A Monotone Scheme on Geodesic Active Contours

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Abstract

We propose a monotone, median-filter–type numerical scheme that converges to the viscosity solution of a level-set PDE whose zero level set evolves as the steepest descent of a weighted curve length. We further show that a modified two-step variant decreases an associated nonlocal total-variation energy. Finally, a numerical experiment—with weights derived from a given image—demonstrates that the scheme achieves effective image segmentation.

1 Problem Formulation

Consider a closed C^2 curve $\gamma : [0,1] \to \mathbb{R}^2$ and a nonnegative function $g : \mathbb{R}^2 \to \mathbb{R}$. Define the weighted length

$$L(\gamma) := \int_0^1 g(\gamma(x))\, |\gamma'(x)|\, dx.$$

Steepest descent of L (see [1]) yields the normal velocity

$$v_n(x) = \nabla g(x) \cdot n + g(x)\, \kappa(x),$$

where n is the outward unit normal and $\kappa(x)$ the scalar curvature at x. Embedding the evolving curve as the zero level set of $\phi : \mathbb{R}^2 \times [0,T] \to \mathbb{R}$ with $n = \nabla\phi/|\nabla\phi|$ leads to the level-set PDE

$$\phi_t = |\nabla\phi|\, v_n = \nabla g \cdot \nabla\phi + g \left\langle \nabla^2\phi \, \tfrac{\nabla^\perp\phi}{|\nabla^\perp\phi|}, \tfrac{\nabla^\perp\phi}{|\nabla^\perp\phi|} \right\rangle. \tag{1}$$

We seek a numerical scheme that converges to the unique viscosity solution.

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2 Monotonicty and Consistency

Within the Barles–Souganidis framework [2], a monotone and consistent numerical scheme satisfying a mild stability condition converges to the unique viscosity solution of (1). When $g \equiv 1$, [3] proves that the median filter fulfils these requirements; we extend their construction to general g.

Let K be a non-negative, radially symmetric kernel of unit mass and rapid decay. For bounded, continuous ϕ and $\lambda \in \mathbb{R}$ set

$$T_\lambda \phi := \{x \mid \phi(x) \geq \lambda\}, \qquad \psi_K^g \phi(x, \lambda) := \int_{T_\lambda \phi} \big(g(y) - g(x) + 2\big) K(x-y)\, dy.$$

If $|g| \leq 1$, then $\psi_K^g \phi(x, \lambda)$ is decreasing and left-continuous in λ. With $c := \lim_{\lambda \to -\infty} \psi_K^g \phi(x, \lambda)$ define

$$M_K^g \phi(x) := \sup\big\{\lambda : \psi_K^g \phi(x, \lambda) \geq c/2\big\}.$$

Lemma 2.1 (Monotonicity). *If $\phi_1 \leq \phi_2$ then $M_K^g \phi_1(x) \leq M_K^g \phi_2(x)$ for all x.*

Proof. Since $\phi_1 \leq \phi_2$, $T_\lambda \phi_1 \subseteq T_\lambda \phi_2$. Thus $\psi_K^g \phi_1(x, \lambda) \leq \psi_K^g \phi_2(x, \lambda)$, for all x and λ. Fixing any x, as $\psi_K^g \phi(x, \lambda)$ is left continuous in λ, we also have $\psi_K^g \phi_1(x, M_K^g \phi_1(x)) \geq \frac{c}{2}$, which implies, by the earlier inequality, that $\psi_K^g \phi_2(x, M_K^g \phi_1(x)) \geq \frac{c}{2}$, and so following the definition of $M_K^g \phi$, we have $M_K^g \phi_1(x) \leq M_K^g \phi_2(x)$ ☐

Lemma 2.2 (Consistency). *Let $g \in C^\infty(\mathbb{R}^2)$ with $0 \leq g \leq 1$ and $\phi \in C_b^\infty(\mathbb{R}^2)$. Fix $x_0 \in \mathbb{R}^2$ such that $\nabla \phi(x_0) \neq 0$ and take $r > 0$ sufficiently small. Let K be the unit-mass distribution supported on the circle of radius r. Then $u(x) := M_K^g \phi(x)$ is the unique solution to*

$$F(r, \lambda) := \int_0^{2\pi} \big(g(x_0 + rv(\theta)) - g(x_0) + 2\big)\, \mathrm{sgn}\big(\phi(x_0 + rv(\theta)) - \lambda\big)\, d\theta = 0,$$

where $v(\theta) = (\cos\theta, \sin\theta)^\top$. Moreover, if $\nabla\phi$ points at angle ψ and redefining $v(\theta) = (\cos(\theta + \psi), \sin(\theta + \psi))^\top$, then

$$\lambda(r) = \phi + \frac{r^2}{2}\Big(\nabla g \cdot \nabla \phi + g \left\langle \nabla^2 \phi \tfrac{\nabla^\perp \phi}{|\nabla^\perp \phi|}, \tfrac{\nabla^\perp \phi}{|\nabla^\perp \phi|} \right\rangle\Big) + \mathcal{O}(r^4). \tag{2}$$

