BEYOND THE ADAMS CONJECTURE

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Abstract. For quasi-split symplectic and even orthogonal groups over a p-adic field, we determine the number of local Arthur packets containing certain fixed tempered representations. As an application, in many cases, we determine the number of local Arthur packets containing the local theta lift of a tempered representation at the first occurrence in the going-up tower. These counts show that the local theta lifts can lie in many more local Arthur packets than those predicted by the Adams conjecture.

Contents

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1. Introduction

Recently, Ben-Zvi, Sakellaridis, and Venkatesh have proposed a series of conjectures which are known as the relative Langlands program ([9]). One of the key ideas in their theory is the notion of duality. The goal of this article is to study a specific manifestation of this duality which we outline below.

The influential Gan-Gross-Prasad (GGP) conjectures seek to study certain branching problems for classical groups ([10, 11]). For example, let $n \in \mathbb{Z}_{\geq 1}$, F be a non-Archimedean local field of characteristic 0, $G = \mathrm{SO}_{2n+1}(F)$ and $H = \mathrm{SO}_{2n}(F)$. Given an irreducible π representation of G, the GGP conjectures aim to study the restriction of π to H, i.e., to determine the irreducible admissible representations π' of H for which

$$\mathrm{Hom}_H(\pi \otimes \pi', \mathbb{C}) \neq 0.$$

The GGP conjectures provide a partial answer in terms of "relevant" pairs of local Arthur packets ([11, Conjectures 6.1 and 7.1]). However, it is known that this relevance condition is not necessary in a sense. Specifically, there exists π and π' as above such that $\mathrm{Hom}_H(\pi \otimes \pi', \mathbb{C}) \neq 0$, but π and π' lie in a pair of non-relevant local Arthur packets ([11, §7]).

The GGP conjectures fit into the theory of the relative Langlands program ([9, §1.5]). In this framework, the GGP conjectures can be considered as the "dual" problem to the Adams conjecture (see Conjecture 1.1 below). We defer to [15, Remark 7.12] for the details of this duality. We are particularly interested in studying the analogues of non-relevant pairs (called non-θ-relevant below) for the Adams conjecture.

Hereinafter, let $m, n \in \mathbb{Z}_{\geq 0}$ and $G_n = \mathrm{Sp}_{2n}(F)$ denote a split symplectic group and $H_m^{\pm} = \mathrm{O}_{2m}^{\pm}(F)$ denote a quasi-split non-split even orthogonal group (see §2.1 for explanation of the notation). We assume that $m > n$ and set $\alpha = 2m - 2n - 1$ which is the difference in the ranks of the dual groups. Let $\Pi(G)$, respectively

$\Pi(H_m^\pm)$, denote the set of equivalence classes of complex irreducible admissible representations of G, respectively $H_m^\pm$. We consider the local theta correspondence defined by Howe ([21]) which we denote as a map $\theta_{-\alpha}^\pm : \Pi(G_n) \to \Pi(H_m^\pm) \cup \{0\}$. The local theta correspondence has proven to be an incredibly useful tool within the Langlands program; however, it does not necessarily preserve L-packets. As an attempt at remedying this non-preservation, Adams proposed that instead of L-packets, the local theta correspondence should preserve local Arthur packets ([1]). This prediction is known as the Adams conjecture and we recall its precise formulation in Conjecture 1.1 below.

We briefly recall several notions related to local Arthur packets and defer to §2.2 for details. For brevity, let $G \in \{G_n, H_m^\pm\}$. A local Arthur parameter of G may be roughly thought of as a homomorphism $\psi : W_F \times \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C}) \to {}^L G$, where W_F is the Weil group of F and ${}^L G$ denotes the L-group of G. We note that we may regard ${}^L G_n = \mathrm{SO}_{2n+1}(\mathbb{C})$ and ${}^L H_m^\pm = \mathrm{O}_{2m}(\mathbb{C})$. Furthermore, we have a natural inclusion of ${}^L G_n \hookrightarrow {}^L H_m^\pm$ since $m > n$. In Arthur's seminal work, to each local Arthur parameter ψ of G, Arthur attaches a local Arthur packet Π_ψ which is a finite subset of $\Pi(G)$ which satisfies the twisted endoscopic character identities ([2, Theorem 1.5.1]). The impact of Arthur's work cannot be understated. Indeed, Arthur's theory implies the local Langlands correspondence for G and also provides a decomposition of discrete spectrum of square-integrable automorphic forms ([2, Theorem 1.5.2]).

Let χ_W denote the trivial representation of W_F and χ_V denote the character of W_F associated to a certain quadratic character related to $H_m^\pm$ (see §2.3) via local class field theory. Given a local Arthur parameter ψ of G_n, we set

$$\psi_\alpha := (\chi_W \chi_V^{-1} \otimes \psi) \oplus \chi_W \otimes S_1 \otimes S_\alpha.$$

The Adams conjecture is the following.

Conjecture 1.1 (The Adams conjecture ([1]))**.** *Suppose that $\pi \in \Pi_\psi$ for some local Arthur parameter ψ of G_n. If $\theta_{-\alpha}^\pm(\pi) \neq 0$, then $\theta_{-\alpha}^\pm(\pi) \in \Pi_{\psi_\alpha}$.*

We remark that the Adams conjecture in this setting is fully understood via [8, 26]. In particular, Mœglin showed that the Adams conjecture is true when $\alpha \gg 0$ ([26, Theorem 6.1]) but it can and does fail otherwise. The exact extent of this failure is determined by Bakić and Hanzer algorithmically ([8, Theorem A]). Moreover, if we fix π and allow ψ to vary, then we understand the extent to which the Adams conjecture holds via [17, Theorem 1.3].

Note that it is the pair of local Arthur parameters (ψ, ψ_α) which are involved in the Adams conjecture. In analogy with the GGP conjectures, we say that a pair of local Arthur parameters (ψ, ψ') of $G_n \times H_m^\pm$ are *θ-relevant* if $\psi' = \psi_\alpha$. Note that representations often lie in many local Arthur packets ([4, 18, 19]). Consequently, we expect to have many non-θ-relevant pairs (ψ, ψ') of local Arthur parameters for which $(\pi, \theta_{-\alpha}^\pm(\pi)) \in \Pi_\psi \times \Pi_{\psi'}$. Our overarching goal is a systematic investigation of these non-θ-relevant pairs given a fixed $\pi \in \Pi(G_n)$.

Problem 1.2. *Determine all non-θ-relevant pairs given a fixed $\pi \in \Pi(G_n)$.*

For a representation $\pi' \in \Pi(G)$, let $\Psi(\pi')$ denote the set of local Arthur parameters ψ of G such that $\pi' \in \Pi_\psi$. Then, for a fixed $\pi \in \Pi(G_n)$, we have that

$(\pi, \theta^{\pm}_{-\alpha}(\pi)) \in \Pi_\psi \times \Pi_{\psi'}$ for any $(\psi, \psi') \in \Psi(\pi) \times \Psi(\theta^{\pm}_{-\alpha}(\pi))$. Consequently, determining the non-θ-relevant pairs reduces to the computation of $\Psi(\theta^{\pm}_{-\alpha}(\pi))$. For $\alpha \gg 0$, it is a straightforward consequence of the theory of intersections of local Arthur packets developed by Liu, Lo, and the first author ([18, 19], see Theorem 3.16) that $\Psi(\theta^{\pm}_{-\alpha}(\pi)) = \{\psi_\alpha \mid \psi \in \Psi(\pi)\}$ and so determining the non-θ-relevant pairs is simple.

We begin our investigation in the following situation. We let $\mathrm{up} \in \{+, -\}$ be the "going-up" tower for $\pi \in \Pi(G_n)$ and $m^{\alpha,\mathrm{up}}(\pi)$ denote a certain odd integer determined by the first occurrence of the local theta lift π in the going-up tower (see §2.1 for terminology). Bakić and Hanzer showed that the Adams conjecture holds at (and above) the first occurrence in the going-up tower ([8, Theorem 2], see Theorem 2.9). If π is supercuspidal, then $\theta^{\mathrm{up}}_{-m^{\alpha,\mathrm{up}}(\pi)}(\pi)$ is also supercuspidal ([23]). Results of Mœglin imply that $|\Psi(\theta^{\mathrm{up}}_{-m^{\alpha,\mathrm{up}}(\pi)}(\pi))| = 3|\Psi(\pi)|$ ([24, 25], see [11, Theorem 7.5]) and so we obtain many non-θ-relevant pairs.

