The Word Problem in the Mapping Class Group of the Five-Times Punctured Sphere

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Grant Koh and Devayani Pradhan
Mentor: Alex Wright

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Abstract

We study the word problem in the mapping class group of the five-times punctured sphere. Using a finite presentation of this mapping class group with half Dehn twists and dihedral generators, we investigate the reducibility of words to geodesic representatives without increasing word length. Our approach combines combinatorial group theory with small cancellation methods, utilizing Van Kampen diagrams and curvature arguments to analyze reductions. We aim to prove that every word equivalent to the identity can be reduced with a sequence of non-length-increasing steps. This is

extended to showing that a word can be reduced to any equivalent geodesic without increasing word length.

1 Introduction

1.1 The Mapping Class Group

The mapping class group of a topological space is the group of symmetries. To formulate this more precisely, we have the following definitions.

Definition 1.1. Let X be a topological space and $f, g : X \to X$ be homeomorphisms. We say that f and g are isotopic if there exists a continuous mapping $F : X \times I \to X$ where $I = [0, 1]$ such that $F(-, 0) = f$, $F(-, 1) = g$ and $F(-, t)$ is a homeomorphism for all $t \in I$.

Definition 1.2. Let X be a topological space. The mapping class group of X, denoted by $\mathrm{MCG}(X)$ is the group of homeomorphisms of X, up to isotopy. The group operation is given by composition.

Definition 1.3. A nontrivial closed curve in a topological space, X, is an injective map $\alpha : I \to X$ such that $\alpha(0) = \alpha(1)$. We often refer to the image of this map as the curve α.

Now, suppose we are given a closed curve, α, on a 2-dimensional surface, S. Let $A = S^1 \times [0, 1]$ denote the annulus. We can find a neighborhood, N, of S, and a homeomorphism $\phi : A \to N$ such that N deformation retracts to α. Formally, N is a tubular neighborhood of α. Note then that we can define a map $\rho : A \to A$ such that $(r, \theta) \mapsto (r, \theta + 2\pi r)$. Then, this map can be extended to a map $\tau_\alpha : S \to S$ such that $\tau = \phi \circ \rho \circ \phi^{-1}$ on N and is the identity outside of N. Note that τ_α does not depend on our choice of N or ϕ; that is, different choices yield isotopic maps. Additionally, τ_α is a homeomorphism so $[\tau_\alpha] \in \mathrm{MCG}(S)$. Moreover, if $\alpha \sim \beta$, then $\tau_\alpha \sim \tau_\beta$.

Definition 1.4. Let S be a surface. Let α be a closed curve in S and τ_α be defined as above. We call τ_α a Dehn twist.

We have the following well-known result [FHnd].

Proposition 1.5. The mapping class group of a compact 2-dimensional surface is generated by Dehn twists.

In this paper, we consider the mapping class group of a non-compact surface, so we will need some modifications to these results. First, let S be some surface with n punctures. More precisely, we can find a surface, S', and $p_1, \cdots, p_n$ distinct points on S' such that $S \cong S' \setminus \{p_1, \cdots, p_n\}$. These points are called punctures. Take two such points, p, q, and take a curve $\beta : [0, 1] \to S'$ such that $\beta(0) = p$ and $\beta(1) = q$. Let α be the closed curve in S' such that $\alpha(t) = \beta(2t)$ for $t \leq \frac{1}{2}$ and $\alpha(t) = \beta(2 - 2t)$ for $t \geq \frac{1}{2}$. Again, we take a neighborhood N of α as specified above. We add the requirement that N does not contain any other punctures. However, we modify ρ such that $\rho' : (r, \theta) \mapsto (r, \theta + \pi(1 - r))$. We now have $\tau'_\alpha : S \to S$ such that $\tau' = \phi \circ \rho' \circ \phi^{-1}$ on N and is the identity outside of N. Note that $\tau'_\alpha(p) = q$ and $\tau'_\alpha(q) = p$. Let $t_{p,q} : S \to S$ be the restriction of τ'_α to S. Note that $t_{p,q}$ is a nontrivial homeomorphism of S.

Definition 1.6. Let S be a surface with punctures. Let p and q be two such punctures and let $t_{p,q}$ be defined as above. We call $t_{p,q}$ a half Dehn twist.

Proposition 1.7. Let S' be a compact 2-dimensional surface. Let $S = S' \setminus \{p_1, \cdots, p_n\}$ where the p_i are distinct points in S'. Then, $\mathrm{MCG}(S)$ is generated by Dehn twists and half Dehn twists.

1.2 The Mapping Class Group of the Five-Times Punctured Sphere

Definition 1.8. Let p_1, p_2, p_3, p_4, p_5 be distinct points on S^2. Then, $S_{0,5} = S^2 \setminus \{p_1, \cdots, p_5\}$ is called the five-times punctured sphere.

Let G be the mapping class group of the five-times punctured sphere. Let $p_1, \cdots, p_5$ denote the five punctured points and e_{ij} denote a line segment connecting point p_i and p_j. We consider the presentation of G with the following generators:

- The five half Dehn twists around the edges $e_{12}, e_{23}, e_{34}, e_{45}, e_{51}$. These are denoted t_1, t_2, t_3, t_4, t_5 respectively.
- The five orientation-reversing homeomorphisms that fix exactly one of the punctures. We call these *dihedral generators* which we denote in permutation notation as follows: $(12)(35), (23)(14), (34)(15), (45)(13), (15)(24)$.
  Note that these generate a subgroup isomorphic to the dihedral group of order ten, denoted D_5.

Let e denote the identity map. We have the following relations:

- For i, j such that $i - j = 1 \mod 5$, let $k = \frac{i+j+5}{2}$, and let n, m denote the other two indices. Then, $t_i t_j t_i = t_j t_i t_j = t_k (ij)(nm)$
- For i, j such that $i - j = 2 \mod 5$, we have $[t_i, t_j] = e$
- For some dihedral generator, σ, we have $\sigma t_i \sigma = t_{\sigma(i)}$
- We have the obvious relations between dihedral generators inherited from D_5.

Recall that G is generated by half Dehn twists. Since the elements listed above generate all possible half Dehn twists, we can be certain that these elements do generate the entire mapping class group.

1.3 The Word Problem

The word problem for a finitely presented group $G = \langle S|R\rangle$ asks if there is an algorithm to determine which words, $s \in S^*$, are equal to the identity in G. Note here that

$$S^* := \bigcup_{n=0}^{\infty} \prod_{i=1}^{n} S$$

We write $w \sim w'$ or $w =_G w'$ if the two words w and w' are equivalent in G.

Definition 1.9. We refer to a word $s \in S^*$ as *geodesic* in G if for all words $s' \in S^*$ with $s \sim s'$, the length of s is less than or equal to the length of s'.

Remark. The word problem may be solved via a brute force search if every word may be reduced to a geodesic representative by a sequence of applications of relators and free reductions where the word length is not increased in each step.

Definition 1.10. For $g \in G$, let $\mathrm{Min}(g)$ be the set of geodesic representatives of g in S^*.

