Realizability Methods for Saturated Transfer Systems Over Finite Cyclic Groups

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Abstract

Equivariant homotopy theory has families of generalizations of commutative monoids with G-actions, where G is a finite group. These generalizations arise from G-operads and differ in which symmetric multiplication operations they admit. A G-transfer system is a combinatorial object that encodes these generalizations. Complex linear isometries operads can be used to turn a collection of irreducible G-representations into a transfer system. However, we are interested in the inverse direction. For which G-transfer systems does such a collection, a "G-universe," exist? Which transfer systems are "realizable"? When G is cyclic, this question reduces to an incidence geometry problem in the subgroup lattice of G. We are especially interested in cyclic groups whose order has at most three distinct prime-power factors. To tackle large families of transfer systems for these groups, our method for realizability involves strategically omitting cosets from the complete set of representations.

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1 Preliminaries

1.1 Transfer Systems

Definition 1.1. Let G be a finite group. A G-**transfer system** is a binary relation $\mathcal{R}$ on the subgroups of G, denoted $\mathrm{Sub}(G)$, that satisfies

1) (Reflexivity) For all $H \in \mathrm{Sub}(G)$, $H \to_{\mathcal{R}} H$
2) (Transitivity) For all $H, K, L \in \mathrm{Sub}(G)$, $(H \to_{\mathcal{R}} K$ and $K \to_{\mathcal{R}} L) \implies H \to_{\mathcal{R}} L$
3) (Refinement) For all $H, K \in \mathrm{Sub}(G)$, $H \to_{\mathcal{R}} K \implies H \leq K$
4) (Conjugation) For all $H, K \in \mathrm{Sub}(G)$ and $g \in G$, $H \to_{\mathcal{R}} K \implies gHg^{-1} \to_{\mathcal{R}} gKg^{-1}$
5) (Restriction) For all $H, K, L \in \mathrm{Sub}(G)$, $(H \to_{\mathcal{R}} K$ and $L \leq K) \implies H \cap L \to_{\mathcal{R}} L$.

Definition 1.2. A G-transfer system is **saturated** if it satisfies

6) (Saturation) For all $H, K, L \in \mathrm{Sub}(G)$, $(H \to_{\mathcal{R}} L$ and $H \leq K \leq L) \implies K \to_{\mathcal{R}} L$.

In this project, we study saturated transfer systems over finite cyclic groups. Throughout, n is a positive integer and $C_n = \mathbb{Z}/n\mathbb{Z}$, written additively. The subgroups of C_n are in order-preserving bijection with the divisors of n,

$$\mathrm{Div}(n) \;=\; \{\, d \in \mathbb{N} \mid dk = n \text{ for some } k \in \mathbb{N} \,\}.$$

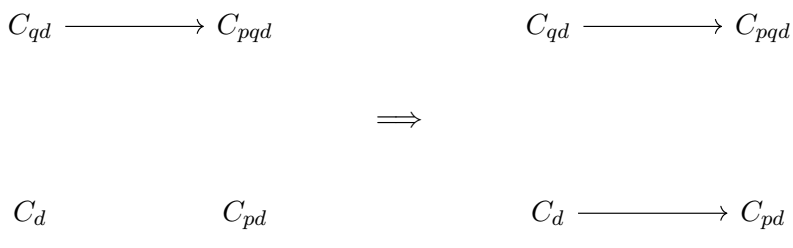
In other words, given $d \mid n$, the subgroup C_d is the unique subgroup of order d, defined as

$$C_d \;=\; \frac{n}{d}\,\mathbb{Z}/n\mathbb{Z} \;\leq\; \mathbb{Z}/n\mathbb{Z},$$

so that $C_d \leq C_e$ if and only if $d \mid e$. Additionally, because cyclic groups are abelian, the conjugation invariance axiom is automatic, giving us a convenient simplification.

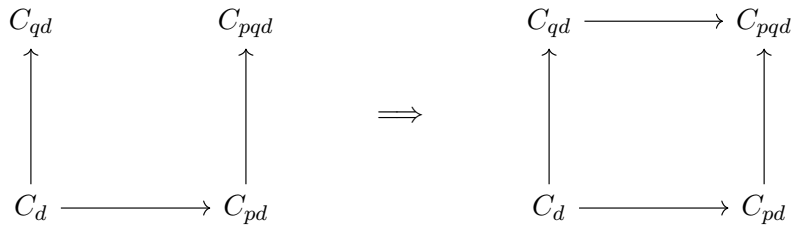
Definition 1.3. A C_n-**transfer graph** is a directed graph $\mathcal{G}$ with vertex set $\mathrm{Sub}(C_n)$ and edge set $E \subseteq \mathrm{Sub}(C_n)^2$ satisfying the following:

1) If $(C_d, C_e) \in E$, then there exists a prime p such that $dp = e$
2) (Rule 1) For distinct primes p, q and $d \in \mathrm{Div}(n)$ such that $pqd \in \mathrm{Div}(n)$ and $(C_{qd}, C_{pqd}) \in E$, we have $(C_d, C_{pd}) \in E$



Visualization of Rule 1

3) (Rule 2) For distinct primes p, q and $d \in \mathrm{Div}(n)$ such that $pqd \in \mathrm{Div}(n)$ and $(C_d, C_{qd}) \in E$ and $(C_{qd}, C_{pqd}) \in E$, we have $(C_{pd}, C_{pqd}) \in E$



Visualization of Rule 2

For a cyclic group C_n, we have the set of saturated C_n-transfer systems, $\mathrm{SatTr}(C_n)$, and the set of C_n-transfer graphs, $\mathrm{TrGr}(C_n)$. Consider the function $f : \mathrm{SatTr}(C_n) \to \mathrm{Sub}(C_n)^2$ defined by

$$f(\mathcal{R}) \;=\; \{(C_d, C_{pd}) \;\mid\; p \text{ is prime},\; d, pd \in \mathrm{Div}(n),\; C_d \to_{\mathcal{R}} C_{pd}\}\,.$$

We can use this map to define a relationship between transfer systems and transfer graphs.

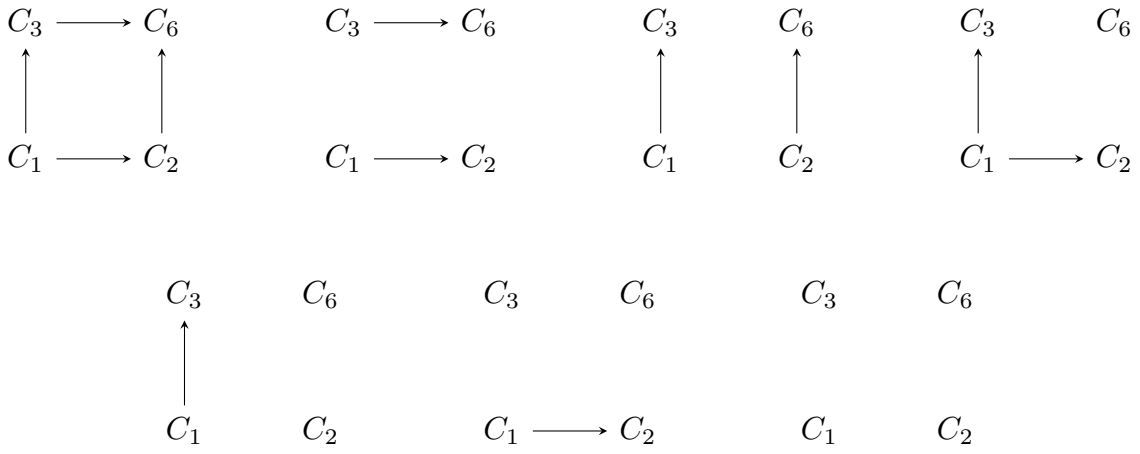
Theorem 1.4. *For every saturated C_n-transfer system $\mathcal{R}$, $f(\mathcal{R})$ is a valid C_n-transfer graph. Moreover, we have a bijection*

$$F \;:\; \mathit{SatTr}(C_n) \;\to\; \mathit{TrGr}(C_n)$$

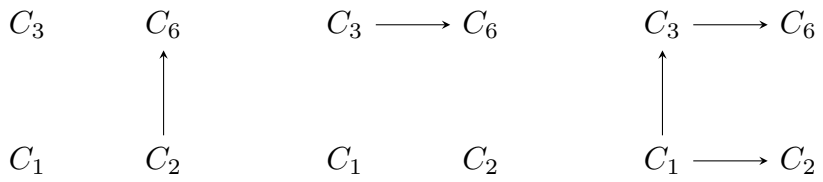
defined by $F(\mathcal{R}) = f(\mathcal{R})$ for all $\mathcal{R} \in \mathit{SatTr}(C_n)$.

This theorem gives us that transfer systems and transfer graphs are equivalent. Therefore, to maintain clarity, from now on we will refer to transfer graphs as **transfer systems.**

Example 1.5. All the saturated C_6-transfer systems.



Example 1.6. Non-examples of saturated transfer systems for C_6 include the graphs below. The leftmost and center graphs fail Rule 1 and the rightmost graph fails Rule 2.



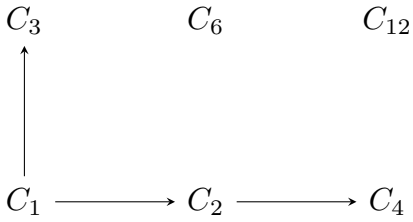
1.2 Cofibrant Subgroups

Definition 1.7. Given a saturated C_n-transfer system $\mathcal{R}$, a subgroup $C_b \leq C_n$ is **$\mathcal{R}$-cofibrant** if for all subgroups $C_a \leq C_b$,

$$C_a \longrightarrow_{\mathcal{R}} C_b \quad \iff \quad a = b.$$

The set of $\mathcal{R}$-cofibrant subgroups is denoted $\mathcal{C}(\mathcal{R})$.

Example 1.8. In this C_{12}-transfer system, the cofibrant subgroups are C_1, C_6, C_{12}.



Definition 1.9. Let $\mathcal{R}$ be a saturated C_n-transfer system and let $C_d \leq C_n$. Define $\chi_{\mathcal{R}}(C_d)$ to be the **maximal $\mathcal{R}$-cofibrant subgroup** of C_d. When dealing with the order of $\chi_{\mathcal{R}}(C_d)$, we write

$$\chi_{\mathcal{R}}(d) := \left|\chi_{\mathcal{R}}(C_d)\right|$$

so that $C_{\chi_{\mathcal{R}}(d)} = \chi_{\mathcal{R}}(C_d)$. Importantly, note that $\chi_{\mathcal{R}}(C_d) = C_d$ when C_d is cofibrant, and $\chi_{\mathcal{R}}(d)$ properly divides d otherwise.