In this article, we provide a conjecture which completely determines the count $|\Psi(\theta^{\mathrm{up}}_{-m^{\alpha,\mathrm{up}}(\pi)}(\pi))|$ when $\pi \in \Pi(G_n)$ is tempered (see Theorem 4.8 and Conjecture 4.11 for details).

Conjecture 1.3. *Suppose that $\pi \in \Pi(G_n)$ is tempered. Then there is a recursive formula for computing $|\Psi(\theta^{\mathrm{up}}_{-m^{\alpha,\mathrm{up}}(\pi)}(\pi))|$.*

We remark that one could compute $\Psi(\theta^{\mathrm{up}}_{-m^{\alpha,\mathrm{up}}(\pi)}(\pi))$ via brute force by algorithmically using algorithms for the local theta correspondence ([5, 8]) and algorithms for determining the local Arthur packets to which the theta correspondence belongs ([19]), but the relation to $\Psi(\pi)$ becomes unclear. Furthermore, this computation by brute force is impractical for theoretical applications. In contrast, the main benefit of Conjecture 1.3 is its explicit use of the structure in $\Psi(\pi)$. For example, when π is supercuspidal, Conjecture 1.3 immediately implies that $|\Psi(\theta^{\mathrm{up}}_{-m^{\alpha,\mathrm{up}}(\pi)}(\pi))| = 3|\Psi(\pi)|$. In contrast, the brute force approach would require computing both $\Psi(\theta^{\mathrm{up}}_{-m^{\alpha,\mathrm{up}}(\pi)}(\pi))$ and $\Psi(\pi)$ explicitly.

One of the main results of this article is the verification of Conjecture 1.3 in many cases which we view as a first step towards Problem 1.2.

Theorem 1.4. *Conjecture 1.3 is true in many cases. Specifically, we verify Theorem 4.8 and Cases 1, 3, 4, 5, and 6 of Conjecture 4.11.*

As a consequence of the above theorem, we also determine a complete result on $|\Psi(\theta^{\mathrm{up}}_{-m^{\alpha,\mathrm{up}}(\pi)}(\pi))|$, when π is anti-tempered (see Theorem 4.16).

Theorem 1.5. *Suppose that $\pi \in \Pi(G_n)$ is anti-tempered. Then there is a recursive formula for computing $|\Psi(\theta^{\mathrm{up}}_{-m^{\alpha,\mathrm{up}}(\pi)}(\pi))|$.*

The proof of Theorem 1.4 largely follows from the following counting problem.

Problem 1.6. *Given a tempered representation $\pi \in \Pi(G_n)$, determine is the size of $\Psi(\pi)$.*

Indeed, in many cases if π is tempered, then $\theta^{\mathrm{up}}_{-m^{\alpha,\mathrm{up}}(\pi)}(\pi)$ is also tempered. For split symplectic and odd special orthogonal groups, the above problem can be answered by reversing [18, Theorem 8.27]; however, this is impractical in practice.

Our second main result is a recursive formula for determining $|\Psi(\pi)|$, where π is a certain kind of tempered representation (see Theorem 4.9).

Theorem 1.7. *For certain tempered representations $\pi \in \Pi(G_n)$, there is a recursive formula for determining $|\Psi(\pi)|$.*

The key idea behind Theorem 1.7 is to decompose the combinatorial data underlying π (specifically a certain tempered extended multi-segment, see Theorems 3.3 and 3.30) into blocks (see Definition 4.4). This decomposition is uniquely determined by π (Lemma 4.5). Theorem 4.9 then relates $|\Psi(\pi)|$ with determining the number of local Arthur packets each block "belongs" to. Theorem 4.7 further gives a recursive formula for determining this count in terms of smaller blocks.

The proofs of the results on this tempered count largely rest upon the parameterization of local Arthur packets using extended multi-segments and their theory of intersections developed in [3, 18, 19]. The results on the local theta correspondence utilize the Adams conjecture ([1]), its specific validity for the going-up tower ([8, 26]), the calculation of the first occurrence on the going-up tower for tempered representations ([6]), and the translation of these results into the language of extended multi-segments ([19]).

Here is the organization of this paper. In §2 and §3, we recall various results on the local theta correspondence and parameterization of local Arthur packets using extended multi-segments. In §4, we state the main results precisely. Then we provide some motivation behind the results in §5. We collect some preliminary definitions and statements in §6. In §7, we study the case of individual blocks. From this theory, we prove the first main result (the number of local Arthur packets containing a given block, Theorem 4.7) in §8. Following this, we prove the several results related to the local theta correspondence in §9. Next, we discuss how blocks may interact with each other in §10. We prove a remaining case of one of the main results in §11. Finally, we discuss some results related to determining $\Psi(\theta^{\mathrm{up}}_{-m^{\alpha,\mathrm{up}}(\pi)}(\pi))$, as opposed to just its size, in §12.

2. Background

Recall that F is a non-Archimedean local field of characteristic 0. For the moment, let G be a reductive group defined over F and $G = \mathrm{G}(F)$. We are primarily concerned with the set of equivalence classes of complex irreducible admissible representations of G, which we denote by $\Pi(G)$.

2.1. **Local theta correspondence.** We let $n \in \mathbb{Z}_{\geq 0}$, fix an additive character ψ_F of F, and let W_{2n} denote the unique (up to isomorphism) symplectic vector space over F of dimension $2n$. Let $m \in \mathbb{Z}_{\geq 1}$, fix $d \in F^\times \setminus (F^\times)^2$, and consider the quadratic spaces over F of dimension $2m$ and discriminant d. Note that up to isomorphism, there are two such quadratic spaces which are distinguished by their Hasse-Witt invariant. We let $V_{2m}^\pm$ be the unique quadratic space over F of dimension $2m$, discriminant d, and Hasse-Witt invariant ± 1. The isometry groups of W_{2n} and $V_{2m}^\pm$, denoted $G_n = G(W_{2n})$ and $H_m^\pm = H(V_{2m}^\pm)$, respectively, are isomorphic to the split group $\mathrm{Sp}_{2n}(F)$ and the quasi-split non-split group $\mathrm{O}_{2m}^\pm(F)$. Here the superscript $\pm$ in $\mathrm{O}_{2m}^\pm(F)$ is used to keep track of the underlying quadratic space as it plays an important role in the local theta correspondence. These symplectic and quadratic spaces are naturally arranged into towers:

$$\mathcal{W} = \{W_{2n} \mid n \in \mathbb{Z}_{\geq 0}\},$$
$$\mathcal{V}^{+} = \{V_{2m}^{+} \mid m \in \mathbb{Z}_{\geq 1}\},$$
$$\mathcal{V}^{-} = \{V_{2m}^{-} \mid m \in \mathbb{Z}_{\geq 1}\}.$$

The pair $(G_n, H_m^{\pm})$ forms a reductive dual pair of a certain metaplectic group. Consequently, given $\pi \in \Pi(G_n)$, we may consider its local theta lift which we denote by $\theta_{W_{2n}, V_{2m}^{\pm}, \psi_F}(\pi)$ which either vanishes or is an element of $\Pi(H_m^{\pm})$. The local theta lift was originally defined by Howe (see [22]) and has found many uses within the Langlands program. For our purposes, it is sufficient to recall several properties of the local theta lift.

Originally conjectured by Howe ([21]), the following theorem was first proven by Waldspurger ([31]) when the residual characteristic of F is not 2 and then in full generality by Gan and Takeda ([14]) and Gan and Sun ([13]).

Theorem 2.1 (Howe Duality)**.** *Let* $\pi_1, \pi_2 \in \Pi(G_n)$.

(1) If $\theta_{W_{2n}, V_{2m}^{\pm}, \psi_F}(\pi_1) \neq 0$, *then* $\theta_{W_{2n}, V_{2m}^{\pm}, \psi_F}(\pi_1)$ *is irreducible.*

(2) If $\pi_1 \ncong \pi_2$ *and both* $\theta_{W_{2n}, V_{2m}^{\pm}, \psi_F}(\pi_1)$ *and* $\theta_{W_{2n}, V_{2m}^{\pm}, \psi_F}(\pi_2)$ *are nonzero, then*

$$\theta_{W_{2n}, V_{2m}^{\pm}, \psi_F}(\pi_1) \ncong \theta_{W_{2n}, V_{2m}^{\pm}, \psi_F}(\pi_2).$$

The following theorem gives the persistence principle (also called the tower property) for the local theta correspondence.