Proof. First, we show uniqueness. Fix any r sufficiently small. Since $\phi(x + rv(\theta))$ is continuous over θ, there exists $m, M \in \mathbb{R}$, such that for all θ, $m \le \phi(x+rv(\theta)) \le M$. Furthermore, for any a, b such that $m \le a < b \le M$. $m(\{\theta \mid a < \phi(x+rv(\theta)) < b\} \ge 0)$. Thus since $W(\theta) \ge 1$

$$F(r,a) - F(r,b) \ge \int_{\{\theta | a < \phi(x+rv(\theta) < b)\}} 2W(\theta)\, dx > 0$$

Therefor $F(r, \lambda)$ is strictly decreasing for λ on $[m, M]$, and as $F(r, M) < 0$ and $F(r, m) > 0$, there exists a unique λ such that $F(r, \lambda) = 0$.

Now we show (2). By implicit function theorem, we can assume that $\lambda = a_0 + a_1 r + \frac{r^2}{2}\mu + \frac{r^3}{6}\gamma + O(r^4)$. With direct computation we obtain $a_0 = \phi(x_0)$, and $a_1 = 0$, so to show (2) to hold, it remains to prove that

$$\mu = |\nabla\phi| \left(\kappa g + \left\langle \nabla g, \frac{\nabla\phi}{|\nabla\phi|} \right\rangle\right) = \nabla g \cdot \nabla\phi + \left\langle \nabla^2\phi \frac{\nabla^\perp\phi}{|\nabla^\perp\phi|}, \frac{\nabla^\perp\phi}{|\nabla^\perp\phi|} \right\rangle$$

, and $\gamma = 0$. For the remainder of the proof we denote $g_0 := g(x_0), \phi_0 := \phi(x_0), \nabla g_0 := \nabla g(x_0), \nabla\phi_0 := \nabla\phi(x_0)$, $H_g := \nabla^2 g$. And we set $W(\theta) := g(x_0 + rv(\theta)) - g_0 + 2$, and $S(\theta) := \phi(x_0 + rv(\theta)) - \lambda$. Then

$$F(r,\lambda) = \int_0^{2\pi} W(\theta) sgn(\cos(\theta))\, d\theta + \int_0^{2\pi} W(\theta)(\mathrm{sgn}(S) - \mathrm{sgn}(\cos\theta))\, d\theta \quad (3)$$

Simplifying the first term of (3) :

$$\begin{aligned}
&\int_0^{2\pi} W(\theta)\ \mathrm{sgn}(\cos\theta)\,\mathrm{d}\theta \\
&= r\int_0^{2\pi} \langle \nabla g_0, v(\theta)\rangle\,\mathrm{sgn}(\theta)\,\mathrm{d}x + \int_0^{2\pi} \frac{r^2}{2}\, v(\theta)^\top H_g(x_0)\, v(\theta)\,\mathrm{d}x + O(r^3) \\
&= r\Big(\int_0^{2\pi} \Big\langle \nabla g_0, \frac{\nabla\phi_0}{|\nabla\phi_0|}\Big\rangle \cos\theta\ \mathrm{sgn}(\cos\theta)\,\mathrm{d}\theta \\
&\quad + \int_0^{2\pi} \Big\langle \nabla g_0, \frac{\nabla\phi_0^\perp}{|\nabla\phi_0^\perp|}\Big\rangle \sin\theta\ \mathrm{sgn}(\cos\theta)\,\mathrm{d}x\Big)\ +\ O(r^3) \\
&= 4r\Big\langle \nabla g_0, \frac{\nabla\phi_0}{|\nabla\phi_0|}\Big\rangle + O(r^3).
\end{aligned}$$

Where we used the fact that

$$\int_0^{2\pi} v(\theta)^\top H v(\theta)\,\mathrm{sgn}(\cos\theta)\ d\theta = 0$$

