in (2.1) becomes Bernstein's Inequality.

When M is an R-module and N is a submodule of M, the fact that we proved in the A_n case still holds that $\dim_R(M) = \max(\dim_R(N), \dim_R(M/N))$. From this it follows that submodules and quotients of minimal dimension modules have minimal dimension.

Just as we showed that holonomic modules are Artinian, we can now do the same for modules of minimal dimension. Note that by (2) of Theorem 2.10, if M is a minimal dimension module, then $\mathrm{grade}_R(\mathrm{Ext}^i_R(M, R)) = \mathrm{dh}(R)$, so $\mathrm{Ext}^{\mathrm{dh}(R)}_R(M, R)$ is also a minimal dimension module. Hence by the biduality formula, we have an antiequivalence from the category of left R-modules of minimum dimension to the category of right R-modules of minimum dimension and vice versa that is given by

$$M \mapsto M^* := \mathrm{Ext}^{\mathrm{dh}(R)}_R(M, R).$$

This dual notion allows a simple proof that the minimal dimension modules are Artinian.

Proposition 2.13. *Minimal dimension R-modules are Artinian.*

Proof. Since we know R is left and right Noetherian, minimal dimension modules are Noetherian as they are finitely generated. Let M be a minimal dimension module. Now suppose we have an infinite descending chain of submodules (all of which have minimal dimension as well)

$$M_0 \supseteq M_1 \supseteq M_2 \supseteq \cdots .$$

Then we have a chain of surjective maps of modules of minimal dimension,

$$M/M_0 \twoheadleftarrow M/M_1 \twoheadleftarrow M/M_2 \twoheadleftarrow \cdots .$$

Applying the dual functor this becomes a chain of inclusions

$$(M/M_0)^* \subseteq (M/M_1)^* \subseteq (M/M_2)^* \subseteq \cdots .$$

Since minimal dimension modules are Noetherian, this chain stabilizes. Applying the dual functor again to this chain shows that the original chain must have stabilized, hence M is Artinian. □

Lastly before we prove the existence of the b-function we need the following two results. These results are what ensure that enough properties and conditions are preserved under localization and the base change from k to $k(s)$ for the proof to follow. The proofs can be found in the paper at [MN91].

Proposition 2.14. [MN91, Proposition 1.2.7] *As before let R be a filtered ring of type* Diff *and let $f \in R_0$. Let M be a finitely generated R_f-module such that $\operatorname{Ext}^i_{R_f}(M, R_f) = 0$ if $i \neq \operatorname{dh}(R)$. Then there exists a minimal dimension R-submodule M' of M such that $M = M'_f$.*

Theorem 2.15. [MN91, Theorem 2.1.4, Theorem 2.2.3] *Let A be a commutative noetherian regular k-algebra also satisfying the conditions given in Section 2.1, so $D_{A/k} = A[\partial_1, \ldots, \partial_n]$ is of type* Diff*. Then the ring $D_{A/k}(s) := k(s) \otimes_k D_{A/k}$ is also of type* Diff*, and $\operatorname{dh}(D_{A/k}(s)) = \operatorname{dh}(D_{A/k})$.*

2.3. **General existence of Bernstein-Sato polynomial.** We will now state and prove the general result of Mebkhout and Narváez-Macarro. Let A be as before a commutative noetherian regular k-algebra satisfying the extra conditions given at the start of Section 2.1. Set $D = D_{A/k}$, and take $f \in D^0_{A/k} = A$. Then set

$$D[s] = k[s] \otimes_k D, \quad D(s) = k(s) \otimes_k D,$$

and $D_f[s]$ and $D_f(s)$ are $k[s] \otimes_k D_f$ and $k(s) \otimes_k D_f$ respectively. Once again we introduce a formal symbol f^s and set

$$\mathbf{F}[s] = A_f[s] \cdot f^s, \quad \mathbf{F}(s) = A_f(s) \cdot f^s.$$

Then $\mathbf{F}[s]$ and $\mathbf{F}(s)$ can be given the structure of $D_f[s]$ and $D_f(s)$ modules respectively via the expected rule that for each $1 \leq i \leq n$,

$$\partial_i(g \cdot f^s) = (\partial_i g + sgf^{-1}(\partial_i f)) \cdot f^s,$$

where $g \in A_f[s]$ or $g \in A_f(s)$.

We can now generalize these constructions with an arbitrary left (or right) D-module M. Define the $D_f[s]$ and $D_f(s)$ modules respectively as

$$M_f[s]f^s = M \otimes_A \mathbf{F}[s],$$
$$M_f(s)f^s = M \otimes_A \mathbf{F}(s) = M_f(s) \otimes_{A_f(s)} \mathbf{F}(s).$$

A general element of $M_f[s]f^s$ is a sum of pure tensors of the form $m \otimes_A (g(s) \cdot f^s)$, where $m \in M$ and $g(s) \in A_f[s]$.

For each integer $\ell \in \mathbb{Z}$ there is a specialization morphism $\Phi_\ell : M_f[s]f^s \to M_f$ that acts as setting the variable $s = \ell$. To be precise,

$$\Phi_\ell(m \otimes (g(s) \cdot f^s)) = g(\ell)f^\ell m.$$

Then if $P(s) \in D_f[s]$ and $v \in M_f[s]f^s$, $\Phi_\ell(P(s)v) = P(\ell)\Phi_\ell(v)$. Note Φ_ℓ is not defined on $M_f(s)f^s$ as there would be issues with poles at ℓ. We can now proceed with the main result.

Theorem 2.16. [MN91, Theorem 3.1.1] *Maintaining the notation as above, fix $u \in M$ where M is of minimal dimension. Then there exists a nonzero polynomial $b(s) \in k[s]$ and an operator $P(s) \in D[s]$ such that*

$$b(s)(u \otimes f^s) = P(s)(u \otimes ff^s),$$

with this equality occurring in $M_f[s]f^s$.

Proof. By Theorem 2.15, $D(s)$ is of type Diff, so if we prove that $\operatorname{Ext}^i_{D_f(s)}(M_f(s)f^s, D_f(s)) = 0$ whenever $i \neq \operatorname{dh}(D(s))$ then we can apply Proposition 2.14. To prove this, we have the following isomorphisms:

$$\begin{aligned}\operatorname{Ext}^i_{D_f(s)}(M_f(s) \otimes_{A_f(s)} \mathbf{F}(s), D_f(s)) &\cong \operatorname{Ext}^i_{D_f(s)}(M_f(s), \operatorname{Hom}_{A_f(s)}(\mathbf{F}(s), D_f(s)) \\ &\cong \operatorname{Ext}^i_{D_f(s)}(M_f(s), D_f(s)) \otimes_{D_f(s)} \operatorname{Hom}_{A_f(s)}(\mathbf{F}(s), D_f(s)) \\ &\cong \operatorname{Ext}^i_{D_f}(M_f, D_f) \otimes_{D_f} \operatorname{Hom}_{A_f(s)}(\mathbf{F}(s), D_f(s)).\end{aligned}$$

The first isomorphism uses that $\mathbf{F}(s)$ is a projective, flat $A_f(s)$-module, so the result follows by derived tensor hom adjunction. The second isomorphism uses that $D_f(s)$ has finite global dimension, and the third uses flat base change for Ext along $D_f \to D_f(s)$.

But since M is of minimal dimension, using that Ext commutes with localization,

$$\operatorname{Ext}^i_{D_f}(M_f, D_f) = \operatorname{Ext}^i_D(M, D) \otimes_A A_f = 0$$

if $i \neq \operatorname{dh}(D)$. Thus $\operatorname{Ext}^i_{D_f(s)}(M_f(s)f^s, D_f(s)) = 0$ if $i \neq \operatorname{dh}(D)$ which equals $\operatorname{dh}(D(s))$ by Theorem 2.15. Then by Proposition 2.14 there exists a $D(s)$-submodule of minimal dimension N of $M_f(s)f^s$ such that $N_f = M_f(s)f^s$.

For any $u \in M$, consider $u \otimes f^s \in M_f(s)f^s$. Then since $N_f = M_f(s)f^s$ there exists an integer $\ell \in \mathbb{N}$ such that $f^\ell(u \otimes f^s) \in N$. Then we have the chain of submodules in N,

$$D(s)f^\ell(u \otimes f^s) \supseteq D(s)f^{\ell+1}(u \otimes f^s) \supseteq \cdots.$$

Since N is of minimal dimension, by Proposition 2.13 this chain stabilizes, so there exists $r \in \mathbb{N}$ such that

$$f^{\ell+r}(u \otimes f^s) \in D(s)f^{\ell+r+1}(u \otimes f^s).$$

Multiplying through to cancel out denominators, we have that there exists a polynomial $b(s) \in k[s]$ and an operator $P(s) \in D[s]$ such that

$$b(s)f^{\ell+r}(u \otimes f^s) = P(s)f^{\ell+r+1}(u \otimes f^s). \tag{2.2}$$

In the proof of Theorem 1.45 we considered the shift function that captured the idea of $s \mapsto s+1$. We have a similar map here, a D_f-linear automorphism $\varphi : M_f[s]f^s \to M_f[s]f^s$ defined by $\varphi(m \otimes g(s)f^s) = m \otimes (g(s+1)f \cdot f^s)$. If we apply $\varphi^{-\ell-r}$ to (2.2) then we have

$$b(s-\ell-r)(u \otimes f^s) = P(s-\ell-r)f(u \otimes f^s).$$

Redefining b and P with the appropriate shifts then gives $b(s) \in k[s]$ an $P(s) \in D[s]$ such that

$$b(s)(u \otimes f^s) = P(s)f(u \otimes f^s). \tag{2.3}$$

□

As a corollary to the existence of the Bernstein-Sato polynomial for $k[X]$ we proved that $k[X, f^{-1}]$ is a finitely generated A_n-module. A similar proof yields that immediate corollary in this more general case as well.

Corollary 2.17. *If M is a D module of minimal dimension, then M_f is finitely generated over D.*

Once again we also define the Bernstein-Sato polynomial of $u \in M$ relative to $f \in A$ as the unique monic generator of the ideal of polynomials $b(s)$ that satisfy (2.3) for some $P(s) \in D[s]$.

2.4. **The example of the Tate algebra.** In this section we'll introduce the Tate Algebra and apply Theorem 2.16 to this setting. We now fix K to be a complete non-archimedean field of characteristic 0. Then the Tate Algebra $K\langle X_1, \ldots, X_n\rangle$ is the ring of convergent power series over the polydisc in K^n. Explicitly we have the definition

$$K\langle X\rangle := K\langle X_1, \ldots, X_n\rangle = \left\{ \sum_{\alpha \in \mathbb{N}^n} c_\alpha X^\alpha : c_\alpha \to 0 \text{ as } |\alpha| \to \infty \right\},$$

with the coefficients $c_\alpha \in K$. The Tate algebra has many properties similar to the polynomial ring. In fact, taking $A = K\langle X\rangle$, then A is a commutative Noetherian regular algebra that also satisfies the conditions (1), (2) and (3) listed at the start of Section 2.1. The explicit elements for condition (3) are $X_1, \ldots, X_n \in A$ and $\partial_1, \ldots, \partial_n \in \mathrm{Der}_K(A)$ defined as one might expect by

$$\partial_i\Big(\sum_{\alpha \in \mathbb{N}^n} c_\alpha X^\alpha\Big) = \sum_{\alpha \in \mathbb{N}^n} \alpha_i c_\alpha X^{\alpha - e_i}.$$

This still lies in the Tate algebra because in a non-archimedean field every integer has norm ≤ 1.

To apply Theorem 2.16 in this case we claim that $A = K\langle X\rangle$ is a module of minimum dimension over $D_{A/K}$. Björk proved in general that for an algebra A satisfying the above properties that $\mathrm{dh}(D_{A/K}) = \dim(A) = n$ (see [Bjo72]). We also know in this case

$$\mathrm{gr}(D_{A/K}) \cong A[\xi_1, \ldots, \xi_n],$$

the polynomial ring in n variables over A, so $\dim \mathrm{gr}(D_{A/K}) = 2n$. Hence to prove $K\langle X\rangle$ has minimal dimension we must prove $\dim_{D_{A/K}}(K\langle X\rangle) = n$. Consider the constant filtration Γ on A, that is $\Gamma_i = A$ for all $i \geq 0$. This is a good filtration as each Γ_i is finitely generated over $D^0_{A/K} = A$ and $\mathrm{gr}^\Gamma A \cong A$ is generated by 1 over $\mathrm{gr}(D_{A/K})$. The annihilator is $(\xi_1, \ldots, \xi_n)$

since multiplication by ξ_i will lift the degree in $\mathrm{gr}^\Gamma A$, but $\mathrm{gr}^\Gamma A$ is concentrated entirely in the zeroth graded piece. Hence

$$\dim_{D_{A/K}}(K\langle X\rangle) = \dim(A[\xi_1,\dots,\xi_n]/(\xi_1,\dots,\xi_n)) = \dim(A) = n.$$

Thus $A = K\langle X\rangle$ is a minimal dimension module over $D_{A/K}$.

Now consider $Y = \mathrm{Sp}(K\langle X\rangle)$ to be the unit polydisc, and $Z = \{f = 0\}$ a hypersurface. Let $\mathcal{O}_Y(*Z)(Y)$ denote the meromorphic functions on the disc that have singularities along Z. In particular, $\mathcal{O}_Y(*Z)(Y) = K\langle X\rangle[f^{-1}]$. Then the existence of the b-function implies (by Corollary 2.17) that $\mathcal{O}_Y(*Z)(Y)$ is finitely generated as a $D_{A/K}$-module.

3. $\widehat{\mathcal{D}}$-modules

3.1. **Locally convex vector spaces, algebras, and modules.** For this next section we need some theory of locally convex vector spaces. Fix K a non-archimedean field, and R the ring of integers. Let V be a K-vector space.

Definition 3.1. A lattice $L \subseteq V$ is an R-submodule such that for all $v \in V$, there exists $a \in K^\times$ such that $av \in L$.

Example 3.2. If $V = K^n$, then $R^n \subseteq K^n$ is a lattice.

Example 3.3. If $V = K\langle x\rangle$ is the Tate algebra in one variable, then $R\langle x\rangle$ is a lattice.

Certain families of lattices form the building blocks of locally convex vector spaces.

Definition 3.4. Let $(L_i)_{i\in I}$ be a family of lattices in V such that the following conditions hold:

(1) For any $i \in I$ and $a \in K^\times$, there exists $j \in I$ such that $L_j \subseteq aL_i$.
(2) For any $i, j \in I$ there exists a $k \in I$ such that $L_k \subseteq L_i \cap L_j$.

The *locally convex topology* on V from the family $(L_i)_{i\in I}$ is then the topology with basis opens $v + L_i$ for $v \in V$.

Proposition 3.5. *The sets of the form $v + L_i$ do in fact form a basis for a topology.*

Proof. These sets clearly cover the entire space V. If two sets $v + L_i$ and $u + L_j$ intersect nontrivially, let w be an element in their intersection. Then $v + L_i = w + L_i$ and $u + L_j = w + L_j$, and

$$(w + L_i) \cap (w + L_j) = w + (L_i \cap L_j).$$

Then applying condition (2) from Definition 3.4 there exists L_k such that $w + L_k \subseteq w + (L_i \cap L_j)$. This shows the sets $v + L_i$ form a basis. □

Then a locally convex vector space V is just a vector space with a locally convex topology. The addition and scalar multiplication maps can be shown to be continuous.

Definition 3.6. A locally convex algebra A is a locally convex vector space such that the multiplication map is continuous.

Proposition 3.7. *Let A be a K-algebra and $(L_i)_{i\in I}$ a family of lattices giving a locally convex topology on A. If each lattice L_i is in fact a R-algebra, then A is a locally convex algebra.*

Lastly in similar fashion we can define a locally convex module.

Definition 3.8. A locally convex module M over a locally convex algebra A is an A-module with a locally convex topology such that the scalar multiplication map $A \times M \to M$ is continuous.

Another way to think about locally convex vector spaces is as arising from a family of seminorms.

Definition 3.9. A seminorm q on a K-vector space V is a map $V \to \mathbb{R}_{\geq 0}$ such that for all $v, w \in V$, $a \in K$,

(1) $q(v+w) \leq \max(q(v), q(w))$,
(2) $q(av) = |a|q(v)$.

