Continuity of Brehm's extension theorem

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1 Introduction

Consider a set of points in $\mathbb{R}^2$, $\{a_1 \dots a_n\}, \{b_1 \dots b_n\}$. Supposing that $|b_i - b_j| \leq |a_i - a_j| \forall i, j$, we have that there exists a piecewise linear mapping i.e. a fold pattern h s.t. $h(a_i) = b_i \forall i$. We show that under certain assumptions, this fold set G is continuous in the data $\{a_1 \dots a_n\}, \{b_1 \dots b_n\}$.

2 Definitions/Setup

Let $G = (V, E, F)$ be a graph in $\mathbb{R}^2$. G is an origami graph with a mapping h that has a crease set equal to E.

h is a piecewise linear mapping on $\mathbb{R}^2$.

Since h is an isometry on each piece we can extend h for any particular F_k to get h_{F_k}.

Let $\Omega(a, b) = \{x : |a - x| < |b - h(x)|\}$

Let $S(a, b)$ be the crease set generated by Brehm's extension theorem mapping a to b.

Let $B(a, b) = \{x : |a - x| = |b - h(x)|\}$ and $C(a, b) = \bigcup_w [a, w]$ s.t. w is a vertex in $B(a, b)$. Brehm's extension theorem gives us that $S(a, b) = B(a, b) \cup C(a, b)$.

Let $L_k = \{[a, h_{F_k}^{-1}(b)]_\perp \cap F_k\}$

3 Lemmas

3.1 Lemma 1

Lemma 1: $B(a,b) = \bigcup_{F_k \in F} L_k(a,b)$ Proof: Since h_{F_k} is a distance-preserving isometry, we have that for $x \in F_k$, $|h(x) - b| = |h_{F_k}(x) - b| = |x - h_{F_k}^{-1}(b)|$

3.2 Lemma 2

Lemma 2: for some $e = F_1 \cap F_2 \in E$, $|a - x| = |b - h(x)|\ \forall x \in e \iff a \in h_{F_1}^{-1}(b) \cup h_{F_2}^{-1}(b)$

Proof: WLOG let e be an edge that goes from $(0,0)$ to $(0,1)$ with F_1 on the right side being the identity and F_2 being a reflection over the y axis. So $h_{F_1}^{-1}(b) = (b_x, b_y)$ and $h_{F_2}^{-1}(b) = (-b_x, b_y)$.

Assume that $a \in (h_{F_1}^{-1}(b) \cup h_{F_2}^{-1}(b))$.
For a point $p_x, p_y \in e$ we have that $|a - p| = \sqrt{(a_x - p_x)^2 + (a_y - p_y)^2}$ and $|b - h(p)| = \sqrt{(b_x - p_x)^2 + (b_y - p_y)^2}$. Since $p_x = 0$, $a_x = \pm b_x$, and $a_y = b_y$ we simplify and obtain $|a - p| = \sqrt{|b_x|^2 + (b_y - p_y)^2} = |b - h(p)|$.

Assume that $\forall x \in e, |a - x| = |b - h(x)|$.
$|a - p| = \sqrt{(a_x - p_x)^2 + (a_y - p_y)^2}$ and $|b - h(p)| = \sqrt{(b_x - p_x)^2 + (b_y - p_y)^2}$ so
$|a - p| = |b - h(p)| \implies \sqrt{a_x^2 + (a_y - p_y)^2} = \sqrt{b_x^2 + (b_y - p_y)^2}$
Since $(0,0) \in e$, $\sqrt{a_x^2 + a_y^2} = \sqrt{b_x^2 + b_y^2} \implies |a| = |b|$.
Since $(0,1) \in e$ $\sqrt{a_x^2 + (a_y - 1)^2} = \sqrt{b_x^2 + (b_y - 1)^2}$.
Squaring both equations implies that $a_x^2 + a_y^2 = b_x^2 + b_y^2$ and $a_x^2 + a_y^2 - 2a_y + 1 = b_x^2 + b_y^2 - 2b_y + 1 \implies -2a_y = -2b_y \implies a_y = b_y$. Since $|a| = |b|$ we have $a_x = \pm b_x$ so $a \in h_{F_1}^{-1}(b) \cup h_{F_2}^{-1}(b)$.