1.4 Main Results

It is known that the word problem is solvable in mapping class groups [FHnd]. However, we attempt to prove a stronger formulation of the word problem. We claim that a word can be reduced to any representative in $\mathrm{Min}(G)$ without increasing word length. Note here that if a dihedral element is inserted, we can apply length-preserving relations to move the element to the end of the word. Then, these dihedral elements can be combined into a single dihedral element. Inserting dihedral elements does not increase the complexity of a brute force search. Thus, in our calculation of word length, we do not count dihedral generators.

Theorem 1.11. Let G be the mapping class group of the five-times punctured sphere. Let $G = \langle\, S|R \,\rangle$ be the presentation described above. If w is a word in G that is equivalent to the identity, then w can be reduced to the identity without increasing word length.

Theorem 1.12. Let G be the mapping class group of the five-times punctured sphere. Let $G = \langle\, S|R \,\rangle$ be the presentation described above. If w is a word in G and $w' \in \mathrm{Min}(G)$ is an equivalent word, then w can be reduced to w' without increasing word length.

For small n, the mapping class group of an n-times punctured sphere is well-understood. The five-times punctured sphere is the smallest n for which the mapping class group is not fully understood. We hope that our results will provide a better understanding of the mapping class groups of punctured spheres and other related surfaces.

2 Van Kampen Diagrams and Curvature

2.1 Introduction

The proofs rely on methods from small cancellation theory, especially on the use of Van Kampen diagrams.

Definition 2.1. Let $G = \langle S|R \rangle$ be a finite presentation for a finitely presented group. A Van Kampen diagram for this presentation is a finite, simply connected, 2-dimensional cell complex that satisfies the following.

- Each 1-cell is a directed edge labeled by $s \in S$
- Given a 2-cell, D, starting at any vertex of ∂D, and reading the edges on ∂D in either direction yields a freely-reduced word in R.

Here, 1-cells are referred to as edges and 2-cells as tiles.

Van Kampen diagrams provide a visualization of the relators applied to reduce a word to the identity.

Proposition 2.2. A word $w \in G = \langle S|R \rangle$ is equivalent to the identity if and only if it is the boundary of a Van Kampen diagram.

Proof Sketch. First, suppose w corresponds to the boundary of a Van Kampen diagram, D. Let v be the vertex where we begin reading w, and suppose we read clockwise. We induct on the area of D where each 2-cell is given area 1. Let C be a 2-cell such that $E = \partial C \cap \partial D$ is non-empty and connected. We also require that $E \setminus v$ is connected. Let $r \in R$ be a word corresponding to reading ∂C clockwise. Let r_1 be the word corresponding to $\partial C \cap \partial D$ and r_2 the word corresponding to $\partial C \setminus \partial D$. Note that $r_1 r_2$ is conjugate to r and $r_1 r_2 \in R$. Consider $D' = D \setminus (\mathrm{Int}(C) \cup (\partial C \cap \partial D)$. Visually, this is the cell-complex obtained from D after "deleting" C. Note that $v \in D'$. Let w' be the word corresponding to reading clockwise around $\partial D'$ starting at v. We see that w' is obtained from w by replacing the subword r_1 with r_2^{-1}. Thus, $w \sim w'$. Now, since D' has less area than D, by the inductive hypothesis, w' is equivalent to the identity. Thus, $w \sim w' \sim e$.
Conversely, suppose w is equivalent to the identity. There is a finite sequence of steps that reduces w to the identity. Let $m_1, \cdots, m_n$ denote a sequence with the minimal number of moves. We will induct on length. Let $w = s_1 \cdots s_n$ where all $s_i \in S$.
Suppose m_1 is an insertion: $s_1 \cdots s_i s_{i+1} \cdots s_n \to s_1 \cdots s_i t t^{-1} s_{i+1} \cdots s_n$ where $t \in S$. Let w' denote the latter word, and note that we can construct a Van Kampen diagram, D', for w'. Let v be the vertex shared by the edges corresponding to s_i and s_{i+1}. Then, let D be the diagram obtained from D' by adding an edge at v labeled t and pointing outwards. Then, D is a Van Kampen diagram for w.
Suppose m_1 corresponds to replacing $s_i \cdots s_j$ by $t_1 \cdots t_k$ where each $t_m \in S$ and $t_1 \cdots t_k s_j^{-1} \cdots s_i^{-1} \in R$. Let $w' = t_1 \cdots t_k s_j^{-1} \cdots s_i^{-1}$. Note that w' has a Van Kampen diagram, D'. Let C be the 2-cell with boundary corresponding to $s_i \cdots s_j t_k^{-1} \cdots t_1^{-1}$. Let D be the diagram obtained by gluing C to D' along $s_i \cdots s_j$. Then, D is a Van Kampen diagram for w.
Finally, suppose m_1 corresponds to a free-reduction. Without loss of generality, we assume $s_{i+1} = s_i^{-1}$, and this step takes $s_1 \cdots s_i s_{i+1} s_n \to s_1 \cdots s_{i-1} s_{i+2} \cdots s_n$. Let w' be the latter word and let D' be a Van Kampen diagram for w'. Let D be the diagram obtained from D' by gluing the edges corresponding to s_i and s_{i+1}. Then, D is a Van Kampen diagram for w. $\square$

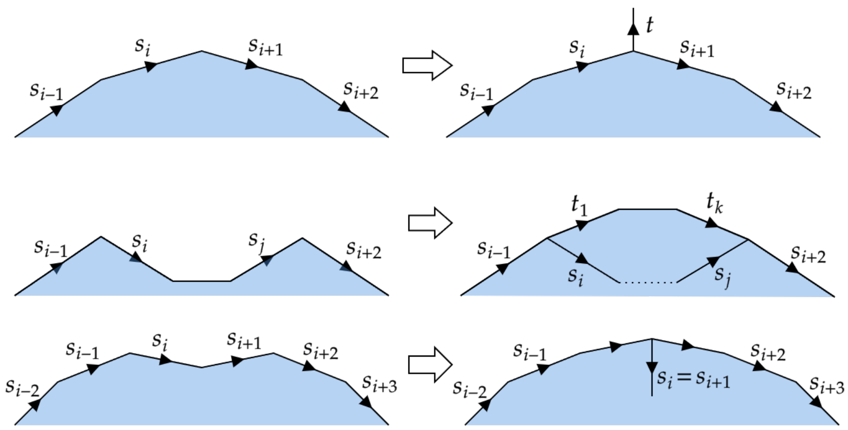

Figure 1: Diagram Modifications Corresponding to Moves

Theorem 2.3 (Combinatorial Gauss-Bonnet for Polyhedra)**.** Let X be a closed surface equipped with a simplicial complex structure. For a vertex, v, let α_v denote the sum of angles at that vertex. Let $\theta_v = 2\pi - \alpha_v$ Then,

$$\sum_v \theta_v = 2\pi\chi(X)$$

where the sum is taken over all vertices of X.