Lemma 1.10. *The two following definitions of $\chi_{\mathcal{R}}(C_d)$ are equivalent:*

1) *$\chi_{\mathcal{R}}(C_d)$ is the maximal $\mathcal{R}$-cofibrant subgroup of C_d*

2) *$\chi_{\mathcal{R}}(C_d)$ is the unique minimal subgroup $C_m \leq C_d$ such that $C_m \to_{\mathcal{R}} C_d$.*

Proof. ($1 \implies 2$) If $C_m = C_d$, then we automatically know that C_d is cofibrant and is thus the unique minimal subgroup related to C_d. Now assume C_m is a proper subgroup of C_d. For any $C_k \leq C_m \leq C_d$, if $C_k \to_{\mathcal{R}} C_d$, then by restriction, $C_k \to_{\mathcal{R}} C_m$. This implies that $C_k = C_m$ because C_m is cofibrant by definition. If there exists some $C_a \neq C_m$ satisfying the same conditions, restriction guarantees that $C_a \cap C_m \to_{\mathcal{R}} C_m$, which contradicts the fact that C_m is cofibrant. Thus, C_m must be unique.
($2 \implies 1$) We know that $C_m \to_{\mathcal{R}} C_d$ for a unique C_m, and also that for all $C_k \lneq C_m$, we have $C_k \not\to_{\mathcal{R}} C_d$. Recall that C_m is cofibrant. If not, then there exists some $C_j \lneq C_m$ such that $C_j \to_{\mathcal{R}} C_m$, which implies that $C_j \to_{\mathcal{R}} C_d$ by transitivity, violating the assumed minimal property of C_m. Note that for all $C_m \leq C_a \leq C_d$, restriction forces $C_m \to_{\mathcal{R}} C_a$, so any subgroup C_a above C_m cannot be cofibrant. □

Proposition 1.11 ([CGMR], [Mac23]). *Let $\mathcal{R}$ be a saturated C_n-transfer system. Some results about cofibrant subgroups:*

1) *C_1 is always cofibrant*

2) *Cofibrant subgroups preserve closure under least common multiples*

3) *Saturated transfer systems are uniquely determined by their cofibrant subgroups.*

1) *Proof.* Since $\mathcal{R}$ is a saturated transfer system, it preserves refinement, meaning that $C_k \leq C_1$, where C_1 is the trivial group. It follows that $C_k = C_1$. □

2) *Proof.* We want to show that if C_{m_1} and C_{m_2} are cofibrant, then $C_{\text{lcm}(m_1,m_2)}$ is also cofibrant. Suppose there exists a subgroup C_k such that $C_k \to_{\mathcal{R}} C_{\text{lcm}(m_1,m_2)}$. By properties of subgroups of cyclic groups, note that $C_{m_1}, C_{m_2} \leq C_{\text{lcm}(m_1,m_2)}$. Since $\mathcal{R}$ is a saturated transfer system, it preserves restriction, meaning that

$$C_k \cap C_{m_1} = C_{\gcd(k,m_1)} \longrightarrow_{\mathcal{R}} C_{m_1}.$$

Since C_{m_1} is cofibrant, we must have that $\gcd(k, m_1) = m_1$, so m_1 divides k. Using the same restriction logic, we deduce that m_2 divides k. It follows that $\text{lcm}(m_1, m_2)$ divides k, and we know by assumption that k divides $\text{lcm}(m_1, m_2)$. Therefore, $k = \text{lcm}(m_1, m_2)$, and $C_{\text{lcm}(m_1,m_2)}$ is thus cofibrant. □

3) *Proof.* Let us define a relation $\sim_{\mathcal{R}}$ on $\text{Sub}(C_n)$, where

$$C_d \sim_{\mathcal{R}} C_e \iff \chi_{\mathcal{R}}(C_d) = \chi_R(C_e).$$

Our first claim is that $\sim_{\mathcal{R}}$ is an equivalence relation. Since $\chi_{\mathcal{R}}(C_d) = \chi_R(C_d)$, we know that $C_d \sim_{\mathcal{R}} C_d$ and $\sim_{\mathcal{R}}$ is reflexive. If we assume that $C_d \sim_{\mathcal{R}} C_e$, then we have $\chi_{\mathcal{R}}(C_d) = \chi_R(C_e)$ and $\chi_{\mathcal{R}}(C_e) = \chi_R(C_d)$, meaning that $C_e \sim_{\mathcal{R}} C_d$ and $\sim_{\mathcal{R}}$ is symmetric. If we assume that $C_d \sim_{\mathcal{R}} C_e$ and $C_e \sim_{\mathcal{R}} C_f$, then $\chi_{\mathcal{R}}(C_d) = \chi_R(C_e)$ and $\chi_{\mathcal{R}}(C_e) = \chi_R(C_f)$, meaning that $\chi_{\mathcal{R}}(C_d) = \chi_R(C_f)$. It follows that $C_d \sim_{\mathcal{R}} C_f$ and $\sim_{\mathcal{R}}$ is transitive. We conclude that $\sim_{\mathcal{R}}$ is an equivalence relation on $\text{Sub}(C_n)$.

Suppose we start with a saturated C_n-transfer system $\mathcal{R}$. To determine the set of $\mathcal{R}$-cofibrant subgroups, take the minimal element of each equivalence class of $\sim_{\mathcal{R}}$. By definition of $\sim_{\mathcal{R}}$, this element is cofibrant. Collecting all such elements gives $\mathcal{C}(\mathcal{R})$.

Conversely, suppose we start with $\mathcal{C}(\mathcal{R})$. Consider a complete graph on the vertices of $\text{Sub}(C_n)$ and intersect this graph with $\sim_{\mathcal{R}}$. That is, $C_a \to_{\mathcal{R}} C_b$ if and only if $\chi_{\mathcal{R}}(C_a) = \chi_{\mathcal{R}}(C_b)$ and $a \mid b$. This action separates the graph into connected components. By Lemma 1.10, the minimal element of each chain is exactly the cofibrant subgroup belonging to $\mathcal{C}(\mathcal{R})$, letting us recover the connected components of the saturated transfer system. □

The relationship between $\mathcal{R}$ and $\mathcal{C}(\mathcal{R})$ lets us conveniently study a saturated transfer system via its cofibrant subgroups, instead of having to draw out the entire transfer system.

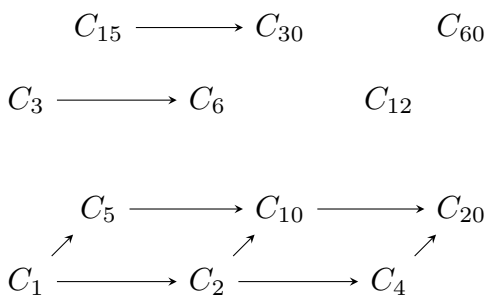
To simplify things, we may "undo" lcm-closure by introducing irreducible cofibrant subgroups.

Definition 1.12. Given a saturated transfer system $\mathcal{R}$, we define an **irreducible cofibrant subgroup** as an element of the set

$$\mathcal{C}^\sharp(\mathcal{R}) = \left\{ C_a \in \mathcal{C}(\mathcal{R}) \;\middle|\; \begin{array}{r} a \neq 1 \text{ and for all } C_b, C_c \in \mathcal{C}(\mathcal{R}), \\ \big(\mathrm{lcm}(b,c) = a\big) \implies \big(a = b \text{ or } a = c\big) \end{array} \right\}.$$

We remove C_1 from the set, as well as any elements that are the least common multiple of two strictly smaller elements, leaving the **minimal generating set** of $\mathcal{C}(\mathcal{R})$.

Example 1.13. In this saturated C_{60}-transfer system, $\mathcal{C}^\sharp(\mathcal{R}) = \{C_3, C_{12}, C_{15}\}$.



2 Realizability Methods

2.1 Complex Linear Isometries Realizability

Definition 2.1. Let $\mathrm{Irr}(G)$ be the set of irreducible G-representations. A G-**universe** is a subset $\mathcal{U} \subseteq \mathrm{Irr}(G)$ containing the trivial representation.

In the cyclic case, this definition simplifies nicely.

Definition 2.2. A C_n-**universe** is, up to isomorphism, a subset $\mathcal{U} \subseteq \{0, \ldots, n-1\}$ containing 0.

So far, we have dealt with cofibrant subgroups of preexisting saturated transfer systems. The definition below outlines what it means to be linear isometries-cofibrant, that is, for saturated transfer systems that lie in the image of the complex linear isometries operad

$$\mathcal{L} : \{C_n\text{-universes}\} \to \mathrm{SatTr}(C_n).$$

Such saturated transfer systems are $\mathcal{L}(\mathcal{U})$ for some C_n-universe $\mathcal{U}$.

Definition 2.3. A subgroup $C_b \leq C_n$ is $\mathcal{L}(\mathcal{U})$-**cofibrant** if for all a properly dividing b, there exists an i mod b such that

1) For all $u \in \mathcal{U}$, $u \not\equiv i \bmod b$

2) There exists $j \in \mathcal{U}$ such that $j \equiv i \bmod a$.

In other words, there exists an residue class that is missing modulo b that is present modulo a.

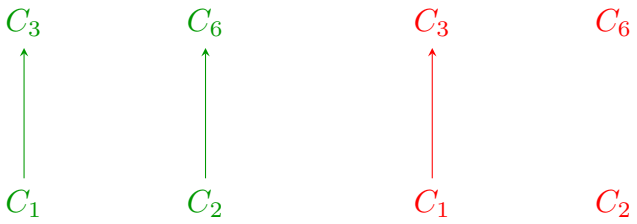
Definition 2.4. A saturated C_n-transfer system $\mathcal{R}$ is **complex linear isometries realizable** if there exists a C_n-universe $\mathcal{U}$ such that $\mathcal{R} = \mathcal{L}(\mathcal{U})$.

Definition 2.5. Let $\mathcal{U}$ be a C_n-universe and let π be the subgroup projection map. For all $a \mid b \mid n$,

$$C_a \to_{\mathcal{L}(\mathcal{U})} C_b \quad \iff \quad \pi_b^n[\mathcal{U}] = (\pi_a^b)^{-1}\big[\pi_a^n[\mathcal{U}]\big].$$

In a perfect world, every saturated C_n-transfer system would be $\mathcal{L}(\mathcal{U})$ for some $\mathcal{U}$. However, not all saturated transfer systems are realizable.

Example 2.6. Consider the two C_6-transfer systems below.



$\mathcal{L}(\mathcal{U})$ for $\mathcal{U} = \{0, 2, 4\}$ Not $\mathcal{L}(\mathcal{U})$ for any $\mathcal{U}$

When dealing with complex linear isometries realizability, the intuition behind the set equalities is as follows: If we project the C_n-universe all the way down to the source of the arrow and lift that projection up to the target of the arrow, the resulting set must be equal to the projection directly to the target of the arrow.