For all matrix symmetric H

Now we need to bound the second term. To do so, we measure how the root of $S(\theta)$ perturbate from $\frac{\pi}{2}$ and $\frac{3\pi}{2}$, which are the roots of $\cos(\theta)$

For this step, we use the implicit function theorem. First, with a redefinition, we let

$$S(\theta, r) = \phi(x_0 + rv(\theta)) - \lambda = a\cos(\theta) + \frac{r}{2} q_2(\cos\theta, \sin\theta) + \frac{r^2}{2} q_3(\cos\theta, \sin\theta) + O(r^3)$$

, where $q_2(x, y) = q_{2,1}x^2 + q_{2,2}xy + q_{2,3}y^2$, and $q_3(x, y) = q_{3,1}x^3 + q_{3,2}x^2y + q_{3,3}xy^2 + q_{3,4}y^3$. Since $S(\theta, r)$ is smooth, $\frac{\partial S(r,\theta)}{\partial\theta}\left(0, \frac{\pi}{2}\right) \neq 0$, $\frac{\partial S(r,\theta)}{\partial\theta}\left(0, \frac{3\pi}{2}\right) \neq 0$. In addition $F\left(0, \frac{\pi}{2}\right) = F\left(0, \frac{3\pi}{2}\right) = 0$. Therefore, by implicit function theorem on a neighborhood around $\left(0, \frac{\pi}{2}\right)$, or a neighborhood around $\left(0, \frac{3\pi}{2}\right)$, there exists a smooth function $\theta(r)$ such that for r sufficiently small, $S(r, \theta(r)) = 0$.

So to obtain a taylor expansion of θ at 0, we calculate $\theta'(0)$ and $\theta''(0)$. We can do so by differentiating $S(r, \theta)$,

$$\frac{d}{dr} S(r, \theta(r)) = \partial_r S(r, \theta(r)) + \partial_\theta S(r, \theta(r))$$

Evaluating both side at $\left(0, \frac{\pi}{2}\right)$, and $\left(0, \frac{3\pi}{2}\right)$. We obtain $\theta'(0)$. Similarly

$$\frac{d^2}{dr^2} S(r, \theta(r)) = \partial_{rr} S(r, \theta(r)) + 2\partial_{r\theta} S(r, \theta(r))\theta'(r) + \partial_{\theta\theta} S(r, \theta(r))\theta'(r)^2 + \partial_\theta S(r, \theta(r))$$

and evaluating both side at $\left(0, \frac{\pi}{2}\right)$ and $\left(0, \frac{3\pi}{2}\right)$ gives us $\theta''(0)$.

After the calculation, we obtain the expansion $\theta^{\pi/2}(r) = \frac{\pi}{2} + \delta_1 r + \delta_2^{(\pi/2)} r^2 + O(r^3)$. And $\theta^{3\pi/2}(r) = \frac{3\pi}{2} - \delta_1 r + \delta_2^{3\pi/2} r^2 + O(r^3)$, where $\delta_1 = \frac{q_{2,3} - \mu}{2a}$, and $\delta_2^{\pi/2} = \frac{q_{3,4} - \gamma}{3a} - \frac{q_{2,2}(q_{2,3} - \mu)}{2a^2}$,$\delta_2^{3\pi/2} = \frac{q_{3,4} + \gamma}{3a} - \frac{q_{2,2}(q_{2,3} - \mu)}{2a^2}$. Note in particular, $q_{2,3} = \left\langle \frac{\nabla\phi^\perp}{|\nabla\phi^\perp|}, H_\phi(x_0)\frac{\nabla\phi^\perp}{|\nabla\phi^\perp|} \right\rangle$.

Now we can seek for a taylor expansion of $\int_0^{2\pi} W(\theta)(\mathrm{sgn}(S) - \mathrm{sgn}(\cos\theta))\, d\theta$ in terms of r, by noting that $\mathrm{sgn}(S) - \mathrm{sgn}(\cos\theta)$ is zero in all places other than the small bands around $\frac{\pi}{2}$ and $\frac{3\pi}{2}$. Assume with out loss of generality $\delta_1 > 0$. We have that

$$
\begin{aligned}
&\int_0^{2\pi} W(\theta)(\operatorname{sgn} S - \operatorname{sgn}(\cos(\theta))) \\
&= \int_{\frac{\pi}{2}}^{\pi/2+\delta_1 r+\delta_2^{(\pi/2)} r^2} 2(2 + r\langle \nabla g_0, v(\theta)\rangle) + \frac{r^2}{2} v(\theta)^\top H_g v(\theta) d\theta \\
&+ \int_{3\pi/2-\delta_1 r+\delta_2^{(3\pi/2)} r^2}^{3\pi/2} 2(2 + r\langle \nabla g_0, v(\theta)\rangle + \frac{r^2}{2} v(\theta)^\top H_g v(\theta) d\theta + O(r^3)
\end{aligned}
$$