Proof. Let V denote the number of vertices, E the number of edges, and F the number of faces of X. Let X_v denote the set of vertices of X, and X_f the set of faces.
First, we note that $\sum_v \alpha_v$ is the sum of all the interior angles of every face. Let β_f denote the sum of interior angles of face f, and let E_f denote the number of edges of face f. Note that $\sum_{f\in X_f} E_f = 2E$. We have that

$$\sum_{v\in X_v} \alpha_v = \sum_{f\in X_f} \beta_f = \sum_{f\in X_f} (\pi E_f - 2\pi) = \pi \sum_{f\in X_f} E_f - 2\pi F = 2\pi(E - F)$$

Then, we have

$$\sum_{v\in X_v} (2\pi - \alpha_v) = 2\pi V - \sum_{v\in X_v} \alpha_v = 2\pi(V - E + F) = 2\pi\chi(X)$$

□

Theorem 2.4 (Combinatorial Gauss-Bonnet for Surfaces with Boundary)**.** Let X be a closed surface with boundary equipped with a simplicial complex structure. For a vertex, v, let α_v denote the sum of angles at that vertex. Let $\theta_v = 2\pi - \alpha_v$ for $v \in \text{Int}(X)$ and $\theta_v = \pi - \alpha_v$ for $v \in \partial X$. Then,

$$\sum_{v\in X} \theta_v = 2\pi\chi(X)$$

Proof. This proof follows the same outline as above. Let X_I denote the set of interior vertices and X_B denote the set of vertices on the boundary. Note that $\sum_{f\in X_f} E_f = 2E - E_B$ where E_B denotes the number of edges on the boundary. Let V_B denote the number of vertices on the boundary and note that $V_B = E_B$. We have that

$$\sum_{v\in X_v} \alpha_v = \sum_{f\in X_f} \beta_f = \sum_{f\in X_f} (\pi E_f - 2\pi) = \pi \sum_{f\in X_f} E_f - 2\pi F = 2\pi(E-F) - \pi E_B$$

Then, we have

$$\sum_{v\in X_I} (2\pi - \alpha_v) + \sum_{v\in X_B} (\pi - \alpha_v) = 2\pi V - \pi V_B - 2\pi(E-F) + \pi E_B = 2\pi(V-E+F) = 2\pi\chi(X)$$

□

We can apply this result to Van Kampen diagrams. Recall that a Van Kampen diagram for a word, w, corresponds to a sequence of steps that reduces w to the identity. Suppose we have a sequence of steps that do not increase word length. If w is freely-reduced, the first step must replace a subword, r_1, by r_2 where $r_1 r_2^{-1} \in R$ and $\text{len}(r_1) \geq \text{len}(r_2)$. This will correspond to deleting a tile that has more edges on the boundary than on the interior. Note that if a diagram is negatively curved, then

$$\sum_{v\in \text{Int}(X)} (2\pi - \alpha_v) < 0$$

Thus, we must have

$$\sum_{v\in \partial X} (2\pi - \alpha_v) > 2\pi$$

2.2 Van Kampen Diagrams for MCG($S_{0,5}$)

Let $G = \langle S|R\rangle$ be the mapping class group of the five-times punctured sphere. We use a slight modification of the presentation given in 1.2. We do not consider dihedral elements of order five as generators. Using the same notation, we have the following relations. For each relation, r, we include conjugates of r by subwords of r.

- $r = t_i t_j t_i t_k^{-1}(ij)(nm)$.
- $r = t_i t_k t_i^{-1} t_k^{-1}$.
- $\sigma t_i \sigma t_{\sigma(i)}$.
- $r = \sigma_1\sigma_2\sigma_3\sigma_4$ for σ_i of order two in D_5.

We give a list of the possible tiles we will allow in a Van Kampen diagram for a word in G. The tiles are classified into four types. We will generally only use dihedral elements of order two (all other dihedral elements may be written using two elements of order two).

- Pentagon pieces: These correspond to relations of the form $t_i t_j t_i t_k^{-1}(ij)(nm) = e$. We will allow pieces to have the central element, t_k, and the dihedral element in any location.
- Commuting square: These correspond to the commuting relations between non-dihedral elements, such as $[t_i, t_k] = e$
- Partial commuting squares: These correspond to relations between a dihedral and non-dihedral element, such as $\sigma t_i \sigma t_{\sigma(i)}^{-1} = e$
- Fully dihedral squares: These correspond to relations between dihedral elements.

Some possible tiles are shown below, along with the angle measures that we assign to interior angles.

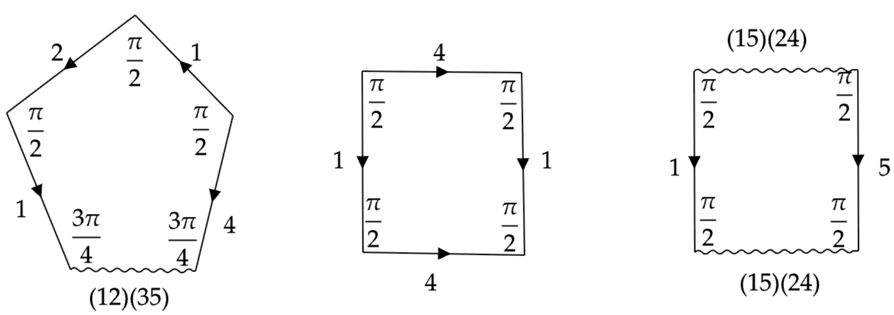

Figure 2: Some Tiles with Angles

In our diagrams, edges corresponding to dihedral elements will be a squiggly line to emphasize that these elements do not contribute to the length.
In certain situations, these angle measures may be modified. Specifically, when we have a pentagon tile with its dihedral edges and at least one adjacent edge both on the boundary, we assign angle measures as follows:

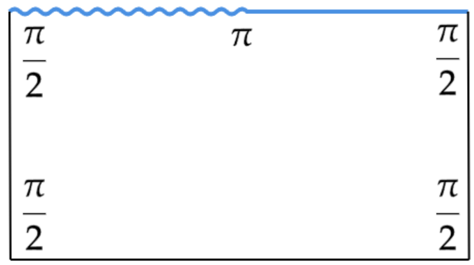

Figure 3: Modified Angle Measures

Note that this does not affect $\sum_{v\in \mathrm{Int}(X)} \alpha_v$ or $\sum_{v\in\partial(X)} \alpha_v$. This modification will be useful in the proof of Proposition 2.14.

When dealing with Van Kampen diagrams, it is useful to have a notion of area. Given a Van Kampen diagram, we declare its area to be (a, b) where a is the number of pentagons in it and b is the number of commuting squares in it. We order area lexicographically. Notice that dihedral commutations and purely dihedral squares do not affect area at all since a Van Kampen diagram composed of dihedral commutation and purely dihedral squares can always be reduced to the identity without increasing word length.
Thus, such a diagram corresponds to a word that can be reduced to the identity without increasing word length. From here onward, we use this notion of area when discussing minimal area diagrams.