Consider $\mathcal{U} = \{0, 2, 4\}$. In the transfer system on the left, we have $C_1 \to_{\mathcal{L}(\mathcal{U})} C_3$ and also $C_2 \to_{\mathcal{L}(\mathcal{U})} C_6$ because projecting $\mathcal{U}$ from C_6 directly to C_3 is the same as projecting from C_6 all the way down to C_1 and lifting back up to C_3, as follows:

$$\pi_3^6[\mathcal{U}] = \{0, 1, 2\} = (\pi_1^3)^{-1}\big[\{0\}\big] = (\pi_1^3)^{-1}\big[\pi_1^6[\mathcal{U}]\big].$$

Similarly, since $\mathcal{U}$ contains only even numbers, projecting this set down to C_2 and lifting back to C_6 is the same as the identity projection on C_6. Thus,

$$\pi_6^6[\mathcal{U}] = \{0, 2, 4\} = (\pi_2^6)^{-1}\big[\{0\}\big] = (\pi_2^6)^{-1}\big[\pi_2^6[\mathcal{U}]\big].$$

Next, notice that $C_1 \not\to_{\mathcal{L}(\mathcal{U})} C_2$ because

$$\pi_2^6[\mathcal{U}] = \{0\} \neq \{0, 1\} = (\pi_1^2)^{-1}\big[\pi_1^6[\mathcal{U}]\big],$$

Finally,

$$\pi_6^6[\mathcal{U}] = \{0, 2, 4\} \neq \{0, 1, 2, 3, 4, 5\} = (\pi_3^6)^{-1}\big[\pi_3^6[\mathcal{U}]\big],$$

so $C_3 \not\to_{\mathcal{L}(\mathcal{U})} C_6$. Thus, $\mathcal{U} = \{0, 2, 4\}$ realizes the left transfer system.

Now, we will attempt to find a universe for the saturated transfer system on the right. We know that there does not exist an arrow between 1 and 2. Note that

$$(\pi_1^2)^{-1}\big[\pi_1^6[\mathcal{U}]\big] = \{0, 1\},$$

so in order for this to work, we must have $\pi_2^6[\mathcal{U}] = \{0\}$. This means that $\mathcal{U}$ must contain only even elements, that is, $\mathcal{U} \subseteq \{0, 2, 4\}$. We also know that $C_1 \to_{\mathcal{L}(\mathcal{U})} C_3$, so we have

$$\pi_3^6[\mathcal{U}] = \{0, 1, 2\} = (\pi_1^3)^{-1}\big[\pi_1^6[\mathcal{U}]\big].$$

This means that $\mathcal{U} \supseteq \{0, 2, 4\}$. Thus, the universe that realizes this transfer system must be $\mathcal{U} = \{0, 2, 4\}$, but this universe already realizes the transfer system on the left.

Another important result relating cofibrant subgroups and realizability is the Layers Theorem.

Theorem 2.7 ([CGMR]). *Suppose $\mathcal{U}$ is a C_n-universe for some even n. If $C_2, \dots, C_{2^k} \in \mathcal{C}^\sharp(\mathcal{L}(\mathcal{U}))$, then for all odd $a, b \mid n$ and $1 \le j \le k$,*

$$C_{2^j a} \longrightarrow_{\mathcal{L}(\mathcal{U})} C_{2^j b} \qquad \iff \qquad C_a \longrightarrow_{\mathcal{L}(\mathcal{U})} C_b.$$

Equivalently, for all odd $d > 1$ and $1 \le j \le k$,$C_{2^j d} \notin \mathcal{C}^\sharp(\mathcal{L}(\mathcal{U}))$.

Corollary 2.8 ([CGMR]). *For all $k \in \mathbb{N}$ and for all odd d, there exists a bijective correspondence between the set of saturated C_d-transfer systems and the set of saturated $C_{2^k \cdot d}$-transfer systems in which $C_2, \dots, C_{2^k}$ are irreducible cofibrant.*

2.2 Bell–Feeney–Harr Realizability

Notation 2.9. We will denote the residue class of a modulo d as $[a]_d$.

Definition 2.10. Given a saturated transfer system $\mathcal{R} = \mathcal{L}(\mathcal{U})$, suppose C_d is $\mathcal{L}(\mathcal{U})$-cofibrant. For a prime $p \mid d$, a p-**representative** is

$$a_p \in \{0, 1, \dots, d-1\} = C_d$$

such that $[a_p]_d \cap \mathcal{U} = \varnothing$ and $[a_p]_{d/p} \cap \mathcal{U} \neq \varnothing$.

Proposition 2.11 ([BFH25]). *Suppose $\mathcal{U}$ is a C_n–universe and C_d is $\mathcal{L}(\mathcal{U})$–cofibrant. For every prime divisor p of d, there exists a p-representative a_p.*

Proof. For each p, we want to find a representative $a_p \in C_d$ for which there exists an $x \in \mathcal{U}$ equivalent to a_p mod $\frac{d}{p}$, but there is no overlap between $\mathcal{U}$ and the class of a_p mod d.

Since our transfer system is $\mathcal{L}(\mathcal{U})$, we have the condition that $C_e \to_{\mathcal{L}(\mathcal{U})} C_d$ if and only if

$$\pi_d^n[\mathcal{U}] = \left(\pi_e^d\right)^{-1}[\pi_e^n[\mathcal{U}]].$$

Recall that RHS always contains LHS, so if C_d is cofibrant, then $d = d/p$, and that means there exists $x \in$ RHS that does not appear in LHS. Note that

$$\left(\pi_{d/p}^d\right)^{-1}[\pi_{d/p}^n[\mathcal{U}]] = \{y \bmod d \mid \exists\, z \in \mathcal{U} \text{ such that } y \equiv z \bmod \tfrac{d}{p}\}$$

$$\pi_d^n[\mathcal{U}] = \{y \bmod d \mid y \in \mathcal{U}\}.$$

We claim that $x = a_p \in \{0, 1, \dots, d-1\}$ and fits the desired conditions.

If $[x]_d \cap \mathcal{U} \neq \varnothing$, then there exists $y \in \mathcal{U}$ such that $y \equiv x \bmod d$. This implies $x \in \pi_d^n[\mathcal{U}]$, which is a contradiction, so $[x]_d$ and $\mathcal{U}$ must be disjoint. Next, note that since

$$x \in \left(\pi_{d/p}^d\right)^{-1}\left[\pi_{d/p}^n[\mathcal{U}]\right],$$

there exists $z \in \mathcal{U}$ such that $z \equiv x \bmod d/p$. This, by definition, gives us that $z \in [x]_{d/p}$, so

$$z \in [x]_{d/p} \cap \mathcal{U}.$$

Thus, the intersection is nonempty. We conclude that $x = a_p$. □

We can then strengthen this result by proving that there exists a k-representative for every proper (not necessarily prime) divisor k of d.

Proposition 2.12 ([BFH25])**.** *Suppose $\mathcal{U}$ is a C_n–universe and C_d is $\mathcal{L}(\mathcal{U})$–cofibrant. For every k properly dividing d, there exists an $a_k \in \{0, 1, \dots, d-1\}$ such that $[a_k]_d \cap \mathcal{U} = \varnothing$ and $[a_k]_k \cap \mathcal{U} \neq \varnothing$.*

Proof. Since k is a proper divisor of d, there exists a prime p such that k divides d/p. By Proposition 2.11, there exists a p-representative a_p such that $[a_p]_d \cap \mathcal{U} = \varnothing$ and $[a_p]_{d/p} \cap \mathcal{U} \neq \varnothing$. Since $k \mid (d/p)$, it follows that $[a_p]_{d/p} \subseteq [a_p]_k$, so

$$[a_p]_k \cap \mathcal{U} \neq \varnothing.$$

Letting $a_k = a_p$ gives the desired conditions. □

We now prove the converse of Proposition 2.12.

Proposition 2.13 ([BFH25])**.** *If for every proper divisor k of d there exists a k-representative a_k, then C_d is $\mathcal{L}(\mathcal{U})$-cofibrant.*

Proof. We claim that for each k,

$$a_k \in (\pi_k^d)^{-1}\big[\pi_k^n[\mathcal{U}]\big] \setminus \pi_d^n[\mathcal{U}].$$

Note that the residue class of $[a_k]_d$, by definition, is $\{x \bmod n \mid x \equiv a_k \bmod d\}$. Similarly, the residue class $[a_k]_k$ is $\{x \bmod n \mid x \equiv a_k \bmod k\}$. We know $[a_k]_k \cap \mathcal{U} \neq \varnothing$, so there exists $x \in \mathcal{U}$ such that $x \equiv a_k \bmod k$, giving us that

$$a_k \in (\pi_k^d)^{-1}\big[\pi_k^n[\mathcal{U}]\big] = \{y \bmod d \mid \exists\, x \in \mathcal{U} \text{ such that } y \equiv x \bmod k\}.$$

We also know $[a_k]_d \cap \mathcal{U} = \varnothing$, so for all $x \in \mathcal{U}$, we have $x \not\equiv a_k \bmod d$, meaning that

$$a_k \notin \pi_d^n[\mathcal{U}] = \{y \bmod d \mid y \in \mathcal{U}\}.$$

Thus, for each proper divisor k of d we have $\pi_d^n[\mathcal{U}] \neq (\pi_k^d)^{-1}[\pi_k^n[\mathcal{U}]]$. As such, $C_k \nrightarrow_{\mathcal{R}} C_d$ for all proper divisors of d. Thus, C_d is $\mathcal{L}(\mathcal{U})$-cofibrant. □

This result gives us a useful equivalent definition of cofibrancy using representatives. The following theorem strengthens this result further by proving the exists a universal d-representative for each $C_d \in \mathcal{C}^\sharp(\mathcal{R})$ that does not depend on the divisors of d.

Theorem 2.14 ([BFH25])**.** *Suppose $\mathcal{U}$ is a C_n-universe and $C_d \in \mathcal{C}^\sharp(\mathcal{L}(\mathcal{U}))$. Then there exists a d-representative $a \in C_d$ such that $[a]_d \cap \mathcal{U} = \varnothing$ and $[a]_{d/p} \cap \mathcal{U} \neq \varnothing$ for all primes $p \mid d$.*

Proof. First, we show that there exists a maximal cofibrant subgroup $C_e \lneq C_d$. That is, C_e is a subgroup such that the only cofibrant proper subgroup of C_d that contains C_e is C_e itself. We begin by expanding $\mathcal{C}^\sharp(\mathcal{L}(\mathcal{U}))$ with respect to lcm-closure, and order the subgroups in increasing order:

$$\mathcal{C}(\mathcal{L}(\mathcal{U})) = \{C_{a_1}, \dots, C_{a_k}\}.$$

Note that $C_{a_1} = C_1$, since C_1 is always cofibrant. If $C_d = C_{a_i}$ for some $i > 2$, then take C_e to be the largest C_{a_j} for which a_j properly divides d. If $C_d = C_{a_2}$, then simply take C_e to be C_1. Thus,

such a C_e always exists, so pick an appropriate subgroup for our given C_d. Now, choose a prime p_e that divides d/e, and choose a representative $a = a_{p_e}$ as in Exercise 18. Note that this gives us $[a]_d \cap \mathcal{U} = \varnothing$ and $[a]_{d/p_e} \cap \mathcal{U} \neq \varnothing$.