The $O(r)$ term of the above expression is $(2\delta_1)(4g) = 4\frac{q_{2,3}-\mu}{|\nabla\phi|}$. By considering the first term of (3), and solve for μ in $4\frac{q_{2,3}-\mu}{|\nabla\phi|} + 4\left\langle \nabla g, \frac{\nabla\phi}{|\nabla\phi|}\right\rangle = 0$, we obtain the desired μ.

It remains to show that γ vanishes. First note that by symmetry:

$$
\int_{\left[\frac{\pi}{2},\frac{\pi}{2}+\delta_1 r\right]\cup\left[\frac{3\pi}{2}-\delta_1 r,\frac{3\pi}{2}\right]} r\langle \nabla g, v(\theta)\rangle \, d\theta = 0r^2 + O(r^3) = O(r^3)
$$

Then the only term we need to consider that contributes with $O(r^2)$ and is the constant 4 integrated over measure $O(r^2)$. This is equivalent to require

$$
m\left(\left[\frac{\pi}{2} + \delta_1 r, \frac{\pi}{2} + \delta_1 r + \delta_2^{(\pi/2)} r^2\right]\right) - m\left(\left[\frac{3\pi}{2} - \delta_1 r, \frac{3\pi}{2} - \delta_1 r + \delta_2^{(3\pi/2)} r^2\right]\right) = 0
$$

, where $m([a,b]) = b - a$. (We are **not** assuming $b \geq a$). Solving for γ (in δ_2), we obtain $\gamma = 0$ □

Remark 1. In the preceding calculation the median uses the weight $g(x_0 + r\,v(\theta)) - g(x_0) + 2$. If instead we use the symmetric weight $g(x_0 + r\,v(\theta)) + g(x_0)$ and repeat the derivation, an additional mobility factor $1/g$ appears in the normal velocity. Consequently, the scheme converges to

$$
\phi_t = |\nabla\phi| \, \frac{1}{g} \big(\nabla g \cdot n + \kappa\, g\big),
$$

which differs from (1) only by the prefactor $1/g$ (after rewriting the κ term via the tangential Laplacian and simplifying).

3 Decrease in Nonlocal Total-Variation Energy

Our starting point is the minimization of the nonlocal total-variation energy

$$\mathcal{E}_K^g(\phi) = \int g(x) \int K(x-y)\,|\phi(x)-\phi(y)|\,dy\,dx. \tag{4}$$

Heuristically, discretize ϕ at points $x_1, \dots, x_n \in \mathbb{R}^n$ and perform coordinate descent on the surrogate

$$\mathcal{E}_K^g(\phi) \approx \sum_i g(x_i) \sum_j K(x_i - x_j)\,|\phi(x_i) - \phi(x_j)|.$$

Updating a single variable $\phi(x_k)$ gives the stationarity condition

$$\frac{\partial \mathcal{E}}{\partial \phi(x_k)} = \sum_j \big(g(x_k) + g(x_j)\big)\, K(x_k - x_j)\, \operatorname{sgn}\!\big(\phi(x_k) - \phi(x_j)\big) = 0.$$

When K is supported on the circle of radius r, the factor $K(x_k - x_j)$ is nonzero only when $\|x_k - x_j\| = r$. Passing to a continuous limit then yields the rule described in Remark 1, with median weights $g(x_k) + g(x_j)$.

Then following from [5], we obtain that

Lemma 3.1. *Let K be positive, radially symmetric and rapidly decaying $S_K^g T_\lambda \phi = T_\lambda M_K^g \phi$, where S_K^g is the weighted threshold dynamics operator, defined in the following algorithm*

Algorithm 1: Weighted Threshold Dynamics

$$u(x) = \frac{\displaystyle\int_\Sigma K(x-y)\big(g(y)+g(x)\big)\,dy}{\displaystyle\int_\Sigma K(x-y)\big(g(y)+g(x)\big)\,dy}, \quad S_K^g(\Sigma) = \{\, x : u(x) \geq \tfrac{1}{2} \}.$$

In [4], it is demonstrated that a two-step threshold dynamics method decrease a certain energy, which when combined with [5], shows that when $g = 1$, the weighted two-step median filter in algorithm 2 decreases the nonlocal total-variation energy. Now we show that this is true for every g such that $0 \leq g \leq 1$.

