Obstructions Toward a Phantom Category on $\mathrm{Bl}_9(\mathbb{F}_3)$

Bo Gao Ari Krishna Sargam Mondal Chunye Yang

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Abstract

We record obstructions to the construction of a phantom subcategory in the bounded derived category of coherent sheaves on the blow–up of the Hirzebruch surface $\mathbb{F}_3$ at nine very general points. Our strategy adapts Krah's construction of a non–full exceptional collection on $\mathrm{Bl}_{10}(\mathbb{P}^2)$ to the surface $X = \mathrm{Bl}_9(\mathbb{F}_3)$.

University of Michigan REU *Mentor: Shengxuan Liu*

1 Background: Derived Categories and Orlov's Blow-up Formula

Here, we start with a summary of the derived-categorical framework that undergirded our research directions this summer.

1.1 Bounded derived category and triangulated structure

Let $\mathcal{A}$ be an abelian category (e.g. $\mathrm{Coh}(X)$ for a smooth projective variety X). Consider the category of bounded complexes $\mathrm{Ch}^b(\mathcal{A})$ and the homotopy category $K^b(\mathcal{A})$ obtained by quotienting chain-homotopies. The *bounded derived category* is the localization

$$D^{\mathrm{b}}(\mathcal{A}) \;=\; K^b(\mathcal{A})\,[\{\text{quasi-isomorphisms}\}^{-1}],$$

which is a *triangulated category*: it comes with a shift endofunctor [1] and a class of distinguished triangles encoding short exact sequences. For X smooth projective we write $D^{\mathrm{b}}(X) := D^{\mathrm{b}}(\mathrm{Coh}(X))$.

Derived functors become honest functors in $D^{\mathrm{b}}(X)$; for instance

$$\mathrm{Ext}^n(\mathcal{F},\mathcal{G}) \;\cong\; \mathrm{Hom}_{D^{\mathrm{b}}(X)}(\mathcal{F},\mathcal{G}[n]).$$

1.2 Exceptional objects, collections, and SODs

An object $E \in D^{\mathrm{b}}(X)$ is *exceptional* if $\mathrm{Hom}^\bullet(E,E) \cong k$ is concentrated in degree 0. A sequence $(E_1,\dots,E_m)$ is an *exceptional collection* if each E_i is exceptional and $\mathrm{Hom}^\bullet(E_j,E_i) = 0$ for all $j > i$. A *semiorthogonal decomposition* (SOD) of a triangulated category $\mathcal{T}$ is a sequence of full triangulated subcategories $\langle \mathcal{A}_1,\dots,\mathcal{A}_n\rangle$ such that $\mathrm{Hom}(\mathcal{A}_j,\mathcal{A}_i) = 0$ for $j > i$ and they generate $\mathcal{T}$. Exceptional collections give SODs with components $\langle E_i\rangle$.

1.3 Orlov's blow-up formula

Let X be smooth projective and $Y \subset X$ smooth of codimension $r \geq 2$. Let $\pi : \widetilde{X} \to X$ be the blow-up with exceptional divisor $i : E = \mathbb{P}(N_{Y/X}) \hookrightarrow \widetilde{X}$. Then Orlov [3] gives an SOD

$$D^{\mathrm{b}}(\widetilde{X}) = \Big\langle\, i_* D^{\mathrm{b}}(Y) \otimes \mathcal{O}_E(1-r),\ \ldots,\ i_* D^{\mathrm{b}}(Y) \otimes \mathcal{O}_E(-1),\ \pi^* D^{\mathrm{b}}(X) \,\Big\rangle.$$

In particular, if $D^{\mathrm{b}}(X)$ and $D^{\mathrm{b}}(Y)$ admit full exceptional collections, then so does $D^{\mathrm{b}}(\widetilde{X})$. This is the source of the length-13 full collection on $\mathrm{Bl}_{10}(\mathbb{P}^2)$ and on $\mathrm{Bl}_9(\mathbb{F}_3)$ used in our arguments.