Theorem 2.2 ([23])**.** *Let* $\pi \in \Pi(G_n)$. *If* $\theta_{W_{2n}, V_{2m}^{\pm}, \psi_F}(\pi) \neq 0$, *then* $\theta_{W_n, V_{2m'}^{\pm}, \psi_F}(\pi) \neq 0$ *for any* $m' \geq m$.

The persistence principle allows us to define the first occurrence as follows.

Definition 2.3. *Let* $\pi \in \Pi(G_n)$. *We define the* first occurrence *of* π *(in* $\mathcal{V}^{\pm}$*) to be*

$$m^{\pm}(\pi) := \min\{2m \mid \theta_{W_{2n}, V_{2m}^{\pm}, \psi_F}(\pi) \neq 0\}.$$

Note that since we have two "target" towers $\mathcal{V}^{\pm}$, we have two first occurrences. They are related by the following theorem which is known as the conservation relation.

Theorem 2.4 ([30])**.** *Let* $\pi \in \Pi(G_n)$. *Then*

$$m^{+}(\pi) + m^{-}(\pi) = 4n + 4.$$

As a consequence, we may choose $\mathrm{up}, \mathrm{down} \in \{\pm\}$ such that $m^{\mathrm{up}}(\pi) \geq 2n+2 \geq m^{\mathrm{down}}(\pi)$. Also, if one inequality is strict, then both inequalities are strict. In this situation, we call the tower whose first occurrence is $m^{\mathrm{up}}(\pi)$ the "going-up" tower for π and denote it by $\mathcal{V}^{\mathrm{up}}$. Similarly, we call the tower whose first occurrence is $m^{\mathrm{down}}(\pi)$ the "going-down" tower for π and denote it by $\mathcal{V}^{\mathrm{down}}$. When $m^{\mathrm{up}}(\pi) = m^{\mathrm{down}}(\pi)$, there is a choice of $\mathrm{up}, \mathrm{down} \in \{\pm\}$, but it will not matter for the Adams conjecture (see Remark 2.10).

It will be convenient for us to repurpose our notation. Fix m and n and let $\alpha = 2m - 2n - 1$ (which is the difference in the ranks of the dual groups). For

$\pi \in \Pi(G_n)$, we let $\theta^{\pm}_{-\alpha}(\pi) = \theta_{W_{2n},V^{\pm}_{2m},\psi_F}(\pi)$. We let $\mathrm{up}, \mathrm{down} \in \{\pm\}$ be such that $m^{\mathrm{up}}(\pi) \geq m^{\mathrm{down}}(\pi)$. This allows us to consider the local theta lifts $\theta^{\mathrm{up}}_{-\alpha}(\pi)$ or $\theta^{\mathrm{down}}_{-\alpha}(\pi)$ for the going-up and going-down towers for π, respectively. We also set $m^{\alpha,\mathrm{up}}(\pi) = m^{\mathrm{up}}(\pi) - 2n - 1$.

2.2. **Local Arthur packets.** In this subsection, recall some results concerning local Arthur packets. We let G denote one of $G_n = \mathrm{Sp}_{2n}(F)$ or $H^{\pm}_m = \mathrm{O}^{\pm}_{2m}(F)$ for brevity. Note that G_n is connected, but $H^{\pm}_m$ is disconnected. Consequently, we will consider the complex dual group and L-group for the identity component $(H^{\pm}_m)^\circ = \mathrm{SO}^{\pm}_{2m}(F)$. We have that the complex dual group $\widehat{G}(\mathbb{C})$ is given by $\mathrm{SO}_{2n+1}(\mathbb{C})$ or $\mathrm{SO}_{2m}(\mathbb{C})$, respectively. Recall that G_n is split while $H^{\pm}_m$ is quasi-split, but not split. Thus, the L-group ${}^LG$ is given by $\mathrm{SO}_{2n+1}(\mathbb{C})$ or $\mathrm{O}_{2m}(\mathbb{C})$, respectively.

A local Arthur parameter of G is a direct sum of irreducible representations

$$\psi : W_F \times \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C}) \to {}^LG$$

$$\psi = \bigoplus_{i=1}^{r} \phi_i |\cdot|^{x_i} \otimes S_{a_i} \otimes S_{b_i}, \tag{2.1}$$

satisfying the following conditions:

(1) $\phi_i(W_F)$ is bounded and consists of semi-simple elements, and $\dim(\phi_i) = d_i$;
(2) $x_i \in \mathbb{R}$ and $|x_i| < \frac{1}{2}$;
(3) the restrictions of ψ to the two copies of $\mathrm{SL}_2(\mathbb{C})$ are analytic, S_k is the k-dimensional irreducible representation of $\mathrm{SL}_2(\mathbb{C})$, and
$$\sum_{i=1}^{r} d_i a_i b_i = N := \begin{cases} 2n+1 & \text{when } G = \mathrm{Sp}_{2n}(F), \\ 2m & \text{when } G = \mathrm{O}^{\pm}_{2m}(F). \end{cases}$$

We remark that the bound $|x_i| < \frac{1}{2}$ follows from the trivial bound of the Ramanujan Conjecture.

Two local Arthur parameters of G are said to be equivalent if they are conjugate under ${}^LG$. When $G = G_n$, this is equivalent to the usual notion of equivalence using conjugation by the complex dual group. However, if $G = H^{\pm}_m$, defining the equivalence using conjugation by the complex dual group would result in equivalence classes of local Arthur parameters for $(H^{\pm}_m)^\circ$, not $H^{\pm}_m$.

We let $\Psi^+(G)$ denote the set of equivalence classes of local Arthur parameters. We will not distinguish a local Arthur parameter ψ and its equivalence class. We let $\Psi(G)$ denote the subset of equivalence classes of bounded local Arthur parameters, i.e., those ψ for which $x_i = 0$ for any $i = 1, \ldots, r$ in the decomposition (2.1).

By the Local Langlands Correspondence for $\mathrm{GL}_d(F)$, any bounded representation ϕ of W_F corresponds to an irreducible unitary supercuspidal representation ρ of $\mathrm{GL}_d(F)$ ([16, 20, 29]). Consequently, we may identify (2.1) as

$$\psi = \bigoplus_{\rho} \left(\bigoplus_{i \in I_\rho} \rho |\cdot|^{x_i} \otimes S_{a_i} \otimes S_{b_i} \right), \tag{2.2}$$

where the first sum runs over a finite set of irreducible unitary supercuspidal representations ρ of $\mathrm{GL}_d(F)$ where $d \in \mathbb{Z}_{\geq 1}$.

For a local Arthur parameter $\psi \in \Psi(G)$, Arthur constructed a finite multi-set Π_ψ consisting of irreducible unitary representations of G that satisfy certain twisted endoscopic character identities ([2]). We call Π_ψ the *local Arthur packet* of ψ. Mœglin further showed that Π_ψ is multiplicity-free ([28]). We do not recall the precise definition of Π_ψ. Instead, it suffices for our purposes to recall a parameterization of Π_ψ using extended multi-segments (see Theorem 3.3).

Mœglin showed that the computation of Π_ψ can be reduced to the "good parity" case (see Theorem 2.6 below). We proceed by recalling this reduction.

Definition 2.5. *Let ψ be a local Arthur parameter as in (2.2), we say that ψ is of* good parity *if $\psi \in \Psi(G)$, i.e., $x_i = 0$ for all i, and every summand $\rho \otimes S_{a_i} \otimes S_{b_i}$ is self-dual and orthogonal. We let $\Psi_{gp}(G)$ denote the subset of $\Psi^+(G)$ consisting of local Arthur parameters of good parity.*

We explicate this condition further. Consider a summand $\rho|\cdot|^x \otimes S_a \otimes S_b$ of ψ as in (2.2). This summand is self-dual and orthogonal if and only if $x = 0$, ρ is orthogonal (resp. symplectic), and $a_i + b_i$ is even (resp. odd).