2.3 The Dual Diagram

Given a Van Kampen diagram for a word in $S_{0,5}$, we construct the dual diagram as follows:

1. In a pentagon, draw lines connecting pairs of opposite non-dihedral edges (i.e. there is another non-dihedral edge between them)

2. In a commuting square, draw lines connecting pairs of opposite edges

3. In a dihedral commuting square, draw a line connecting the two non-dihedral edges.

Definition 2.5. We call lines in the dual diagram "dual lines". The intersection points of the dual lines are called "dual vertices". If v is a dual vertex, we let T_v denote the corresponding tile in the Van Kampen diagram.

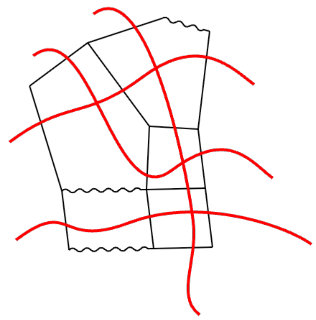

Figure 4: A Simple Dual Diagram

Definition 2.6. A closed region, Q, in the dual diagram is a subdiagram satisfying the following:

- Q is bounded by dual lines
- $\partial Q \cap \partial X = \emptyset$
- Q has no dual vertices in its interior

For each dual line, ℓ, bounding Q, we call $\ell \cap \partial Q$ a dual edge of Q. We say Q is a closed dual n-gon if Q has n edges.

Definition 2.7. An open region, U, in the dual diagram is a subdiagram satisfying the following:

- U is bounded by dual lines or the edge of the Van Kampen diagram
- $\partial U \cap \partial X \neq \emptyset$
- U has no dual vertices in its interior

For each dual line, ℓ, bounding U, we call $\ell \cap \partial U$ a dual edge of U. We say U is an open dual n-gon if U has $n - 1$ dual edges.

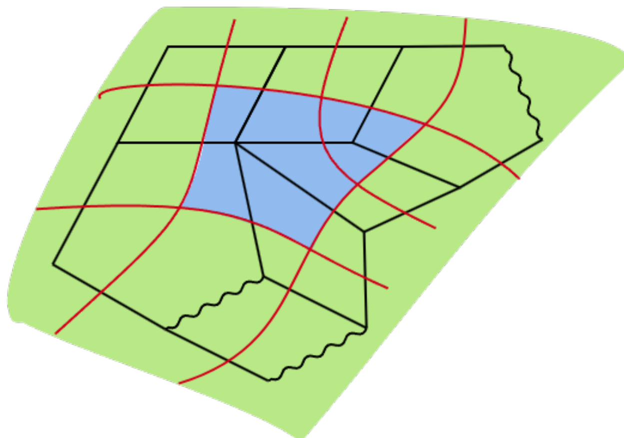

Figure 5: Closed Regions Are Denoted in Blue; Open Regions Are Denoted in Green

Proposition 2.8 (Pushing Tiles to the Boundary)**.** Let D be a Van Kampen diagram for a word w that has an open 3-gon, A. Then, there is an equal-area diagram D' for a word $w' =_G w$ where the following holds:

- D' and D agree except on tiles that intersect A
- Let A' denote the corresponding open dual 3-gon in D'. Let v' be the sole vertex of A'. Then $T_{v'}$ has at least two edges on the boundary.

Proof. We make the moves depicted in Figure 6, highlighting the boundary in blue and the dual diagram in red. Here, a move refers to a re-tiling, turning D into D', where D and D' have the same area. We leave the bottom of the tile pushed to the boundary ambiguous to emphasize that it may be a pentagon or a square.

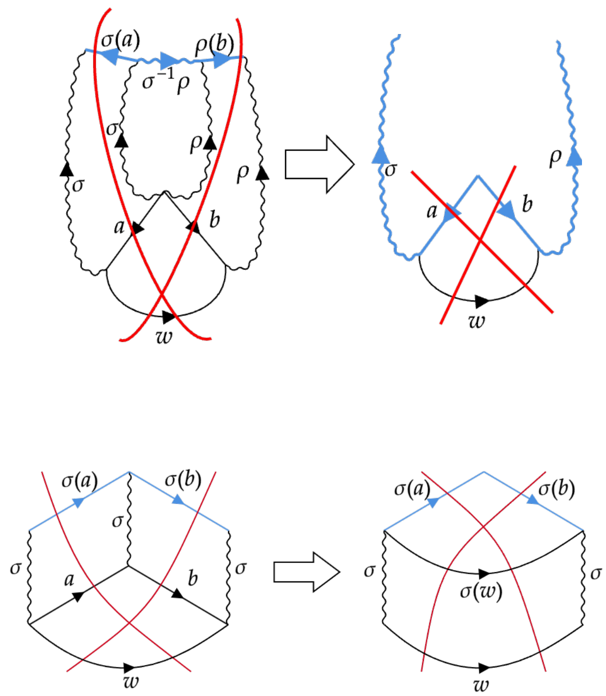

Figure 6: Pushing a Tile to the Boundary

□

Lemma 2.9. Let Q be a dual n-gon. Let V be the set of interior vertices in the Van Kampen diagram dual to Q. Then,

$$\sum_{v \in V} \theta_v = 2\pi - \frac{n\pi}{2}$$

Definition 2.10. Consider a Van Kampen diagram, X, with dual diagram, D. For each dual n-gon, Q, let $\Theta_Q = 2\pi - \frac{n\pi}{2}$. Let the *interior dual curvature* of X be $\mathcal{C}(X) := \sum \Theta_Q$ where the sum is taken over all closed n-gons of D.

Proposition 2.11. If we can only find two (or less) open 3-gons, then we must have a bigon (i.e. a closed 2-gon) in the diagram.

Proof Sketch. We induct on the number of vertices – the base case of a dual diagram with one vertex is trivial. Let D be a dual diagram with two or fewer open 3-gons. If a line $\ell \in D$ self-intersects, let R be the region bounded solely by ℓ. Some other line, ℓ' must enter and exit R, so it follows that ℓ and ℓ' form a bigon. Thus, we may assume that none of the lines in D self intersect. Consider the set L of dual lines that have nonempty intersection with ∂D, and pick a line $\ell \in L$ that does not form two dual 3-gons. If such a line does not exist, all dual lines on the boundary form open 3-gons, so there must be exactly two dual lines on the boundary (if there was only one, a line would intersect itself) and we must have a bigon. Since ℓ enters and exits the dual diagram without self-intersecting, ℓ divides the dual diagram into at least two subdiagrams. If any subdiagram D' contains neither of the open 3-gons from D, then it has exactly 2 open 3-gons (both formed by ℓ), so D' has a bigon by induction. Otherwise, we must have that ℓ splits D into exactly two subdiagrams, in which case it follows from the fact that $\ell \cap \partial D \neq \emptyset$ that ℓ forms one of the dual 3-gons in D, A. In this case, the subdiagram containing A will have exactly two dual 3-gons. Thus it will contain a bigon by induction.