2 Krah's Proof of a Phantom on $\mathrm{Bl}_{10}(\mathbb{P}^2)$

In this section, we detail Krah's proof of the existence of a phantom on the blowup of $\mathbb{P}^2$ at ten general points.

2.1 Geometry and the Picard involution

Let $X = \mathrm{Bl}_{10}(\mathbb{P}^2)$ at ten very general points with classes $H, E_1, \ldots, E_{10}$, so

$$K_X = -3H + \sum_{i=1}^{10} E_i, \qquad K_X^2 = -1.$$

Write $\langle \cdot, \cdot \rangle$ for the intersection form. Consider the orthogonal transformation

$$\iota : \mathrm{Pic}(X) \to \mathrm{Pic}(X), \qquad \iota(D) := -D - \langle D, K_X \rangle\, K_X,$$

which fixes K_X and acts as $-\mathrm{id}$ on $K_X^{\perp}$. Define the classes (Krah's notation)

$$D_i := \iota(E_i), \qquad F := \iota(H),$$

so explicitly $D_i = -6H + 2\sum_{j=1}^{10} E_j - E_i$ and $F = -19H + 6\sum_{i=1}^{10} E_i$.

2.2 The candidate sequence and numerical semiorthogonality

Krah exhibits the 13-term sequence of line bundles

$$\mathcal{E}_{\mathrm{Krah}} = (\ \mathcal{O}_X,\ \mathcal{O}_X(D_1), \ldots, \mathcal{O}_X(D_{10}),\ \mathcal{O}_X(F),\ \mathcal{O}_X(2F)\ ).$$

Using Riemann–Roch on surfaces,

$$\chi(\mathcal{O}_X(D)) = \frac{1}{2}(D^2 - D \cdot K_X) + 1,$$

one checks that for all $j > i$,

$$\chi(\mathcal{O}_X(D_j), \mathcal{O}_X(D_i)) = \chi(\mathcal{O}_X(D_i - D_j)) = 0,$$

so the sequence is numerically semiorthogonal.

2.3 Upgrading to exceptionality

For line bundles, $\mathrm{Ext}^p(\mathcal{O}_X(A), \mathcal{O}_X(B)) \cong H^p(X, \mathcal{O}_X(B-A))$. Thus exceptionality is equivalent to

$$H^0(X, \mathcal{O}_X(D_j - D_i)) = H^2(X, \mathcal{O}_X(D_j - D_i)) = 0 \quad (j > i),$$

and by Serre duality $H^2(\mathcal{O}_X(D))^\vee \cong H^0(\mathcal{O}_X(K_X - D))$. By examining differences of divisors $D_j - D_i$:

- many are clearly ineffective (e.g. of type $E_i - E_i$);
- or fall under known cases of the SHGH conjecture on linear systems (here, for such linear systems $|D|$, there is no (-1)-curve C with $D \cdot C \leq -2$).

This yields $\mathrm{Ext}^i = 0$ in the required directions for $i = 0, 2$, hence $\mathcal{E}_{\mathrm{Krah}}$ is exceptional.

2.4 Non–fullness via anticanonical pseudoheight (Kuznetsov)

Kuznetsov devised an invariant, the *anticanonical pseudoheight*, which gives rise to sufficient conditions for non-fullness; Krah leveraged this in his construction.