Let $\psi \in \Psi^+(G)$. Since ψ is self-dual, we have a decomposition

$$\psi = \psi_{ngp} + \psi_{gp} + \psi_{ngp}^\vee,$$

where $\psi_{ngp}^\vee$ denotes the dual of ψ_{ngp}, $\psi_{gp} \in \Psi_{gp}(G)$, and ψ_{gp} is maximal for this decomposition, i.e., if we decompose $\psi_{ngp} + \psi_{ngp}^\vee$ as in (2.2), then any irreducible summand $\rho|\cdot|^x \otimes S_a \otimes S_b$ is not of good parity. Note that ψ_{gp} is uniquely determined by this decomposition, but ψ_{ngp} is not necessarily unique. Mœglin showed that the local Arthur packet Π_ψ can be constructed from $\Pi_{\psi_{gp}}$.

Theorem 2.6 ([27, Proposition 5.1])**.** *Let $\psi \in \Psi^+(G)$ with decomposition $\psi = \psi_{ngp} + \psi_{gp} + \psi_{ngp}^\vee$ as above. Then, there exists $\tau \in \Pi(\mathrm{GL}_d(F))$ (determined by ψ_{ngp}) such that for any $\pi_{gp} \in \Pi_{\psi_{gp}}$, the normalized parabolic induction $\tau \rtimes \pi_{gp}$ is irreducible and*

$$\Pi_\psi = \{\tau \rtimes \pi_{gp} \mid \pi_{gp} \in \Pi_{\psi_{gp}}\}. \tag{2.3}$$

2.3. **The Adams conjecture.** In this subsection, we recall the Adams conjecture (see Conjecture 2.7 below) along with some relevant results.

As in [12, §3.2], we fix a pair of characters χ_W, χ_V associated to W_n and $V_m^\pm$ respectively. More specifically, we have that χ_W is the trivial character of $F^\times$ and χ_V is the quadratic character associated to $F(\sqrt{d})/F$, where d is the discriminant of $V_m^\pm$. Recall also that $\alpha = 2m - 2n - 1$.

It is known that the theta correspondence does not preserve L-packets. As a remedy, Adams proposed that instead, it should preserve local Arthur packets ([1]). We recall Adams' conjecture below.

Conjecture 2.7. *Assume that $m > n$. Suppose that $\pi \in \Pi(G_n)$ lies in a local Arthur packet Π_ψ for some $\psi \in \Psi^+(G_n)$. If $\theta_{-\alpha}^\pm(\pi) \neq 0$, then $\theta_{-\alpha}^\pm(\pi) \in \Pi_{\psi_\alpha}$ where*

$$\psi_\alpha = (\chi_W \chi_V^{-1} \otimes \psi) \oplus \chi_W \otimes S_1 \otimes S_\alpha. \tag{2.4}$$

Mœglin verified that when α is large, the Adams' conjecture is true.

Theorem 2.8 ([26, Theorem 6.1]). *Suppose that* $\pi \in \Pi(G_n)$ *lies in a local Arthur packet* Π_ψ *for some* $\psi \in \Psi^+(G_n)$. *For* $\alpha \gg 0$, *we have* $\theta^{\pm}_{-\alpha}(\pi) \in \Pi_{\psi_\alpha}$.

However, Mœglin also showed that the Adams conjecture can and does fail in general. The failure of the Adams conjecture in this case is well-understood through the works of [8, 17]. We are particularly concerned with the case of the going-up tower for π in which case Bakić and Hanzer showed that the Adams conjecture always holds.

Theorem 2.9 ([8, Theorem 2]). *Suppose that* $\pi \in \Pi(G_n)$ *lies in a local Arthur packet* Π_ψ *for some* $\psi \in \Psi^+(G_n)$. *If* $\theta^{\mathrm{up}}_{-\alpha}(\pi) \neq 0$, *then* $\theta^{\mathrm{up}}_{-\alpha}(\pi) \in \Pi_{\psi_\alpha}$.

We remark that while [8, Theorem 2] is stated for $\psi \in \Psi(G_n)$; however, the extension to $\Psi^+(G_n)$ follows directly from [17, Lemma 2.33].

Remark 2.10. *We remark that if* $m^{up}(\pi) = m^{down}(\pi)$ *then the Adams conjecture is always true. That is, for any odd positive integer* α *and* $\pi \in \Pi_\psi$, *we have* $\theta^{\pm}_{-\alpha}(\pi) \in \Pi_{\psi_\alpha}$ *(see also* [8, p. 15]*).*

3. Extended multi-segments

Let $G \in \{G_n, H^{\pm}_m\}$. In light of the Theorem 2.6, it is desirable to parameterize Π_ψ for $\psi \in \Psi_{gp}(G)$. In this section, we recall such a parameterization using extended multi-segments (Theorem 3.3). Furthermore, we also recall some results on the theory of intersections of local Arthur packets from [18, 19]. We begin by recalling some notions related for extended multi-segments.

We fix the following notation throughout this subsection. Let $\psi \in \Psi_{gp}(G)$ with decomposition

$$\psi = \bigoplus_{\rho} \bigoplus_{i \in I_\rho} \rho \otimes S_{a_i} \otimes S_{b_i}.$$

We set $A_i = \frac{a_i+b_i}{2} - 1$ and $B_i = \frac{a_i - b_i}{2}$ for $i \in I_\rho$.

We say that a total order $>_\psi$ on I_ρ is *admissible* if satisfies:

(P) For $i, j \in I_\rho$, if $A_i > A_j$ and $B_i > B_j$, then $i >_\psi j$.

Sometimes we consider an order $>_\psi$ on I_ρ satisfying:

(P') For $i, j \in I_\rho$, if $B_i > B_j$, then $i >_\psi j$.

Note that (P') implies (P). For brevity, we often write $>$ instead of $>_\psi$ when it is clear that we are working with a fixed admissible order.

Suppose now that we have fixed an admissible order for ψ. We define the *support* of ψ to be the collection of ordered multi-sets

$$\mathrm{supp}(\psi) := \cup_\rho \{[A_i, B_i]_\rho\}_{i \in (I_\rho, >)}.$$

Note that $\mathrm{supp}(\psi)$ depends implicitly on the fixed admissible order.

We recall the definition of extended multi-segments.

Definition 3.1. *(Extended multi-segments)*

(1) An extended segment *is a triple* $([A, B]_\rho, l, \eta)$, *where*
- $[A, B]_\rho = \{\rho|\cdot|^A, \rho|\cdot|^{A-1}, \ldots, \rho|\cdot|^B\}$ *is a segment for an irreducible unitary supercuspidal representation* ρ *of some* $\mathrm{GL}_d(F)$;

- $l \in \mathbb{Z}$ *with* $0 \leq l \leq \frac{b}{2}$, *where* $b = \#[A, B]_\rho = A - B + 1$*;*
- $\eta \in \{\pm 1\}$.

(2) An extended multi-segment *for* G *is an equivalence class (via the equivalence defined below) of multi-sets of extended segments*

$$\mathcal{E} = \cup_\rho \{([A_i, B_i]_\rho, l_i, \eta_i)\}_{i \in (I_\rho, >)}$$

such that

- I_ρ *is a totally ordered finite set with a fixed admissible total order* $>$*;*
- $A_i + B_i \geq 0$ *for all* ρ *and* $i \in I_\rho$*;*
- *we have that*

$$\psi_{\mathcal{E}} = \bigoplus_\rho \bigoplus_{i \in I_\rho} \rho \otimes S_{a_i} \otimes S_{b_i}.$$

  where $(a_i, b_i) = (A_i + B_i + 1, A_i - B_i + 1)$*, is a local Arthur parameter for* G *of good parity.*
- *The sign condition*

$$\prod_\rho \prod_{i \in I_\rho} (-1)^{[\frac{b_i}{2}] + l_i} \eta_i^{b_i} = \epsilon_G \tag{3.1}$$

  holds. Here $\epsilon_G = 1$ *if* $G = \mathrm{Sp}_{2n}(F)$ *or* $G = \mathrm{O}_{2m}^+(F)$ *and* $\epsilon_G = -1$ *otherwise.*

(3) Two extended segments $([A, B]_\rho, l, \eta)$ *and* $([A', B']_{\rho'}, l', \eta')$ *are* weakly equivalent *if*

- $[A, B]_\rho = [A', B']_{\rho'}$*;*
- $l = l'$*; and*
- $\eta = \eta'$ *whenever* $l = l' < \frac{b}{2}$.