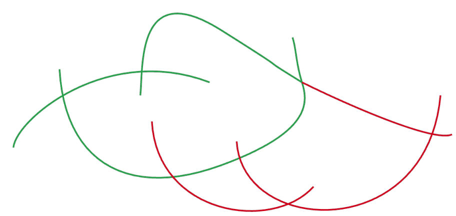

Figure 7: Finding a Subdiagram With Two Open 3-gons

□

2.4 Proof Sketch

We state and prove a series of statements that will allow us to prove our main result.

Proposition 2.12. Suppose for all words, $w \in G$, equivalent to the identity, we can find a minimal area Van Kampen diagram for w such that there are three open dual 3-gons. Then, any such word can be reduced to the identity without increasing word length.

Proof Sketch. Fix a word $w \in G$ equivalent to the identity. Suppose such a Van Kampen diagram, call it D, does exist for w. Let $w = a_1 \cdots a_n$. We induct on word length and the area of D.

First, we address the base case. If D has area 0, then D is composed of commuting-dihedral squares and purely dihedral squares. Any tile with a non-dihedral edge on the boundary can be deleted without increasing word length. A word composed only of dihedral elements can be reduced to the identity without increasing word length.

By Proposition 2.8, we can find an equivalent diagram D' which has a tile, T, with two edges on the boundary. Since there are three 3-gons, there are three possible equivalent diagrams with tiles, T_1, T_2, T_3, respectively such that the $\partial T_i \cap \partial X$ are distinct. Thus, we can guarantee that ∂T does not contain the edges corresponding to $a_n a_1$. Additionally, note that T is either a pentagon tile or a commuting square. Then, deleting T corresponds to applying a relation which does not increase word length. Note that the resulting diagram, D'', has less area. Thus, by induction, D'' corresponds to a word that can be reduced to

the identity without increasing word length. This in turn implies that w can be reduced to the identity without increasing word length.
Note here that when edges corresponding to s and s^{-1} are adjacent along the boundary, a minimal area diagram may collapse these edges. We must address the issue of when this collapsing occurs at the ends of w, as this would not correspond to any non-length-increasing relation on w. Suppose we take the maximal i such that $a_1 \cdots a_i = a_n^{-1} \cdots a_{n-i+1}^{-1}$. Then, let $w' = a_{i+1} \cdots a_{n-i}$. Note that w' can be reduced to the identity without increasing word length by the inductive hypothesis. Since w and w' are conjugates, it follows that w can be reduced to the identity without increasing word length.

□

Proposition 2.13. Suppose for all words, $w \in G$, equivalent to the identity, we can find a minimal area Van Kampen diagram for w such that there are three open dual 3-gons. Then, for any two equivalent geodesics, w and v, we can find a sequence of non-length-increasing steps to transform w to v.

Proof Sketch. Fix a geodesic $w \in \mathrm{Min}(G)$ and suppose $w \sim v$ with $v \in \mathrm{Min}(G)$. Then, wv^{-1} is equivalent to the identity. For each open 3-gon, we have an equivalent diagram with a tile with two edges on the boundary. Let $w = a_1 \cdots a_n$ and let $v = b_1 \cdots b_m$. We can guarantee that at least one tile, T, does not intersect the edges corresponding to $a_n b_1^{-1}$ or $b_m^{-1} a_1$. Thus, deleting T corresponds to applying a non-length-increasing relation to either w or v^{-1}. The resulting diagram has strictly less area. Inductively, we obtain a sequence of steps reducing wv^{-1} to the identity. Let $w_0 = w$ and $v_0 = v$. Each step does one of the following:

- A non-length-increasing relation (or cancellation) is applied purely to w_i. More rigorously, $w_i v_j^{-1}$ is changed to $w_{i+1} v_j^{-1}$ where w_i and w_{i+1} are equivalent and differ only by one relation. Note that $\mathrm{len}(w_i) = \mathrm{len}(w_{i+1})$ so w_{i+1} is geodesic. This corresponds to deleting a tile bordering only w.

- A non-length-increasing relation (or cancellation) is applied purely to v_j. More rigorously, $w_i v_j^{-1}$ is changed to $w_i v_{j+1}^{-1}$ where v_j and v_{j+1} are equivalent and differ only by one relation. Note that $\mathrm{len}(v_j) = \mathrm{len}(v_{j+1})$ so v_{j+1} is geodesic. This corresponds to deleting a tile bordering only v.

- A cancellation is applied to the end of w and v^{-1}. Letting $w_i = c_1 \cdots c_p$ and $v_j = d_1 \cdots d_q$, we either have $c_1 = d_1$ or $c_p = d_q$. This corresponds to collapsing edges when constructing a minimal area diagram.

Now, we induct on the length of w_i. The first two types of steps correspond to applying non-length-increasing relations on w_i and v_j, so they preserve the length of w_i. Since these steps reduce wv^{-1} to the identity, we must apply the last type of step at least once.
Suppose we have reached $w_i v_j^{-1}$ and we must apply the last step. Letting $w_i = c_1 \cdots c_p$ and $v_j = d_1 \cdots d_q$, suppose we have $c_1 = d_1$. Then, it suffices to show that $w_i' = c_2 \cdots c_p$ can be transformed to $v_j' = d_2 \cdots d_q$ without increasing word length. Note that this is guaranteed

by the inductive hypothesis, since w'_i and v'_j are also geodesic.
To summarize, we have obtained a sequence of length preserving steps transforming w to w_i and v to v_j such that $w = sw'$ and $v = sv'$ where $s \in S$. Inductively, we have a sequence of non-length-increasing steps transforming w' to v'. Thus, we transform w to sw', then transform sw' to sv'. The steps transforming v to sv' can be applied in reverse to transform sv' to v. Thus, we have a sequence of length preserving steps transforming w to v. $\square$

Proposition 2.14. Suppose for all words, $w \in G$, equivalent to the identity, we can find a minimal area Van Kampen diagram for w such that there are three open dual 3-gons. Then, for any word w and any geodesic v equivalent to w, we can find a sequence of non-length-increasing steps to transform w to v.

Proof Sketch. We follow the same outline as above. Now, we have the following steps

- A non-length-increasing relation (or cancellation) is applied purely to w_i. More rigorously, $w_i v_j^{-1}$ is changed to $w_{i+1} v_j^{-1}$ where w_i and w_{i+1} are equivalent and differ only by one relation. Note that $\text{len}(w_i) \geq \text{len}(w_{i+1})$. This corresponds to deleting a tile bordering only w.

- A non-length-increasing relation (or cancellation) is applied purely to v_j. More rigorously, $w_i v_j^{-1}$ is changed to $w_i v_{j+1}^{-1}$ where v_j and v_{j+1} are equivalent and differ only by one relation. Note that $\text{len}(v_j) = \text{len}(v_{j+1})$ so v_{j+1} is geodesic. This corresponds to deleting a tile bordering only v.

- A cancellation is applied to the end of w and v^{-1}. Letting $w_i = c_1 \cdots c_p$ and $v_j = d_1 \cdots d_q$, we either have $c_1 = d_1$ or $c_p = d_q$. This corresponds to collapsing edges when constructing a minimal area diagram.