Let $(q_i)_{i \in I}$ be a family of seminorms on V. Then the induced topology on V is the coarsest topology such that each map $q_i : V \to \mathbb{R}$ is continuous and translation maps $w \mapsto v + w$ are continuous on V. This turns out to be the same (see [Sch02]) as the topology given by the family of lattices of the form

$$U(q_{i_1}, \ldots, q_{i_n}; \varepsilon) = \{v \in V \mid q_{i_j}(v) \leq \varepsilon \text{ for each } 1 \leq j \leq n\}.$$

3.2. **The completed ring of differential operators $\widehat{\mathcal{D}}$.** Let $X = \mathrm{Sp}A$ be a smooth affinoid variety with free tangent sheaf, i.e. there exists $\partial_1, \ldots, \partial_m \in \mathrm{Der}_K(A)$ that form a basis for $\mathrm{Der}_K(A)$. Then we have the ring of differential operators

$$D_{A/K} = \left\{ \sum_{\alpha \in \mathbb{N}^m} g_\alpha \partial^\alpha \mid g_\alpha \in A, \text{ all but finitely many } g_\alpha \text{ are } 0 \right\}.$$

Let $\| \cdot \|$ be a Banach norm on A arising from writing A as a quotient of a Tate algebra with the Gauss norm. Fix $\pi \in K$ with $0 < |\pi| < 1$. Let $D := D_{A/K}$. We can define a family of submultiplicative Banach norms on D indexed by $\mathbb{N}$ by

$$\left\| \sum_\alpha g_\alpha \partial^\alpha \right\|_n = \sup_\alpha \|g_\alpha\| |\pi|^{-n|\alpha|}.$$

Then we have a unit-ball lattice $\Lambda_n = \{F \in D \mid \|F\|_n \leq 1\}$. Completing D with respect to this norm we obtain the algebra

$$D_n(X) = \left\{ \sum_{\alpha \in \mathbb{N}^m} g_\alpha \partial^\alpha \mid g_\alpha \in A, \|g_\alpha\| |\pi|^{-n|\alpha|} \to 0 \text{ as } |\alpha| \to \infty \right\}.$$

Note we have inclusion maps $D_{n+1}(X) \to D_n(X)$. Then the completed ring of differential operators on X is

$$\widehat{\mathcal{D}}(X) = \varprojlim D_n(X) = \bigcap_{n \geq 0} D_n(X) = \left\{ \sum_{\alpha \in \mathbb{N}^m} g_\alpha \partial^\alpha \mid \|g_\alpha\| r^{|\alpha|} \to 0 \text{ as } |\alpha| \to \infty \text{ for all } r > 0 \right\},$$

where the $g_\alpha \in A$. This is the ring of global sections of the more general completed sheaf of differential operators $\widehat{\mathcal{D}}_X$. For affinoid subdomains $Y \subseteq X$, the ring $\widehat{\mathcal{D}}(Y)$ has the same form as $\widehat{\mathcal{D}}(X)$ above except the functions g_α are now in $\mathcal{O}_X(Y)$. For modules over $\widehat{\mathcal{D}}(X)$ there exists a finiteness condition known as coadmissibility. The definition of a coadmissible module will arise from the structure of $\widehat{\mathcal{D}}(X)$ as a Fréchet-Stein algebra.

3.3. **Fréchet-Stein algebras and coadmissibility.**

Definition 3.10. A topological k-algebra A is a two-sided Fréchet-Stein algebra if there exist two-sided Noetherian Banach k-algebras $A_1, A_2, \dots$ such that $A = \varprojlim A_n$ and the morphisms $A_{n+1} \to A_n$ have dense image and make A_n into a flat A_{n+1}-module on both sides.

Proposition 3.11. *$\widehat{\mathcal{D}}(X)$ is a Fréchet-Stein algebra.*

Proof. A full proof can be found in [AW19, Section 6]. Below we will just show part of the statement, specifically that the maps $D_{n+1}(X) \to D_n(X)$ given by inclusion have dense image. These inclusion maps are clearly contractions hence continuous.

Note for all $n \in \mathbb{N}$, $D_n(X)$ contains all finite sums of differential operators, so $\mathcal{D}(X) \subseteq D_n(X)$. If $s = \sum_{\alpha \in \mathbb{N}^m} g_\alpha \partial^\alpha \in D_n(X)$, then define $s_\ell = \sum_{|\alpha| \leq \ell} g_\alpha \partial^\alpha \in \mathcal{D}(X)$. It is clear that $s_\ell \to s$ in $D_n(X)$, and each s_ℓ is in the image of the inclusion $D_{n+1}(X) \to D_n(X)$, thus this inclusion has dense image. □

From this Fréchet-Stein structure there is a canonical topology on $\widehat{\mathcal{D}}(X)$. The norms on each D_n induce a norm on $\widehat{\mathcal{D}}(X)$. Then the locally convex topology on $\widehat{\mathcal{D}}(X)$ comes from this family of norms. Specifically as we described before the topology is generated by lattices of the form

$$U(n_1, \dots, n_t; \varepsilon) = \{F \in \widehat{\mathcal{D}}(X) \mid \|F\|_{n_i} \leq \varepsilon \text{ for each } 1 \leq i \leq t\}.$$

Definition 3.12. A module M over a Fréchet-Stein algebra $A = \varprojlim A_n$ is coadmissible if for all n,

$$M_n := A_n \otimes_A M$$

is finitely generated over A_n, and $M = \varprojlim M_n$.

In [ST03, Corollary 3.5] they show that the category of coadmissible modules is a full abelian subcategory of $\mathrm{Mod}(A)$. In particular it is closed under cokernels, so since A^ℓ is always coadmissible any finitely presented A-module is coadmissible.

Now let $f \in A$ and $Z = \{f = 0\}$ be a hypersurface in X. Let $U = X - Z$, and let $j : U \hookrightarrow X$ be the inclusion map. Take $U_n = \{x \in X \mid |f(x)| \geq |\pi|^n\}$, then $\mathcal{O}_X(U_n) = A\langle \pi^n f^{-1} \rangle$, and $\mathcal{O}_X(U) = \varprojlim A\langle \pi^n f^{-1} \rangle$. The sheaf $\widehat{\mathcal{D}}_U$ has global sections

$$\widehat{\mathcal{D}}(U) = \bigcap_{n \geq 0} \widehat{\mathcal{D}}(U_n) = \left\{ \sum_\alpha g_\alpha \partial^\alpha \mid g_\alpha \in \mathcal{O}_X(U), \|g_\alpha\|_{U_n} r^{|\alpha|} \to 0 \text{ for all } n \text{ and } r > 0 \right\}$$

Then $\widehat{\mathcal{D}}(U)$ is also a Fréchet-Stein algebra. We have $\widehat{\mathcal{D}}(U) = \varprojlim D_n(U_n)$, where $D_n(U_n)$ is defined similarly to $D_n(X)$ except the functions lie in $\mathcal{O}(U_n)$, and the Banach norm is $\|\cdot\|_{U_n}$ on $\mathcal{O}(U_n)$ arising from a presentation of $\mathcal{O}(U_n)$ as an affinoid algebra.

3.4. **Extending meromorphic connections.** Let N be a module over $D_{A[f^{-1}]/K} = D[f^{-1}]$ that is finitely generated as an $A[f^{-1}]$-module with generating set $m_1, \dots, m_k$. Such N is called a meromorphic connection with singularities along Z. The ring $A[f^{-1}] = A[t]/(1 - ft)$ is still commutative, Noetherian and regular, as well as satisfies conditions (1)-(3) in Section 2.1 by extending the derivations on A to $A[f^{-1}]$ via the quotient rule. Thus the ring $D[f^{-1}]$ is of type Diff, and $\mathrm{gr}(D[f^{-1}]) = A_f[\xi_1, \dots, \xi_m]$, the symbols of $\partial_1, \dots, \partial_m$. Since N is

finitely generated over $(D[f^{-1}])^0 = A[f^{-1}]$, the constant filtration $\Gamma_i = N$ is a good filtration on N over $D[f^{-1}]$. Then $\mathrm{gr}^\Gamma N$ is concentrated in degree zero, so $(\xi_1, \dots, \xi_m)$ annihilates $\mathrm{gr}^\Gamma N$. Thus

$$\dim_{D[f^{-1}]} N \leq \dim(A[f^{-1}][\xi_1, \dots, \xi_m]/(\xi_1, \dots, \xi_m)) = \dim A[f^{-1}] = m.$$

Since $\mathrm{dh}(D) = \mathrm{dh}(D[f^{-1}]) = m$, by Theorem 2.10, $\dim_{D[f^{-1}]} N \geq 2m - m = m$, so the above must be an equality. Thus N is of minimal dimension over $D[f^{-1}]$, so there exists a D-submodule $N' \subseteq N$ of minimal dimension over D such that $N'[f^{-1}] = N$. There exists $\ell_i \in \mathbb{Z}$ such that $f^{\ell_i} m_i \in N'$, then replacing m_i with $f^{\ell_i} m_i$, Theorem 2.16 implies that there exist b-functions $b_i'(s) \in K[s]$ and a differential operator $P_i'(s) \in D[s]$ such that $P_i'(s) f^{s+1} m_i = b_i'(s) f^s m_i$. This does not change the m_i generating N.

To match the conventions of Bitoun and Bode, we will now define $b_i(s) = b_i'(-s-1)$, $P_i(s) = P_i'(-s-1)$ and work with $b_i(s)$ as the b-functions, so that

$$P_i(s) f^{-s} m_i = b_i(s) f^{-s-1} m_i$$

The existence of these b-functions shows that N is finitely generated as a D-module. Since D is Noetherian, N is also finitely presented as a D-module. Tensoring with $\widehat{\mathcal{D}}(X)$ gives a finitely presented and therefore coadmissible $\widehat{\mathcal{D}}(X)$-module, $\widehat{\mathcal{D}}(X) \otimes_D N$. This is a sort of completion of N.

Because of the natural morphism $D[f^{-1}] \to \widehat{\mathcal{D}}(U)$, we can define another "completion" of N by extending scalars as

$$M := \widehat{\mathcal{D}}(U) \otimes_{D[f^{-1}]} N.$$

Although M has the structure of a $\widehat{\mathcal{D}}(U)$-module, as a $\mathcal{O}(U)$-module M is just $\mathcal{O}(U) \otimes_{A[f^{-1}]} N$.

Since there is a restriction map $\widehat{\mathcal{D}}(X) \to \widehat{\mathcal{D}}(U)$, M also has the structure of a $\widehat{\mathcal{D}}(X)$-module, and it is this structure that is of interest. There is a canonical morphism of D-modules $N \to M$ just by sending $n \mapsto 1 \otimes n$. But since M is a $\widehat{\mathcal{D}}(X)$-module, we can extend this to a morphism of $\widehat{\mathcal{D}}(X)$-modules

$$\theta : \widehat{\mathcal{D}}(X) \otimes_D N \to M = \widehat{\mathcal{D}}(U) \otimes_{D[f^{-1}]} N.$$

The paper by Bitoun and Bode tries to answer the question of when M is coadmissible as a $\widehat{\mathcal{D}}(X)$-module. This is done by studying the morphism θ, and they prove in fact in [BB21] that M is coadmissible if and only if θ is an isomorphism. The main result of the paper gives a sufficient condition for θ to be an isomorphism that is based on the "type" of the roots of the polynomials b_i. We will now define type.

Definition 3.13. Let $\lambda \in K$. Say λ has positive type if $\lambda \in \mathbb{Z}_{\geq 0}$ or if there exists $r \in \mathbb{Z}$ such that

$$\frac{\pi^{\ell r}}{\prod_{i=0}^{\ell-1}(\lambda - i)} \to 0, \text{ as } \ell \to \infty.$$

If λ does not have positive type then say it has type zero.

It is a fact that λ has positive type if and only if $\lambda - n$ has positive type for $n \in \mathbb{Z}$.

We can now state the main theorem of Bitoun and Bode.

Theorem 3.14. [BB21, Theorem 4.2] *If the roots of $b_i(s)$ all have positive type for each $1 \leq i \leq k$, then the morphism*

$$\theta : \widehat{\mathcal{D}}(X) \otimes_D N \to M = \widehat{\mathcal{D}}(U) \otimes_{D[f^{-1}]} N$$

is an isomorphism. In particular M is a coadmissible $\widehat{\mathcal{D}}(X)$-module.

As before let N be generated by $m_1, \dots, m_k$ as an $A[f^{-1}]$-module. Each polynomial b_i has finitely many roots, so there exists $r \in \mathbb{Z}_{\geq 0}$ such that $b_i(\ell) \neq 0$ for all $\ell \in \mathbb{Z}$ and $\ell \geq r$. Hence replacing m_i with $f^{-r}m_i$, we can always assume that $b_i(\ell) \neq 0$ for all $\ell \in \mathbb{Z}_{\geq 0}$. This does not change the type of the roots because of the integer shift invariance. Then since we can divide by $b_i(\ell)$, we can obtain all negative powers of f, so $m_1, \dots, m_k$ generate N as a D-module.

If we set $s = \ell$ for any $\ell \in \mathbb{Z}_{\geq 0}$, then

$$P_i(\ell)f^{-\ell}m_i = b_i(\ell)f^{-\ell-1}m_i \implies \frac{P_i(\ell)}{b_i(\ell)}f^{-\ell}m_i = f^{-\ell-1}m_i.$$

Taking $\ell = 0$,

$$\frac{P_i(0)}{b_i(0)}m_i = f^{-1}m_i.$$

Inductively we see that

$$\frac{P_i(\ell-1)\cdots P_i(0)}{b_i(\ell-1)\cdots b_i(0)}m_i = f^{-\ell}m_i, \tag{3.1}$$

with this equality taking place in N.

Lemma 3.15. *Fix $n \in \mathbb{N}, 1 \leq i \leq k$. If all roots of $b_i(s)$ are positive type, then there exists $r \in \mathbb{N}$ such that*

$$\pi^{\ell r}\frac{P_i(\ell-1)\cdots P_i(0)}{b_i(\ell-1)\cdots b_i(0)} \in \Lambda_n$$

for all $\ell \geq 1$.

Proof. Recall Λ_n is the unit ball of the submultiplicative norm $\|\cdot\|_n$ on D. This norm can be extended to a submultiplicative norm on $\mathcal{D}(X)[s]$ by setting $|s| = 1$. Then the evaluation morphism $\mathcal{D}(X)[s] \to \mathcal{D}(X)$ defined by $s \mapsto t \in \mathbb{Z}$ is a contraction map because $|t| \leq 1$. So for each integer t, $\|P_i(t)\|_n \leq \|P_i(s)\|_n$. Now pick an integer r' such that $\|P_i(s)\|_n \leq |\pi|^{r'}$. Then for any $\ell \geq 1$,

$$\|P_i(\ell-1)\cdots P_i(0)\|_n \leq |\pi|^{\ell r'}.$$

To obtain bounds on the denominator, in the algebraic closure of K let $b_i(s)$ factor as $b_i(s) = \prod_{k=1}^{d}(s - \beta_k)$. Then

$$b_i(\ell-1)\cdots b_i(0) = \prod_{j=0}^{\ell-1}(j-\beta_1)\cdots\prod_{j=0}^{\ell-1}(j-\beta_d).$$

By assumption each β_k is of positive type, so let $r_k \in \mathbb{Z}$ be such that

$$\frac{\pi^{\ell r_k}}{\prod_{j=0}^{\ell-1}(j-\beta_k)} \to 0, \text{ as } \ell \to \infty.$$

Then

$$\frac{\pi^{\ell(r_1+\cdots+r_d)}}{b_i(\ell-1)\cdots b_i(0)} \to 0, \text{ as } \ell \to \infty.$$

This sequence above converges to 0 so in particular it must be bounded. Let the maximum absolute value of any term be $C \geq 0$. Then choose an integer $r'' \in \mathbb{Z}$ large enough so that for all $\ell \geq 1$,

$$\left|\pi^{\ell(r''+r'-(r_1+\cdots+r_d))}\right| \leq \frac{1}{C}.$$

Then for all $\ell \geq 1$,

$$\left|\frac{\pi^{\ell(r''+r')}}{b_i(\ell-1)\cdots b_i(0)}\right| \leq \frac{1}{C}\cdot C = 1,$$

so

$$\left|\frac{\pi^{\ell r''}}{b_i(\ell-1)\cdots b_i(0)}\right| \leq |\pi|^{-\ell r'}.$$

Hence lastly for all $\ell \geq 1$ this implies

$$\left\|\pi^{\ell r''}\frac{P_i(\ell-1)\cdots P_i(0)}{b_i(\ell-1)\cdots b_i(0)}\right\|_n \leq 1.$$

□

In the proof of Theorem 3.14 we will need to consider two different locally convex topologies on N. First recall that D has a canonical topology from the family of norms $\{\|\cdot\|_n\}_{n\in\mathbb{N}}$ with unit balls Λ_n.