Two extended multi-segments $\mathcal{E} = \cup_\rho \{([A_i, B_i]_\rho, l_i, \eta_i)\}_{i \in (I_\rho, >)}$ *and* $\mathcal{E}' = \cup_\rho \{([A'_i, B'_i]_\rho, l'_i, \eta'_i)\}_{i \in (I_\rho, >)}$ *are* weakly equivalent *if for any* ρ *and* $i \in I_\rho$*, the extended segments* $([A_i, B_i]_\rho, l_i, \eta_i)$ *and* $([A'_i, B'_i]_\rho, l'_i, \eta'_i)$ *are weakly equivalent.*

Note that if $l = \frac{b}{2}$*, then* η *is arbitrary. In this case, we always take* $\eta = 1$ *(unless explicitly stated otherwise, e.g., in Definition 3.11).*

(4) We define the support *of* $\mathcal{E}$ *to be the collection of ordered multi-sets*

$$\mathrm{supp}(\mathcal{E}) = \cup_\rho \{[A_i, B_i]_\rho\}_{i \in (I_\rho, >)}.$$

We implicitly include the admissible order $>$ *in* $\mathrm{supp}(\mathcal{E})$.

(5) We let $\mathrm{Eseg}(G)$ *denote the set of all extended multi-segments of* G *up to weak equivalence.*

If the admissible order $>$ is clear in the context, for $k \in I_\rho$, we often let $k+1 \in I_\rho$ be the unique element adjacent with k and $k+1 > k$.

The data in an extended multi-segment can be cumbersome to list out in detail. Instead, we attach a symbol to each extended multi-segment by the same way in [3, Section 3]. We give an example to explain this.

Example 3.2. *Let ρ be an orthogonal representation of $\mathrm{GL}_d(F)$. The symbol*

$$\mathcal{E} = \begin{array}{c} \begin{array}{cccccc} -1 & 0 & 1 & 2 & 3 & 4 \end{array} \\ \left(\begin{array}{cccccc} \lhd & \lhd & \oplus & \ominus & \rhd & \rhd \\ & & & \lhd & \rhd & \\ & & & & & \ominus \end{array} \right)_{\rho} \end{array}$$

corresponds to $\mathcal{E} = \{([A_i, B_i]_\rho, l_i, \eta_i)\}_{i \in (1<2<3)}$ of $\mathrm{Sp}_{44d}(F)$ where the data is given as follows.

- *$([A_1, B_1]_\rho, [A_2, B_2]_\rho, [A_3, B_3]_\rho) = ([4, -1]_\rho, [3, 2]_\rho, [4, 4]_\rho)$ specify the "support" of each row.*
- *$(l_1, l_2, l_3) = (2, 1, 0)$ counts the number of pairs of triangles in each row.*
- *$(\eta_1, \eta_2, \eta_3) = (1, 1, -1)$ records the sign of the first circle in each row. Note that setting $\eta_2 = \pm 1$ results in weakly equivalent extended multi-segments.*

The associated local Arthur parameter is

$$\psi_{\mathcal{E}} = \rho \otimes S_4 \otimes S_6 + \rho \otimes S_6 \otimes S_2 + \rho \otimes S_9 \otimes S_1.$$

With the above symbol in mind, we often say that an extended segment $r = ([A, B]_\rho, l, \eta)$ is a *row* of $\mathcal{E}$ if $r \in \mathcal{E}$. Furthermore, we define the support of r to be $\mathrm{supp}(r) = [A, B]_\rho$ and let $A(r) = A$, $B(r) = B$, $a(r) = A(r) + B(r) + 1$, $b(r) = A(r) - B(r) + 1$, $l(r) = l$, and $\eta(r) = \eta$.

Let $\mathcal{E} \in \mathrm{Eseg}(G)$. If $G = \mathrm{Sp}_{2n}(F)$, then we attach a representation $\pi(\mathcal{E})$ of G as in [3, §3.2]. This is done using the theory of derivatives developed by Atobe and Mínguez in [7]. We have that either $\pi(\mathcal{E})$ vanishes or $\pi(\mathcal{E}) \in \Pi(G)$. If $G = \mathrm{O}^{\pm}_{2m}(F)$, then we attach a representation $\pi(\mathcal{E})$ of G as in [19, Definition 6.3]. This definition avoids the use of derivatives by using the Adams conjecture as established by Mœglin (see Theorem 2.8). Again, we have that either $\pi(\mathcal{E})$ vanishes or $\pi(\mathcal{E}) \in \Pi(G)$.

In either case, we have that extended multi-segments parameterize local Arthur packets.

Theorem 3.3. *Suppose $\psi = \bigoplus_\rho \bigoplus_{i \in I_\rho} \rho \otimes S_{a_i} \otimes S_{b_i}$ is a local Arthur parameter of good parity of G. Choose an admissible order $>$ on I_ρ for each ρ that satisfies (P') if $\frac{a_i - b_i}{2} < 0$ for some $i \in I_\rho$. Then*

$$\bigoplus_{\pi \in \Pi_\psi} \pi = \bigoplus_{\mathcal{E}} \pi(\mathcal{E}),$$

where $\mathcal{E}$ runs over all extended multi-segments with $\mathrm{supp}(\mathcal{E}) = \mathrm{supp}(\psi)$ and $\pi(\mathcal{E}) \neq 0$.

When $G = \mathrm{Sp}_{2n}(F)$, the above theorem was proven by Atobe ([3, Theorem 3.3]). When $G = \mathrm{O}^{\pm}_{2m}(F)$, the above theorem was proven in [19, Theorem 6.10]. As a direct consequence of the above theorem and that fact that local Arthur packets are multiplicity-free ([28]), we obtain the following corollary.

Corollary 3.4. *Let $\mathcal{E}_1, \mathcal{E}_2 \in \mathrm{Eseg}(G)$ and suppose that $\mathrm{supp}(\mathcal{E}_1) = \mathrm{supp}(\mathcal{E}_2)$. If $\frac{a_i - b_i}{2} < 0$ for some $i \in I_\rho$, then we also require that the order on I_ρ is (P'). If $\pi(\mathcal{E}_1) = \pi(\mathcal{E}_2) \neq 0$, then $\mathcal{E}_1 = \mathcal{E}_2$.*

Recall that $\mathrm{supp}(\mathcal{E})$ implicitly records the admissible order on $\mathcal{E} \in \mathrm{Eseg}(G)$. Thus, the hypothesis $\mathrm{supp}(\mathcal{E}_1) = \mathrm{supp}(\mathcal{E}_2)$ in the above corollary also asserts that the admissible orders on $\mathcal{E}_1$ and $\mathcal{E}_2$ agree.

3.1. **Intersections.** In this subsection, we recall the theory of intersections of local Arthur packets as developed in [18, 19]. We begin by recalling various operators which are used in the classification of these intersections (see Theorem 3.16), along with some other useful operators and their properties.

We note that the effect of an operator often only depends on a fixed ρ. To simplify the definitions of the operators, we introduce the following notation.

Definition 3.5. *Let $\mathcal{E} = \cup_\rho \{([A_i, B_i]_\rho, l_i, \eta_i)\}_{i \in (I_\rho, >)} \in \mathrm{Eseg}(G)$. We set*

$$\mathcal{E}_\rho = \{([A_i, B_i]_\rho, l_i, \eta_i)\}_{i \in (I_\rho, >)}, \ \ \mathcal{E}^\rho = \cup_{\rho' \not\cong \rho} \{([A_i, B_i]_{\rho'}, l_i, \eta_i)\}_{i \in (I_{\rho'}, >)}.$$

We let $\mathrm{Vseg}(G)$ denote the set of elements of the form

$$\mathcal{F} = \cup_{\rho'} \mathcal{E}_{\rho'}$$

where the union is over a finite set of irreducible self-dual supercuspidal representations ρ' of $\mathrm{GL}_d(F)$, $d \geq 1$, and $\mathcal{E} \in \mathrm{Eseg}(G)$. Essentially, $\mathcal{F}$ is an extended multi-segment (for some group) except that we not enforce the sign condition (3.1). We say that $\mathcal{F}$ is a virtual extended multi-segment*.*

Next, we recall the shift and add operators.