Now, we induct on the length of w. The first two types of steps correspond to applying non-length-increasing relations on w_i and v_j, so they either preserve or decrease the length of w_i. Since these steps reduce wv^{-1} to the identity, we must apply the last type of step at least once, as that is the only type of step that modifies the length of v^{-1}.
Suppose we have reached $w_i v_j^{-1}$ and we must apply the last step. Letting $w_i = c_1 \cdots c_p$ and $v_j = d_1 \cdots d_q$, suppose we have $c_1 = d_1$. Then, it suffices to show that $w'_i = c_2 \cdots c_p$ can be transformed to $v'_j = d_2 \cdots d_q$ without increasing word length. Note that this is guaranteed by the inductive hypothesis, since v'_j is also geodesic.
To summarize, we have obtained a sequence of non-length-increasing steps transforming w to w' and v to v' such that $w = sw'$ and $v = sv'$ where $s \in S$. Inductively, we have a sequence of non-length-increasing steps transforming w' to v'. Thus, we transform w to sw', then transform sw' to sv'. The steps transforming v to sv' can be applied in reverse to transform cv' to v since they do not modify length. Thus, we have a sequence of non-length-increasing steps transforming w to v. $\square$

Theorem 2.15. For all words, $w \in G$, equivalent to the identity, we can find a minimal area Van Kampen diagram, X, with three dual 3-gons.

The following sections are devoted to developing tools we hope will be useful in proving this statement.

3 Basic Reductions

The configurations of one pentagon with two squares, and of three pentagons at a single vertex, immediately reduce, as seen in Figure 8 and Figure 9. The reduction of other configurations of three pentagons at a vertex is similar and straightforward.

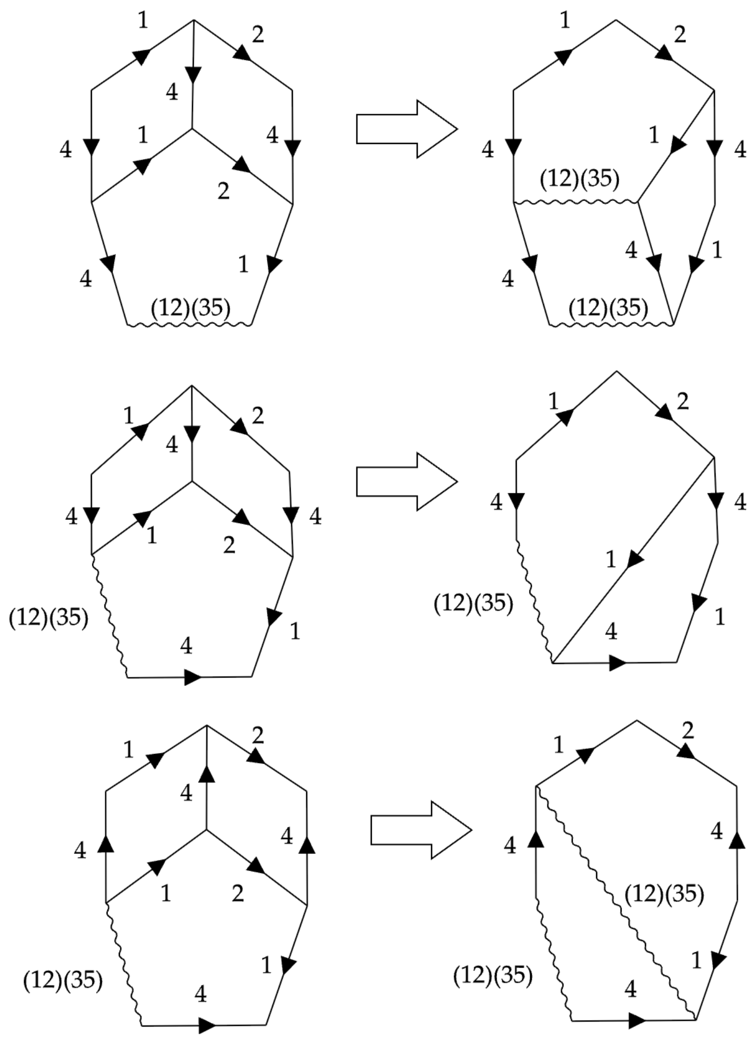

Figure 8: Reductions of One Pentagon and Two Squares Configurations

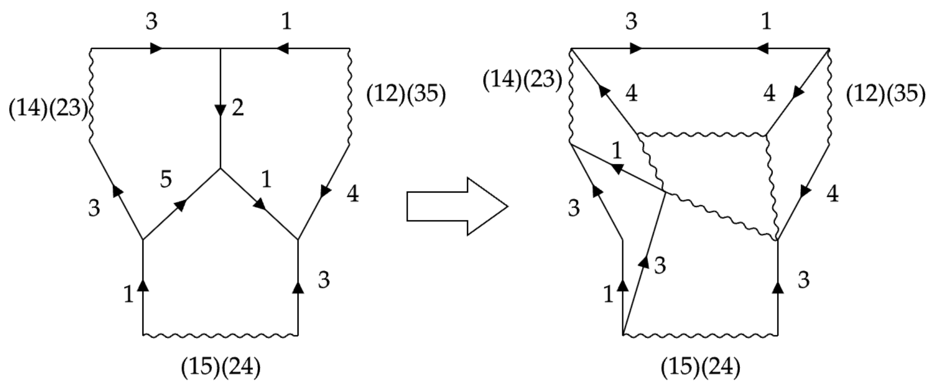

Figure 9: Reduction of a Three Pentagon Configuration

Here, we also briefly address the issue of closed 2-gons. These will appear as in Figure 10. We introduce a pseudo-tile and redraw the dual diagram as shown in the figure below, assigning angles as shown.

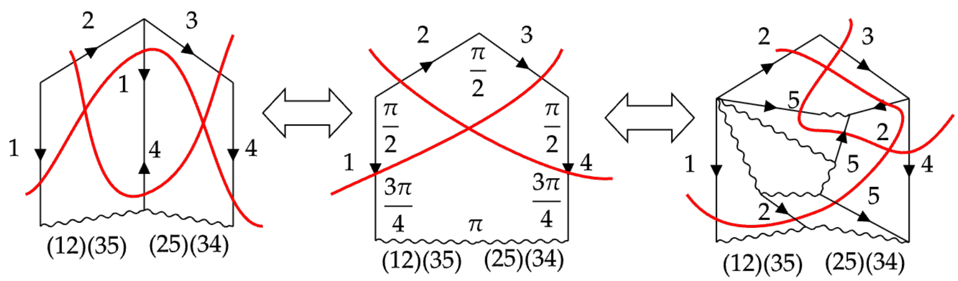

Figure 10: A Mini Bigon

Note that the pseudo-tile can be split into pentagon tiles in two different ways. One can verify that this allows us to treat this pseudo-tile as a pentagon tile in most basic reductions.