Let T_1 be the quotient topology on N coming from the surjection $D^k \twoheadrightarrow N$. Then concretely the topology T_1 is generated by unit balls of the form

$$\sum \Lambda_{n_i}\cdot m_i$$

for some $n_i \in \mathbb{N}$.

Similarly let T_2 be the quotient topology on N coming from the surjection $D[f^{-1}]^k \twoheadrightarrow N$. The topology on $D[f^{-1}]$ is generated by unit balls of the form

$$\Lambda_n[\pi^n f^{-1}].$$

This means these balls and their scalings and intersections form a basis of neighborhoods of 0. Then the topology T_2 on N is generated by unit balls

$$\sum \Lambda_{n_i}[\pi^{n_i} f^{-1}]\cdot m_i$$

Let $\widehat{N}$ be the Hausdorff completion of N with respect to T_1. If we consider the presentation where the map $D^k \to N$ is strict,

$$D^\ell \longrightarrow D^k \longrightarrow N \longrightarrow 0,$$

then tensoring with $\widehat{\mathcal{D}}(X)$ gives a presentation of $\widehat{\mathcal{D}}(X)\otimes_D N$,

$$\widehat{\mathcal{D}}(X)^\ell \longrightarrow \widehat{\mathcal{D}}(X)^k \longrightarrow \widehat{\mathcal{D}}(X)\otimes_D N \longrightarrow 0.$$

But Hausdorff completion is a left adjoint and thus a right exact functor, so we have the same presentation

$$\widehat{\mathcal{D}}(X)^\ell \longrightarrow \widehat{\mathcal{D}}(X)^k \longrightarrow \widehat{N} \longrightarrow 0.$$

Thus $\widehat{N} = \widehat{\mathcal{D}}(X) \otimes_D N$. A similar proof shows that the completion of N with respect to T_2 gives $\widehat{\mathcal{D}}(U) \otimes_{D[f^{-1}]} N$. Thus the strategy of the proof of Theorem 3.14 will be to show T_1 and T_2 are equivalent so that their completions are isomorphic.

Proof of Theorem 3.14. Since $\Lambda_n \subseteq \Lambda_n[\pi^n f^{-1}]$, T_1 is finer than T_2. Thus if we consider the surjection $D^k \twoheadrightarrow (N, T_2)$, then it must be continuous. We will show now that it is strict as well, which will immediately imply that $T_1 = T_2$.

A continuous surjection is strict if and only if it is open. If we show that for each lattice Λ_n that $\Lambda_n \cdot m_i$ contains some $\Lambda_t[\pi^t f^{-1}] \cdot m_i$ this would imply that the image of each unit ball is open so ϕ is open. So fix n, and let $r \in \mathbb{N}$ be such that

$$\pi^{\ell r} \frac{P_i(\ell-1)\cdots P_i(0)}{b_i(\ell-1)\cdots b_i(0)} \in \Lambda_n$$

for all $\ell \geq 0$.

Now consider $x \in \Lambda_t[\pi^t f^{-1}] \cdot m_i$ for some $t \in \mathbb{N}$. Then x is a sum of terms $g_{\ell,\alpha}(\pi^t \partial)^\alpha (\pi^t f^{-1})^\ell \cdot m_i$ where $g_{\ell,\alpha} \in A$ and $\|g\| \leq 1$. Thus we can write

$$x = \left(\sum_{\alpha \in \mathbb{N}^m, \ell \geq 0} \pi^{t(|\alpha|+\ell)} g_{\ell,\alpha} \partial^\alpha f^{-\ell} \right) \cdot m_i,$$

where the sum is finite. Applying the identity in equation (3.1) we have

$$x = \left(\sum_{\alpha,\ell} \pi^{t(|\alpha|+\ell)} g_{\ell,\alpha} \partial^\alpha \frac{P_i(\ell-1)\cdots P_i(0)}{b_i(\ell-1)\cdots b_i(0)} \right) \cdot m_i. \tag{3.2}$$

The operator acting on m_i now lies in D. Set $t = n + r$. Then rearranging,

$$\pi^{t(|\alpha|+\ell)} g_{\ell,\alpha} \partial^\alpha \frac{P_i(\ell-1)\cdots P_i(0)}{b_i(\ell-1)\cdots b_i(0)} = \pi^{(n+r)|\alpha|+n\ell} g_{\ell,\alpha} \partial^\alpha \pi^{r\ell} \frac{P_i(\ell-1)\cdots P_i(0)}{b_i(\ell-1)\cdots b_i(0)}.$$

By the definition of r, $\pi^{r\ell} \frac{P_i(\ell-1)\cdots P_i(0)}{b_i(\ell-1)\cdots b_i(0)} \in \Lambda_n$ for all $\ell \geq 0$. Also $(n+r)|\alpha| + n\ell \geq n|\alpha|$, so $(\pi^{(n+r)|\alpha|+n\ell} g_{\ell,\alpha} \partial^\alpha) \in \Lambda_n$, hence the differential operator acting on m_i in (3.2) is in Λ_n. Then $x \in \Lambda_n \cdot m_i$, so

$$\Lambda_{n+r}[\pi^{n+r} f^{-1}] \cdot m_i \subseteq \Lambda_n \cdot m_i.$$

So ϕ is strict and $T_1 = T_2$.

The identity $(N, T_1) \to (N, T_2)$ as a map of locally convex vector spaces is now an isomorphism. Thus when we pass to the completions, the map

$$\theta : \widehat{\mathcal{D}}(X) \otimes_D N \to M = \widehat{\mathcal{D}}(U) \otimes_{D[f^{-1}]} N$$

must be an isomorphism. In particular since $\widehat{\mathcal{D}}(X) \otimes_D N$ is a coadmissible $\widehat{\mathcal{D}}(X)$-module, $\widehat{\mathcal{D}}(U) \otimes_{D[f^{-1}]} N$ is as well. □

Bitoun and Bode in their paper [BB21] give a family of $D[x^{-1}]$-modules N_λ that are finitely generated over $A[x^{-1}]$ such that $\widehat{\mathcal{D}}(U) \otimes_{D[x^{-1}]} N_\lambda$ is not a coadmissible $\widehat{\mathcal{D}}(X)$-module. This example which we detail below shows that the positive type condition is not just a condition required for the proof, but it is a condition that is at least in some cases actually necessary.

Let $X = \mathrm{Sp} K\langle x \rangle$ and $U = X - \{0\}$ the punctured disc. Fix $\lambda \in K^\times$. Let x^λ be a formal symbol and $N_\lambda = A[x^{-1}] \cdot x^\lambda$ where given $g \cdot x^\lambda$ D acts on N_λ by

$$\partial(g \cdot x^\lambda) = (\partial(g) + g\lambda x^{-1}) x^\lambda.$$

Then set

$$M_\lambda = \widehat{\mathcal{D}}(U) \otimes_{D[x^{-1}]} N_\lambda = \mathcal{O}(U) \cdot x^\lambda.$$

Theorem 3.16. [BB21, Theorem 5.2] *M_λ is coadmissible as a $\widehat{\mathcal{D}}(X)$-module if and only if λ is of positive type.*

Proof. The full proof can be found in the paper, we will highlight the idea of the proof though. First, since shifts $\lambda \mapsto \lambda - n$ do not affect the type, we can assume that x^λ generates N_λ as a D-module. Then

$$\partial x^{-s} \cdot x^\lambda = (\lambda - s) x^{-s-1} \cdot x^\lambda,$$

so the b-function has its only root as λ. If λ is of positive type, then by Theorem 3.14 M_λ is coadmissible.

The interesting part is then the converse, see [BB21] for the proof. □

Part II. Beilinson-Bernstein Localization

4. Lie Groups and Algebras

A Lie group is a smooth manifold with a group structure, such that the multiplication and inverse maps are differentiable. A variant of this that we consider is an algebraic group. Throughout we will take all of our varieties to be over $\mathbb{C}$.

Definition 4.1. An affine algebraic group G is an affine algebraic variety with a group structure such that the multiplication map and inverse maps are morphisms. We will assume all our algebraic groups are connected.

Any algebraic group over $\mathbb{C}$ also has the structure of a Lie group. Many examples of Lie/algebraic groups are given by groups of matrix operators, such as $\mathrm{GL}(n, \mathbb{C})$ and $\mathrm{SL}(n, \mathbb{C})$. In fact, any affine algebraic group, also known as linear algebraic group, can be embedded into $\mathrm{GL}(n, \mathbb{C})$. If $g \in G$, then multiplication by g is an isomorphism $G \to G$, and since a general point is smooth, G must be a smooth variety, so all algebraic groups are smooth. Note this holds since we are working just with varieties, and could fail when the algebraic groups are non-reduced group schemes.

As groups, Lie groups are often very large and complicated. To simplify the study of a Lie group, and in particular its representations, we can associate to a Lie group a corresponding Lie algebra which is often easier to study. Instead of studying a Lie group, we can consider its "linearization" and work with the vector space T_eG, the tangent space of G at the identity e.

Let $\mathfrak{g}$ denote the tangent space T_eG, then $\mathfrak{g}$ is naturally a representation of G. In particular, given $g \in G$, consider the conjugation map $G \to G$, $h \mapsto ghg^{-1}$. Differentiating gives a map, denoted $\mathrm{Ad}(g)$ from $\mathfrak{g} \to \mathfrak{g}$. So $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ is a representation of G. Going one step further, take the differential of this map at the identity. Since $\mathrm{GL}(\mathfrak{g})$ is open in $\mathrm{End}(\mathfrak{g})$, its tangent space is $\mathrm{End}(\mathfrak{g})$, so we get a map, which we denote ad,

$$\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g}).$$

Given $x, y \in \mathfrak{g}$, the element $\mathrm{ad}(x)(y) \in \mathfrak{g}$ is denoted $[x, y]$, where $[\cdot, \cdot]$ is called the Lie bracket. It can be shown that the Lie bracket is alternating and satisfies Jacobi's identity. We detail these explicitly below in the general definition of a Lie algebra, of which $\mathfrak{g} = \mathrm{Lie}(G)$ is a specific example.

Remark 4.2. In the reverse direction, there is an exponential map taking an element of the Lie algebra back to the group. Each $x \in \mathfrak{g}$ determines a one parameter subgroup of G,

$$t \mapsto \exp(tx), \quad t \in \mathbb{C}.$$

This is a homomorphism from $(\mathbb{C}, +)$ to G, and the derivative at $t = 0$ recovers x. If $G \subset \mathrm{GL}(n)$ is a matrix group, then this recovers the standard exponential map for matrices. We will use later the facts that since G is connected it is generated by elements of the form $\exp(tx)$, $x \in \mathfrak{g}$, $t \in \mathbb{C}$, and secondly that $\mathrm{Ad}(\exp(tx)) = \exp(t\,\mathrm{ad}(x))$, where the right hand side is the usual operator exponential in $\mathfrak{g}$. This second fact follows from the fact that both are the unique solution to the linear ODE in $\mathfrak{g}$, $\gamma'(t) = \mathrm{ad}(x)\gamma(t)$.

We now introduce the general theory of Lie algebras that will be relevant in this paper. Many proofs and more in-depth explanation can be found in standard references on Lie algebras such as [Hum72] and [FH91].

Definition 4.3. A Lie algebra $\mathfrak{g}$ over the field k is a k-vector space with a bilinear operation known as the Lie bracket

$$[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

such that for all $x, y, z \in \mathfrak{g}$,

(1) $[x, y] = -[y, x]$.
(2) $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$.

The condition (2) above is known as Jacobi's Identity.

Throughout this paper all of our Lie algebras are considered to be finite dimensional.

Example 4.4. When $G \subset \mathrm{GL}(n)$ is a matrix group, the Lie bracket on $\mathrm{Lie}(G)$ equals the commutator bracket, $[X, Y] = XY - YX$.

Example 4.5. The corresponding Lie algebras to the earlier mentioned Lie groups $\mathrm{GL}(n)$ and $\mathrm{SL}(n)$ are:

(1) $\mathfrak{gl}_n = \{n \times n \text{ matrices}\}$.
(2) $\mathfrak{sl}_n = \{n \times n \text{ matrices with trace } 0\}$.

The Lie bracket on these algebras is the commutator.

If a subspace of a Lie algebra $\mathfrak{g}$ is closed under the Lie bracket, then we call it a Lie subalgebra, and it is itself a Lie algebra. We say a Lie algebra $\mathfrak{g}$ is abelian if $[x, y] = 0$ for all $x, y \in \mathfrak{g}$.

Definition 4.6. A homomorphism of Lie algebras $\varphi : \mathfrak{g} \to \mathfrak{h}$ is a linear map that respects the Lie bracket. i.e. $\varphi([x, y]) = [\varphi(x), \varphi(y)]$.

Definition 4.7. A representation of a Lie algebra $\mathfrak{g}$ is a vector space V and a homomorphism of Lie algebras $\mathfrak{g} \to \mathrm{End}(V)$.

Example 4.8. A Lie algebra $\mathfrak{g}$ is a representation of itself via the map

$$\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g}),$$
$$\mathrm{ad}(x)(y) = [x, y].$$

Jacobi's identity shows that this is indeed a map of Lie algebras.

4.1. **Semisimple Lie algebras.**

Definition 4.9. The derived series of a Lie algebra $\mathfrak{g}$ is defined as $D^0\mathfrak{g} = \mathfrak{g}$, and

$$D^{n+1}\mathfrak{g} = [D^n\mathfrak{g}, D^n\mathfrak{g}]$$

for $n \geq 0$.

Definition 4.10. A Lie algebra $\mathfrak{g}$ is called solvable if there exists $n \in \mathbb{N}$ such that $D^n\mathfrak{g} = 0$.

Definition 4.11. A subspace $\mathfrak{h} \subset \mathfrak{g}$ is an ideal if $[\mathfrak{g}, \mathfrak{h}] \subset \mathfrak{h}$.

In what follows we will restrict ourselves to a specific class of Lie algebras known as semisimple.

Definition 4.12. A Lie algebra $\mathfrak{g}$ is semisimple if it has no nonzero solvable ideals.

Example 4.13. The Lie algebra $\mathfrak{gl}_n$ is not semisimple. Consider the center $\mathfrak{z} = \{\lambda\mathbb{I} \mid \lambda \in \mathbb{C}\}$. Then $[\mathfrak{gl}_n, \mathfrak{z}] = 0$, so $\mathfrak{z}$ is an ideal. But $D^1\mathfrak{z} = 0$, so $\mathfrak{z}$ is solvable, proving $\mathfrak{gl}_n$ is not semisimple.

On the other hand, for all n, $\mathfrak{sl}_n$ is semisimple. The most important example we will return to throughout of a semisimple Lie algebra is $\mathfrak{sl}_2$. In fact, every finite dimensional complex semisimple Lie algebra is in a sense built from copies of $\mathfrak{sl}_2$ glued together.

An important object to study semisimple Lie algebras will be the Killing form.

Definition 4.14. The Killing form $K : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ is a bilinear form on $\mathfrak{g}$ defined as

$$K(x, y) = \mathrm{tr}(\mathrm{ad}(x)\mathrm{ad}(y)).$$

The Killing form will give us a useful way to identify $\mathfrak{g} \cong \mathfrak{g}^*$.

Now we will look at how a semisimple Lie algebra can be decomposed in a nice form with respect to a Cartan subalgebra.