Next, take the minimal subgroup $C_f \leq C_d$ for which $[a]_f \cap \mathcal{U} = \varnothing$. Note that since C_f is the minimal subgroup with this property, for every proper divisor k of f we have $[a]_k \cap \mathcal{U} \neq \varnothing$. By Exercise 18, it follows that f is cofibrant. Now we show that f does not divide e. Given that e is a proper divisor of d, by Proposition 2.11 we know that $[a]_e \cap \mathcal{U} \neq \varnothing$. If $f \mid e$, then

$$\varnothing \neq [a]_e \cap \mathcal{U} \subseteq [a]_f \cap \mathcal{U},$$

contradicting the assumption that $[a]_f \cap \mathcal{U} = \varnothing$, giving us that $f \nmid e$.

Suppose $e \leq f \leq d$ does not hold. Since d divides f we have that e does not divide f. Then $\gcd(e, f)$ properly divides e. By lcm-closure $\operatorname{lcm}(e, f)$ is cofibrant, and since d is an irreducible cofibrant $\operatorname{lcm}(e, f)$ must properly divide d. This means that e is a proper divisor of $\operatorname{lcm}(e, f)$, which contradicts the maximality of e. Thus, $C_e \leq C_f \leq C_d$. Since C_e is a maximal proper cofibrant subgroup of C_d, it follows that $C_f = C_d$.

Using this, we want to conclude that $[a]_{d/p} \cap \mathcal{U} \neq \varnothing$ for all primes $p \mid d$. Since C_f is the minimal subgroup of C_d for which $[a]_f \cap \mathcal{U} = \varnothing$, we know that any proper divisor m of d, that is, of f, will have the property that

$$[a]_m \cap \mathcal{U} \neq \varnothing.$$

□

Definition 2.15. Let $\mathcal{R}$ be a saturated C_n-transfer system. A **BFH-witness** for $\mathcal{R}$ is an indexed family of d-representatives

$$(r_d)_{C_d \in \mathcal{C}^\sharp(\mathcal{R})}$$

where each $r_d \in C_d$ and $r_d \neq 0$. This sequence satisfies two witness conditions:

1) For each $C_d \in \mathcal{C}^\sharp(\mathcal{R})$ and every prime divisor $p \mid d$,

$$[r_d]_{d/p} \nsubseteq \bigcup_{C_e \in \mathcal{C}^\sharp(\mathcal{R})} [r_e]_e$$

2) For every $f \mid n$ and every $b \in C_f$,

$$[b]_f \subseteq \bigcup_{C_e \in \mathcal{C}^\sharp(\mathcal{R})} [r_e]_e \implies [b]_{\chi_{\mathcal{R}}(f)} \subseteq \bigcup_{C_e \in \mathcal{C}^\sharp(\mathcal{R})} [r_e]_e.$$

Definition 2.16. A saturated C_n-transfer system $\mathcal{R}$ is **BFH-realizable** if there exists a BFH-witness $(r_d)_{C_d \in \mathcal{C}^\sharp(\mathcal{R})}$ for $\mathcal{R}$.

The following are some relevant results regarding realizability.

Theorem 2.17 ([BFH25])**.** *For all odd n, every saturated C_n-transfer system is BFH-realizable.*

Theorem 2.18 ([Mac23]; cf. [BFH25])**.** *For all odd n, every saturated C_n-transfer system is complex linear isometries-realizable.*

We establish a relationship between BFH realizability and complex linear isometries realizability.

Theorem 2.19. *Let $\mathcal{R}$ be a saturated C_n-transfer system for any n. Suppose $\mathcal{R}$ is BFH-realizable with witness $(r_d)_{C_d \in \mathcal{C}^\sharp(\mathcal{R})}$. The* ***BFH-universe***

$$\mathcal{U} \;=\; \{0, 1, \ldots, n-1\} \;\; \setminus \bigcup_{C_d \in \mathcal{C}^\sharp(\mathcal{R})} [r_d]_d$$

satisfies $\mathcal{L}(\mathcal{U}) = \mathcal{R}$. That is, $\mathcal{R}$ is complex linear isometries-realizable using a BFH-universe.

Proof. We will show that the cofibrants of $\mathcal{R}$ are exactly the cofibrants of $\mathcal{L}(\mathcal{U})$ by showing that $\mathcal{C}^\sharp(\mathcal{R}) \subseteq \mathcal{C}(\mathcal{L}(\mathcal{U}))$ and $\mathcal{C}(\mathcal{L}(\mathcal{U})) \subseteq \mathcal{C}(\mathcal{R})$, which would imply that $\mathcal{C}^\sharp(\mathcal{R}) = \mathcal{C}^\sharp(\mathcal{L}(\mathcal{U}))$. Fix $C_d \in \mathcal{C}^\sharp(\mathcal{R})$ and let r_d be the representative of C_d in the witnessing family. Since r_d is a witness, we know that for every prime p dividing d,

$$[r_d]_{d/p} \;\; \nsubseteq \bigcup_{C_e \in \mathcal{C}^\sharp(\mathcal{R})} [r_e]_e.$$

This means that $[r_d]_{d/p} \cap \mathcal{U} \neq \varnothing$ and $[r_d]_d \cap \mathcal{U} = \varnothing$, implying that C_d is $\mathcal{L}(\mathcal{U})$-cofibrant.

For the other direction, we will prove the contrapositive. If C_d is not $\mathcal{R}$-cofibrant, then $\chi_{\mathcal{R}}(d)$ must be a proper divisor of d. We show that $C_{\chi_{\mathcal{R}}(d)} \to_{\mathcal{L}(\mathcal{U})} C_d$, that is, for all $0 \leq r < d$, either

1) There exists some $j \in \mathcal{U}$ such that $j \equiv r \bmod d$, or

2) For all $i \in \mathcal{U}$, $i \not\equiv r \bmod \chi_{\mathcal{R}}(d)$.

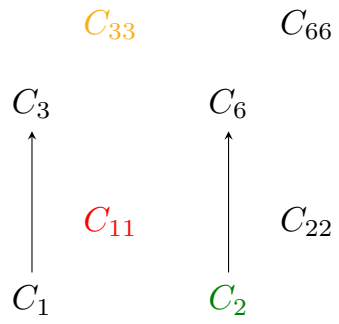
For our purposes, fix $0 \leq r < d$ and suppose that condition 1) does not hold, that is, for all $j \in \mathcal{U}$, $j \not\equiv r \bmod d$. This means that $[r]_d \subseteq \bigcup_{C_e \in \mathcal{C}^\sharp(\mathcal{R})} [r_e]_e$, which gives us

$$[r]_{\chi_{\mathcal{R}}(d)} \;\; \subseteq \bigcup_{C_e \in \mathcal{C}^\sharp(\mathcal{R})} [r_e]_e.$$

It follows that for all $i \in \mathcal{U}$, we have $i \not\equiv r \bmod \chi_{\mathcal{R}}(d)$. Thus, C_d is not $\mathcal{L}(\mathcal{U})$-cofibrant. □

We will now illustrate how this theorem works.

Example 2.20. Consider the following transfer system $\mathcal{R}$ for C_{66}.



The irreducible cofibrant set of this transfer system is $\mathcal{C}^\sharp(\mathcal{R}) = \{C_2, C_{11}, C_{33}\}$. We begin by creating a grid with axes mod 2, mod 3, and mod 11. We use the Chinese Remainder Theorem to place each element of C_{66} uniquely within this grid with respect to its residue class mod 2, 3, and 11. Note that this grid is actually three-dimensional because 66 is a product of three prime factors, however, we place the mod 2 classes side by side to improve visibility.

mod 11

mod 3 \	0	1	2	3	4	5	6	7	8	9	10	mod 2
0	0	12	24	36	48	60	6	18	30	42	54	
1	22	34	46	58	4	16	28	40	52	64	10	0
2	44	56	2	14	26	38	50	62	8	20	32	
0	33	45	57	3	15	27	39	51	63	9	21	
1	55	1	13	25	37	49	61	7	19	31	43	1
2	11	23	35	47	59	5	17	29	41	53	65	

We now represent the irreducible cofibrant subgroups of $\mathcal{R}$. For each $C_d \in \mathcal{C}^\sharp(\mathcal{R})$, we will remove a coset mod d from the grid. The remaining portion of the grid is the C_{66}-universe that realizes $\mathcal{R}$. Our first irreducible cofibrant subgroup is C_2. By definition, a C_{66}-universe must always contain zero, so we cannot delete the 0 mod 2 class. It follows that $r_2 = 1$, so we delete $[1]_2$.

mod 11

mod 3 \	0	1	2	3	4	5	6	7	8	9	10	mod 2
0	0	12	24	36	48	60	6	18	30	42	54	
1	22	34	46	58	4	16	28	40	52	64	10	0
2	44	56	2	14	26	38	50	62	8	20	32	
0	33	45	57	3	15	27	39	51	63	9	21	
1	55	1	13	25	37	49	61	7	19	31	43	1
2	11	23	35	47	59	5	17	29	41	53	65	

We repeat the process for C_{11}. In the scope of this example, we are free to pick any nonzero coset, so we choose $r_{11} = 10$ and delete $[10]_{11}$.

mod 11

mod 3 \	0	1	2	3	4	5	6	7	8	9	10	mod 2
0	0	12	24	36	48	60	6	18	30	42	54	
1	22	34	46	58	4	16	28	40	52	64	10	0
2	44	56	2	14	26	38	50	62	8	20	32	
0	33	45	57	3	15	27	39	51	63	9	21	
1	55	1	13	25	37	49	61	7	19	31	43	1
2	11	23	35	47	59	5	17	29	41	53	65	

Finally, we deal with C_{33}. We again have a choice of which class we delete, as long as it is nonzero and does not fully overlap with anything we have already deleted. We let $r_{33} = 20$ and remove $[20]_{33}$.

mod 3 \ mod 11	0	1	2	3	4	5	6	7	8	9	10	mod 2
0	0	12	24	36	48	60	6	18	30	42	54	
1	22	34	46	58	4	16	28	40	52	64	10	0
2	44	56	2	14	26	38	50	62	8	20	32	
0	33	45	57	3	15	27	39	51	63	9	21	
1	55	1	13	25	37	49	61	7	19	31	43	1
2	11	23	35	47	59	5	17	29	41	53	65	

Now that all cosets pertaining to $\mathcal{C}^\sharp(\mathcal{R})$ have been deleted from the grid, we recover the C_{66}-universe by looking at the complement of the deleted portion.

mod 3 \ mod 11	0	1	2	3	4	5	6	7	8	9	10	mod 2
0	0	12	24	36	48	60	6	18	30	42	54	
1	22	34	46	58	4	16	28	40	52	64	10	0
2	44	56	2	14	26	38	50	62	8	20	32	
0	33	45	57	3	15	27	39	51	63	9	21	
1	55	1	13	25	37	49	61	7	19	31	43	1
2	11	23	35	47	59	5	17	29	41	53	65	