Definition 3.6. *Let $\mathcal{E} = \cup_\rho \{([A_i, B_i]_\rho, l_i, \eta_i)\}_{i \in (I_\rho, >)}$ be an extended multi-segment. For $j \in I_{\rho'}$ and $d \in \mathbb{Z}$, we define the following operators. It is immediate that the operators commute with each other and so we denote the composition by summation.*

1. *$sh_j^d(\mathcal{E}) = \cup_\rho \{([A_i', B_i']_\rho, l_i, \eta_i)\}_{i \in (I_\rho, >)}$ with*
$$[A_i', B_i']_\rho = \begin{cases} [A_i + d, B_i + d]_\rho & \text{if } \rho = \rho' \text{ and } i = j, \\ [A_i, B_i]_\rho & \text{otherwise,} \end{cases}$$
and $sh_{\rho'}^d = \sum_{j \in I_{\rho'}} sh_j^d$. Also, we define $sh^d := \sum_\rho sh_\rho^d$.
2. *$add_j^d(\mathcal{E}) = \cup_\rho \{([A_i', B_i']_\rho, l_i', \eta_i)\}_{i \in (I_\rho, >)}$ with*
$$([A_i', B_i']_\rho, l_i') = \begin{cases} ([A_i + d, B_i - d]_\rho, l_i + d) & \text{if } \rho = \rho' \text{ and } i = j, \\ ([A_i, B_i]_\rho, l_i) & \text{otherwise,} \end{cases}$$
and $add_{\rho'}^d = \sum_{j \in I_{\rho'}} add_j^d$. Also, we define $add^d := \sum_\rho add_\rho^d$.

We use these notations in the case that the resulting object is still an extended multi-segment.

The next operator we recall is the row exchange operator. Its effect is to change the admissible order on an extended multi-segment. We first recall the definition of a symbol.

Definition 3.7. *A* symbol *is a multi-set of extended segments*

$$\mathcal{E} = \cup_\rho \{([A_i, B_i]_\rho, l_i, \eta_i)\}_{i \in (I_\rho >)},$$

which satisfies the same conditions in Definition 3.1(2) except we drop the condition $0 \leq l_i \leq \frac{b_i}{2}$, for each $i \in I_\rho$.

Any change of admissible orders can be derived from a composition of the row exchange operators R_k which we recall from [18, Definition 3.15].

Definition 3.8 (Row exchange). *Suppose $\mathcal{E}$ is a symbol where*

$$\mathcal{E}_\rho = \{([A_i, B_i]_\rho, l_i, \eta_i)\}_{i \in (I_\rho, >)}.$$

For $k < k+1 \in I_\rho$, let $\gg$ be the total order on I_ρ defined by $k \gg k+1$ and if $(i,j) \neq (k, k+1)$, then $i \gg j$ if and only if $i > j$.

Suppose $\gg$ is not an admissible order on I_ρ, then we define $R_k(\mathcal{E}) = \mathcal{E}$. Otherwise, we define

$$R_k(\mathcal{E}_\rho) = \{([A_i, B_i]_\rho, l'_i, \eta'_i)\}_{i \in (I_\rho, \gg)},$$

where $(l'_i, \eta'_i) = (l_i, \eta_i)$ for $i \neq k, k+1$, and (l'_k, η'_k) and (l'_{k+1}, η'_{k+1}) are given as follows: Denote $\epsilon = (-1)^{A_k - B_k}\eta_k\eta_{k+1}$.

Case 1. $[A_k, B_k]_\rho \supset [A_{k+1}, B_{k+1}]_\rho$:

In this case, we set $(l'_{k+1}, \eta'_{k+1}) = (l_{k+1}, (-1)^{A_k - B_k}\eta_{k+1})$, and

(a) If $\epsilon = 1$ and $b_k - 2l_k < 2(b_{k+1} - 2l_{k+1})$, then

$$(l'_k, \eta'_k) = (b_k - (l_k + (b_{k+1} - 2l_{k+1})), (-1)^{A_{k+1} - B_{k+1}}\eta_k).$$

(b) If $\epsilon = 1$ and $b_k - 2l_k \geq 2(b_{k+1} - 2l_{k+1})$, then

$$(l'_k, \eta'_k) = (l_k + (b_{k+1} - 2l_{k+1}), (-1)^{A_{k+1} - B_{k+1} + 1}\eta_k).$$

(c) If $\epsilon = -1$, then

$$(l'_k, \eta'_k) = (l_k - (b_{k+1} - 2l_{k+1}), (-1)^{A_{k+1} - B_{k+1} + 1}\eta_k).$$

Case 2. $[A_k, B_k]_\rho \subset [A_{k+1}, B_{k+1}]_\rho$:

In this case, we set $(l'_k, \eta'_k) = (l_k, (-1)^{A_{k+1} - B_{k+1}}\eta_k)$, and

(a) If $\epsilon = 1$ and $b_{k+1} - 2l_{k+1} < 2(b_k - 2l_k)$, then

$$(l'_{k+1}, \eta'_{k+1}) = (b_{k+1} - (l_{k+1} + (b_k - 2l_k)), (-1)^{A_k - B_k}\eta_{k+1}).$$

(b) If $\epsilon = 1$ and $b_{k+1} - 2l_{k+1} \geq 2(b_k - 2l_k)$, then

$$(l'_{k+1}, \eta'_{k+1}) = (l_{k+1} + (b_k - 2l_k), (-1)^{A_k - B_k + 1}\eta_{k+1}).$$

(c) If $\epsilon = -1$, then

$$(l'_{k+1}, \eta'_{k+1}) = (l_{k+1} - (b_k - 2l_k), (-1)^{A_k - B_k + 1}\eta_{k+1}).$$

Finally, we define $R_k(\mathcal{E}) = \mathcal{E}^\rho \cup R_k(\mathcal{E}_\rho)$.

We remark that there is another definition of row exchange is given in [3, Section 4.2]; however, these definitions agree when $\pi(\mathcal{E}) \neq 0$.

The next operator we recall is known as union-intersection.

Definition 3.9 (union-intersection). *Let $\mathcal{E} \in \mathrm{Eseg}(G)$. For $k < k+1 \in I_\rho$, we define an operator ui_k, called union-intersection, on $\mathcal{E}$ as follows. Write*

$$\mathcal{E}_\rho = \{([A_i, B_i]_\rho, l_i, \eta_i)\}_{i \in (I_\rho, >)}.$$

Denote $\epsilon = (-1)^{A_k - B_k}\eta_k\eta_{k+1}$. If $A_{k+1} > A_k$, $B_{k+1} > B_k$ and any of the following cases holds:

Case 1. $\epsilon = 1$ *and* $A_{k+1} - l_{k+1} = A_k - l_k$,
Case 2. $\epsilon = 1$ *and* $B_{k+1} + l_{k+1} = B_k + l_k$,
Case 3. $\epsilon = -1$ *and* $B_{k+1} + l_{k+1} = A_k - l_k + 1$,

we define

$$ui_k(\mathcal{E}_\rho) = \{([A'_i, B'_i]_\rho, l'_i, \eta'_i)\}_{i \in (I_\rho, >)},$$

where $([A'_i, B'_i]_\rho, l'_i, \eta'_i) = ([A_i, B_i]_\rho, l_i, \eta_i)$ *for* $i \neq k, k+1$, *and* $[A'_k, B'_k]_\rho = [A_{k+1}, B_k]_\rho$, $[A'_{k+1}, B'_{k+1}]_\rho = [A_k, B_{k+1}]_\rho$, *and* $(l'_k, \eta'_k, l'_{k+1}, \eta'_{k+1})$ *are given case by case as follows:*

(1) *in Case 1,* $(l'_k, \eta'_k, l'_{k+1}, \eta'_{k+1}) = (l_k, \eta_k, l_{k+1} - (A_{k+1} - A_k), (-1)^{A_{k+1}-A_k}\eta_{k+1})$;

(2) *in Case 2, if* $b_k - 2l_k \geq A_{k+1} - A_k$, *then*

$$(l'_k, \eta'_k, l'_{k+1}, \eta'_{k+1}) = (l_k + (A_{k+1} - A_k), \eta_k, l_{k+1}, (-1)^{A_{k+1}-A_k}\eta_{k+1}),$$

if $b_k - 2l_k < A_{k+1} - A_k$, *then*

$$(l'_k, \eta'_k, l'_{k+1}, \eta'_{k+1}) = (b_k - l_k, -\eta_k, l_{k+1}, (-1)^{A_{k+1}-A_k}\eta_{k+1});$$

(3) *in Case 3, if* $l_{k+1} \leq l_k$, *then*

$$(l'_k, \eta'_k, l'_{k+1}, \eta'_{k+1}) = (l_k, \eta_k, l_{k+1}, (-1)^{A_{k+1}-A_k}\eta_{k+1}),$$

if $l_{k+1} > l_k$, *then*

$$(l'_k, \eta'_k, l'_{k+1}, \eta'_{k+1}) = (l_k, \eta_k, l_k, (-1)^{A_{k+1}-A_k+1}\eta_{k+1});$$

(3′) *if we are in Case 3 and* $l_k = l_{k+1} = 0$, *then we delete* $([A'_{k+1}, B'_{k+1}]_\rho, l'_{k+1}, \eta'_{k+1})$ *from* $ui_k(\mathcal{E}_\rho)$.