Definition 4.15. A subalgebra $\mathfrak{h} \subset \mathfrak{g}$ of a semisimple Lie algebra is called Cartan if it is a maximal abelian subalgebra, and for all $h \in \mathfrak{h}$, $\mathrm{ad}(h) : \mathfrak{g} \to \mathfrak{g}$ is diagonalizable.

Theorem 4.16. *Every semisimple Lie algebra contains a Cartan subalgebra.*

Let $\mathfrak{g}$ be a semisimple Lie algebra and fix a Cartan subalgebra $\mathfrak{h}$. Since $\mathfrak{h}$ is abelian, all of the $\mathrm{ad}(h)$ for $h \in \mathfrak{h}$ commute, so they are simultaneously diagonalizable. Thus $\mathfrak{g}$ decomposes into a direct sum of subspaces $\mathfrak{g}_\alpha$, indexed by $\alpha \in \mathfrak{h}^*$, such that if $x \in \mathfrak{g}_\alpha$ and $h \in \mathfrak{h}$ then

$$\mathrm{ad}(h)(x) = \alpha(h)x.$$

Concretely we have the following.

Theorem 4.17. *Let $\mathfrak{g}$ be a semisimple Lie algebra, and $\mathfrak{h}$ a Cartan subalgebra. Then $\mathfrak{g}$ decomposes as*

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \mathfrak{h}^* - \{0\}} \mathfrak{g}_\alpha.$$

Furthermore, $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$, each $\mathfrak{g}_\alpha$ is one dimensional, and $K(\mathfrak{g}_\alpha, g_\beta) = 0$ unless $\alpha + \beta = 0$.

Let $R \subset \mathfrak{h}^*$ be the finite set $R = \{\alpha \in \mathfrak{h}^* - \{0\} \mid \mathfrak{g}_\alpha \neq 0\}$. Then R is called the set of roots, and we will write the decomposition as $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$. We will fix a subset $\Delta^+ \subset R$ known as a positive root system with the properties

(1) $\Delta^+ \cup (-\Delta^+) = R$,

(2) $\Delta^+ \cap (-\Delta^+) = \emptyset$,
(3) If $\alpha, \beta \in \Delta^+$ and $\alpha + \beta \in R$, then $\alpha + \beta \in \Delta^+$.

Definition 4.18. A positive root $\alpha \in \Delta^+$ is simple if it cannot be written as a sum of two positive roots.

The set of positive simple roots form a basis of $\mathfrak{h}^*$, in particular there are $\dim \mathfrak{h}$ simple roots. If $\alpha_1, \ldots, \alpha_r$ are the simple roots, any positive root α decomposes uniquely as

$$\alpha = \sum_{i=1}^{r} n_i \alpha_i, \quad n_i \in \mathbb{Z}_{\geq 0}.$$

If we have a specified Cartan $\mathfrak{h}$ and positive root system $\Delta^+ \subset \mathfrak{h}^*$, then we split the decomposition of $\mathfrak{g}$ into three subalgebras as $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}$ where

$$\mathfrak{n}^- = \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_{-\alpha}, \quad \mathfrak{n} = \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_{\alpha}.$$

4.2. **Universal enveloping algebra.**

Definition 4.19. The universal enveloping algebra $U(\mathfrak{g})$ is an associative algebra and a map $i : \mathfrak{g} \to U(\mathfrak{g})$ of Lie algebras that is universal with respect to this property. Concretely, if $j : \mathfrak{g} \to A$ is any map of Lie algebras such that A is an associative algebra, then j factors uniquely through i via a morphism of associative algebras $U(\mathfrak{g}) \to A$.

Theorem 4.20 (Poincaré-Birkhoff-Witt)**.** *If $\mathfrak{g}$ is finite dimensional with basis $v_1, \ldots, v_n$, then the elements*

$$i(v_1)^{\alpha_1} \cdots i(v_n)^{\alpha_n}, \quad \alpha \in \mathbb{N}^n$$

form a basis for $U(\mathfrak{g})$ over k.

This allows us to naturally define a filtration on $U(\mathfrak{g})$. We define it as $F_0 U(\mathfrak{g}) = \mathbb{C}$, and $F_m U(\mathfrak{g})$ for $m \geq 1$ is the span of the terms $i(v_1)^{\alpha_1} \cdots i(v_n)^{\alpha_n}$ as above where $\alpha_1 + \cdots + \alpha_n \leq m$.

We also have the following explicit description of $U(\mathfrak{g})$.

Proposition 4.21. *The universal enveloping algebra $U(\mathfrak{g})$ is isomorphic to the tensor algebra $T(\mathfrak{g})$ quotiented by the two-sided ideal generated by elements $xy - yx - [x, y]$ for all $x, y \in \mathfrak{g}$.*

Passing from a Lie group to its Lie algebra turned the problem of studying representations of G by the easier problem of the representations of $\mathfrak{g}$. The universal enveloping algebra lets us take this one step further, replacing representations of $\mathfrak{g}$ by $U(\mathfrak{g})$-modules

If we are given a Lie algebra representation $\rho : \mathfrak{g} \to \mathrm{End}(V)$, then by the universal property there is a unique morphism of associative algebras $\tilde{\rho} : U(\mathfrak{g}) \to \mathrm{End}(V)$ such that $\rho = \tilde{\rho} \circ i$. Conversely given a morphism of algebras $\varphi : U(\mathfrak{g}) \to \mathrm{End}(V)$, then $\varphi \circ i : \mathfrak{g} \to \mathrm{End}(V)$ gives a Lie algebra representation of $\mathfrak{g}$. This is summarized by the following.

Proposition 4.22. *Restriction along $i \colon \mathfrak{g} \to U(\mathfrak{g})$ gives an isomorphism of categories*

$$\mathrm{Mod}(U(\mathfrak{g})) \xrightarrow{\sim} \mathrm{Rep}(\mathfrak{g}).$$

Lastly, let $Z(\mathfrak{g})$ denote the center of $U(\mathfrak{g})$, then we prove the following result which we will need later.

Proposition 4.23. *The center $Z(\mathfrak{g})$ is equal to the elements of $U(\mathfrak{g})$ fixed under the adjoint action of G, denoted $U(\mathfrak{g})^G$.*

Proof. Recall that G acts on $\mathfrak{g}$ via $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$, then this extends multiplicatively to a map $\rho : G \to \mathrm{Aut}(U(\mathfrak{g}))$. Just as differentiating Ad gives ad, a computation shows that for $x \in \mathfrak{g}$, $u \in U(\mathfrak{g})$,

$$d\rho(x)(u) = [x, u].$$

Now suppose that $u \in U(\mathfrak{g})^G$, that is $\rho(g)(u) = u$ for all $g \in G$. In particular for $x \in \mathfrak{g}$, $t \in \mathbb{C}$, $\rho(\exp(tx))(u) = u$. Then differentiating at $t = 0$ gives

$$d\rho(x)(u) = 0 = [x, u].$$

Since u commutes with $\mathfrak{g}$, it commutes with all of $U(\mathfrak{g})$, so $u \in Z(\mathfrak{g})$.

For the other direction, take $u \in Z(\mathfrak{g})$. Then for $x \in \mathfrak{g}$, $[x, u] = d\rho(x)(u) = 0$, so a computation shows $t \mapsto \rho(\exp(tx))(u)$ is constant. At $t = 0$ it equals u, hence it equals u for all t. Since G is generated by elements of the form $\exp(tx)$, u is fixed by all of G. □

4.3. **Borel subgroups.** Having decomposed a semisimple Lie algebra via its root system, we can now consider a class of subgroups of the algebraic group known as Borel subgroups which will allow us to construct the geometric object on which our D-modules will live.

Definition 4.24. A Borel subgroup B of G is a maximal, connected, solvable subgroup.

The meaning of solvable here is that the derived series, $B \supseteq [B, B] \supseteq [[B, B], [B, B]] \supseteq \dots$ terminates at the trivial group. This is equivalent to the associated Lie algebra $\mathfrak{b} = \mathrm{Lie}(B)$ being solvable, so we make the following definition.

Definition 4.25. A Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ is a maximal solvable subalgebra.

There is an exact correspondence between Borel subalgebras of $\mathfrak{g}$ and Borel subgroups of G, so we will use them interchangeably.

One of the most common examples of a Borel subgroup B is the subgroup of upper triangular matrices in $\mathrm{SL}(n)$. The first term of the derived series $[B, B]$ will be the subgroup with all 1's on the diagonal. The next term $[[B, B], [B, B]]$ is the subgroup of all 1's on the diagonal and 0's on the superdiagonal. The derived series will continue to contract in this fashion until it equals the trivial group. The corresponding Borel subalgebra $\mathfrak{b} = \mathrm{Lie}(B)$ is then just the upper triangular matrices with trace 0.

Proposition 4.26. *Any two Borel subgroups are conjugate to each other.*

This is one of the most important facts about Borel subgroups. We can also translate it into the following version for Borel subalgebras.

Corollary 4.27. *The adjoint action of G is transitive on the set of Borel subalgebras of $\mathfrak{g}$.*

Proof. Consider $\mathfrak{b} = \mathrm{Lie}(B)$ and $\mathfrak{b}' = \mathrm{Lie}(B')$. Then there is some $g \in G$ such that $gBg^{-1} = B'$. So the conjugation by g map $G \to G$ restricts to a bijection $B \to B'$. Taking the derivative of this map shows that $\mathrm{Ad}(g)\mathfrak{b} = \mathfrak{b}'$. □

Another important related fact is the following.

Proposition 4.28. *Borel subgroups are self-normalizing, that is*

$$N_G(B) = \{g \in G \mid gBg^{-1} = B\} = B.$$

For Borel subalgebras, this says that

$$N_{\mathfrak{g}}(\mathfrak{b}) = \{x \in \mathfrak{g} \mid [x, \mathfrak{b}] \subseteq \mathfrak{b}\} = \mathfrak{b}.$$

If we have a given decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}$ as before, then $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$ is a Borel subalgebra. Furthermore all Borel subalgebras are given by such a construction. Then $[\mathfrak{b}, \mathfrak{b}] = \mathfrak{n}$ is an ideal in $\mathfrak{b}$. This follows from the fact in Theorem 4.17 that $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$.

4.4. **Flag variety.** Let G be a semisimple algebraic group. Fix B a Borel subgroup. Then G/B is a smooth projective variety ([Bor91, Theorem 11.1]) known as the flag variety.

Example 4.29. Consider $G = \mathrm{SL}(n, \mathbb{C})$. Then the upper triangular matrices in $\mathrm{SL}(n, \mathbb{C})$ form a Borel subgroup of G. If the matrices are written in say a basis $e_1, \dots, e_n \in \mathbb{C}^n$, the matrices that fix the complete flag

$$0 \subseteq \mathrm{span}(e_1) \subseteq \mathrm{span}(e_1, e_2) \subseteq \cdots \subseteq \mathrm{span}(e_1, \dots, e_n) = \mathbb{C}^n$$

are precisely the set of upper triangular matrices. Since G acts transitively on the set of complete flags, we have by orbit-stabilizer that

$$G/B \xrightarrow{\sim} \{\text{complete flags in } \mathbb{C}^n\}.$$

This explains the title the flag variety, and it can be viewed as a subvariety of a product of Grassmannians.

Generally the tangent space at identity eB of G/B can be identified with $\mathfrak{g}/\mathfrak{b}$. In particular it has dimension $\dim \mathfrak{n}$, so $\dim(G/B) = \dim \mathfrak{n}$.

Another view of the flag variety which will arise is viewing G/B as the set of Borel subgroups of G. If we consider G acting on this set by conjugation, then by Proposition 4.26 this action is transitive and the stabilizer of B is just B itself, so we can identify G/B with the set of Borel subgroups. Equivalently this is the space of all Borel subalgebras of $\mathfrak{g}$.

Throughout the rest of the paper we will fix a Borel subgroup B and the corresponding Borel subalgebra $\mathfrak{b}$, with $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$ and the decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}$.

4.5. **Geometric vector bundles.** A key ingredient in the proof of Beilinson-Bernstein Localization is the identification of two vector bundles over the flag variety as isomorphic. We recap here the definition of a geometric vector bundle and give as well that of a G-equivariant vector bundle.

Definition 4.30. A geometric vector bundle of rank n over a variety X is a morphism of varieties $\pi : E \to X$ with an open cover $\{U_i\}$ of X and isomorphisms

$$\psi_i : \pi^{-1}(U_i) \xrightarrow{\sim} U_i \times \mathbb{A}^n$$

over U_i that commute with projections to U_i and on each overlap $U_i \cap U_j$ the transition map

$$\psi_j \circ \psi_i^{-1} : (U_i \cap U_j) \times \mathbb{A}^n \longrightarrow (U_i \cap U_j) \times \mathbb{A}^n$$

has the form $(x, v) \mapsto (x,\, g_{ij}(x)\, v)$ for some morphism $g_{ij} : U_i \cap U_j \to \mathrm{GL}(n, \mathbb{C})$.

The fiber $E_x = \pi^{-1}(x)$ then has the structure of an n-dimensional vector space.

Let G be a semisimple affine algebraic group acting on a smooth variety X.

Definition 4.31. Let $\pi : E \to X$ be a geometric vector bundle. We say E is a G-equivariant vector bundle if G acts on E by a morphism $G \times E \to E$ such that

(1) π is G-equivariant, i.e. $\pi(g \cdot e) = g \cdot \pi(e)$ for all $g \in G$, $e \in E$,
(2) for all $g \in G$ and $x \in X$, the induced map on fibers $E_x \to E_{gx}$ is linear.

The following proposition will be important later to identify certain G-equivariant vector bundles on the flag variety $X = G/B$. Intuitively, the result is true because the action of G is transitive on X, so the entire bundle is determined by the fiber E_{eB} and how B acts on it.

Proposition 4.32. *There is an equivalence of categories between finite dimensional representations of B and G-equivariant vector bundles on X, given by*

$$V \mapsto G \times^B V$$

where $b \cdot (g, v) = (gb^{-1}, bv)$.

The inverse functor above maps a G-equivariant vector bundle E to the representation of B that is the fiber E_{eB}. Thus given two G-equivariant vector bundles E, E' on G/B, they are isomorphic if and only if E_{eB} and E'_{eB} are isomorphic as B-representations.

4.6. **Tangent and cotangent bundles on the flag variety.** Let TX and T^*X be the tangent and cotangent bundles on the flag variety respectively. For $g \in G$, taking the differential of the map $X \to X$, $x \mapsto g \cdot x$ at some $y \in X$ gives a linear isomorphism

$$L_g : T_y X \to T_{g \cdot y} X.$$

These maps combine to form a morphism $G \times TX \to TX$, which shows that TX is a G-equivariant vector bundle on the flag variety. Taking instead the duals of the L_g maps shows that the cotangent bundle T^*X is a G-equivariant vector bundle on the flag variety.

By Proposition 4.32, TX and T^*X are determined by their fibers over the identity eB. The differential of the quotient map $G \to X$ gives a linear map $\mathfrak{g} \to T_{eB}X$ which has kernel $\mathfrak{b}$, so $T_{eB}X \cong \mathfrak{g}/\mathfrak{b}$ as representations of B. Thus as G-equivariant vector bundles,

$$TX \cong G \times^B (\mathfrak{g}/\mathfrak{b}),$$

with the action of B given by $b \cdot (g, v) = (gb^{-1}, bv)$.

Taking duals we have

$$T^*X \cong G \times^B (\mathfrak{g}/\mathfrak{b})^*.$$

If $\lambda \in (\mathfrak{g}/\mathfrak{b})^*$, the action of $b \in B$ is given by

$$(b \cdot \lambda)(x + \mathfrak{b}) = \lambda(\mathrm{Ad}(b^{-1})x + \mathfrak{b}).$$

Thus if we take the natural isomorphism $(\mathfrak{g}/\mathfrak{b})^* \cong \mathfrak{b}^0 \subseteq \mathfrak{g}^*$ where $\mathfrak{b}^0$ is the annihilator, then this is B-equivariant so it holds as B-representations as well. Then the Killing form gives an isomorphism of vector spaces

$$\mathfrak{b}^0 \cong \mathfrak{b}^\perp.$$

By Theorem 4.17, $\mathfrak{n} \subseteq \mathfrak{b}^\perp$. But also

$$\dim \mathfrak{b}^\perp = \dim \mathfrak{g} - \dim \mathfrak{b} = \dim \mathfrak{n},$$

thus $\mathfrak{b}^\perp = \mathfrak{n}$.