Let $e(E, F) = \inf\{k : \mathrm{Ext}^k(E, F) \neq 0\}$. Following Kuznetsov [4], the *anticanonical pseudoheight* of a collection $E_\bullet$ is

$$\mathrm{ph}_{ac}(E_\bullet) = \min_{a_0 < \cdots < a_p} \left(e(E_{a_0}, E_{a_1}) + \cdots + e(E_{a_{p-1}}, E_{a_p}) + e(E_{a_p}, E_{a_0} \otimes \omega_X^{-1}) - p \right).$$

Importantly, $\mathrm{ph} = \mathrm{ph}_{ac} + \dim X$, the (normal/Hochschild) height h satisfies $h \geq \mathrm{ph}$, and if $h > 0$ (equivalently, the pseudoheight is at least $-\dim X$) then the collection is not full. In his paper, Krah computes $\mathrm{ph}_{ac}(\mathcal{E}_{\mathrm{Krah}}) > -2$ on the surface, hence $\mathcal{E}_{\mathrm{Krah}}$ is not full.

2.5 From non–full to phantom

On the other hand, Orlov's blow-up formula gives a full exceptional collection of length 13 on $D^{\mathrm{b}}(X)$ (ten exceptional curves plus a full collection from $\mathbb{P}^2$). Hence $K_0(D^{\mathrm{b}}(X)) \cong \mathbb{Z}^{13}$. Since $\mathcal{E}_{\mathrm{Krah}}$ also has length 13 but is not full, we have a semiorthogonal decomposition

$$D^{\mathrm{b}}(X) = \langle \mathcal{A}^\perp, \ \langle \mathcal{E}_{\mathrm{Krah}} \rangle \rangle,$$

and additivity of K_0 on semiorthogonal decompositions forces $K_0(\mathcal{A}^\perp) = 0$ while $\mathcal{A}^\perp \neq 0$. Thus $\mathcal{A}^\perp$ is a phantom subcategory.

3 Our setup on $X = \mathrm{Bl}_9(\mathbb{F}_3)$

Hereafter, we attempt to mimic Krah's strategy on a particular blowup of a Hirzebruch surface, and describe the obstructions we ran into.

3.1 Geometry and notation

Let $\mathbb{F}_3 = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(3))$ with $\mathrm{Pic}(\mathbb{F}_3) = \mathbb{Z}S \oplus \mathbb{Z}F$, $S^2 = -3$, $F^2 = 0$, $S \cdot F = 1$, $K_{\mathbb{F}_3} = -2S - 5F$. Blow up nine very general points to obtain

$$X = \mathrm{Bl}_9(\mathbb{F}_3), \qquad \mathrm{Pic}(X) = \mathbb{Z}S \oplus \mathbb{Z}F \oplus \bigoplus_{i=1}^{9} \mathbb{Z}E_i,$$

$$K_X = -2S - 5F + \sum_{i=1}^{9} E_i, \qquad K_X^2 = 8 - 9 = -1.$$

By Orlov, $D^{\mathrm{b}}(X)$ admits a full exceptional collection of length $4 + 9 = 13$.

3.2 Picard involution and the 13-term candidate

Because in our case we also have $K_X^2 = -1$, we furnish the involution

$$\iota(D) := -D - 2(D \cdot K_X)K_X,$$

which fixes K_X and acts by $-\mathrm{id}$ on $K_X^{\perp}$. Direct computation yields:

$$\begin{aligned}
\iota(F) &= -8S - 21F + 4\sum_{i=1}^{9} E_i, \\
\iota(S) &= \quad 3S + 10F - 2\sum_{i=1}^{9} E_i, \\
\iota(E_i) &= -4S - 10F + 2\sum_{j=1}^{9} E_j - E_i.
\end{aligned}$$

Set

$$D_0 := 0, \quad D_i := \iota(E_i)\ (1 \le i \le 9), \quad D_{10} := \iota(F), \quad D_{11} := \iota(S), \quad D_{12} := \iota(S + F),$$

and consider the sequence

$$\mathcal{E} = (\ \mathcal{O}_X,\ \mathcal{O}_X(D_1), \dots, \mathcal{O}_X(D_9),\ \mathcal{O}_X(D_{10}),\ \mathcal{O}_X(D_{11}),\ \mathcal{O}_X(D_{12})\ ).$$

3.3 Numerical semiorthogonality (proved)

Using $\chi(\mathcal{O}_X(D)) = \frac{1}{2}(D^2 - D \cdot K_X) + 1$, we verified that $\chi(\mathcal{O}_X(D_i), \mathcal{O}_X(D_j)) = 0$ for all $i > j$. Hence $\mathcal{E}$ is numerically semiorthogonal by a similar argument to Krah's (or by manually using Riemann–Roch for surfaces); this follows from the fact that ι acts by 1 on K_X and by -1 on its orthogonal complement.