Algorithm 2: Weighted Two-Step Median Filter

Step 1. Grow super level-sets:

$$\phi^{n+\frac{1}{2}} \;=\; \max\big\{\phi^n,\; M_K^g\phi^n\big\}.$$

Step 2. Shrink super level-sets:

$$\phi^{n+1} \;=\; \min\big\{\phi^{n+\frac{1}{2}},\; M_K^g\phi^{n+\frac{1}{2}}\big\}.$$

Algorithm 3: Weighted Two-Step Threshold Dynamics

Expansion step:

$\Sigma^{n+\frac{1}{2}} \leftarrow \Sigma^n \cup S_K^g(\Sigma^n)$

Shrinking step:

$\Sigma^{n+1} \leftarrow \Sigma^{n+\frac{1}{2}} \cap S_K^g\big(\Sigma^{n+\frac{1}{2}}\big)$

Theorem 3.2 (Two step Median Filter Decrease Energy)**.** *Let K be a non-negative, radially symmetric function with sufficient decay. Then the two step median filter method with weight, g, $0 \le g \le 1$, decreases the energy $\mathcal{E}_{K,g}(\phi) = \int g(x) \int K(x-y)|\phi(x)-\phi(y)|\, dy\, dx$*

Proof. As will be seen later in the proof, we let the corresponding energy of a set Σ be defined as

$$\begin{aligned} E_K(\Sigma) &:= \int_{\Sigma^c} K * (g\mathbf{1}_\Sigma)\, dx + \int_\Sigma K * (g\mathbf{1}_{\Sigma^c})\, dx \\ &= \int_\Sigma \int_{\Sigma^c} g(x)K(x-y)\, dy\, dx + \int_{\Sigma^c}\int_\Sigma g(y)K(x-y)\, dx\, dy \end{aligned}$$

Now to write with $|\phi(x)-\phi(y)|$ in "level-set form", we can use the identity

$$\int_{\mathbb{R}} \mathbf{1}_{T_\lambda\phi}(y)(1-\mathbf{1}_{T_\lambda\phi}(x))\, d\lambda = (\phi(y)-\phi(x))_+$$

For later in the proof, we write g as a subscript for notational simplicity. In particular, we let $M_{K,g}^+$ be a growing half step of the weighted median filter (i.e $M_{K,g}^+\phi^n = \phi^{n+1/2}$), and $S_{K,g}^+$ be a corresponding growing half step of threshold dynamics. Then we obtain

$$
\begin{aligned}
\mathcal{E}_{K,g}(M^+_{K,g}\phi) &= \iint g(x)K(x-y)\int_{\mathbb{R}} \mathbf{1}_{T_\lambda M^+_{K,g}\phi}(x)(1-\mathbf{1}_{T_\lambda M^+_{K,g}\phi}(y)), d\lambda\, dx\, dy \\
&+ \iint g(x)K(x-y)\int_{\mathbb{R}} \mathbf{1}_{T_\lambda M^+_{K,g}}(y)(1-\mathbf{1}_{T_\lambda M^+_{K,g}}(x))\, d\lambda\, dx\, dy \\
&= \iint g(x)K(x-y)\int_{\mathbb{R}} \mathbf{1}_{S^+_{K,g}T_\lambda\phi}(x)(1-\mathbf{1}_{S^+_{K,g}T_\lambda\phi}(y)), d\lambda\, dx\, dy \\
&+ \iint g(x)K(x-y)\int_{\mathbb{R}} \mathbf{1}_{S^+_{K,g}T_\lambda\phi}(y)(1-\mathbf{1}_{S^+_{K,g}T_\lambda\phi}(x))\, d\lambda\, dx\, dy \\
&= \int_{\mathbb{R}} E_K(S^+_{K,g}T_\lambda\phi)\, d\lambda
\end{aligned}
$$