Lastly the Killing form is G-invariant, so this isomorphism is B-equivariant, yielding

$$T^*X \cong G \times^B \mathfrak{n}.$$

For the purposes of simplifying later steps, we will actually work with the identification $\mathfrak{b}^0 \cong \mathfrak{b}^\perp$ coming from the negative of the Killing form.

4.7. **Infinitesimal character.** Consider a representation $\rho : \mathfrak{g} \to \mathrm{End}(V)$. Extend this to an algebra homomorphism $\tilde{\rho} : U(\mathfrak{g}) \to \mathrm{End}(V)$, then consider the restriction of this map to the center of $U(\mathfrak{g})$, i.e. $Z(\mathfrak{g}) \to \mathrm{End}(V)$.

If every element of the center $Z(\mathfrak{g})$ acts by a scalar on V, then we say the representation has an infinitesimal character, that character being the above map which we denote $\chi : Z(\mathfrak{g}) \to \mathbb{C}$.

Example 4.33. Let V be one-dimensional and consider the trivial representation $\mathfrak{g} \to \mathrm{End}(V)$, that is every $x \in \mathfrak{g}$ maps to zero. The corresponding infinitesimal character $\chi_0 : Z(\mathfrak{g}) \to \mathbb{C}$ is then defined by being the identity on $\mathbb{C}$ and zero on any products of elements in $\mathfrak{g}$. The kernel of χ_0 is $Z(\mathfrak{g})_+$, that is all elements with no constant term.

If a representation $\mathfrak{g} \to \mathrm{End}(V)$ has infinitesimal character χ, then the kernel of $U(\mathfrak{g}) \to \mathrm{End}(V)$ contains the ideal generated by $\ker \chi$, so it factors to a map $U(\mathfrak{g})/(\ker \chi) \to \mathrm{End}(V)$. Furthermore given a map $U(\mathfrak{g})/(\ker \chi) \to \mathrm{End}(V)$ the composition with the projection from $U(\mathfrak{g})$ gives representation $\mathfrak{g} \to \mathrm{End}(V)$ with infinitesimal character χ. So in general we have a correspondence

$$\left\{ \begin{array}{c} \text{representations of } \mathfrak{g} \\ \text{with infinitesimal} \\ \text{character } \chi \end{array} \right\} \xleftrightarrow{\sim} \mathrm{Mod}(U(\mathfrak{g})/(\ker \chi))$$

In future we will denote the left hand side above as $\mathrm{Rep}_\chi(\mathfrak{g})$.

5. Sheaf of Differential Operators

In Section 1.3 we defined the ring of differential operators over a k-algebra R. In Section 3 we then also talked about the differential operators over X or $U \subseteq X$. These ideas are generalized in this section, as we must understand the sheaf of differential operators for the Beilinson-Bernstein correspondence.

Let Y be a smooth irreducible algebraic variety over $\mathbb{C}$. We will give two equivalent definitions.

Definition 1: Consider $\mathcal{E}nd_{\mathbb{C}}(\mathcal{O}_Y)$, the sheaf of $\mathbb{C}$-linear endomorphisms of the structure sheaf on Y. Then $\mathcal{O}_Y$ and the tangent sheaf $\mathcal{T}_Y$ are subsheaves of $\mathcal{E}nd_{\mathbb{C}}(\mathcal{O}_Y)$, $\mathcal{O}_Y$ acting by left multiplication as usual and $\mathcal{T}_Y$ acting by derivations. Define $\mathcal{D}_Y$ to be the subsheaf of $\mathcal{E}nd_{\mathbb{C}}(\mathcal{O}_Y)$ generated by $\mathcal{O}_Y$ and $\mathcal{T}_Y$.

Definition 2: This definition reflects closely the construction of the ring of differential operators in Section 1.3. Define inductively the sheaves $\mathcal{D}_Y^{\leq n}$ of differential operators of order $\leq n$ by,

(1) $\mathcal{D}_Y^{\leq 0} = \mathcal{O}_Y$ viewed as a subsheaf of $\mathcal{E}nd_{\mathbb{C}}(\mathcal{O}_Y)$,
(2) For $n > 0$, $\mathcal{D}_Y^{\leq n}$ is the subsheaf of $\mathcal{E}nd_{\mathbb{C}}(\mathcal{O}_Y)$ defined by $P \in \mathcal{E}nd_{\mathbb{C}}(\mathcal{O}_Y)$ is in $\mathcal{D}_Y^{\leq n}$ if $[P, \mathcal{O}_Y] \in \mathcal{D}_Y^{\leq n-1}$.

Then define the sheaf of differential operators as

$$\mathcal{D}_Y = \bigcup_{n \geq 0} \mathcal{D}_Y^{\leq n}.$$

The second definition illustrates the filtration on $\mathcal{D}_Y$, known as the order filtration. Inductively it is clear $\mathcal{D}_Y^{\leq n-1}$ is a subsheaf of $\mathcal{D}_Y^{\leq n}$.

One can show that $\mathcal{D}_Y$ is generated by its first filtered piece $\mathcal{D}_Y^{\leq 1}$, which is in fact just equal to $\mathcal{O}_Y \oplus \mathcal{T}_Y$.

We can then consider the associated graded sheaf

$$\operatorname{gr} \mathcal{D}_Y := \bigoplus_{n \geq 0} \mathcal{D}_Y^{\leq n} / \mathcal{D}_Y^{\leq n-1},$$

where by convention $\mathcal{D}_Y^{\leq -1} = 0$.

Remark 5.1. Another way to understand the differential operators is that if $U = \operatorname{Spec} R \hookrightarrow Y$ is an affine open subvariety, then $\mathcal{D}_Y(U) = D_{R/\mathbb{C}}$, the ring of differential operators over R.

In Theorem 1.3, we proved that for the order filtration on a ring of differential operators that $[D_{R/k}^n, D_{R/k}^m] \subseteq D_{R/k}^{n+m-1}$. Combining this with our first definition for $\mathcal{D}_Y$ above shows that

$$[\mathcal{D}_Y^{\leq n}, \mathcal{D}_Y^{\leq m}] \subseteq \mathcal{D}_Y^{\leq n+m-1},$$

so that $\operatorname{gr} \mathcal{D}_Y$ is a sheaf of commutative rings.

Example 5.2. Consider $Y = \mathbb{A}^n$. Then the global sections $\Gamma(Y, \mathcal{D}_Y)$ is just the Weyl algebra

$$A_n = \bigoplus_{\alpha \in \mathbb{N}^n} \mathbb{C}[x_1, \ldots, x_n] \partial^\alpha.$$

On an affine $U \subseteq \mathbb{A}^n$, then

$$\mathcal{D}_Y(U) = \bigoplus_{\alpha \in \mathbb{N}^n} \mathcal{O}_Y(U) \partial^\alpha.$$

The graded sheaf $\operatorname{gr} \mathcal{D}_Y$ has global sections equal to the polynomial ring $(\mathbb{C}[x_1, \ldots, x_n])[y_1, \ldots, y_n]$ as from Remark 1.10.

Intuitively, taking the associated graded sheaf of the $\mathcal{D}_Y$ leaves us with homogeneous polynomials in the partial derivatives ∂_i. This is reflected in the following standard fact.

Proposition 5.3. *Let $\pi : T^*Y \to Y$ be the cotangent bundle. Then $\operatorname{gr}(\mathcal{D}_Y) \cong \pi_* \mathcal{O}_{T^*Y} \cong \operatorname{Sym}^\bullet(\mathcal{T}_Y)$ are isomorphic as sheaves of graded $\mathcal{O}_Y$-algebras.*

Proof. See [HTT08, pg. 17] □

This isomorphism also preserves the gradings, so

$$\operatorname{gr}^n \mathcal{D}_Y = \mathcal{D}_Y^{\leq n} / \mathcal{D}_Y^{\leq n-1} \cong \operatorname{Sym}^n \mathcal{T}_Y.$$

When we talk about $\mathcal{D}_Y$-modules in the future, we will mean a sheaf of left $\mathcal{D}_Y$-modules. There is also the notion of quasicoherence.

Definition 5.4. A $\mathcal{D}_Y$-module is quasicoherent if it is quasicoherent as an $\mathcal{O}_Y$-module.

Let $\operatorname{QCoh}(\mathcal{D}_Y)$ be the category of quasicoherent $\mathcal{D}_Y$-modules. This will be the site for the correspondence of Beilinson-Bernstein localization given in the next section.

Lastly, let G be an algebraic group which acts on Y. The action of G on $\mathcal{O}_Y$ is shown at the start of Section 6.2. We then have an action of G on $\mathcal{D}_Y$. If $D \in \mathcal{D}_Y(U)$ and $f \in \mathcal{O}_Y(g \cdot U)$, then

$$(g \cdot D)(f) = g \cdot D(g^{-1} \cdot f) \in \mathcal{O}_Y(g \cdot U),$$

so $g \cdot D$ is a differential operator in $\mathcal{D}_Y(g \cdot U)$.

6. Beilinson-Bernstein Localization

6.1. **The statement.** Let G be a semisimple algebraic group, $\mathfrak{g} = \mathrm{Lie}(G)$. Fix a Borel subgroup B and let $X = G/B$ be the flag variety. As in the previous section let $\chi_0 : Z(\mathfrak{g}) \to \mathbb{C}$ denote the trivial infinitesimal character.

Theorem 6.1 (Beilinson-Bernstein Localization)**.** *There is an equivalence of categories* $\mathrm{QCoh}(\mathcal{D}_X) \xrightarrow{\sim} \mathrm{Rep}_{\chi_0}(\mathfrak{g})$. *That is between* $\mathcal{D}_X$*-modules that are quasicoherent as* $\mathcal{O}_X$*-modules, and representations of* $\mathfrak{g}$ *with trivial infinitesimal character.*

The proof of this statement can be broken down into two steps. The first step, given below is the one that we will focus on.

Theorem 6.2. *There is an isomorphism* $\Phi_0 : U(\mathfrak{g})/(Z(\mathfrak{g})_+) \xrightarrow{\sim} \Gamma(X,\ D_X)$.

Since $\mathrm{Rep}_{\chi_0}(\mathfrak{g})$ is equivalent to $\mathrm{Mod}(U_0)$ where $U_0 = U(\mathfrak{g})/(Z(\mathfrak{g})_+)$, this theorem would imply that $\mathrm{Rep}_{\chi_0}(\mathfrak{g})$ is equivalent to $\mathrm{Mod}(\Gamma(X, \mathcal{D}_X))$. Part two of the theorem is then to prove that X is $\mathcal{D}$-affine.

Theorem 6.3 (D-Affine)**.** *There is an equivalence of categories*

$$\Gamma(X, -) : \mathrm{QCoh}(\mathcal{D}_X) \xrightarrow{\sim} \mathrm{Mod}(\Gamma(X, \mathcal{D}_X))$$

by taking global sections.

Putting these two results together proves Theorem 6.1. The actual theorem proven by Beilinson and Bernstein is more general. Instead of just considering representations of $\mathfrak{g}$ with trivial infinitesimal character, their result handles other infinitesimal characters as well, and the equivalence is then with a twisted sheaf of differential operators.

6.2. **Constructing the map.** Let $X = G/B$ be the flag variety. Then G acts transitively on X, so in particular X is smooth. For each $g \in G$, the automorphism $X \to X$, $x \mapsto g \cdot x$ induces maps on sections

$$\mathcal{O}_X(U) \to \mathcal{O}_X(gU), \quad p \mapsto g \cdot p.$$

where

$$(g \cdot p)(x) = p(g^{-1}x), \quad x \in gU.$$

These maps are compatible with restriction, so they define an action of G on the sheaf $\mathcal{O}_X$.

Differentiating the induced action on $\mathcal{O}_X$ gives a map $\mathfrak{g} \to \Gamma(X, \mathcal{T}_X)$, where $\mathcal{T}_X$ is the tangent sheaf. For $Y \in \mathfrak{g}$, the action of Y locally on functions is given by

$$(Y \cdot p)(x) = \left.\frac{d}{dr}\right|_{r=0} p(\exp(-rY)x). \tag{6.1}$$

To check Leibniz rule, we have that

$$\begin{aligned} Y(pq)(x) &= \left.\frac{d}{dr}\right|_{r=0} (pq)(\exp(-rY)x) \\ &= \left.\frac{d}{dr}\right|_{r=0} p(\exp(-rY)x)q(\exp(-rY)x) \\ &= (Yp)(x)q(x) + p(x)(Yq)(x). \end{aligned}$$

The map $\mathfrak{g} \to \Gamma(X, \mathcal{T}_X)$ is in fact a homomorphism of Lie algebras. Composing with the inclusion $\Gamma(X, \mathcal{T}_X) \hookrightarrow \Gamma(X, \mathcal{D}_X)$ we get a map $\mathfrak{g} \to \Gamma(X, \mathcal{D}_X)$. Note this maps $\mathfrak{g}$ into the

first filtered piece. The universal property of the universal enveloping algebra then extends this to a map

$$\Phi : U(\mathfrak{g}) \to \Gamma(X, \mathcal{D}_X).$$

Since $\mathfrak{g}$ maps into the first filtered piece of $\Gamma(X, \mathcal{D}_X)$, this map preserves the filtration on either side.

Proposition 6.4. *The map Φ factors through to $\Phi_0 : U(\mathfrak{g})/(Z(\mathfrak{g})_+) \to \Gamma(X, \mathcal{D}_X)$.*

Proof. We first will show that Φ is G-equivariant. Recall from the end of Section 5 that for $D \in \Gamma(X, \mathcal{D}_X)$, the action of $g \in G$ is given by

$$g \cdot D = g \circ D \circ g^{-1}.$$

Consider $Y \in \mathfrak{g}$, and take a section $f \in \mathcal{O}_X(U)$ over some open $U \subseteq X$ and $x \in U$. Then

$$\begin{aligned}
((g \cdot \Phi(Y))f)(x) &= (g \cdot (\Phi(Y)(g^{-1}f)))(x) \\
&= (\Phi(Y)(g^{-1}f))(g^{-1}x) \\
&= \left.\frac{d}{dr}\right|_{r=0} f(g \exp(-rY)g^{-1}x) \\
&= \left.\frac{d}{dr}\right|_{r=0} f(\exp(-r\mathrm{Ad}(g)Y)x) \\
&= (\Phi(\mathrm{Ad}(g)Y) \cdot f)(x)
\end{aligned}$$

showing the G-equivariance for $Y \in \mathfrak{g}$. This extends to all of $U(\mathfrak{g})$ because for $u_1, u_2 \in U(\mathfrak{g})$, $\mathrm{Ad}(g)\cdot(u_1u_2) = (\mathrm{Ad}(g)u_1)(\mathrm{Ad}(g)u_2)$, and for $D_1, D_2 \in \Gamma(X, \mathcal{D}_X)$, $g\cdot(D_1D_2) = (g\cdot D_1)(g\cdot D_2)$.

By Proposition 4.23, if $Y \in Z(\mathfrak{g}) = U(\mathfrak{g})^G$ then $\Phi(Y) \in \Gamma(X, \mathcal{D}_X)^G$. This allows us to leverage the following fact.

Lemma 6.5. $\Gamma(X, \mathcal{D}_X)^G = \mathbb{C} \cdot \mathrm{id}$.