3.4 From numerical to genuine: difference tables

For exceptionality we must have $H^0(X, \mathcal{O}_X(D_j - D_i)) = H^0(X, \mathcal{O}_X(K_X - (D_j - D_i))) = 0$ for all $j > i$. Below are the complete lists of differences to check.

Table A: differences $D_j - D_i$ (for $j > i$).

From $D_0 = 0$	Expression in $\mathrm{Pic}(X)$
$D_0 - D_k$ $(1 \le k \le 9)$	$-\,\iota(E_k) = 4S + 10F - 2\sum_{m=1}^{9} E_m - E_k$
$D_0 - D_{10}$	$-\,\iota(F) = 8S + 21F - 4\sum_{m=1}^{9} E_m$
$D_0 - D_{11}$	$-\,\iota(S) = -3S - 10F + 2\sum_{m=1}^{9} E_m$
$D_0 - D_{12}$	$-\,\iota(S{+}F) = 5S + 11F - 2\sum_{m=1}^{9} E_m$
Between $1 \le i < j \le 9$	$D_i - D_j = E_j - E_i$
$D_i - D_{10}$	$4S + 11F - 2\sum_{m=1}^{9} E_m - E_i$
$D_i - D_{11}$	$-7S - 20F + 4\sum_{m=1}^{9} E_m + E_i$
$D_i - D_{12}$	$S + F - E_i$
$D_{10} - D_{11}$	$-11S - 31F + 6\sum_{m=1}^{9} E_m$
$D_{10} - D_{12}$	$3S + 10F - 2\sum_{m=1}^{9} E_m = -\,\iota(S)$
$D_{11} - D_{12}$	$-8S - 21F + 4\sum_{m=1}^{9} E_m = -\,\iota(F)$

Table B: Serre-dual differences $K_X - (D_j - D_i)$.

	Expression in $\mathrm{Pic}(X)$
$K_X - (D_k - D_0)$	$-6S - 15F + 3\sum_{m=1}^{9} E_m + E_k$
$K_X - (D_{10} - D_0)$	$-10S - 26F + 5\sum_{m=1}^{9} E_m$
$K_X - (D_{11} - D_0)$	$S + 5F - \sum_{m=1}^{9} E_m$
$K_X - (D_{12} - D_0)$	$-7S - 16F + 3\sum_{m=1}^{9} E_m$
$K_X - (D_j - D_i)$	$-2S - 5F + \sum_{m=1}^{9} E_m - E_j + E_i$ (for $1 \le i < j \le 9$)
$K_X - (D_{10} - D_i)$	$-6S - 16F + 3\sum_{m=1}^{9} E_m + E_i$
$K_X - (D_{11} - D_i)$	$5S + 15F - 3\sum_{m=1}^{9} E_m - E_i$
$K_X - (D_{12} - D_i)$	$-3S - 6F + \sum_{m=1}^{9} E_m + E_i$
$K_X - (D_{11} - D_{10})$	$9S + 26F - 5\sum_{m=1}^{9} E_m$
$K_X - (D_{12} - D_{10})$	$-5S - 15F + 3\sum_{m=1}^{9} E_m$
$K_X - (D_{12} - D_{11})$	$6S + 16F - 3\sum_{m=1}^{9} E_m$