We conclude that a BFH-universe that realizes $\mathcal{R}$ is

$$\mathcal{U} = \left\{ \begin{matrix} 0, 2, 4, 6, 8, 12, 14, 16, 18, 22, 24, 26, 28, 30, 34, \\ 36, 38, 40, 42, 44, 46, 48, 50, 52, 56, 58, 60, 62, 64 \end{matrix} \right\}.$$

3 Saturated Transfer Systems for Cyclic Groups of Single-Prime Power Order

3.1 Realizability

Theorem 3.1. *Let p be a prime. Every saturated C_{p^k}-transfer system $\mathcal{R}$ is BFH-realizable.*

Proof. Given a saturated C_{p^k}-transfer system $\mathcal{R}$, the irreducible cofibrants are $\mathcal{C}^\sharp(\mathcal{R}) = \{p^{i_1}, \dots, p^{i_\ell}\}$. For each $1 \leq j \leq \ell$, let the representative be $r_j = p^{i_j - 1}$. We want to show that each r_j satisfies the two witness conditions. For each p^{i_j} the only prime divisor is p, so

$$[r_j]_{p^{i_j-1}} = \{0 \leq x < p^k \mid x \equiv 0 \bmod p^{i_j-1}\}.$$

Since each r_j is nonzero for all $1 \leq j \leq \ell$ we know that $0 \notin \bigcup_{j=1}^{\ell} [r_j]_{p^{i_j}}$. Therefore,

$$[r_j]_{p^{i_j-1}} \quad \not\subseteq \quad \bigcup_{j=1}^{\ell} [r_j]_{p^{i_j}}.$$

Thus, the first witness condition is satisfied. Now, fix $0 \leq e \leq k$ and $b \in C_{p^e}$ such that

$$[b]_{p^e} \quad \subseteq \quad \bigcup_{j=1}^{\ell} [r_j]_{p^{i_j}}.$$

If C_{p^e} is cofibrant, then the second witness condition is trivially satisfied as $C_{p^e} = \chi_{\mathcal{R}}(C_{p^e})$. If C_{p^e} is not cofibrant, then $\chi_{\mathcal{R}}(C_{p^e}) = C_{p^{i_d}}$ with $i_d < e$. If $b \equiv r_a \bmod p^{i_a}$ for some $a < d$, then for all $x \in [b]_{p^{i_d}}$ we have $x \equiv r_a \bmod p^{i_a}$. As such,

$$[b]_{p^{i_d}} \quad \subseteq \quad [r_a]_{p^{i_a}} \quad \subseteq \quad \bigcup_{j=1}^{\ell} [r_j]_{p^{i_j}},$$

as such the second witnessing condition is satisfied. Now, consider the case when $b \not\equiv r_j \bmod p^{i_j}$ for all $j < d$. We will show that $b \equiv r_d \bmod p^{i_d}$. Note that $[x]_{p^{i_d}}$ has p^{k-i_d} many elements for any $0 \leq x < p^{i_d}$. If $x \not\equiv p^{i_j-1} \bmod p^{i_j}$ for each $1 \leq j \leq d$, then the largest number of elements that we can remove from that class is

$$\sum_{j=d+1}^{\ell} p^{k-i_j}.$$

Observe the following:

$$\sum_{j=d+1}^{\ell} p^{k-i_j} \quad \leq \quad \sum_{s=0}^{k-i_d-1} p^s \quad < \quad p^{k-i_d} \quad = \quad \left|[x]_{p^{i_d}}\right|.$$

The case $d = \ell$ gives the empty sum, which still follows the given inequalities. In other words, it is impossible to remove a class mod p^{i_d} by just removing classes mod the irreducible cofibrants strictly greater than p^{i_d}. This means that the only class mod p^{i_d} that is included exactly in $\bigcup_{j=1}^{\ell} [r_j]_{p^{i_j}}$ is $[r_d]_{p^{i_d}}$. Furthermore, we have

$$\sum_{j=d+1}^{\ell} p^{k-i_j} \quad \leq \quad \sum_{s=0}^{k-e-1} p^s \quad < \quad p^{k-e} \quad = \quad \left|[x]_{p^e}\right|.$$

The case of $d = \ell$ gives the empty sum, which still follows the given inequalities. Just as above, this means that we cannot remove a class modulo p^e just by removing all of the cofibrant classes finer than p^e. Therefore, by assumption, $[b]_{p^e}$ must be contained in a class modulo p^{i_d}, meaning that $b \equiv r_d \bmod p^{i_d}$. Thus, $[b]_{\chi_{\mathcal{R}}(p^e)} = [r_d]_{p^{i_d}} \subseteq \bigcup_{j=1}^{\ell} [r_j]_{p^{i_j}}$ and we are done. □

3.2 Enumeration

Theorem 3.2. *Let p be a prime. The number of saturated C_{p^n}-transfer systems is given by $a_n = 2^n$.*

Proof. Note that for each saturated C_{p^n}-transfer system $\mathcal{R}$, its irreducible cofibrant set $\mathcal{C}^\sharp(\mathcal{R})$ is contained in the set

$$\{A \subseteq \mathrm{Sub}(C_{p^n}) \mid C_1 \notin A\}.$$

Fix such a subset $A \subseteq \mathrm{Sub}(C_{p^n})$. Note that A is of the form $\{C_{p^{i_1}}, \ldots, C_{p^{i_k}}\}$ for distinct $i_1, \ldots, i_k$. Since the powers are distinct, A is lcm−minimal, meaning that we can generate a corresponding lcm-closed set A'. By Proposition 1.11, we may interpret A' as a set of cofibrant subgroups for some saturated transfer system $\mathcal{R}$. Therefore,

$$\{A \subseteq \mathrm{Sub}(C_{p^n}) \mid C_1 \notin A\} \;\subseteq\; \{\mathcal{C}^\sharp(\mathcal{R}) \mid \mathcal{R} \text{ is a saturated } C_{p^n}\text{-transfer system}\}.$$

We have double containment, so the sets are equal. Since $|\{A \subseteq \mathrm{Sub}(C_{p^n}) \mid C_1 \notin A\}| = 2^n$, we conclude that there are 2^n many saturated C_{p^n} transfer systems. □

4 Saturated Transfer Systems for Cyclic Groups of Two-Prime Power Order

4.1 Realizability

Theorem 4.1. *Let p be a prime greater than 2. Let $\mathcal{R}$ be a saturated $C_{2\cdot p^k}$-transfer system. Then $\mathcal{R}$ is BFH-realizable if and only if $C_2 \notin \mathcal{C}^\sharp(\mathcal{R})$, or $C_{2\cdot p^j} \notin \mathcal{C}^\sharp(\mathcal{R})$ for all $1 \leq j \leq k$.*

Proof. We first deal with the case when C_2 is cofibrant. By lcm-minimality, we know that our irreducible cofibrant set must be of the form

$$\mathcal{C}^\sharp(\mathcal{R}) = \{C_2, C_{p^{i_1}}, \ldots, C_{p^{i_\ell}}\}.$$

Define our representatives by $r_2 = 1$ and $r_j = p^{i_j - 1}$ for each $1 \leq j \leq \ell$. We check the first witness condition. For $C_2 \in \mathcal{C}^\sharp(\mathcal{R})$, the only prime divisor of 2 is itself. Since zero must always remain in the universe, we know

$$0 \;\notin\; [r_2]_2 \cup \bigcup_{j=1}^{\ell} [r_j]_{p^{i_j}}.$$

Since $r_2 \equiv 0 \bmod 1$, it immediately follows that

$$[r_2]_1 \;\not\subseteq\; [r_2]_2 \cup \bigcup_{j=1}^{\ell} [r_j]_{p^{i_j}},$$

so our 2-representative satisfies the first witness condition. Next, fix $1 \leq j \leq \ell$ and consider $p^{i_j} \in \mathcal{C}^\sharp(\mathcal{R})$. Notice that $r_j \equiv 0 \bmod p^{i_j - 1}$, so $0 \in [r_j]_{p^{i_j - 1}}$. As established above, 0 must remain in the universe, so

$$[r_j]_{p^{i_j - 1}} \;\not\subseteq\; [r_2]_2 \cup \bigcup_{j=1}^{\ell} [r_j]_{p^{i_j}}.$$

Thus, the first witness condition holds for every p^{i_j}-representative. We now move on to the second witness condition. Fix $f \mid 2 \cdot p^k$ and $b \in C_f$ such that

$$[b]_f \;\subseteq\; [r_2]_2 \cup \bigcup_{j=1}^{\ell} [r_j]_{p^{i_j}}.$$

If f is cofibrant, then $\chi_{\mathcal{R}}(f) = f$, so the second condition is satisfied trivially. When f is not cofibrant, we first consider the case when $\chi_{\mathcal{R}}(f)$ is an irreducible cofibrant. Then $\chi_{\mathcal{R}}(f) = 2$ or $\chi_{\mathcal{R}}(f) = p^{i_a}$ for some $1 \leq i_a \leq \ell$. If $\chi_{\mathcal{R}}(f) = 2$ and f is properly divided by 2, then we know that $i_1 > \nu_p(f)$ due to lcm-closure of $\mathcal{C}(\mathcal{R})$. We will now show that $b \equiv r_2 \bmod 2$. Suppose for contradiction that $b \not\equiv r_2 \bmod 2$. Then, we have

$$[b]_f \;\subseteq\; \bigcup_{j=1}^{\ell} [r_j]_{p^{i_j}}$$

since $r_2 \not\equiv b \bmod 2$. Note that each coset $[r_j]_{p^{i_j}}$ contributes p^{k-i_j} many even elements, so the total number of even elements removed by all such classes is

$$\sum_{j=1}^{\ell} p^{k-i_j}.$$

Since $i_1 > \nu_p(f)$, we know that

$$\{i_1, \ldots, i_\ell\} \;\subseteq\; \{\nu_p(f)+1, \ldots, k\},$$

which gives us

$$\sum_{j=1}^{\ell} p^{k-i_j} \;\leq \sum_{e=\nu_p(f)+1}^{k} p^{k-e}.$$

Using basic properties of summation and the fact that $p > 2$, it follows that

$$\sum_{e=\nu_p(f)+1}^{k} p^{k-e} \;=\; \frac{p^{k-\nu_p(f)}-1}{p-1} \;<\; p^{k-\nu_p(f)} - 1.$$

However, note that $p^{k-\nu_p(f)}$ is the size of the class $[b]_f$, so we conclude that we are deleting strictly less than the size of $[b]_f$, which leads to a contradiction. Thus, we must have $b \equiv r_2 \bmod 2$, so

$$[b]_2 \;=\; [b]_{\chi_{\mathcal{R}}(f)} \;\subseteq\; [r_2]_2 \cup \bigcup_{j=1}^{\ell} [r_j]_{p^{i_j}}.$$