3.3 Lemma 3

Assume our graph G is a single vertex $v = (0,0)$ with $\theta_1, \theta_2 \ldots \theta_{2k}$.
We have some $a, b \in \mathbb{R}^2$ with $a = r_a(\cos\alpha, \sin\alpha)$ and $b = r_b(\cos\phi_1, \sin\phi_1)$.
Notably $\phi_{i+1} = 2\theta_i - \phi_i$.
Let p_i be the point on the line $(\cos\theta_i, \sin\theta_i)t$ where $|a - p_i| = |b - h(p_i)|$ and $t \geq 0$.
We have the following equation for p_i:
$|p_i| = \frac{(r_b^2 - r_a^2)}{2(r_b\cos(\phi_i - \theta_i) - r_a\cos(\alpha - \theta_i))} \geq 0$ and $p_i = |p_i|(\cos\theta_i, \sin\theta_i)$

Lemma 3: Assuming non degeneracy, for some a, b where $|p_1| = |p_2| \cdots = |p_{2k}| = 0$ and $\epsilon > 0$ there exists a δ such that $\sum_{|p_i| \geq 0} |p_i| < \epsilon$
With our non degeneracy assumption, we have $2(r_b\cos(\phi_i - \theta_i) - r_a\cos(\alpha - \theta_i)) \neq$

0, as a result this equation is continuous near a, b, thus we can find a δ satisfying the above.

4 Continuity proof part 1

Consider a fixed fold pattern G and $a_0, b_0 \in \mathbb{R}^2$ s.t. $a_0 \neq h_{F_k}^{-1}(b_0) \forall F_k \in F$. We show that $S(a,b)$ is continuous at a_0, b_0 under the distance metric $|X - Y| = \max\left\{\sup_{x\in X} \inf_{y\in Y} |x-y|, \ \sup_{y\in Y} \inf_{x\in X} |x-y|\right\}$. This is iff the non degeneracy hypothesis holds

Let $D = \min_{F_k \in F}(|a_0 - h_{F_k}^{-1}(b_0)|)$.
We can choose $\delta < \frac{D}{2}$ so that we never violate the degeneracy hypothesis.

4.1 Continuity of $B(a,b)$

Note that for any points $c, d \in \mathbb{R}^2$ the line $[c,d]_\perp$ is of the form $(\frac{c_x+d_x}{2}, \frac{c_y+d_y}{2}) + (d_y - c_y, c_x - d_x)t$. Inside a ball of radius R around c, d, this line is continuous under the distance metric. As a result the line $[a, h_{F_k}^{-1}(b)]$ is likewise continuous in $\mathbb{R}^2$.

We assumed that $a_0 \neq h_{F_k}^{-1}(b_0) \forall F_k \in F$. As a result, for any edge $e \in F_k$, $L_k(a_0, b_0) \neq e$. As a result, $L_k(a,b)$ is continuous.

Consider a point y on $B(a,b)$ with $y \in e = F_1 \cap F_2$.
Assume WLOG that e is the edge from $(0,0)$ to $(0,1)$ and that $F_1 \in \mathbb{R}^2_+$ is the identity.

We thus have the following formula for y: $y = (0, \frac{a_x^2 + a_y^2 - b_x^2 - b_y^2}{2(a_y - b_y)})$.
Since we assumed the existence of such a y, either $a_y - b_y = 0$ and $|a| - |b| = 0 \implies a_x = \pm b_x$ or $a_y - b_y \neq 0$ and $|a| - |b| \neq 0$. Under the non-degeneracy hypothesis and Lemma 2, it is clear that the latter must be true. As a result, the motion of y is continuous at (a_0, b_0).

Now consider $v \in V$ s.t. $v \in B(a_0, b_0)$.
We know that $\Omega(a,b)$ is star-shaped. As a result there must be some face F_k with $v \in F_k$ s.t. $|L_k(a_0, b_0)| \neq 0$. Let there be two points $p_1, p_2 \in L_k(a_0, b_0)$. The location of these points uniquely determines $L_k(a,b) \cap \partial F_k$. By the previous, we know that the motion of p_1, p_2 is continuous with respect to a, b. Thus, the motion of $L_k(a,b) \cap \partial F_k$ is continuous at a_0, b_0.

4.2 Continuity of $C(a,b)$

By the above, we have that the motion of w, a vertex of $B(a,b)$, is continuous at (a_0, b_0). Evidently, since a is also continuous at a_0, each distinct line $[a, w]$ must be continuous at (a_0, b_0) as well.

4.3 Continuity of $S(a, b)$

Clearly, $B(a, b)$ and $C(a, b)$ being continuous in the distance metric at a_0, b_0 implies that $S(a, b)$ is also continuous at a_0, b_0 under the distance metric. Thus, for a fixed fold pattern, the fold pattern generated by Brehm's extension theorem is continuous under the distance metric.