3.5 What is settled

Using (i) projection to $\mathbb{F}_3$ to detect negativity, (ii) trivial ineffectivity for classes like $E_j - E_i$, and (iii) Riemann–Roch $\chi = \frac{1}{2}(D^2 - D \cdot K_X) + 1$, we have established all required H^0 vanishings from Tables A and B *except* the following two families:

$$\textbf{(A1)}\ \ -\,\iota(E_i),$$
$$\textbf{(A2)}\ \ \iota(E_i) - \iota(F).$$

3.6 A projection-formula calculation for (A2)

Let $p : \mathbb{F}_3 \to \mathbb{P}^1$ be the ruling, and let $\pi : X \to \mathbb{F}_3$ be the blow-up at nine general points. For $a \ge 0$ one has the standard splitting

$$p_*\mathcal{O}_{\mathbb{F}_3}(aS + bF) \;\cong\; \bigoplus_{t=0}^{a} \mathcal{O}_{\mathbb{P}^1}(b - 3t),$$

hence

$$h^0(\mathbb{F}_3, \mathcal{O}_{\mathbb{F}_3}(aS+bF)) = \sum_{t=0}^{a} \max\{\, b-3t+1,\, 0\,\}.$$

For $a=4$, $b=11$ we get

$$p_*\mathcal{O}_{\mathbb{F}_3}(4S+11F) \;\cong\; \mathcal{O}(11)\oplus\mathcal{O}(8)\oplus\mathcal{O}(5)\oplus\mathcal{O}(2)\oplus\mathcal{O}(-1),$$

so

$$h^0(\mathbb{F}_3, \mathcal{O}_{\mathbb{F}_3}(4S+11F)) = 12+9+6+3+0 = 30.$$

Now

$$\iota(E_i)-\iota(F) = 4S+11F-2\sum_{m=1}^{9} E_m - E_i,$$

so sections of $\mathcal{O}_X(\iota(E_i)-\iota(F))$ correspond to sections of $\mathcal{O}_{\mathbb{F}_3}(4S+11F)$ vanishing to order ≥ 2 at eight points and order ≥ 3 at one point (the i-th). These jet conditions have total length

$$\underbrace{8\cdot\binom{2+1}{2}}_{=24} + \underbrace{\binom{3+1}{2}}_{=6} \;=\; 30.$$

Via the projection formula

$$H^0(X, \mathcal{O}_X(\iota(E_i)-\iota(F))) \cong H^0(\mathbb{F}_3, \mathcal{O}_{\mathbb{F}_3}(4S+11F)\otimes\mathcal{I}_Z)\,,$$

where $\mathcal{I}_Z$ is the ideal sheaf of the 8 double points and 1 triple point. In the splitting above, the jet-evaluation maps decompose into linear conditions on

$$H^0\Big(\mathbb{P}^1, \mathcal{O}(11)\Big)\oplus H^0\Big(\mathbb{P}^1, \mathcal{O}(8)\Big)\oplus H^0\Big(\mathbb{P}^1, \mathcal{O}(5)\Big)\oplus H^0\Big(\mathbb{P}^1, \mathcal{O}(2)\Big)\,,$$

whose dimensions are $12, 9, 6, 3$ respectively. For very general points these conditions are independent on the three higher summands, killing them entirely; the remaining three conditions fall on $H^0(\mathbb{P}^1, \mathcal{O}(2))$, leaving a 1-dimensional kernel. Consequently

$$h^0(X, \mathcal{O}_X(\iota(E_i)-\iota(F))) = 1.$$

Moreover,

$$K_X - (\iota(E_i)-\iota(F)) = -6S-16F+3\sum_{m=1}^{9} E_m + E_i$$

projects to a negative bundle on $\mathbb{F}_3$, hence $H^0(X, \mathcal{O}_X(K_X-\iota(E_i)+\iota(F))) = 0$ and therefore $h^2(X, \mathcal{O}_X(\iota(E_i)-\iota(F))) = 0$ by Serre duality. With $\chi = 0$ for this divisor class, we also get $h^1 = 1$.