So it remains to show $E_K(S^+_{K,g}T_\lambda\phi) \leq E_K(T_\lambda\phi)$, which gives us $\mathcal{E}_{K,g}(M^+_{K,g}\phi) \leq \mathcal{E}(\phi)$. Fixing a set Σ, and denote the expansion half step threshold dynamics evolution of Σ be $\Sigma_{1/2}$. We let $u(x) := \mathbf{1}_\Sigma(x)$ and $\phi(x) = \mathbf{1}_{\Sigma^{1/2}}(x) - \mathbf{1}_\Sigma(x)$.

$$
\begin{aligned}
&E_{K,g}\big(\Sigma_{\frac{1}{2}}\big) - E_{K,g}(\Sigma) \\
&= \iint g(x)\, K(x-y)\, \big[(u(x)+\phi(x))(1-u(y)-\phi(y)) - u(x)(1-u(y))\big]\, \mathrm{d}x\, \mathrm{d}y \\
&\quad + \iint g(x)\, K(x-y)\, \big[(u(y)+\phi(y))(1-u(x)-\phi(x)) - u(y)(1-u(x))\big]\, \mathrm{d}x\, \mathrm{d}y \\
&= -\iint gt(x)\, K(y-x)\, u(x)\, \phi(y)\, \mathrm{d}x\, \mathrm{d}y + \iint g(x)\, K(x-y)\, \phi(x)\, (1-u(y))\, \mathrm{d}x\, \mathrm{d}y \\
&\quad - \iint g(x)\, K(x-y)\, u(y)\, \phi(x)\, \mathrm{d}x\, \mathrm{d}y + \iint g(x)\, K(y-x)\, \phi(y)\, \big(1-(u(x))\big)\, \mathrm{d}x\, \mathrm{d}y \\
&\quad - 2\iint g(x)\, K(x-y)\, \phi(x)\, \phi(y)\, \mathrm{d}x\, \mathrm{d}y \\
&= \int \phi\, \big[\, g\, (K * \mathbf{1}_{\Sigma^c} - K * \mathbf{1}_\Sigma) + K * (g\, \mathbf{1}_{\Sigma^c} - g\, \mathbf{1}_\Sigma)\, \big]\, \mathrm{d}x - 2\int (K * g\phi)\, \phi\, \mathrm{d}x \;\leq 0.
\end{aligned}
$$

The last step can be justified by noting $\phi, g, K \geq 0$ so the second term is non-positive, whereas when $\phi > 0$, by the definition of the modified threshold dynamics algorithm

$$
g(K * \mathbf{1}_{\Sigma^c} - K * \mathbf{1}_\Sigma) + K * (g\mathbf{1}_{\Sigma^c} - g\mathbf{1}_\Sigma) \leq 0
$$

, so the first term is also non-positive.

The proof for the shrinking step follows similarly □

4 Numerical Experiments

We apply the weighted median algorithm—using the weight $g(x_0 + r\,v(\theta)) + g(x_0)$ with a suitable radius r—to an image of vegetables on a solid-colored background. The function g (Figure 1) is chosen to be close to 0 near vegetable boundaries (where colors vary sharply) and close to 1 elsewhere. Starting from an initial function whose zero level set is a rectangular box, the zero level set contracts over iterations, stabilizes, and delineates the vegetables enclosed by the initial level-set curve (Figure 2).

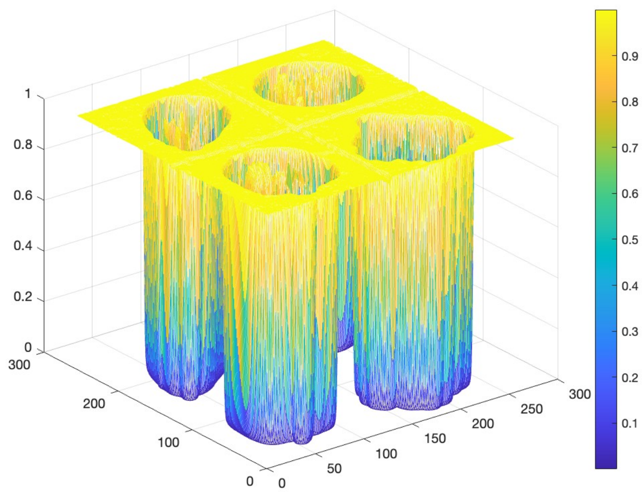

Figure 1: The function g used for numerical simulations

Figure 2: Evolution of Level Sets in Weighted Median Filter

Acknowledgment

The author thanks Professor Selim Esedoglu for mentorship, Dr. Jiajia Guo for valuable help and for providing numerical simulation examples, the University of Michigan for hosting the REU, and the NSF for funding.

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