4.4 Iff

Suppose that the non degeneracy hypothesis fails to hold. That is, we have some F_k s.t. $a = h_{F_k}^{-1}(b)$. We thus have that on an edge e, $L_k(a, b) = e$. Let F_j share this edge. For some small $\delta > 0$ consider moving b in the direction $b - a$ so $b' = b + \delta(b - a)$. This will make $L_k(a, b)$ move further into F_k clearly preserving continuity. However, now consider moving b in the direction $a - b$ so $b' = b + \delta(a - b)$. Evidently $L_j(a, b)$ is not near the edge e since $h_{F_j}^{-1}(b) = Ref(h_{F_k}^{-1}(b))$ where Ref is the reflection across the edge e. Thus, at a_0, b_0, the line $L_k(a, b)$ vanishes for any $\delta > 0$ with $b' = b + \delta(a - b)$. Thus if the non degeneracy hypothesis fails, so too does continuity under the distance metric.

5 Continuity proof part 2

Consider 4 points, $a, b, c, d \in \mathbb{R}^2$. Let our initial fold pattern and map be $\tilde{G}$ and $\tilde{h}$. We then run Brehm's extension theorem to map a to b and create G and h. We fix c and d to be c_0, d_0 under the non-degeneracy hypothesis.
We show that the set $S(c_0, d_0)$ is continuous under the variation of a, b at a specific point (a_0, b_0).

5.1 Continuity of $B(c, d)$

Evidently the portion of $B(c, d)$ that is outside of $\Omega(a, b)$ is continuous by the logic in section 4.

We know that the fold pattern generated by mapping a_0 to b_0 is a pie-shaped region around a_0. Let the faces around a_0, at a_0, b_0, be $F_1, F_2 \dots F_{2n}$. Notably, at a_0, b_0, due to the non degeneracy hypothesis, we have that for some F_i, $\exists \tilde{F}_j$ s.t. $F_i \cap \tilde{F}_j$ is not a single point. As a result, we can find some δ to limit this line to stay within $\tilde{F}_j$.

By section 4 we have that $B(c, d)$ will vary continuously in each of $F_1 \dots F_{2n}$ at a_0, b_0. The issue manifests when for some vertex w on $B(c, d)$ at a_0, b_0, $w \in \tilde{V}$. Imagine we have some new vertices $w_1, w_2 \dots$ being formed at w. Via lemma 3, these new vertices can be bounded within some distance from w.

Additionally we know that the amount of added vertices must be an even number for Kawasaki to hold. Since we are adding an even amount, the addition of these new faces does not change the value of $h^{-1}_{F'_i}(d)$ for any other F'_i.

Since we are adding a very small angle in the limit, we can choose δ to keep $L_k(a, b)$ for a newly added F_k approximating a straight line. This gives us that in the construction of new faces at a_0, b_0, we have continuity in $B(c, d)$.

5.2 Continuity of $C(c, d)$

By the above we have that the motion of w, a vertex of $B(c, d)$ is continuous at a_0, b_0. This is supported by Lemma 3. Evidently since a is continuous at a_0, each distinct line $[a, w]$ must be continuous at (a_0, b_0).

5.3 Continuity of $S(c, d)$

Clearly, $B(c, d)$ and $C(c, d)$ being continuous in the distance metric at a_0, b_0 implies that $S(c, d)$ is also continuous at a_0, b_0 under the distance metric. Thus, for two consecutive iterations of Brehm's extension theorem, first on a, b then on c, d, the fold set generated by c, d for a fixed c_0, d_0 is continuous in the data a, b so long as the non-degeneracy hypothesis is satisfied.

6 Conclusion

We have that for a fixed fold pattern, the fold set generated by Brehm's extension theorem on a, b is continuous in a, b. We also have that for two consecutive iterations, a, b and c, d, the fold set generated by c, d is continuous in a, b. By induction, we are able to propagate this continuity forward to further iterations of Brehm so long as the non-degeneracy hypothesis continues to be satisfied.

As such we conclude the following. We have initial data $a_1, a_2 \dots a_n$ and $b_1 \dots b_n$ and consecutively constructed $h_0 = I_d$, $h_1, h_2 \dots h_n$ where $h_i(a_i) = b_i$. The crease set G_n at h_n is continuous in the input data iff at no intermediate step i was it the case that $a_{i+1} = h_{i,F_k}^{-1}(b_{i+1})$ for some $F_k \in G_i$.

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