Otherwise, we define $ui_k(\mathcal{E}_\rho) = \mathcal{E}_\rho$. *In any case, we define* $ui_k(\mathcal{E}) = \mathcal{E}^\rho \cup ui_k(\mathcal{E}_\rho)$.

We say ui_k *is applicable on* $\mathcal{E}$ *or* $\mathcal{E}_\rho$ *if* $ui_k(\mathcal{E}) \neq \mathcal{E}$. *We say this* ui_k *is of type 1 (resp. 2, 3, 3') if* $\mathcal{E}_\rho$ *is in Case 1 (resp. 2, 3, 3').*

We also consider the composition of the row exchange and union-intersection operators as follows.

Definition 3.10. *Suppose* $\mathcal{E} \in \mathrm{Eseg}(G)$ *and write*

$$\mathcal{E}_\rho = \{([A_i, B_i]_\rho, l_i, \eta_i)\}_{i \in (I_\rho, >)}.$$

Given $i, j \in I_\rho$, *we define* $ui_{i,j}(\mathcal{E}_\rho) = \mathcal{E}_\rho$ *unless*

1. *We have* $A_i < A_j$, $B_i < B_j$ *and* $(j, i, >')$ *is an adjacent pair for some admissible order* $>'$ *on* I_ρ.
2. ui_i *is applicable on* $\mathcal{E}_{\rho,>'}$

In this case, we define $ui_{i,j}(\mathcal{E}_\rho) := (ui_i(\mathcal{E}_{\rho,>'}))_>$, *so that the admissible order of* $ui_{i,j}(\mathcal{E}_\rho)$ *and* $\mathcal{E}_\rho$ *are the same. (If the* ui_i *is of type 3', then we delete the* j*-th row.) Finally, we define* $ui_{i,j}(\mathcal{E}) = \mathcal{E}^\rho \cup ui_{i,j}(\mathcal{E}_\rho)$.

We say $ui_{i,j}$ *is applicable on* $\mathcal{E}$ *if* $ui_{i,j}(\mathcal{E}) \neq \mathcal{E}$. *Furthermore, we say that* $ui_{i,j}$ *is of type 1, 2, 3, or 3' if the operation* ui_i *is of type 1, 2, 3, or 3', respectively, in Definition 3.9.*

The next operator is called the dual operator.

Definition 3.11 (dual)**.** *Let* $\mathcal{E} = \cup_\rho \{([A_i, B_i]_\rho, l_i, \eta_i)\}_{i \in (I_\rho, >)}$ *be an extended multi-segment such that the admissible order* $>$ *on* I_ρ *satisfies (P') for all* ρ. *We define*

$$dual(\mathcal{E}) = \cup_\rho \{([A_i, -B_i]_\rho, l'_i, \eta'_i)\}_{i \in (I_\rho, >')}$$

as follows:

(1) *The order $>'$ is defined by $i >' j$ if and only if $j > i$.*
(2) *We set*

$$l_i' = \begin{cases} l_i + B_i & \text{if } B_i \in \mathbb{Z}, \\ l_i + B_i + \frac{1}{2}(-1)^{\alpha_i}\eta_i & \text{if } B_i \notin \mathbb{Z}, \end{cases}$$

and

$$\eta_i' = \begin{cases} (-1)^{\alpha_i+\beta_i}\eta_i & \text{if } B_i \in \mathbb{Z}, \\ (-1)^{\alpha_i+\beta_i+1}\eta_i & \text{if } B_i \notin \mathbb{Z}, \end{cases}$$

where $\alpha_i = \sum_{j \in I_\rho, j<i} a_j$, and $\beta_i = \sum_{j \in I_\rho, j>i} b_j$, $a_j = A_j + B_j + 1$, $b_j = A_j - B_j + 1$.
(3) *When $B_i \notin \mathbb{Z}$ and $l_i = \frac{b_i}{2}$, we set $\eta_i = (-1)^{\alpha_i+1}$.*

If $\mathcal{F} = \mathcal{E}_\rho$, we define $dual(\mathcal{F}) := (dual(\mathcal{E}))_\rho$.

As a shorthand, for each $i \in I_\rho$, let $r_i = ([A_i, B_i]_\rho, l_i, \eta_i)$. Then we let $\widehat{r_i}$ denote the effect of the dual operator on $\mathcal{E}$ on this row, i.e.,

$$\widehat{r_i} = ([A_i, -B_i]_\rho, l_i', \eta_i').$$

We note that if $\pi(\mathcal{E}) \neq 0$, then $\pi(dual(\mathcal{E}))$ is the Aubert-Zelevinsky dual of $\pi(\mathcal{E})$ ([3, Theorem 6.2] if $G = \mathrm{Sp}_{2n}(F)$ and [19, Proposition 6.11] if $G = \mathrm{O}^{\pm}_{2m}(F)$). For our purposes, it is sufficient to use the following direct implication.

Lemma 3.12. *Let $\mathcal{E} \in \mathrm{Eseg}(G)$ be such that $\pi(\mathcal{E}) \neq 0$. Then $\pi(dual(\mathcal{E})) \neq 0$. Moreover, if $\mathcal{E}' \in \mathrm{Eseg}(G)$ is such that $\pi(\mathcal{E}') = \pi(\mathcal{E})$, then $\pi(dual(\mathcal{E}')) = \pi(dual(\mathcal{E}))$.*

From [18, 19], we understand the inverse of union-intersections not of type 3'.

Lemma 3.13. *If ui is not of type 3', then its inverse is of the form $dual \circ ui \circ dual$ of the same type.*

The next operator is known as the partial dual operator.

Definition 3.14 (partial dual)**.** *Suppose $\mathcal{E}$ satisfies (P') and*

$$\mathcal{E}_\rho = \mathcal{F} = \{([A_i, B_i]_\rho, l_i, \eta_i)\}_{i \in (I_\rho, >)}.$$

For $i \in I_\rho$, denote

$$\alpha_i = \sum_{j<i}(A_j + B_j + 1), \ \beta_i = \sum_{j>i}(A_j - B_j + 1).$$

Suppose there exists $k \in I_\rho$ such that

1. *$B_k = 1/2, l_k = 0$,*
2. *$(-1)^{\alpha_k}\eta_k = -1$,*
3. *for any $i < k$, $B_i < 1/2$.*

Then we define $dual_k^+(\mathcal{F})$ as follows. We write the decomposition

$$\mathcal{F} = \mathcal{F}_1 + \{([A_k, 1/2]_\rho, 0, \eta_k)\} + \mathcal{F}_2,$$

where $\mathcal{F}_1 = \mathcal{F}_{<1/2}$, and

$$dual(\mathcal{F}) = \widetilde{\mathcal{F}_2} + \{([A_k, -1/2]_\rho, 0, (-1)^{\beta_k})\} + \widetilde{\mathcal{F}_1},$$

where $\widetilde{\mathcal{F}_1} = (dual(\mathcal{F}))_{>-1/2}$. *Finally, write*

$$dual(\widetilde{\mathcal{F}_2} + \{([A_k, 1/2]_\rho, 0, (-1)^{\beta_k+1})\} + \widetilde{\mathcal{F}_1}) = \widetilde{\widetilde{\mathcal{F}_1}} + \{([A_k, -1/2]_\rho, 0, -\eta_k)\} + \widetilde{\widetilde{\mathcal{F}_2}},$$

where $\widetilde{\widetilde{\mathcal{F}_2}} = (dual(\widetilde{\mathcal{F}_2} + \{([A_k, 1/2]_\rho, 0, (-1)^{\beta_k+1})\} + \widetilde{\mathcal{F}_1}))_{>-1/2}$. *Then we define*

$$dual_k^+(\mathcal{F}) = \widetilde{\widetilde{\mathcal{F}_1}} + \{([A_k, -1/2]_\rho, 0, -\eta_k)\} + \mathcal{F}_2,$$

and say $dual_k^+$ *is applicable on* $\mathcal{F}$.