It is also easy to see that there is no positively curved configuration involving three squares, as such a configuration would necessitate a triple t_i, t_j, t_k of half Dehn twists that pairwise commute, and no such triple exists.

4 Elephants

There is one main way to create positive curvature around a vertex that does not admit simple reduction, which we will call an *elephant*, as seen in Figure 11.

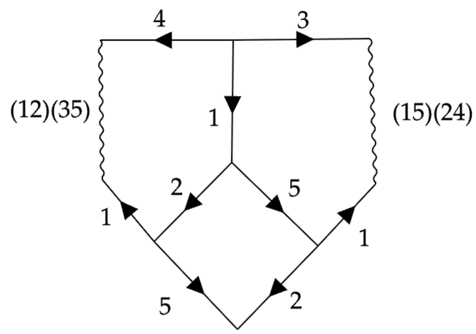

Figure 11: An Elephant

Clearly, one can construct more such diagrams by attaching and absorbing dihedral commutations to the elephant shown. However, commuting the central elements in either pentagon with some other non-dihedral element is not permissible. This would lead to an immediate area reduction as demonstrated in Figure 13.

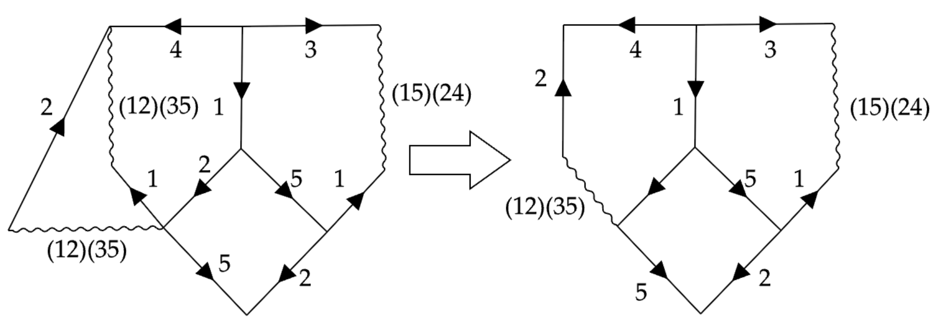

Figure 12: New Elephants From Old

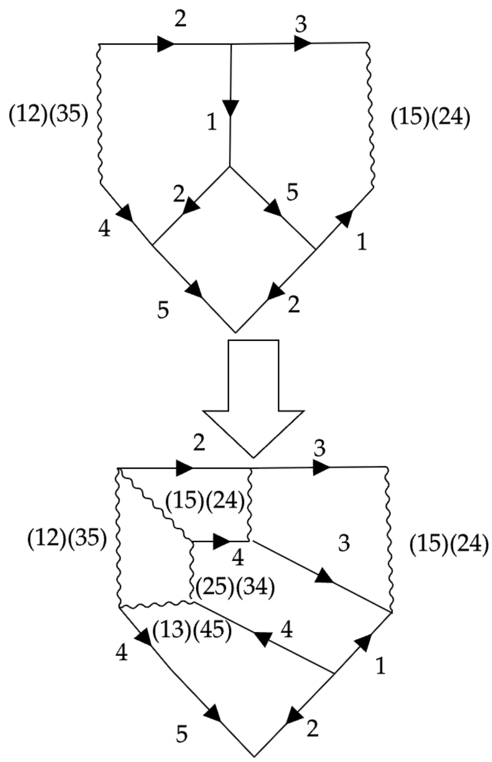

Figure 13: Commutation Destroys Elephants

Remark. In certain cases, it is necessary to use several dihedral commutation tiles to turn a three-tile region of positive curvature into an elephant, as seen in Figure 14.

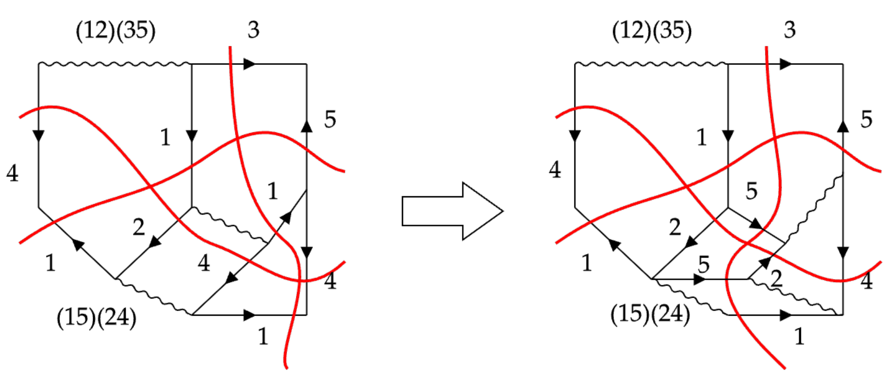

Figure 14: Re-Tiling Reveals Elephants

Lemma 4.1. Elephants correspond exactly to closed 3-gons in the dual diagram of a minimal area Van Kampen diagram.

Proof. From the previous section, we see that elephants are the only configuration that result in positive curvature and do not admit an area reduction. □

Although we can have positive curvature in a minimal area Van Kampen diagram, this local positive curvature is immediately canceled out by negative curvature that is "sufficiently" close. We will first show that this occurs in small regions near the positive curvature. To do this, we rigorously define these regions and call them *sectors*.

Suppose we have an elephant, composed of tiles, T_1, T_2, T_3. Let v_1, v_2, v_3 be the corresponding dual vertices. Note that these points are the vertices of a triangle in the dual diagram. Let ℓ_1, ℓ_2, ℓ_3 be the lines forming this triangle. Let V be the set of dual vertices adjacent to one of v_1, v_2, v_3 including the v_i. Let G be the induced subgraph in the dual graph with vertices in V and edges with both endpoints in V. Let $e_i = \ell_i \cap V$; that is, the part of ℓ_i contained in V.

Definition 4.2 (Sector)**.** Consider a maximal subgraph, X, of G, such that at least one of the e_i is on the boundary. Let D be the subdiagram consisting of tiles with their dual vertices in X. We call D a sector.

Definition 4.3 (Full Sector)**.** In a full sector, there is exactly one e_i on the boundary, and this e_i consists of at least three edges.

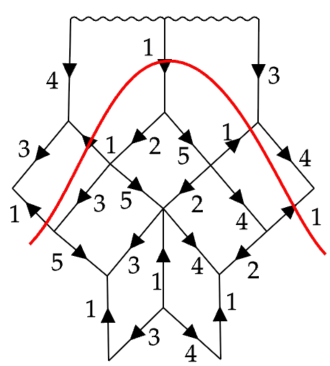

Figure 15: A Full Sector with the Bounding Dual Line

Proposition 4.4. In a minimal area Van Kampen diagram, all full sectors of all elephants are non-positively curved. More precisely, the corresponding dual subdiagram has dual curvature ≤ 0.