Proof. Let $D \in \Gamma(X, \mathcal{D}_X)^G$, and suppose that D has order less than or equal to $n > 0$. The filtration and the symbol map are G-equivariant, so $\sigma_n(D) \in \Gamma(X, \mathrm{Sym}^n \mathcal{T}_X)^G$. Then $\sigma_n(D)$ is a G-invariant section of the G-equivariant vector bundle $\mathrm{Sym}^n \mathcal{T}_X$, it is determined by its value in the fiber over eB. This is fixed by B, so

$$\sigma_n(D)_{eB} \in (\mathrm{Sym}^n(T_{eB}X))^B = (\mathrm{Sym}^n(\mathfrak{g}/\mathfrak{b}))^B.$$

Note $\mathfrak{g}/\mathfrak{b} \cong \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_{-\alpha}$ as representations of $\mathfrak{h}$. Then given a monomial $x_{\alpha_1} \cdots x_{\alpha_n} \in \mathrm{Sym}^n(\mathfrak{g}/\mathfrak{b})$ where $x_{\alpha_i} \in \mathfrak{g}_{-\alpha_i}$, then the weight of the action of $\mathfrak{h}$ on the monomial is $-(\alpha_1 + \cdots + \alpha_n)$. In particular it is nonzero since it is the negative of a sum of positive roots, so no nonzero elements of $\mathrm{Sym}^n(\mathfrak{g}/\mathfrak{b})$ are killed by all of $\mathfrak{h}$. Since $\mathfrak{h} \subseteq \mathfrak{b}$, the same holds for $\mathfrak{b}$, so $(\mathrm{Sym}^n(\mathfrak{g}/\mathfrak{b}))^B = 0$. Thus $\sigma_n(D) = 0$, so D has order less than or equal to $n-1$. Inducting down we see D has order zero, so D is multiplication by some $f \in \Gamma(X, \mathcal{O}_X)^G$. Since the action of G on X is transitive, f must be a constant function, so $D \in \mathbb{C} \cdot \mathrm{id}$. □

If $Y \in \mathfrak{g}$, then $\Phi(Y) \in \Gamma(X, \mathcal{T}_X)$ acts as a derivation on $\mathcal{O}_X$, so it acts as zero on constants. In particular if $u \in Z(\mathfrak{g})_+$, then $\Phi(u)$ acts as zero on $1 \in \Gamma(X, \mathcal{O}_X)$. But by the above proposition, $\Phi(u) \in \mathbb{C} \cdot \mathrm{id}$, so necessarily $\Phi(u) = 0$. This shows $(Z(\mathfrak{g})_+) \subseteq \ker \Phi$, so Φ factors through to a map $\Phi_0 : U(\mathfrak{g})/(Z(\mathfrak{g})_+) \to \Gamma(X, D_X)$. □

To prove that the map Φ_0 is in fact an isomorphism, we will study the associated graded map,

$$\text{gr } \Phi_0 : \text{gr}(U(\mathfrak{g})/(Z(\mathfrak{g})_+)) \to \text{gr } \Gamma(X, \mathcal{D}_X).$$

They key to studying this map is showing that it arises as the map on functions induced by a geometric map known as the Springer resolution.

6.3. **The Springer resolution.** Define the nilpotent cone $\mathcal{N} \subset \mathfrak{g}$ as

$$\mathcal{N} = \{x \in \mathfrak{g} \mid \text{ad}(x) \text{ is nilpotent}\}.$$

This is a closed subvariety of $\mathfrak{g}$, but it is singular. This section will show a resolution of singularities of $\mathcal{N}$. In our proof of Beilinson-Bernstein localization the import of $\mathcal{N}$ comes from the fact we will prove that $\mathbb{C}[\mathcal{N}] \cong \text{gr}(U(\mathfrak{g})/(Z(\mathfrak{g})_+)$.

Proposition 6.6. *$\mathcal{N}$ is a closed algebraic subvariety of $\mathfrak{g}$.*

Proof. The operator $\text{ad}(x) : \mathfrak{g} \to \mathfrak{g}$ is nilpotent if and only if it's characteristic polynomial $\det(t\mathbb{I} - \text{ad}(x))$ is just some power of t. If we write

$$\det(t\mathbb{I} - \text{ad}(x)) = t^n + f_{n-1}(x)t^{n-1} + \cdots + f_0(x)$$

then the $f_i \in \mathbb{C}[\mathfrak{g}]$ and $\mathcal{N} = V(f_0, \ldots, f_{n-1})$. □

Let $\mathbb{C}[\mathfrak{g}]^G_+$ denote the G-invariant elements of $\mathbb{C}[\mathfrak{g}]$ that have no constant term.

Proposition 6.7. *The nilcone $\mathcal{N}$ is the vanishing locus of $\mathbb{C}[\mathfrak{g}]^G_+$.*

Proof. Let $f_{n-1}, \ldots, f_0$ be from the previous proposition. Observe that $\det(t\mathbb{I} - \text{ad}(0)) = t^n$, so $f_i(0) = 0$, that is, each f_i has no constant term. If $g \in G$, then

$$\det(t\mathbb{I} - \text{ad}(\text{Ad}(g)x)) = \det(t\mathbb{I} - \text{Ad}(g)\text{ad}(x)\text{Ad}(g)^{-1}) = \det(t\mathbb{I} - \text{ad}(x)).$$

Thus $f_i(x) = f_i(gx)$ for each i, showing $f_i \in \mathbb{C}[\mathfrak{g}]^G_+$. Thus if $x \in V(\mathbb{C}[g]^G_+)$, then in particular $f_i(x) = 0$ for each i, so $x \in \mathcal{N} = V(f_0, \ldots, f_{n-1})$.

For the reverse containment, let $x \in \mathcal{N}$. We claim that there exists some $g \in G$ such that $g \cdot x = cx$ where $c \in \mathbb{C}$ is not a root of unity. Once we have proven this, then for $F \in \mathbb{C}[\mathfrak{g}]^G_+$ any homogeneous polynomial,

$$F(x) = F(gx) = F(cx) = c^{\deg F} F(x).$$

Since $\deg F \geq 1$, this implies $F(x) = 0$, so $\mathcal{N} \subseteq V(\mathbb{C}[\mathfrak{g}]^G_+)$. It suffices to consider homogeneous polynomials since the action of G preserves degree so $(\mathbb{C}[\mathfrak{g}]^G_+)$ is generated by homogeneous polynomials.

To prove the claim, consider the centralizer of x, denoted

$$C(x) = \{y \in \mathfrak{g} \mid [x, y] = 0\}.$$

Then by rank-nullity on the map $\text{ad}(x) : \mathfrak{g} \to \mathfrak{g}$ we have that $\dim[\mathfrak{g}, x] + \dim C(x) = \dim \mathfrak{g}$. Now if $y \in C(x)$, then $\text{ad}(x)$ and $\text{ad}(y)$ commute, so $\text{ad}(x)\text{ad}(y)$ is nilpotent, and $K(x, y) = 0$. Thus $x \perp C(x)$. Furthermore if z is any element of $\mathfrak{g}$, the Killing form being invariant on $\mathfrak{g}$ implies

$$K([z, x], y) = K(z, [y, x]) = 0.$$

So $[\mathfrak{g}, x] \subseteq C(x)^\perp$. But $\dim C(x)^\perp = \dim[\mathfrak{g}, x]$, so we have equality. Since $x \in C(x)^\perp$, there exists $u \in \mathfrak{g}$ such that $[u, x] = x$. Pick $t \in \mathbb{C}$ such that e^t is not a root of unity, then applying the second fact from Remark 4.2,

$$\text{Ad}(\exp(tu)) \cdot x = \exp(t\text{ad}(u)) \cdot x = e^t x,$$

finishing the proof. □

Kostant proved the following powerful theorem that allows us to identify the coordinate ring of $\mathcal{N}$.

Theorem 6.8 (Kostant)**.** *The ideal generated by $\mathbb{C}[\mathfrak{g}]^G_+$, denoted $(\mathbb{C}[\mathfrak{g}]^G_+)$, is radical.*

It follows from this that $\mathbb{C}[\mathcal{N}] = \mathbb{C}[\mathfrak{g}]/(\mathbb{C}[\mathfrak{g}]^G_+)$. We will now construct the Springer resolution map.

Proposition 6.9. *The image of the adjoint action map $G \times \mathfrak{n} \to \mathfrak{g}$ is the nilcone $\mathcal{N}$.*

Proof. If $x \in \mathcal{N}$ and $g \in G$, then

$$\mathrm{ad}(\mathrm{Ad}(g)x) = \mathrm{Ad}(g)\mathrm{ad}(x)\mathrm{Ad}(g)^{-1}$$

is nilpotent because $\mathrm{ad}(x)$ is, so $gx \in \mathcal{N}$. Since every element $\mathfrak{n}$ is nilpotent, this shows that the image of the action map is contained in $\mathcal{N}$. To show the reverse inclusion we need the following.

Claim 6.10. $\mathcal{N} \cap \mathfrak{b} = \mathfrak{n}$.

Proof. Since every element of $\mathfrak{n}$ is nilpotent, $\mathfrak{n} \subseteq \mathcal{N} \cap \mathfrak{b}$. With the positive roots Δ^+ of $\mathfrak{g}$, we have

$$\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha.$$

Consider the decomposition $x = x_h + x_n \in \mathfrak{b} \cap \mathcal{N}$ where $x_h \in \mathfrak{h}$, $x_n \in \mathfrak{n}$. Then $\mathrm{ad}(x) = \mathrm{ad}(x_h) + \mathrm{ad}(x_n)$. The action of $\mathrm{ad}(x_n)$ on $\mathfrak{g}_\alpha$ strictly increases the root height, whereas $\mathrm{ad}(x_h)$ acts by a scalar on each $\mathfrak{g}_\alpha$. Thus if we order the roots from largest height to least, then on the space $\mathfrak{b}$, $\mathrm{ad}(x_n)$ is strictly upper triangular and $\mathrm{ad}(x_h)$ is diagonal. Since $\mathrm{ad}(x)$ is nilpotent, all diagonal entries are zero so $\alpha(x_h) = 0$ for each $\alpha \in \Delta^+$. Since Δ^+ spans $\mathfrak{h}^*$, this implies $x_h = 0$, so $x \in \mathfrak{n}$. □

Consider any $x \in \mathcal{N}$. Since x is nilpotent, there is some Borel subalgebra $\mathfrak{b}' \subset \mathfrak{g}$ such that $x \in \mathfrak{b}'$. Since G acts transitively on the Borel subalgebras, there is some $g \in G$ such that $g \cdot \mathfrak{b} = \mathfrak{b}'$. Then $g^{-1}x \in \mathfrak{b}$ and $g^{-1}x$ is nilpotent. By the claim then $g^{-1}x \in \mathfrak{n}$, and x is the image of $(g, g^{-1}x)$, proving surjectivity onto $\mathcal{N}$. □

Corollary 6.11. *$\mathcal{N}$ is irreducible.*

Proof. It is the image of $G \times \mathfrak{n}$ which is irreducible. □

The adjoint action of the Borel subgroup B fixes $\mathfrak{n}$ because $\mathfrak{n}$ is an ideal in $\mathfrak{b}$. Thus we can consider B acting on $G \times \mathfrak{n}$ as

$$b \cdot (g, x) = (gb^{-1}, \mathrm{Ad}(b)x)$$

The action map $G \times \mathfrak{n} \to \mathcal{N}$ is clearly constant on B-orbits, so it factors through to a map

$$\mu : G \times^B \mathfrak{n} \to \mathcal{N},$$

where $G \times^B \mathfrak{n} = (G \times \mathfrak{n})/B$. We saw in Section 4.6 that $G \times^B \mathfrak{n}$ is isomorphic as a G-equivariant vector bundle to the cotangent bundle T^*X. It follows that $G \times^B \mathfrak{n}$ is in particular a smooth variety.

To show μ is birational, we will look at the map on the preimage of $\mathcal{N}^{\mathrm{reg}}$, the regular elements of $\mathcal{N}$.

Definition 6.12. An element $x \in \mathfrak{g}$ is regular if $\dim C(x) = \operatorname{rank} \mathfrak{g}$. Equivalently if $\operatorname{rank}(\operatorname{ad}(x)) = \dim \mathfrak{g} - \operatorname{rank} \mathfrak{g}$.

It is a general fact that

$$\dim C(x) \geq \operatorname{rank} \mathfrak{g},$$

see [Dix77, Proposition 1.11.13(iii)]. The map $x \mapsto \operatorname{rank}(\operatorname{ad}(x))$ is lower semicontinuous because $\{x \in \mathfrak{g} \mid \operatorname{rank}(\operatorname{ad}(x)) \geq m\}$ is the union of the non-vanishing locus of the $m \times m$ minors of $\operatorname{ad}(x)$, thus a Zariski open. In particular then, $\{x \in \mathfrak{g} \mid \dim C(x) \leq \operatorname{rank} \mathfrak{g}\}$ is open, and $\dim C(x) \leq \operatorname{rank} \mathfrak{g}$ is equivalent to $\dim C(x) = \operatorname{rank} \mathfrak{g}$, showing the set of regular elements, denoted $\mathfrak{g}^{\mathrm{reg}}$ is Zariski open. The role of the regular elements in this paper is that $\mathfrak{g}^{\mathrm{reg}} \cap \mathcal{N}$ is the dense open set on which we will prove the Springer resolution is an isomorphism.

Proposition 6.13. *The map $G \times^B \mathfrak{n} \to \mathcal{N}$ is proper and birational.*

Proof. Consider the map $G \times^B \mathfrak{n} \to \mathfrak{g} \times X$ of vector bundles over X given by $[g, x] \mapsto (\operatorname{Ad}(g)x, gB)$. If we identify X as the set of Borel subalgebras of $\mathfrak{g}$, then on the fiber over $\mathfrak{b}' = \operatorname{Ad}(g)\mathfrak{b}$ the map is given by the inclusion

$$\mathfrak{n}_{\mathfrak{b}'} = \operatorname{Ad}(g)\mathfrak{n} \hookrightarrow \mathfrak{g}.$$

Over every fiber the map is injective, so by Proposition 6.9, $G \times^B \mathfrak{n} \hookrightarrow \mathfrak{g} \times X$ is a closed immersion into the subset

$$\{(x, \mathfrak{b}') \in \mathcal{N} \times X \mid x \in \mathfrak{b}'\}.$$

Since X is a projective variety, the projection $\mathrm{pr}_1 : \mathcal{N} \times X \to \mathcal{N}$ is projective and thus proper. The Springer resolution $\mu : G \times^B \mathfrak{n} \to \mathcal{N}$ is then given by the composition of a closed immersion and projective map,

$$G \times^B \mathfrak{n} \hookrightarrow \mathcal{N} \times X \xrightarrow{pr_1} \mathcal{N} \qquad (\mu) \tag{6.2}$$

so μ is proper.

For birationality we will first show that $\mathcal{N}^{\mathrm{reg}} = \mathcal{N} \cap \mathfrak{g}^{\mathrm{reg}}$ is a dense Zariski open in $\mathcal{N}$. Since $\mathcal{N}$ is irreducible, it suffices to show $\mathcal{N}^{\mathrm{reg}}$ is nonempty.

Claim 6.14. For $\alpha \in \Delta^+$ a simple root, let e_α be a nonzero vector in $\mathfrak{g}_\alpha$. Then $e = \sum_{\alpha \text{ simple}} e_\alpha$ is in $\mathcal{N}^{\mathrm{reg}}$.

Proof. First $e \in \mathfrak{n} \subseteq \mathcal{N}$, so we must prove that e is regular. We omit this proof as it uses some ideas not introduced in this paper, see [Eti23, Lemma 17.2] for the proof. □

With this element e, we prove now that the regular locus $\mathcal{N}^{\mathrm{reg}}$ is a single orbit of $\mathfrak{g}$ under the adjoint action of G. First observe that by the fiber dimension formula,

$$\dim(G \cdot e) = \dim G - \dim G_e,$$

where $G_e \subset G$ is the stabilizer of e. The Lie algebra of G_e then is just $C(e)$, so

$$\dim(G \cdot e) = \dim \mathfrak{g} - \dim C(e) = \dim \mathfrak{g} - r = 2 \dim \mathfrak{n}.$$

Recall that since $\mathcal{N}$ is stable under the G action that $G \cdot e \subseteq \mathcal{N}$. Since μ is a surjective morphism,

$$\dim(G \cdot e) = 2 \dim \mathfrak{n} \leq \dim \mathcal{N} \leq \dim(G \times^B \mathfrak{n}) = 2 \dim \mathfrak{n}.$$

Hence $\dim \mathcal{N} = 2 \dim \mathfrak{n}$. Since $\mathcal{N}$ is irreducible, $\overline{G \cdot e} = \mathcal{N}$. It is a general fact that a G-orbit in $\mathfrak{g}$ is open in its closure (see [Bor91, Proposition I.1.8]), so $G \cdot e$ is a dense open in $\mathcal{N}$. If $x \in \mathcal{N}^{\text{reg}}$ is another regular nilpotent element and $x \neq e$, then the same computation above shows $G \cdot x$ is a dense open in $\mathcal{N}$. Then $G \cdot e$ and $G \cdot x$ must intersect, but different orbits are disjoint so in fact $G \cdot x = G \cdot e$.