Now consider the case where $\chi_{\mathcal{R}}(f) = p^{i_a}$ for some $1 \leq a \leq \ell$. Then from lcm-closure we deduce that f is odd. If $b \equiv r_j \bmod p^{i_j}$ for some $1 \leq j < a$, then we immediately conclude that

$$[b]_{\chi_{\mathcal{R}}(f)} \;\subseteq\; [r_2]_2 \cup \bigcup_{j=1}^{\ell} [r_j]_{p^{i_j}}$$

since p^{i_j} divides $\chi_{\mathcal{R}}(f)$. As such, we deal with the case where $b \not\equiv r_j \bmod p^{i_j}$ for all $1 \leq j < a$. If $a = \ell$, then we immediately know $b \equiv r_a \bmod p^{i_a}$ because all the cosets $[r_j]_{p^{i_j}}$ are disjoint. So, we consider the case where $a \neq \ell$. Suppose for contradiction that $b \not\equiv r_a \bmod p^{i_a}$ as well. Then,

$$[b]_f \;\subseteq\; [r_2]_2 \cup \bigcup_{j=a+1}^{\ell} [r_j]_{p^{i_j}}.$$

Note that the class $[r_2]_2$ contributes all of the odd elements in $[b]_f$. This means that the even elements contributed by each $[r_j]_{p^{i_j}}$ class must cover the entire even subset of $[b]_f$, which has size $p^{k-\nu_p(f)}$. Note that each $[r_j]_{p^{i_j}}$ contributes p^{k-i_j} many even elements, so in total we remove

$$\sum_{j=a+1}^{\ell} p^{k-i_j}$$

many even elements. Since the lcm-closure of $\mathcal{C}(\mathcal{R})$ lets us deduce that $\nu_p(f) < i_{a+1}$, we know

$$\{i_{a+1}, \ldots, i_\ell\} \;\subseteq\; \{\nu_p(f)+1, \ldots, k\}.$$

Using similar algebraic manipulations as before, it follows that

$$\sum_{j=a+1}^{\ell} p^{k-i_j} \;\leq\; \sum_{e=\nu_p(f)+1}^{k} p^{k-e} \;<\; p^{k-\nu_p(f)} - 1.$$

However, there are $p^{k-\nu_p(f)}$ many even elements in $[b]_f$, so we cannot entirely cover the even elements in $[b]_f$ using the other cosets. Thus, we arrive at a contradiction and conclude that $b \equiv r_a \bmod p^{i_a}$. Therefore, we must have

$$[b]_{\chi_{\mathcal{R}}(f)} \;\subseteq\; [r_2]_2 \cup \bigcup_{j=1}^{\ell} [r_j]_{p^{i_j}}.$$

This concludes the case when $\chi_{\mathcal{R}}(f)$ is a proper divisor of f and $\chi_{\mathcal{R}}(C_f)$ is irreducible.

We now move on to the case when $\chi_{\mathcal{R}}(C_f) \notin \mathcal{C}^\sharp(\mathcal{R})$. Since C_2 is cofibrant, by lcm-closure of $\mathcal{C}(\mathcal{R})$ we must have that $\chi_{\mathcal{R}}(f) = 2 \cdot p^{i_a}$ for some $1 \leq a \leq \ell$. The case when $b \equiv 1 \bmod 2$ immediately guarantees the second condition because $\chi_{\mathcal{R}}(f)$ is divisible by 2. As such, we consider the case when $b \equiv 0 \bmod 2$. Just as before, the second witness condition is immediate if $b \equiv r_j \bmod p^{i_j}$ for some $1 \leq j < a$. We must deal with the case where $b \not\equiv r_j \bmod p^{i_j}$ for all $1 \leq j < a$. If $a = \ell$, then we immediately know $b \equiv r_a \bmod p^{i_a}$ because all the cosets $[r_j]_{p^{i_j}}$ are disjoint. So, we consider the case where $a \neq \ell$. Suppose for contradiction that $b \not\equiv r_a \bmod p^{i_a}$ as well. Then

$$[b]_f \;\subseteq\; \bigcup_{j=a+1}^{\ell} [r_j]_{p^{i_j}}.$$

Each of the $[r_j]_{p^{i_j}}$ classes contribute p^{k-i_j} many even elements, so their total contribution is

$$\sum_{j=a+1}^{\ell} p^{k-i_j}$$

many elements. Once again, lcm-closure of $\mathcal{C}(\mathcal{R})$ lets us deduce that $\nu_p(f) < i_{a+1}$, hence

$$\{i_{a+1}, \dots, i_\ell\} \ \subseteq \ \{\nu_p(f)+1, \dots, k\}.$$

Using similar algebraic manipulation gives

$$\sum_{j=a+1}^{\ell} p^{k-i_j} \ < \ p^{k-\nu_p(f)}.$$

Since $[b]_f$ has cardinality $p^{k-\nu_p(f)}$, it is impossible to entirely cover $[b]_f$ using other cosets. We thus arrive at a contradiction and conclude that $b \equiv r_a \bmod p^{i_a}$. Therefore,

$$[b]_{\chi_R(f)} \ = \ [b]_{2p^{i_a}} \ \subseteq \ [b]_{p^{i_a}} \ \subseteq \ [r_2]_2 \cup \bigcup_{j=1}^{\ell} [r_j]_{p^{i_j}},$$

resolving the case when $\chi_{\mathcal{R}}(f)$ is a proper divisor of f and $\chi_{\mathcal{R}}(C_f)$ is not irreducible. These cases show that the second witness condition is satisfied for any choice of divisor f and element $b \in C_f$, so we conclude that the family

$$(r_2) \cup (r_j)_{j=1}^{\ell}$$

is a witness for a saturated $C_{2\cdot p^k}$-transfer system in which C_2 is cofibrant.

Now, we switch gears and deal with the case when C_2 is *not* cofibrant. For every $C_d \in \mathcal{C}^\sharp(\mathcal{R})$, define the corresponding representative r_d as follows: For each $1 \le j \le k$,

1) If $d = p^j$, then r_d is the unique element in C_{p^j} such that $r_d \equiv (p-1)p^{j-1} \bmod p^j$
2) If $d = 2 \cdot p^j$, then r_d is the unique element in $C_{2\cdot p^j}$ such that $r_d \equiv 1 \bmod 2$ and also $r_d \equiv (p-2)p^{j-1} \bmod p^j$.

We check the first witness condition. Fix $C_d \in \mathcal{C}^\sharp(\mathcal{R})$. When $d = p^j$ for some $1 \le j \le k$, the only prime divisor of d is p. We know $r_d \equiv 0 \bmod p^{j-1}$, so $0 \in [r_d]_{d/p}$. For our universe to remain valid, it must contain 0, which in turn guarantees that

$$[r_d]_{d/p} \ \not\subseteq \bigcup_{C_e \in \mathcal{C}^\sharp(\mathcal{R})} [r_e]_e.$$

Now we consider the case when $d = 2\cdot p^j$ for some $1 \le j \le k$. The prime divisors of d are now 2 and p. We deal with p first. Note that $r_d \equiv 1 \bmod 2$ and $r_d \equiv 0 \bmod p^{j-1}$, meaning that $p^k \in [r_d]_{d/p}$. Since $r_e \not\equiv 0 \bmod p^k$ for all $C_e \in \mathcal{C}^\sharp(\mathcal{R})$, we know that $p^k \notin \bigcup_{C_e \in \mathcal{C}^\sharp(\mathcal{R})} [r_e]_e$, and thus,

$$[r_d]_{d/p} \ \not\subseteq \bigcup_{C_e \in \mathcal{C}^\sharp(\mathcal{R})} [r_e]_e.$$

As for the divisor 2 of $2\cdot p^j$, consider $x \in C_{2\cdot p^j}$ satisfying $x \equiv 0 \bmod 2$ and $x \equiv (p-2)p^{j-1} \bmod p^j$. Notice that $[x]_{p^j} \subseteq [r_d]_{d/2}$ and also $x \notin \bigcup_{C_e \in \mathcal{C}^\sharp(\mathcal{R})} [r_e]_e$. It follows that

$$[r_d]_{d/2} \ \not\subseteq \bigcup_{C_e \in \mathcal{C}^\sharp(\mathcal{R})} [r_e]_e.$$

Thus, the first witness condition is satisfied for all $C_d \in \mathcal{C}^\sharp(\mathcal{R})$.

We move on to the second witness condition. Fix $f \mid 2 \cdot p^k$ and $b \in C_f$ such that

$$[b]_f \;\subseteq \bigcup_{C_e \in \mathcal{C}^\sharp(\mathcal{R})} [r_e]_e$$

Once again, if C_f is cofibrant, then $\chi_{\mathcal{R}}(f) = f$ so the second witness condition is satisfied trivially. For the case that C_f is not cofibrant, we first consider the case when $\chi_{\mathcal{R}}(C_f) \in \mathcal{C}^\sharp(\mathcal{R})$. If there exists some $C_{d_0} \in \mathcal{C}^\sharp(\mathcal{R})$ such that d_0 properly divides $\chi_{\mathcal{R}}(f)$ and $b \equiv r_{d_0} \bmod d_0$, then

$$[b]_{\chi_{\mathcal{R}}(f)} \;\subseteq\; [b]_{d_0} \;=\; [r_{d_0}]_{d_0} \;\subseteq \bigcup_{C_e \in \mathcal{C}^\sharp(\mathcal{R})} [r_e]_e,$$

so the second witness condition is trivial once again. Now, consider the case where $b \not\equiv r_d \bmod d$ for all $C_d \in \mathcal{C}^\sharp(\mathcal{R})$ where d properly divides $\chi_{\mathcal{R}}(f)$. First suppose that f is even. In this case, we know that no $C_d \in \mathcal{C}^\sharp(\mathcal{R})$ satisfies

$$\nu_p(\chi_{\mathcal{R}}(f)) \;<\; \nu_p(d) \;<\; \nu_p(f).$$

For this case, note that

$$\{\, C_d \in \mathcal{C}^\sharp(\mathcal{R}) \;\mid\; d \nmid \chi_{\mathcal{R}}(f) \,\} \;\subseteq\; \{\, C_e \leq C_{2\cdot p^k} \;\mid\; \nu_p(f) < \nu_p(e) \leq k \,\}$$

Since the set $\mathcal{C}^\sharp(\mathcal{R})$ has to be lcm-minimal, p^j and $2 \cdot p^j$ cannot coexist in $\mathcal{C}^\sharp(\mathcal{R})$ for any $j > \nu_p(f)$. Now note that the total contribution of $[r_d]_d$ for each $C_d \in \mathcal{C}^\sharp(\mathcal{R})$ such that $d \nmid \chi_{\mathcal{R}}(f)$ is

$$\sum_{d \text{ even}} p^{k-\nu_p(d)} + \sum_{d \text{ odd}} 2 \cdot p^{k-\nu_p(d)} \;\leq\; \sum_{d} 2 \cdot p^{k-\nu_p(d)}.$$