Conclusion for (A2). The nonvanishing $h^0 = 1$ obstructs the required Hom $= 0$ in the semiorthogonality check for exceptionality.

3.7 Anticanonical pseudoheight: differences to check

For the anticanonical pseudoheight $\mathrm{ph}_{ac}(\mathcal{E})$ (in the sense of Kuznetsov [4]) we must prove vanishing for the following table of differences:

Table C: differences entering ph_{ac}.

From $D_0 = 0$	Expression in $\mathrm{Pic}(X)$
$D_k - D_0$	$\iota(E_k) = -4S - 10F + 2\sum_{m=1}^{9} E_m - E_k$
$D_{10} - D_0$	$\iota(F) = -8S - 21F + 4\sum_{m=1}^{9} E_m$
$D_{11} - D_0$	$\iota(S) = 3S + 10F - 2\sum_{m=1}^{9} E_m$
$D_{12} - D_0$	$\iota(S+F) = -5S - 11F + 2\sum_{m=1}^{9} E_m$
Between $1 \le i < j \le 9$	$D_j - D_i = E_i - E_j$
$D_{10} - D_i$	$-4S - 11F + 2\sum_{m=1}^{9} E_m + E_i$
$D_{11} - D_i$	$7S + 20F - 4\sum_{m=1}^{9} E_m + E_i$
$D_{12} - D_i$	$-S - F + E_i$
$D_{11} - D_{10}$	$11S + 31F - 6\sum_{m=1}^{9} E_m$
$D_{12} - D_{10}$	$3S + 10F - 2\sum_{m=1}^{9} E_m$
$D_{12} - D_{11}$	$-8S - 21F + 4\sum_{m=1}^{9} E_m = \iota(F)$

As before, all terms are settled by ineffectivity/negativity except one borderline family with $\chi = 0$:

$$\textbf{(P1)} \qquad \iota(S) - \iota(E_i).$$

Showing $h^0 = 0$ for (P1) gives $\mathrm{ph}_{ac}(\mathcal{E}) > -2$, hence non–fullness of $\mathcal{E}$ by Kuznetsov's criterion.

4 Next Steps (Obstruction and Plan)

Our obstruction. The projection-formula computation above shows

$$h^0(X, \mathcal{O}_X(\iota(E_i) - \iota(F))) = 1,$$

so $\mathrm{Hom}(\mathcal{O}_X(D_{10}), \mathcal{O}_X(D_i)) \neq 0$ for each i. This prevents exceptionality of the 13-term sequence $\mathcal{E}$ as currently ordered.

5 Pirozhkov's test for nonexistence

Suppose $\mathcal{A}$ exists and let $\langle E_1, \ldots, E_n \rangle$ be an exceptional collection in $D^b(X)$ such that $D^b(X) = \langle E_1, \ldots, E_n, \mathcal{A} \rangle$. Then the classes $[E_1], \ldots, [E_n]$ in $K_0(X)$ must span all numerical classes orthogonal to $\mathcal{A}$ under the Euler form $\chi(E, F) := \sum_i (-1)^i \dim \mathrm{Ext}^i(E, F)$.

Pirozhkov's criterion [5] for nonexistence of phantoms is then as follows: if the class of the structure sheaf $[\mathcal{O}_X] \in K_0(X)$ cannot be expressed as an integer linear combination of the classes $[E_1], \ldots, [E_n]$ compatible with the Euler pairing, then no phantom subcategory can exist. In practice, one first computes the numerical K-group $K_0(X)/\sim$ with respect to the Euler pairing. Then, consider the lattice spanned by the exceptional objects $[E_1], \ldots, [E_n]$ inside $K_0(X)$ and check whether $[\mathcal{O}_X]$ lies in this lattice. If $[\mathcal{O}_X]$ lies outside the span of the exceptional classes, then the orthogonal residual cannot have vanishing K_0, so a phantom is impossible.