This shows $\mathcal{N}^{\text{reg}} \subset G \cdot e$, equality follows from the fact that regularity is invariant under $\text{Ad}(g)$ for $g \in G$ since $\text{Ad}(g)$ is an invertible map $\mathfrak{g} \to \mathfrak{g}$ and the Lie bracket is G-invariant, so

$$[x, y] = 0 \text{ for all } y \in \mathfrak{g} \iff [\text{Ad}(g)x, \text{Ad}(g)y] = 0 \text{ for all } y \in \mathfrak{g}.$$

We claim now that for this Zariski dense $\mathcal{N}^{\text{reg}}$, the fibers of μ all contain just a single element. By the decomposition in (6.2), it suffices to consider the fibers of $\mathcal{N}^{\text{reg}}$ over the projection $\text{pr}_1 : \mathcal{N} \times X \to \mathcal{N}$ restricted to the subset

$$\{(x, \mathfrak{b}') \in \mathfrak{g} \times X \mid x \in \mathfrak{b}'\}.$$

Since μ is a surjective morphism between irreducible varieties of equal dimension, the general fiber over $x \in \mathcal{N}$ is zero-dimensional, i.e. finite. Since $\mathcal{N}^{\text{reg}} \subset \mathcal{N}$ is dense and open, there exists $x = \text{Ad}(g) \cdot e \in \mathcal{N}^{\text{reg}} = G \cdot e$ such that the fiber over x is finite, equivalently x is contained in finitely many Borel subalgebras. The map $\mathfrak{b}' \mapsto \text{Ad}(g)\mathfrak{b}'$ gives a bijection between the Borel subalgebras containing e and those containing x, so the fiber over e is also finite.

Let $\alpha_1, \ldots, \alpha_r$ be the simple roots. They form a basis of $\mathfrak{h}^*$, so there exists $h \in \mathfrak{h}$ such that $\alpha_i(h) = 1$ for each $1 \leq i \leq r$. Then

$$[h, e] = \sum_{i=1}^{r} [h, e_{\alpha_i}] = \sum_{i=1}^{r} \alpha_i(h) e_{\alpha_i} = \sum_{i=1}^{r} e_{\alpha_i} = e.$$

By the second fact in Remark 4.2 we have that for any $t \in \mathbb{C}$,

$$\text{Ad}(\exp(th)) \cdot e = \exp(t \, \text{ad}(h)) \cdot e = \exp(t) \cdot e.$$

Thus if we apply $\text{Ad}(\exp(th))$ to the set of Borel subalgebras containing e, the set is preserved. A continuous one parameter family acting on a discrete set must all act the same, and at $t = 0$, $\text{Ad}(\exp(th))$ is just the identity. So if $\mathfrak{b}'$ is a Borel containing e, then for all $t \in \mathbb{C}$,

$$\text{Ad}(\exp(th)) \cdot \mathfrak{b}' = \mathfrak{b}'.$$

If we fix $y \in \mathfrak{b}'$, then differentiating this at $t = 0$ tells us that $[h, y] \in \mathfrak{b}'$, so $[h, \mathfrak{b}'] \subseteq \mathfrak{b}'$. As we stated in Proposition 4.28, $\mathfrak{b}'$ is self-normalizing, so $h \in \mathfrak{b}'$. We claim that h is also regular. On any positive root $\mathfrak{g}_\alpha$, where $\alpha = \sum_{i=1}^{r} n_i \alpha_i$, $\text{ad}(h)$ acts just by the scalar $\sum n_i > 0$, and on the negative space $\mathfrak{g}_{-\alpha}$, $\text{ad}(h)$ will act by $-\sum n_i$. Thus $C(h) = \mathfrak{h}$, so $\dim C(h) = \text{rank}\, \mathfrak{g}$, and h is regular. Since $\mathfrak{h} \subseteq \mathfrak{b}'$, $\mathfrak{b}'$ is specified by a choice of positive root system of the roots corresponding to $\mathfrak{h}$. But we know that $e = \sum_{i=1}^{r} e_{\alpha_i} \in \mathfrak{b}'$, so each root α_i must be chosen positive. These are the simple roots for the Borel $\mathfrak{b}'$, and every positive root is a sum of simple roots, so we must have in fact $\mathfrak{b}' = \mathfrak{b}$. This proves e is only contained in one Borel, so the fiber of μ over e, and thus over each point of $\mathcal{N}^{\text{reg}}$ is a single point.

To finish, we have shown μ is a surjective (thus dominant) morphism between irreducible varieties of the same dimension. The cardinality of a general fiber of $\mathcal{N}$ is then equal to $[\mathbb{C}(G \times^B \mathfrak{n}) : \mathbb{C}(\mathcal{N})]$, but we have shown explicitly it is equal to 1. Thus $[\mathbb{C}(G \times^B \mathfrak{n}) : \mathbb{C}(\mathcal{N})] = 1$, so $\mathbb{C}(\mathcal{N}) \to \mathbb{C}(G \times^B \mathfrak{n})$ is an isomorphism, proving μ is birational. $\square$

This justifies the title Springer resolution, showing that $\mu : G \times^B \mathfrak{n} \to \mathcal{N}$ is in fact a resolution of singularities. A fact from algebraic geometry that we present below will tell us $\mathbb{C}[\mathcal{N}]$ and $\mathbb{C}[T^*X]$ are isomorphic. Recall that an affine variety is normal if its coordinate ring is integrally closed.

Proposition 6.15. *Let Z be an irreducible normal variety. A resolution of singularities is a proper birational morphism $p : Y \to Z$ where Y is smooth. A resolution of singularities then gives an isomorphism $p^* : \Gamma(Z, \mathcal{O}_Z) \to \Gamma(Y, \mathcal{O}_Y)$.*

Proof. See [Stacks, Tag 0AY8]. □

In our setting this yields an isomorphism $\mu^* : \mathbb{C}[\mathcal{N}] \to \mathbb{C}[T^*X]$ because of the following.

Proposition 6.16. *$\mathcal{N}$ is normal.*

Proof. See [Eti23, Corollary 27.13] □

We wish to relate the Springer resolution to the original map $\Phi : U(\mathfrak{g}) \to \Gamma(X, \mathcal{D}_X)$. To do this we can replace $\mathbb{C}[T^*X]$ with $\Gamma(X, \operatorname{gr} \mathcal{D}_X)$ via Proposition 5.3. Then

$$\mathbb{C}[T^*X] \cong \Gamma(X, \pi_* \mathcal{O}_{T^*X}) \cong \Gamma(X, \operatorname{gr} \mathcal{D}_X).$$

When dealing with the map $\operatorname{gr} \Phi$, the codomain is $\operatorname{gr} \Gamma(X, \mathcal{D}_X)$, not $\Gamma(X, \operatorname{gr} \mathcal{D}_X)$. The following will allow us to identify them.

Proposition 6.17. $\Gamma(X, \operatorname{gr} \mathcal{D}_X) \cong \operatorname{gr} \Gamma(X, \mathcal{D}_X)$

Proof. The flag variety X is a projective variety and thus a Noetherian topological space, so by [Har77, Remark III.2.9.1], taking global sections commutes with infinite direct sums, so

$$\Gamma(X, \operatorname{gr} \mathcal{D}_X) \cong \Gamma(X, \bigoplus_{n \geq 0} \operatorname{gr}^n \mathcal{D}_X) \cong \bigoplus_{n \geq 0} \Gamma(X, \operatorname{gr}^n \mathcal{D}_X).$$

where $\operatorname{gr}^n \mathcal{D}_X = \mathcal{D}_X^{\leq n} / \mathcal{D}_X^{\leq n-1}$. We will show now that $\Gamma(X, \operatorname{gr}^n \mathcal{D}_X) \cong \Gamma(X, \mathcal{D}_X^{\leq n}) / \Gamma(X, \mathcal{D}_X^{\leq n-1})$. We have the exact sequence,

$$0 \longrightarrow \mathcal{D}_X^{\leq n-1} \longrightarrow \mathcal{D}_X^{\leq n} \longrightarrow \operatorname{gr}^n \mathcal{D}_X \longrightarrow 0$$

which then extends to the long exact sequence

$$0 \longrightarrow \Gamma(X, \mathcal{D}_X^{\leq n-1}) \longrightarrow \Gamma(X, \mathcal{D}_X^{\leq n}) \longrightarrow \Gamma(X, \operatorname{gr}^n \mathcal{D}_X)$$
$$H^1(X, \mathcal{D}_X^{\leq n-1}) \longrightarrow H^1(X, \mathcal{D}_X^{\leq n}) \longrightarrow H^1(X, \operatorname{gr}^n \mathcal{D}_X)$$

Lemma 6.18. *$H^1(X, \operatorname{gr}^n(\mathcal{D}_X)) = 0$ for all $n \geq 0$.*

Proof. Recall $\operatorname{gr}^n(\mathcal{D}_X) \cong \operatorname{Sym}^n(\mathcal{T}_X)$, also $\pi : T^*X \to X$ is affine and X is Noetherian, so

$$H^i(T^*X, \mathcal{O}_{T^*X}) = H^i(X, \pi_* \mathcal{O}_{T^*X}) = \bigoplus_{n \geq 0} H^i(X, \operatorname{Sym}^n \mathcal{T}_X).$$

Thus it suffices to prove $H^1(T^*X, \mathcal{O}_{T^*X}) = 0$. Hesselink proved in 1976 that $\mathcal{N}$ has rational singularities. In particular this implies $R^i \mu_* \mathcal{O}_{T^*X} = 0$ for $i > 0$, [Hes76]. The Leray spectral sequence for μ then implies

$$H^1(T^*X, \mathcal{O}_{T^*X}) \cong H^1(\mathcal{N}, \mu_* \mathcal{O}_{T^*X}).$$

Lastly $H^1(\mathcal{N}, \mu_* \mathcal{O}_{T^*X}) = 0$ because $\mathcal{N}$ is affine. □

Now we can induct to prove $H^1(X, \mathcal{D}_X^{\leq n}) = 0$. As a base case, $H^1(X, \mathcal{O}_X) = H^1(X, \mathrm{gr}^0 \mathcal{D}_X) = 0$. If $H^1(X, \mathcal{D}_X^{\leq n-1}) = 0$, then since

$$H^1(X, \mathcal{D}_X^{\leq n-1}) = 0 \longrightarrow H^1(X, \mathcal{D}_X^{\leq n}) \longrightarrow H^1(X, \mathrm{gr}^n \mathcal{D}_X) = 0$$

is exact, necessarily $H^1(X, \mathcal{D}_X^{\leq n}) = 0$. This shows then the sequence

$$0 \longrightarrow \Gamma(X, \mathcal{D}_X^{\leq n-1}) \longrightarrow \Gamma(X, \mathcal{D}_X^{\leq n}) \longrightarrow \Gamma(X, \mathrm{gr}^n \mathcal{D}_X) \longrightarrow 0$$

is exact, so

$$\Gamma(X, \mathrm{gr}^n \mathcal{D}_X) \cong \Gamma(X, \mathcal{D}_X^{\leq n}) / \Gamma(X, \mathcal{D}_X^{\leq n-1}).$$

Hence $\Gamma(X, \mathrm{gr}\, \mathcal{D}_X) \cong \mathrm{gr}\, \Gamma(X, \mathcal{D}_X)$. □

6.4. **Descent from grading.** Recall we had constructed the map $\Phi : U(\mathfrak{g}) \to \Gamma(X, \mathcal{D}_X)$, then this factored through to the map $\Phi_0 : U(\mathfrak{g})/(Z(\mathfrak{g})_+) \to \Gamma(X, \mathcal{D}_X)$. We will now show that the associated graded map of Φ_0 is precisely the Springer resolution map.

By the Poincaré-Birkhoff-Witt theorem Theorem 4.20, the associated graded algebra $\mathrm{gr}(U(\mathfrak{g})) \cong S(\mathfrak{g})$ the symmetric algebra. Using the Killing form we will then identify $S(\mathfrak{g}) = \mathbb{C}[\mathfrak{g}^*] \cong \mathbb{C}[\mathfrak{g}]$.

Proposition 6.19. *The graded map* $\mathrm{gr}\, \Phi : \mathbb{C}[\mathfrak{g}] \to \mathrm{gr}\, \Gamma(X, \mathcal{D}_X)$ *is equal to the projection* $\pi : \mathbb{C}[\mathfrak{g}] \to \mathbb{C}[\mathcal{N}]$ *composed with* $\mu^* : \mathbb{C}[\mathcal{N}] \to \mathbb{C}[T^*X] = \mathrm{gr}\, \Gamma(X, \mathcal{D}_X)$.

Proof. Both maps $\mathbb{C}[\mathfrak{g}] \to \mathrm{gr}\, \Gamma(X, \mathcal{D}_X)$ in consideration are graded maps of $\mathbb{C}$-algebras, since $\mathbb{C}[\mathfrak{g}]$ is generated by the degree 1 piece as a $\mathbb{C}$-algebra, suffices to prove the maps agree on degree 1.

Note that $\Gamma(X, \mathcal{T}_X) = \Gamma(X, \mathrm{gr}^1 \mathcal{D}_X) \subset \mathrm{gr}\, \Gamma(X, \mathcal{D}_X) = \mathbb{C}[T^*X]$. A cotangent vector $\xi \in T_p^* X$ is then evaluated at $v \in \Gamma(X, \mathcal{T}_X)$ via

$$v(p, \xi) = \langle v_p, \xi \rangle.$$

Consider the vector field $\sigma_1(\Phi(Y)) \in \Gamma(X, \mathcal{T}_X) = \Gamma(X, \mathrm{gr}^1 \mathcal{D}_X)$. Pick a point $p \in X$, then differentiating the map $G \to X$, $h \mapsto h \cdot p$ gives the map $a_p : \mathfrak{g} \to T_p X \cong \mathfrak{g}/\mathfrak{b}'$, where p corresponds to $\mathfrak{b}'$. Thus $\sigma_1(\Phi(Y))_p = -Y \pmod{\mathfrak{b}'}$. For a cotangent vector $[g, x] \in T_p^* X$, evaluating $\sigma_1(\Phi(Y))$ at $[g, x]$ equals

$$\langle \sigma_1(\Phi(Y))_p, [g, x] \rangle = -K(\mathrm{Ad}(g)x, -Y) = K(\mu([g, x]), Y),$$

where we used the fact that we took the negative of the Killing form in Section 4.6

The first degree piece of $\mathrm{gr}\, U(\mathfrak{g}) = \mathbb{C}[\mathfrak{g}]$ is just $\mathfrak{g}$. Given $Y \in \mathfrak{g}$, this is viewed as an element of $\mathbb{C}[\mathfrak{g}]$ via the function $K(-, Y)$. If we then apply the projection and μ^*, we get $K(\mu(-), Y) \in \mathbb{C}[T^*X]$, which agrees with the element of $\mathbb{C}[T^*X]$ computed in the previous paragraph. □

Proposition 6.20. *With the induced filtration from* $U(\mathfrak{g})$, $\mathrm{gr}((Z(\mathfrak{g})_+)) = (\mathbb{C}[\mathfrak{g}]_+^G)$.