From the constraint mentioned above and because $p \geq 3$, we know that

$$\sum_{d} 2 \cdot p^{k-\nu_p(d)} \;\leq \sum_{e=\nu_p(f)+1}^{k} 2 \cdot p^{k-e} \;<\; p^{k-\nu_p(f)}.$$

Since $[b]_f$ has $p^{k-\nu_p(f)}$ many elements, we conclude that

$$[b]_f \;\not\subseteq \bigcup_{\substack{C_d \in \mathcal{C}^\sharp(\mathcal{R}) \\ d \nmid \chi_{\mathcal{R}}(f)}} [r_d]_d.$$

By the initial assumption, we must have $[b]_f \subseteq [r_{\chi_{\mathcal{R}}(f)}]_{\chi_{\mathcal{R}}(f)}$. Thus,

$$[b]_{\chi_{\mathcal{R}}(f)} = [r_{\chi_{\mathcal{R}}(f)}]_{\chi_{\mathcal{R}}(f)} \;\subseteq \bigcup_{C_d \in \mathcal{C}^\sharp(\mathcal{R})} [r_d]_d,$$

satisfying the second witness condition for even f. Now, we consider the case where f is odd. In this case, we know that d must be even if $\nu_p(\chi_{\mathcal{R}}(f)) < \nu_p(d) < \nu_p(f)$ holds. Note that $b \not\equiv r_d \bmod d$ for any d such that $\nu_p(\chi_{\mathcal{R}}(f)) < \nu_p(d) < \nu_p(f)$, so $[r_d]_d$ would exclude any even element of $[b]_f$ because d is even. It is therefore enough to consider the cofibrant subgroups of the form

$$\{\, C_d \in \mathcal{C}^\sharp(\mathcal{R}) \;\mid\; d \nmid f \,\} \;\subseteq\; \{\, C_e \leq C_{2\cdot p^k} \;\mid\; \nu_p(f) < \nu_p(e) \leq k \,\} \cup \{\, C_{2f} \,\}$$

If $\{C_d \in \mathcal{C}^\sharp(\mathcal{R}) \mid d \nmid f\}$ is also a subset of $\{C_{p^n} \mid \nu_p(f) < n \leq k\}$, then the total contribution by these subgroups is

$$\sum_{d \nmid f} 2 \cdot p^{k-\nu_p(d)} \quad \leq \quad \sum_{e=\nu_p(f)+1}^{k} 2 \cdot p^{k-e} \quad < \quad p^{k-\nu_p(f)}.$$

Since $[b]_f$ has $2 \cdot p^{k-\nu_p(f)}$ many elements, we conclude that

$$[b]_f \quad \not\subseteq \bigcup_{\substack{C_d \in \mathcal{C}^\sharp(\mathcal{R}) \\ d \nmid \chi_{\mathcal{R}}(f)}} [r_d]_d,$$

so by the initial assumption, we must have $[b]_f \subseteq [r_{\chi_{\mathcal{R}}(f)}]_{\chi_{\mathcal{R}}(f)}$. Hence,

$$[b]_{\chi_{\mathcal{R}}(f)} = \; [r_{\chi_{\mathcal{R}}(f)}]_{\chi_{\mathcal{R}}(f)} \quad \subseteq \bigcup_{C_d \in \mathcal{C}^\sharp(\mathcal{R})} [r_d]_d,$$

proving the second witness condition for this case. Now, if there exists an even $d_0 \nmid f$ such that $C_{d_0} \in \mathcal{C}^\sharp(\mathcal{R})$, then C_{p^j} and $C_{2 \cdot p^j}$ cannot coexist in $\mathcal{C}^\sharp(\mathcal{R})$ for any $j > \nu_p(d_0)$ because $\mathcal{C}^\sharp(\mathcal{R})$ is lcm-minimal. The total contribution of $[r_d]_d$ for each $C_d \in \mathcal{C}^\sharp(\mathcal{R})$ such that $d \nmid f$ is

$$\sum_{\nu_p(d) \leq \nu_p(d_0)} 2 \cdot p^{k-\nu_p(d)} \quad + \quad p^{k-\nu_p(d_0)} \quad + \sum_{\substack{d \text{ even} \\ \nu_p(d) > \nu_p(d_0)}} p^{k-\nu_p(d)} \quad + \sum_{\substack{d \text{ odd} \\ \nu_p(d) > \nu_p(d_0)}} 2 \cdot p^{k-\nu_p(d)}.$$

Together, the lcm-minimality constraint of $\mathcal{C}^\sharp(\mathcal{R})$ and the fact that $\nu_p(d_0) \geq \nu_p(f)$ imply that the sum above is less than or equal to

$$p^{k-\nu_p(f)} \quad + \sum_{d \neq d_0} 2 \cdot p^{k-\nu_p(d)} \quad \leq \quad p^{k-\nu_p(f)} \quad + \sum_{e=\nu_p(f)+1}^{k} 2 \cdot p^{k-e} \quad < \quad 2 \cdot p^{k-\nu_p(f)}$$

Since $[b]_f$ has $2 \cdot p^{k-\nu_p(f)}$ many elements, we conclude that

$$[b]_f \quad \not\subseteq \bigcup_{\substack{C_d \in \mathcal{C}^\sharp(\mathcal{R}) \\ d \nmid \chi_{\mathcal{R}}(f)}} [r_d]_d$$

so by the initial assumption, we must have $[b]_f \subseteq [r_{\chi_{\mathcal{R}}(f)}]_{\chi_{\mathcal{R}}(f)}$. Hence

$$[b]_{\chi_{\mathcal{R}}(f)} = [r_{\chi_{\mathcal{R}}(f)}]_{\chi_{\mathcal{R}}(f)} \subseteq \bigcup_{C_d \in \mathcal{C}^\sharp(\mathcal{R})} [r_d]_d.$$

That concludes all of the possible casework for $\chi_{\mathcal{R}}(C_f) \in \mathcal{C}^\sharp(\mathcal{R})$.

Now we consider the case $\chi_{\mathcal{R}}(C_f) \in \mathcal{C}(\mathcal{R}) \setminus \mathcal{C}^\sharp(\mathcal{R})$. Then we know that $\chi_{\mathcal{R}}(f)$ must be even, that $C_{\chi_{\mathcal{R}}(f)/2} \in \mathcal{C}^\sharp(\mathcal{R})$, and that there exists some even-order $C_d \in \mathcal{C}^\sharp(\mathcal{R})$. Also, it is important to note that there is no $C_e \in \mathcal{C}^\sharp(\mathcal{R})$ satisfying

$$\nu_p(\chi_{\mathcal{R}}(f)) \; < \; \nu_p(e) \; < \; \nu_p(f).$$

If there exists some $C_{d_0} \in \mathcal{C}^\sharp(\mathcal{R})$ such that d_0 properly divides $\frac{\chi_{\mathcal{R}}(f)}{2}$ and $b \equiv r_{d_0} \bmod d_0$, then

$$[b]_{\chi_{\mathcal{R}}(f)} \; \subseteq \; [b]_{d_0} \; = \; [r_{d_0}]_{d_0} \quad \subseteq \bigcup_{C_e \in \mathcal{C}^\sharp(\mathcal{R})} [r_e]_e,$$

so the second witness condition is trivial once again. As such, we assume that all proper divisors d of $\chi_{\mathcal{R}}(f)$ that are not $\frac{\chi_{\mathcal{R}}(f)}{2}$ satisfy the constraint $b \not\equiv r_d \bmod d$. In this nontrivial case, we consider the parity of b. First of all, if b is even, then $[b]_f$ consists of only even elements due to the Chinese Remainder Theorem. As such $[b]_f \not\subseteq [r_d]_d$ for odd d, so it is enough to consider even order $C_d \in \mathcal{C}^\sharp(\mathcal{R})$. Note that the odd-order subgroups $C_d \in \mathcal{C}^\sharp(\mathcal{R})$ that do not properly divide $\frac{\chi_{\mathcal{R}}(f)}{2}$ contribute

$$\sum_{d \text{ odd}} p^{k-\nu_p(d)}$$

many elements. Since

$$\left\{ C_d \in \mathcal{C}^\sharp(\mathcal{R}) \;\middle|\; d \text{ is odd, } d \nmid \tfrac{\chi_{\mathcal{R}}(f)}{2} \right\} \;\subseteq\; \left\{ C_{p^e} \;\middle|\; \nu_p(\chi_{\mathcal{R}}(f)) < e \leq k \right\},$$

the sum above is less than or equal to

$$\sum_{e=\nu_p(f)+1}^{k} p^k - e,$$

which is itself less than $p^{k-\nu_p(f)}$. Since $[b]_f$ has $p^{k-\nu_p(f)}$ many elements, we conclude that

$$[b]_f \;\not\subseteq \bigcup_{\substack{C_d \in \mathcal{C}^\sharp(\mathcal{R}) \\ d \nmid (\chi_{\mathcal{R}}(f)/2)}} [r_d]_d.$$

By the initial assumption, we must have $[b]_f \subseteq [r_{\frac{\chi_{\mathcal{R}}(f)}{2}}]_{\frac{\chi_{\mathcal{R}}(f)}{2}}$. Hence

$$[b]_{\chi_{\mathcal{R}}(f)} \;\subseteq\; [r_{\frac{\chi_{\mathcal{R}}(f)}{2}}]_{\frac{\chi_{\mathcal{R}}(f)}{2}} \;\subseteq \bigcup_{C_d \in \mathcal{C}^\sharp(\mathcal{R})} [r_d]_d.$$

The above holds for all even $b \in C_f$. Now, we consider when b is odd. If b is odd, then the entire class of $[b]_f$ consists of odd elements, so we consider the odd subsets of $[r_d]_d$ for all subgroups $C_d \in \mathcal{C}^\sharp(\mathcal{R})$ such that $d \nmid \chi_{\mathcal{R}}(f)$. Note that

$$\{\, C_d \in \mathcal{C}^\sharp(\mathcal{R}) \;\mid\; d \nmid \chi_{\mathcal{R}}(f) \,\} \;\subseteq\; \{\, C_\ell \leq C_{2 \cdot p^k} \;\mid\; \nu_p(f) < \nu_p(\ell) \leq k \,\},$$

and since there exists a $C_{d_0} \in \mathcal{C}^\sharp(\mathcal{R})$ of even order with $\nu_p(d_0) < \nu_p(\chi_{\mathcal{R}}(f))$, we have an additional constraint that both p^j and $2 \cdot p^j$ cannot coexist in $\mathcal{C}^\sharp(\mathcal{R})$ for $j > \nu_p(f)$. As such, the contribution of even elements in $\{C_d \in \mathcal{C}^\sharp(\mathcal{R}) \mid d \nmid \chi_{\mathcal{R}}(f)\}$ is

$$\sum_{d \mid \chi_{\mathcal{R}}(f)} p^{k-\nu_p(d)}$$

and is less than or equal to

$$\sum_{e=\nu_p(f)+1}^{k} p^{k-e}.$$

The above sum is itself less than $p^{k-\nu_p(f)}$, which is exactly the size of $[b]_f$. We conclude that

$$[b]_f \;\not\subseteq \bigcup_{\substack{C_d \in \mathcal{C}^\sharp(\mathcal{R}) \\ d \mid \frac{\chi_{\mathcal{R}}(f)}{2}}} [r_d]_d$$

so by the initial assumption, we must have $[b]_f \subseteq [r_{\frac{\chi_{\mathcal{R}}(f)}{2}}]_{\frac{\chi_{\mathcal{R}}(f)}{2}}$. Hence

$$[b]_{\chi_{\mathcal{R}}(f)} \subseteq [r_{\frac{\chi_{\mathcal{R}}(f)}{2}}]_{\frac{\chi_{\mathcal{R}}(f)}{2}} \subseteq \bigcup_{C_d \in \mathcal{C}^\sharp(\mathcal{R})} [r_d]_d.$$