This proposition can be verified via a brute-force generation of all ways to surround an elephant. Note that in particular this implies that each triangle in the dual diagram is adjacent to two pentagons in the dual diagram. Figure 16 depicts an acceptable configuration of the dual diagram. The bottom sector of Figure 17 is positively curved, and thus such a dual diagram is not present in a minimal area diagram.

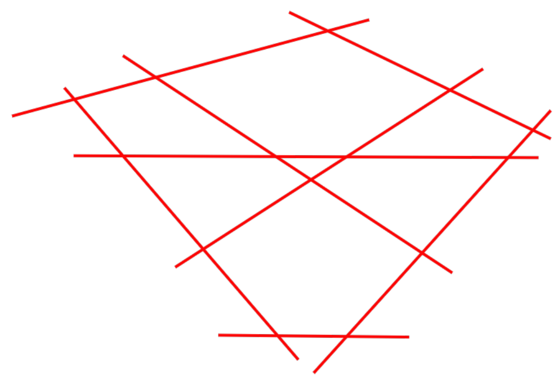

Figure 16: An Allowable Dual Diagram

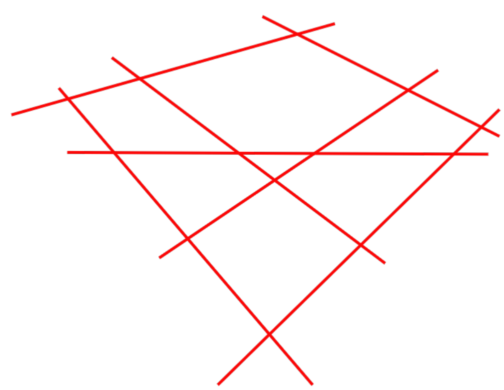

Figure 17: A Non-Allowable Dual Diagram

Notice that the non-positive curvature of all sectors implies that there are negatively curved features near all sides of an elephant except for sides on the boundary).

5 Disproving Existence of Bigons

We will now describe the tools we hope will be sufficient to prove that every word has a minimal area diagram that has at least three open 3-gon regions. Note that elephants correspond to triangles in the dual diagrams. Any negative curvature corresponds to an n-gon for $n \geq 5$.

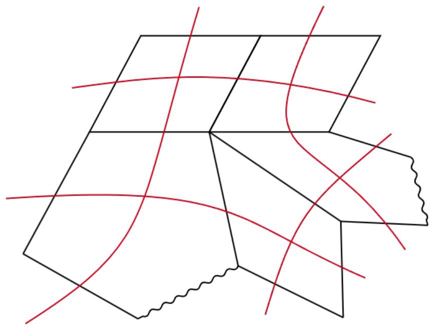

Figure 18: Dual 5-gon Region

Given an n-gon in the dual diagram, notice that there are $2n$ places where there may lie another polygon (a polygon may be attached on each vertex and each edge of the n-gon).

Conjecture 5.1. Consider an n-gon in the dual diagram to a minimal area Van Kampen diagram where $n \geq 5$. Any pair of triangles adjacent to the n-gon are in one of the following configurations.

- Both are attached at a vertex.
- Both are attached to edges, but the edges are not adjacent.
- One is attached to an edge; the other is attached to a vertex. The vertex and edge are not adjacent.

Conjecture 5.2. Let D be a dual diagram to a minimal area Van Kampen diagram. Suppose D contains one of the dual diagrams depicted in Figure 19. If the open blue-shaded region in the subdiagrams in Figure 19 is closed in D, then it is a closed n-gon in D where $n \geq 5$.

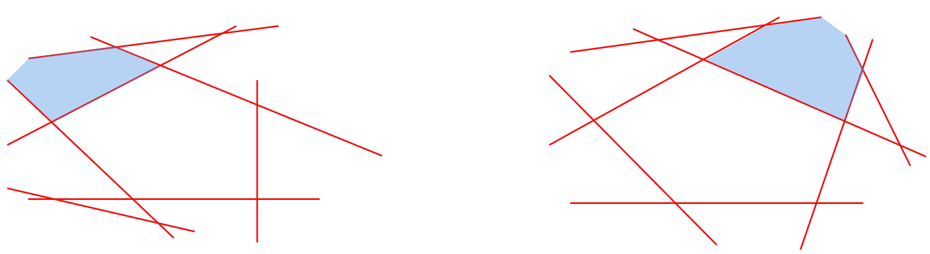

Figure 19: Extending Dual Diagrams Makes n-gons

Recall from Proposition 2.11 that any word with less than three open dual 3-gons will have a bigon in its dual diagram. We hope to proceed by showing that there are no bigons in the dual diagram for a minimal area Van Kampen diagram.

Definition 5.3. A bigon in the dual diagram is *skinny* if there are no vertices in its interior.

We have the following conjecture which seems to hold upon examining the ways one might add tiles to an elephant in order to eventually create a skinny bigon.

Conjecture 5.4. There are no skinny bigons in a minimal area Van Kampen diagram.

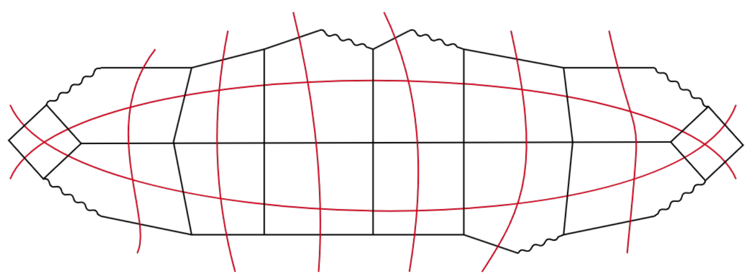

Figure 20: A Skinny Bigon

It remains to consider the case of a non-skinny bigon. Notice that by Gauss-Bonnet applied to the Van Kampen diagram, the total curvature of the bigon is $+\pi$ and thus, $\sum_{D_n \in \mathcal{D}}(4-n) = 2$ where $\mathcal{D}$ is the set of closed n-gons, D_n, contained in the bigon. We will proceed by showing that maintaining this balance of triangles and n-gons will necessitate packing elephants too tightly, using the results of Conjecture 5.2. It may be necessary to develop similar results for other configurations in the dual diagram. Thus, such configurations will admit reduction.

6 Acknowledgments

We would like to thank our mentor, Alex Wright. This project would not have been possible without his guidance and support. We also thank the University of Michigan Math Department and their REU program.

References

[CM17] Matt Clay and Dan Margalit, editors. *Office Hours with a Geometric Group Theorist.* Princeton University Press, Princeton, NJ, 2017.

[FHnd] Shintaro Fushida-Hardy. Finite generation of mapping class groups. Unpublished lecture notes, n.d.

[Rie17] Charlotte Rieder. Finitely generating the mapping class group with dehn twists. Technical report, University of Chicago REU, 2017. REU paper; dated August 28, 2017.

[Wis21] Daniel T. Wise. *From Riches to RAAGs: 3-Manifolds, Right-Angled Artin Groups, and Cubical Geometry*, volume 117 of *CBMS Regional Conference Series in Mathematics*. American Mathematical Society, Providence, RI, 2021.