Proof. We showed that $(Z(\mathfrak{g})_+) \subset \ker \Phi$, thus

$$\mathrm{gr}((Z(\mathfrak{g})_+)) \subseteq \ker \mathrm{gr}\, \Phi = \ker(\mu^* \circ \pi) = (\mathbb{C}[\mathfrak{g}]_+^G).$$

To show the reverse inclusion, work with the identification $(\mathbb{C}[\mathfrak{g}]_+^G) \cong (S(\mathfrak{g})_+^G)$ via the Killing form which is G-invariant. Since $\mathrm{gr}((Z(\mathfrak{g})_+))$ is a graded ideal in $S(\mathfrak{g})$ and $(S(\mathfrak{g})_+^G)$

is generated by $S^n(\mathfrak{g})^G$ for $n \geq 1$, it suffices to prove that $S^n(\mathfrak{g})^G \subseteq \mathrm{gr}((Z(\mathfrak{g})_+))$ for all such n. Let $\omega : S^n(\mathfrak{g}) \to F_nU(\mathfrak{g})$ be the symmetrization map, defined on products via

$$\omega(x_1 \cdots x_n) = \frac{1}{n!} \sum_{\sigma \in S_n} x_{\sigma(1)} \cdots x_{\sigma(n)}.$$

Note that the map is G-equivariant. Also recall the symbol map

$$\sigma_n : F_nU(\mathfrak{g}) \to F_nU(\mathfrak{g})/F_{n-1}U(\mathfrak{g}) \cong S^n(\mathfrak{g}).$$

If $f \in S^n(\mathfrak{g})^G$, set $u = \omega(f) \in F_nU(\mathfrak{g})$, then clearly $\sigma_n(u) = f$, and $u \in U(\mathfrak{g})^G = Z(\mathfrak{g})$ (Proposition 4.23). Lastly, $u - \chi_0(u) \in Z(\mathfrak{g})_+$, and since $n \geq 1$, $\sigma_n(u - \chi_0(u)) = \sigma_n(u) = f$. This shows the symbol map $\sigma_n : F_nU(\mathfrak{g}) \cap Z(\mathfrak{g})_+ \to S^n(\mathfrak{g})^G$ is surjective, so $(S(\mathfrak{g})_+^G) \subseteq \mathrm{gr}((Z(\mathfrak{g})_+))$. □

Putting together the previous two propositions, we finally obtain our desired result.

Theorem 6.21. *The map $\Phi_0 : U(\mathfrak{g})/(Z(\mathfrak{g})_+) \to \Gamma(X, \mathcal{D}_X)$ is an isomorphism.*

Proof. By Proposition 6.20, the source of $\mathrm{gr}\,\Phi_0$ is $\mathbb{C}[\mathfrak{g}]/(\mathbb{C}[\mathfrak{g}]_+^G)$, then by Proposition 6.19,

$$\mathrm{gr}\,\Phi_0 : \mathbb{C}[\mathfrak{g}]/(\mathbb{C}[\mathfrak{g}]_+^G) \to \mathrm{gr}\,\Gamma(X, \mathcal{D}_X)$$

is equal to $\mu^* : \mathbb{C}[\mathcal{N}] \to \mathbb{C}[T^*X]$. In particular, $\mathrm{gr}\,\Phi_0$ is an isomorphism. By Proposition 1.12, then Φ_0 must have been an isomorphism. □

This completes the proof Theorem 6.2. Coupling the result with Theorem 6.3 yields the the full result of Beilinson-Bernstein Localization for the trivial infinitesimal character.

6.5. **Example of $\mathfrak{sl}_2$.** Working through this isomorphism for $G = SL(2, \mathbb{C})$ helps make clear the map. First recall that

$$\mathfrak{sl}_2(\mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} : a, b, c \in \mathbb{C} \right\}$$

has a basis of $e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, and $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. The Borel subgroup of G is the upper triangular matrices in $\mathrm{SL}(2, \mathbb{C})$, concretely

$$B = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} : a, b \in \mathbb{C}, a \neq 0 \right\}.$$

From Example 4.29 we see that the flag variety $X = G/B$ is the complete flags in $\mathbb{C}^2$, which is just $X = \mathbb{P}^1$. With this identification, if $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$, then for $[x, y] \in \mathbb{P}^1$, the action is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot [x, y] = [ax + by, cx + dy].$$

Let $U_0 = \mathbb{P}^1 - \{[0, 1]\}$, $U_\infty = \mathbb{P}^1 - \{[1, 0]\}$ be the two affine charts covering $\mathbb{P}^1$. Let z be the coordinate on U_0, defined by $z([x, y]) = y/x$, and w the coordinate on U_∞ defined by $w([x, y]) = x/y$. Then on the overlap $U_0 \cap U_\infty$, $w = z^{-1}$.

Now let us compute the map $\Phi : U(\mathfrak{sl}_2) \to \Gamma(X, \mathcal{D}_X)$. First take the U_0 chart and $p(z) \in \mathcal{O}_X(U_0)$. We can compute the action of $\Phi(e)$ on $\mathcal{O}_X(U_0)$ using (6.1):

$$\begin{aligned}(\Phi(e) \cdot p)(z) &= \left.\frac{d}{dr}\right|_{r=0} p(\exp(-re)z) \\ &= \left.\frac{d}{dr}\right|_{r=0} p\left(\begin{pmatrix} 1 & -r \\ 0 & 1 \end{pmatrix} \cdot z\right) \\ &= \left.\frac{d}{dr}\right|_{r=0} p\left(\frac{z}{1-rz}\right) \\ &= \left[p'\left(\frac{z}{1-rz}\right) \cdot \left(\frac{z}{1-rz}\right)^2\right]_{r=0} \\ &= z^2 p'(z).\end{aligned}$$

Thus $\Phi(e)$ restricted to $\mathcal{D}_X(U_0)$ equals $z^2\partial_z$. On the intersection $U_0 \cap U_\infty$, the identity $z = 1/w$ implies $-z^2\partial_z = \partial_w$, so $\Phi(e)$ restricted to $\mathcal{D}_X(U_\infty)$ equals $-\partial_w$. Then $\Phi(e)$ equals the pair

$$\Phi(e) = (z^2\partial_z, -\partial_w) \in \mathcal{D}_X(U_0) \times_{\mathcal{D}_X(U_0 \cap U_\infty)} \mathcal{D}_X(U_\infty) \cong \Gamma(X, \mathcal{D}_X).$$

Similar computations as above then show

$$\Phi(f) = (-\partial_z, w^2\partial_w), \quad \Phi(h) = (2z\partial_z, -2w\partial_w).$$

The map Φ is then

$$\begin{aligned}\Phi : U(\mathfrak{sl}_2(\mathbb{C})) &\to \Gamma(X, \mathcal{D}_X), \\ e \mapsto z^2\partial_z,\ f &\mapsto -\partial_z,\ h \mapsto 2z\partial_z,\end{aligned}$$

where we only wrote the U_0-chart of the differential operators. Note that not every element of $\mathcal{D}_X(U_0)$ extends to all of $\mathbb{P}^1$. For example $z \in \mathcal{D}_X(U_0)$, but $z = w^{-1}$ is not defined on all of the U_∞ chart.

The center of $U(\mathfrak{sl}_2(\mathbb{C}))$ is the algebra generated by the Casimir operator $\Delta = \frac{1}{2}h^2 + ef + fe$, see [CG10, Page 132] for a proof using the Harish-Chandra isomorphism. As we can check by hand, Δ is in the kernel of Φ. On the U_0 chart:

$$\Phi(\Delta)|_{U_0} = 2z\partial_z z\partial_z - z^2\partial_z^2 - \partial_z z^2\partial_z = 0.$$

Thus Φ factors through to

$$\Phi_0 : U(\mathfrak{sl}_2(\mathbb{C}))/U(\mathfrak{sl}_2(\mathbb{C}))\Delta \to \Gamma(X, \mathcal{D}_X).$$

Theorem 6.2 tells us that Φ_0 is in fact an isomorphism.

6.6. **Rigid analytic setting.** In Section 3 we introduced the completed sheaf of differential operators $\widehat{\mathcal{D}}$ on a rigid analytic variety. Ardakov and Wadsley proved an analogous version of Beilinson-Bernstein localization in this setting that we will present now, and examine a specific consequence of.

Let $K/\mathbb{Q}_p$ be a finite extension, and G a connected split semisimple algebraic group over K, with Lie algebra $\mathfrak{g}$. Let B be a Borel subgroup, and $X = (G/B)^{\mathrm{an}}$, the rigid analytic flag variety. To match the correspondence with the completed differential operators, we will also consider a Fréchet completion of the universal enveloping algebra. We can define a family of

seminorms on $U(\mathfrak{g})$ similar to those on $\mathcal{D}(X)$ in Section 3.2, completing $U(\mathfrak{g})$ with respect to these seminorms gives the Arens-Michael envelope, $\widehat{U(\mathfrak{g})}$. If $x_1, \ldots, x_n$ is a basis of $\mathfrak{g}$, then

$$\widehat{U(\mathfrak{g})} = \left\{ \sum_{\alpha \in \mathbb{N}^n} c_\alpha x^\alpha : c_\alpha \in K, |c_\alpha| r^{|\alpha|} \to 0 \text{ as } |\alpha| \to \infty, \forall r > 0 \right\},$$

where the relations between the x_i are the same as in $U(\mathfrak{g})$. Analogously to $\widehat{\mathcal{D}}(X)$, the Arens-Michael envelope is a Fréchet-Stein algebra (Definition 3.10). The result of Ardakov and Wadsley is then the following:

Theorem 6.22. [AW19, Theorem E] *There is an equivalence of categories via localization with inverse given by global sections,*

$$\left\{ \begin{array}{c} \text{coadmissible } \widehat{U(\mathfrak{g})}\text{-modules} \\ \text{killed by } Z(\mathfrak{g})_+ \end{array} \right\} \xleftrightarrow{\sim} \left\{ \begin{array}{c} \text{coadmissible sheaves} \\ \text{of } \widehat{\mathcal{D}}_X\text{-modules} \end{array} \right\}.$$

The condition of being killed by $Z(\mathfrak{g})_+$ is basically that of having trivial infinitesimal character. Part of the proof of this theorem in specific is an isomorphism

$$\Phi_0 : \widehat{U(\mathfrak{g})}/\widehat{U(\mathfrak{g})} Z(\mathfrak{g})_+ \xrightarrow{\sim} \Gamma(X, \widehat{\mathcal{D}}_X).$$

Now consider the case of $G = \mathrm{SL}_2$. The flag variety is then $X = \mathbb{P}^{1,\mathrm{an}}$, the rigid analytic projective line. Let $U = \mathbb{P}^{1,\mathrm{an}} - \{0, \infty\}$ and $j : U \hookrightarrow X$ the inclusion. Let $\lambda \in K$, and as before consider the $\widehat{\mathcal{D}}_U$-modules $\mathcal{M}_\lambda = \mathcal{O}_U \cdot z^\lambda$. Then the pushforward $j_* \mathcal{M}_\lambda$ is a $\widehat{\mathcal{D}}_X$-module. Leveraging Theorem 3.16 ([BB21, Theorem 5.2]) lets us conclude when $j_* \mathcal{M}_\lambda$ is coadmissible as a $\widehat{\mathcal{D}}_X$-module

Theorem 6.23. *The pushforward $j_* \mathcal{M}_\lambda$ is coadmissible as a $\widehat{\mathcal{D}}_X$-module if and only if λ and $-\lambda$ are of positive type.*

Proof. The rigid analytic projective line $\mathbb{P}^{1,\mathrm{an}}$ is covered by two affinoid unit discs $X_0 = \mathrm{Sp}(K\langle z \rangle) = \{[x, 1] \mid |x| \leq 1\}$ and $X_\infty = \mathrm{Sp}(K\langle w \rangle) = \{[1, y] \mid |y| \leq 1\}$ glued along their boundaries via $z = w^{-1}$. In this setting, for $j_* \mathcal{M}_\lambda$ to be coadmissible is equivalent to for $i \in \{0, \infty\}$ that $j_* \mathcal{M}_\lambda(X_i)$ is a coadmissible module over $\widehat{\mathcal{D}}_X(X_i)$.

For $i = 0$, this is the statement that $\mathcal{M}(X_0 - \{0\})$ is coadmissible over $\widehat{\mathcal{D}}_X(X_0)$. This is precisely the setup of Bitoun and Bode in Theorem 3.16, so it holds if and only if λ is of positive type.

For the X_∞ chart, we have $X_\infty = \mathrm{Sp}(K\langle w \rangle)$. The setup is the same as before but now $z^\lambda = w^{-\lambda}$. Thus by Theorem 3.16, $j_* \mathcal{M}_\lambda(X_\infty)$ to be coadmissible over $\widehat{\mathcal{D}}_X(X_\infty)$ is equivalent to $-\lambda$ is of positive type. □

Remark 6.24. It is not enough for only λ to be of positive type as there exist p-adic numbers λ of positive type such that $-\lambda$ is of type zero.

Just as we saw in Section 6.5, the images of e, f, h under the isomorphism $\widehat{U(\mathfrak{g})}/\widehat{U(\mathfrak{g})}\Delta \xrightarrow{\sim} \Gamma(X, \widehat{\mathcal{D}}_X)$ are given by

$$\Phi_0(e) = (z^2 \partial_z, -\partial_w),\ \Phi_0(f) = (-\partial_z, w^2 \partial_w),\ \Phi_0(h) = (2z\partial_z, -2w\partial_w).$$

We wrote the elements of $\Gamma(X, \widehat{\mathcal{D}}_X)$ as pairs given by their restrictions to X_0 and X_∞.

There is a natural map

$$\Gamma(X, \widehat{\mathcal{D}}_X) \to \Gamma(X, j_*\mathcal{M}_\lambda) = \mathcal{M}_\lambda(U),$$
$$P \mapsto P \cdot z^\lambda.$$

This maps $z\partial - \lambda$ to zero. Under the identification $\Gamma(X, \widehat{\mathcal{D}}_X) \cong \widehat{U(\mathfrak{g})}/\widehat{U(\mathfrak{g})\Delta}$, the element $h \in \mathfrak{g}$ corresponds to $2z\partial$, thus the above factors to give a natural map

$$\widehat{U(\mathfrak{g})}/\widehat{U(\mathfrak{g})}(\Delta, h - 2\lambda) \to \Gamma(X, j_*\mathcal{M}_\lambda). \tag{6.3}$$

Theorem 6.25. *If $\lambda \notin \mathbb{Z}$ and both λ and $-\lambda$ are of positive type then (6.3) is an isomorphism.*

Proof. Consider now just the finite order differential operators $\mathcal{D}_X$. The map

$$\mathcal{D}_X(X_0) \to \mathcal{O}_X(X_0)[z^{-1}] \cdot z^\lambda,\ P \mapsto P \cdot z^\lambda.$$

This map has kernel exactly $\mathcal{D}_X(X_0)(z\partial - \lambda)$, so there is an exact sequence

$$\mathcal{D}_X(X_0) \xrightarrow{\times(z\partial-\lambda)} \mathcal{D}_X(X_0) \longrightarrow \mathcal{O}_X(X_0)[z^{-1}] \cdot z^\lambda \longrightarrow 0.$$

There is a similar exact sequence over the X_∞ chart. If we let $\mathcal{N}$ be the sheaf on X obtained by gluing $\mathcal{O}_X(X_0)[z^{-1}] \cdot z^\lambda$ and $\mathcal{O}_X(X_\infty)[w^{-1}] \cdot w^{-\lambda}$ along $X_0 \cap X_\infty$, then we have an exact sequence

$$\mathcal{D}_X \xrightarrow{\times(z\partial_z-\lambda)} \mathcal{D}_X \longrightarrow \mathcal{N} \longrightarrow 0.$$

By [Bod19, Lemma 4.14], we then have an exact sequence of sheaves

$$\widehat{\mathcal{D}}_X \xrightarrow{\times(z\partial-\lambda)} \widehat{\mathcal{D}}_X \longrightarrow \widehat{\mathcal{D}}_X \otimes_{\mathcal{D}_X} \mathcal{N} \longrightarrow 0.$$

Looking at the two charts X_0 and X_∞ and applying the main result of Bitoun and Bode [BB21], there is an isomorphism $\widehat{\mathcal{D}}_X \otimes_{\mathcal{D}_X} \mathcal{N} \cong j_*\mathcal{M}_\lambda$. By Theorem 6.22, the global sections functor $\Gamma(X, -)$ is quasi inverse to localization, thus it is exact, giving the exact sequence

$$\widehat{U(\mathfrak{g})}/\widehat{U(\mathfrak{g})}\Delta \xrightarrow{\times(h-2\lambda)} \widehat{U(\mathfrak{g})}/\widehat{U(\mathfrak{g})}\Delta \longrightarrow \Gamma(X, j_*\mathcal{M}_\lambda) \longrightarrow 0.$$

This proves the result. □

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