This concludes the case when $\chi_{\mathcal{R}}(C_f) \in \mathcal{C}(\mathcal{R}) \setminus \mathcal{C}^\sharp(\mathcal{R})$. Thus, the second witness condition is proven for the case when C_2 is not cofibrant. As such, the list $(r_d)_{d \in \mathcal{C}^\sharp(\mathcal{R})}$ is a witnessing family for $\mathcal{R}$, so the BFH-universe

$$\mathcal{U} \;=\; \{0, 1, \ldots, 2 \cdot p^k - 1\} \quad \setminus \bigcup_{C_d \in \mathcal{C}^\sharp(\mathcal{R})} [r_d]_d$$

realizes $\mathcal{R}$.

To prove the converse direction, suppose $\mathcal{R}$ is a saturated $C_{2 \cdot p^k}$-transfer system such that $C_2 \in \mathcal{C}^\sharp(\mathcal{R})$ and $C_{2 \cdot p^{j_0}} \in \mathcal{C}^\sharp(\mathcal{R})$ for some $1 \leq j_0 \leq k$. Suppose $\mathcal{R}$ is BFH-realizable by the witnessing family $(r_d)_{C_d \in \mathcal{C}^\sharp(\mathcal{R})}$. Since r_2 cannot be 0 by definition, we have $r_2 = 1$. Now consider $r_{2 \cdot p^j}$ for each $1 \leq j \leq k$. We show that $r_{2 \cdot p^j}$ can neither be even nor odd, which gives us the desired contradiction. Suppose $r_{2 \cdot p^j} \equiv 0 \bmod 2$. Then

$$[r_{2 \cdot p^j}]_{p^j} \;=\; [r_{2 \cdot p^j}]_{2 \cdot p^j} \cup [r_{2 \cdot p^j} + p^j]_{2 \cdot p^j} \;\subseteq \bigcup_{C_d \in \mathcal{C}^\sharp(\mathcal{R})} [r_d]_d$$

because $r_{2 \cdot p^j} + p^j \equiv 1 \bmod 2$. Therefore, if must be true that $r_{2 \cdot p^j} \equiv 1 \bmod 2$. However, this implies

$$[r_{2 \cdot p^j}]_{2 \cdot p^{j-1}} \;\subseteq\; [r_2]_2 \;\subseteq \bigcup_{C_d \in \mathcal{C}^\sharp(\mathcal{R})} [r_d]_d,$$

so $r_{2 \cdot p^j} \not\equiv 1 \bmod 2$. Therefore, the first witnessing condition cannot be satisfied, yielding a contradiction. Hence, $\mathcal{R}$ is not BFH-realizable. □

4.2 Enumeration

Theorem 4.2. *Let p be a prime greater than 2. Let a_n be the number of saturated $C_{2 \cdot p^n}$-transfer systems where C_2 is not cofibrant, and let b_n be the number of saturated $C_{2 \cdot p^n}$-transfer systems where C_2 is cofibrant. Then $a_n = 3a_{n-1} + 2^{n-1}$ for all $n \geq 1$, with $a_0 = 1$ and $b_n = 2^n$.*

Proof. We first show that $b_n = 2^n$. Since we assume that C_2 is cofibrant, we use the Layers Theorem to say that every saturated transfer system with this condition has a correspondence with the C_{p^n} case. By Theorem 3.1, there are 2^n saturated transfer systems of this form.

We will prove the recurrence for a_n by induction. For $n = 0$, we are dealing with C_2-transfer systems. There is exactly one in which C_2 is not cofibrant, giving $a_0 = 1$. Now suppose the recurrence holds for an arbitrary n. Going from a $C_{2 \cdot p^n}$-transfer system $\mathcal{R}$ to a $C_{2 \cdot p^{n+1}}$-transfer system $\mathcal{R}'$ involves unioning $\mathcal{C}^\sharp(\mathcal{R})$ with one of the following:

$$\varnothing, \qquad \{p^{n+1}\} \qquad \{2 \cdot p^{n+1}\}, \qquad \{p^{n+1},\, 2 \cdot p^{n+1}\}.$$

The sets $\varnothing$, $\{p^{n+1}\}$, and $\{2 \cdot p^{n+1}\}$ can be appended freely without violating lcm-closure with the irreducible cofibrants for the $C_{2 \cdot p^n}$-transfer system. We may only add $\{p^{n+1},\, 2 \cdot p^{n+1}\}$ when no element of the form $2 \cdot p^k$ is already irreducible cofibrant for $1 \leq k \leq n$. Such sets are exactly the elements of the set

$$\{A \subseteq \mathrm{Sub}(C_{p^n}) \mid C_1 \notin A\}$$

so there are 2^n saturated transfer systems with this constraint, meaning that $a_{n+1} = 3a_n + 2^n$, thus concluding the inductive step. □

5 Saturated Transfer Systems for Cyclic Groups of Three-Prime Power Order

5.1 Realizability

Conjecture 5.1. *Let $\mathcal{R}$ be a saturated $C_{2\cdot 3\cdot p^n}$-transfer system, where $p \geq 5$. If*

$$C_3, C_6 \in \mathcal{C}^\sharp(\mathcal{R}) \quad \text{and} \quad C_2 \notin \mathcal{C}^\sharp(\mathcal{R}),$$

then $\mathcal{R}$ is BFH-realizable.

5.2 Enumeration

Theorem 5.2. *Let p be a prime greater than 3. The number of saturated $C_{2\cdot 3\cdot p^n}$-transfer systems in which $C_3, C_6 \in \mathcal{C}^\sharp(\mathcal{R})$ and $C_2 \notin \mathcal{C}^\sharp(\mathcal{R})$ is given by the recurrence $a_n = 6a_{n-1} + 4^{n-1}$ where $a_0 = 1$.*

Proof. We will use induction. For $n = 0$, the only set that satisfies the assumptions is $\{3, 6\}$, so $a_0 = 1$. Now suppose the recurrence holds for a fixed n, and consider a_{n+1}. Going from a $C_{2\cdot 3\cdot p^n}$-transfer system to a $C_{2\cdot 3\cdot p^{n+1}}$-transfer system involves appending some subcollection of

$$\begin{gathered}
\varnothing, \quad \{p^{n+1}\}, \quad \{2\cdot p^{n+1}\}, \quad \{3\cdot p^{n+1}\}, \quad \{6\cdot p^{n+1}\} \\
\{p^{n+1}, 2\cdot p^{n+1}\}, \quad \{p^{n+1}, 3\cdot p^{n+1}\}, \quad \{p^{n+1}, 6\cdot p^{n+1}\}, \\
\{2\cdot p^{n+1}, 3\cdot p^{n+1}\}, \quad \{2\cdot p^{n+1}, 6\cdot p^{n+1}\}, \quad \{3\cdot p^{n+1}, 6\cdot p^{n+1}\}, \\
\{p^{n+1}, 2\cdot p^{n+1}, 3\cdot p^{n+1}\}, \quad \{p^{n+1}, 2\cdot p^{n+1}, 6\cdot p^{n+1}\}, \\
\{p^{n+1}, 3\cdot p^{n+1}, 6\cdot p^{n+1}\}, \quad \{2\cdot p^{n+1}, 3\cdot p^{n+1}, 6\cdot p^{n+1}\}, \\
\{p^{n+1}, 2\cdot p^{n+1}, 3\cdot p^{n+1}, 6\cdot p^{n+1}\}
\end{gathered}$$

to the set of cofibrants. Given the above possibilities, the following seven options do not immediately violate lcm-minimality with 3 and 6:

$$\varnothing, \quad \{p^{n+1}\}, \quad \{2\cdot p^{n+1}\}, \quad \{3\cdot p^{n+1}\}, \quad \{6\cdot p^{n+1}\}, \quad \{p^{n+1}, 2\cdot p^{n+1}\}, \quad \{2\cdot p^{n+1}, 3\cdot p^{n+1}\}.$$

Within these, the sets

$$\varnothing, \quad \{p^{n+1}\}, \quad \{2\cdot p^{n+1}\}, \quad \{3\cdot p^{n+1}\}, \quad \{6\cdot p^{n+1}\}, \quad \{2\cdot p^{n+1}, 3\cdot p^{n+1}\}$$

can be added without violating lcm-minimality with the preexisting cofibrants, and $\{p^{n+1}, 2\cdot p^{n+1}\}$ has the condition that no element of the form $C_{2\cdot p^k}$ can be irreducible cofibrant for any k. We will count how many of those sets exist. Suppose that number is b_n. We will show that $b_n = 4^n$ by induction and recurrence, with $b_0 = 1$.

For the initial condition $n = 0$, we have exactly 1 set, which is $\{3, 6\}$. Now suppose the recurrence holds for a fixed n and consider b_{n+1}. Adding to a set for b_n to create a set that satisfies conditions for b_{n+1} involves unioning with some subcollection of

$$\begin{gathered}
\varnothing, \quad \{p^{n+1}\}, \quad \{3\cdot p^{n+1}\}, \quad \{6\cdot p^{n+1}\}, \quad \{p^{n+1}, 3\cdot p^{n+1}\}, \\
\{p^{n+1}, 6\cdot p^{n+1}\}, \quad \{3\cdot p^{n+1}, 6\cdot p^{n+1}\}, \quad \{p^{n+1}, 3\cdot p^{n+1}, 6\cdot p^{n+1}\}.
\end{gathered}$$

Of the above options, only

$$\varnothing, \quad \{p^{n+1}\}, \quad \{3\cdot p^{n+1}\}, \quad \{6\cdot p^{n+1}\}$$

can be appended without any violation of lcm-minimality, so $b_{n+1} = 4b_n$. This tells us that $b_n = 4^n$. We conclude that $a_{n+1} = 6a_n + 4^n$. □

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