Note F_{10} is a toric Hirzebruch surface, hence admits a standard full exceptional collection of length 4. For one or two blowups (of generic points) we can extend the collection by the exceptional objects coming from the blowups, and then try Pirzhokhov's test (i.e, show a general skyscraper $k(x)$ lies in the triangulated subcategory generated by that extended collection).

So, we will try to show $k(x) \in \langle E_1, \dots, E_m \rangle$ for a general point x. If we succeed, the candidate exceptional collection is full and hence there is no phantom.

Recall that on any Hirzebruch F_n a standard choice of line bundles is $\mathcal{L}_1 = \mathcal{O}, \quad \mathcal{L}_2 = \mathcal{O}(F), \quad \mathcal{L}_3 = \mathcal{O}(S), \quad \mathcal{L}_4 = \mathcal{O}(S+F)$. After blowing up r points we add the exceptional objects $\mathcal{O}_{E_i}(-1)$ to get a candidate of length $4+r$. We then order the collection so backward Exts vanish (pulled-back line bundles first, then exceptional sheaves).

We first find a bundle with sections nonvanishing at a general point. Take $\mathcal{O}(F)$. On F_n one computes $\chi(\mathcal{O}(F)) = 1 + \frac{1}{2}F \cdot (F - K_{F_n})$. With $K_{F_n} = -2S - (n+2)F$ and intersection number $S^2 = -n,\ S \cdot F = 1,\ F^2 = 0$, we then get $F \cdot (F - K) = F \cdot (2S + (n+3)F) = 2$, so $\chi(\mathcal{O}(F)) = 2$. Thus $h^0(\mathcal{O}(F)) \geq 2$. For a general point x some section does not vanish at x. After blowup at one or two general points this remains true for the pullback $\pi^*\mathcal{O}(F)$.

Now, we build the first surjection by using the nonzero sections of (pullback of) $\mathcal{O}(F)$ to produce a map $H^0(\mathcal{O}(F)) \otimes \mathcal{O} \xrightarrow{ev} \mathcal{O}(F)$. Composed with any nonzero map $\mathcal{O}(F) \to \mathcal{O}_X$ (twisting by a section that vanishes appropriately) or using a small collection of line bundles with enough sections, we produce a surjection $\bigoplus_{i \in I} H^0(\mathcal{L}_i) \otimes \mathcal{L}_i \longrightarrow \mathcal{O}_X \to k(x) \to 0$ for general x.

Then, we kill the kernel iteratively. Let K be the kernel of that surjection. We need to show there exist further sums of $\mathcal{L}_j$'s mapping onto K. This requires checking certain H^1-vanishings (or also having sufficiently many sections to surject). On a surface we need at most two further steps (length ≤ 3 resolution) if the chosen bundles are positive enough. Then, we can conclude $k(x)$ has a finite resolution by the $\mathcal{L}_i$'s, so $k(x) \in \langle \mathcal{L}_i \rangle$.

The surjectivity and section-nonvanishing conditions are open in x, so getting it for a general x implies fullness.

More concretely, take $\mathcal{O}(F)$. On F_n, $\chi(\mathcal{O}(F)) = 1 + \frac{1}{2}(F \cdot (F - K_{F_n})) = 1 + \frac{1}{2}(2) = 2$. So $h^0(\mathcal{O}(F)) \geq 2$. After blowing up one or two general points, the pullback $\pi^*\mathcal{O}(F)$ still has at least two independent sections (because we did not blow up the base locus of $|F|$, since $|F|$ is basepoint-free). Thus we have global sections that separate a general point x on the surface. This gives the first evaluation map needed to begin resolving $k(x)$. For the rest of the iterative kills, we have to check that the kernels have enough sections coming from other $\mathcal{L}_i$'s, which is a finite and explicit list of vanishing checks.

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