Suppose $dual(\mathcal{F})$ *satisfies above condition, then we define*

$$dual_k^-(\mathcal{F}) = dual \circ dual_k^+ \circ dual(\mathcal{F}),$$

and say $dual_k^-$ *is applicable on* $\mathcal{F}$.

We call this operator partial dual, and use $dual_k$ *to denote* $dual_k^+$ *or* $dual_k^-$ *if it is clear from the context.*

Finally, we define $dual_k(\mathcal{E}) = \mathcal{E}^\rho \cup dual_k(\mathcal{E}_\rho)$.

We remark that, by the definition of $dual$, it follows that $dual_k^-$ is applicable on $\mathcal{F}$ if and only if $B_k = -1/2$, $l_k = 0$ and $B_j > -1/2$ for all $j > k$.

We distinguish certain operators that have a key role in the intersection of local Arthur packets (see Theorem 3.16 below).

Definition 3.15. *The operators* R_k, $ui_{i,j}$, $dual \circ ui_{j,i} \circ dual$, *or* $dual_k$ *are known as* basic operators.

Basic operators fully determine intersections of local Arthur packets as follows.

Theorem 3.16. *Let* $\mathcal{E} \in \mathrm{Eseg}(G)$ *be such that* $\pi(\mathcal{E}) \neq 0$. *Then the following hold.*

(1) If T *is a basic operator, then* $\pi(\mathcal{E}) = \pi(T(\mathcal{E}))$.
(2) If $\mathcal{E}'$ *is another extended multi-segment for which* $\pi(\mathcal{E}) = \pi(\mathcal{E}')$, *then* $\mathcal{E}$ *and* $\mathcal{E}'$ *are related by a finite composition of basic operators and their inverses.*

We remark the above theorem was proven when $G = \mathrm{Sp}_{2n}(F)$ in [18, Theorem 1.4] and when $G = \mathrm{O}_{2m}^{\pm}(F)$ in [19, Theorems 6.13 and 6.15]. With the above theorem in mind, for $\mathcal{E}, \mathcal{E}' \in \mathrm{Eseg}(G)$, we write $\mathcal{E} \sim \mathcal{E}'$, and say that they are (strongly) equivalent, if $\mathcal{E}$ and $\mathcal{E}'$ are related by a finite composition of basic operators and their inverses.

A special class of the basic operators are called raising operators. They elucidate a structure on the set

$$\Psi(\pi) = \{\psi \in \Psi(G) \mid \pi \in \Pi_\psi\}.$$

Definition 3.17. *The operators* $dual \circ ui \circ dual$, ui^{-1}, *and* $dual_k^-$ *are called* raising operators. *Given local Arthur parameters* $\psi_1, \psi_2 \in \Psi(\pi)$, *we write* $\psi_1 \geq_O \psi_2$ *if there exists a sequence of raising operators* $(T_i)_{i=1}^l$ *such that*

$$\mathcal{E}_1 = (T_l \circ T_{l-1} \circ \cdots \circ T_1)(\mathcal{E}_2),$$

where $\mathcal{E}_j \in \mathrm{Eseg}(G)$ *is such that* $\psi_{\mathcal{E}_j} = \psi_j$ *and* $\pi(\mathcal{E}_j) = \pi$ *for* $j = 1, 2$.

Remarkably, the partial order $\geq_O$ on $\Psi(\pi)$ has unique maximal and minimal elements.

Theorem 3.18. *Let $\pi \in \Pi(G)$ be of Arthur type. Then there exists unique maximal and minimal elements of $\Psi(\pi)$ with respect to $\geq_O$. We denote these elements by $\psi^{max}(\pi)$ and $\psi^{min}(\pi)$, respectively.*

The above theorem was proven when $G = \mathrm{Sp}_{2n}(F)$ in [18, Theorem 1.7] and when $G = \mathrm{O}_{2m}^{\pm}(F)$ in [19, Theorem 7.3].

We record several explicit relations on the operators which will be useful later.

Lemma 3.19. *Let $\mathcal{E} \in \mathrm{Eseg}$ and suppose that $\pi(\mathcal{E}) \neq 0$. If $\frac{a_i - b_i}{2} < 0$ for some $i \in I_\rho$, then we also require that the admissible orders below on I_ρ satisfy (P'). Then the following hold:*

1. *Let $i, j, k, l \in I_\rho$. Then $(R_i \circ R_j)(\mathcal{E}) = (R_k \circ R_l)(\mathcal{E})$ provided that the resulting admissible orders agree.*
2. *Let $i, j \in I_\rho$. Then $(dual \circ R_i)(\mathcal{E}) = (R_j \circ dual)(\mathcal{E})$ provided that the resulting admissible orders agree.*
3. *Let $i, j \in I_\rho$. Then $(ui_i \circ R_j)(\mathcal{E}) = (R_j \circ ui_i)(\mathcal{E})$ provided that the resulting admissible orders agree.*

Proof. The proof of these claims follow the same pattern. For brevity, we only give the details of the proof of Part (2) that $(dual \circ R_i)(\mathcal{E}) = (R_j \circ dual)(\mathcal{E})$ provided that the resulting admissible orders are the agree.

By Lemma 3.12 and Theorem 3.16(1), we have that $\pi((dual \circ R_i)(\mathcal{E})) = \pi((R_j \circ dual)(\mathcal{E}))$. Moreover, $\mathrm{supp}((dual \circ R_i)(\mathcal{E})) = \mathrm{supp}((R_j \circ dual)(\mathcal{E}))$. If the resulting admissible orders are the agree, then we obtain that $(dual \circ R_i)(\mathcal{E}) = (R_j \circ dual)(\mathcal{E})$ from Corollary 3.4. This completes the proof of Part (2). ☐

Next, we recall some results which determines whether $\pi(\mathcal{E})$ vanishes or not, for some $\mathcal{E} \in \mathrm{Eseg}(G)$, purely combinatorially.

Proposition 3.20 ([32, Lemma 5.5, 5.6, 5.7])**.** *Let $\mathcal{E} = \cup_\rho \{([A_i, B_i]_\rho, l_i, \eta_i)\}_{i \in (I_\rho, >)}$ be an extended multi-segment of G_n whose admissible order satisfies (P') if $B_i < 0$ for some $i \in I_\rho$. Denote $\epsilon = (-1)^{A_k - B_k}\eta_k \eta_{k+1}$. We have $\pi(\mathcal{E}) \neq 0$ only if the following conditions hold for all $k < k+1 \in I_\rho$.*

1. *If $A_k \leq A_{k+1}$, $B_k \leq B_{k+1}$, then*
$$\begin{cases} \epsilon = 1 & \Rightarrow B_k + l_k \leq B_{k+1} + l_{k+1}, A_k - l_k \leq A_{k+1} - l_{k+1}, \\ \epsilon = -1 & \Rightarrow A_k - l_k < B_{k+1} + l_{k+1}. \end{cases}$$
2. *If $[A_k, B_k]_\rho \subset [A_{k+1}, B_{k+1}]_\rho$, then*
$$\begin{cases} \epsilon = 1 & \Rightarrow 0 \leq l_{k+1} - l_k \leq b_{k+1} - b_k, \\ \epsilon = -1 & \Rightarrow l_k + l_{k+1} \geq b_k. \end{cases}$$
3. *If $[A_k, B_k]_\rho \supset [A_{k+1}, B_{k+1}]_\rho$, then*
$$\begin{cases} \epsilon = 1 & \Rightarrow 0 \leq l_k - l_{k+1} \leq b_k - b_{k+1}, \\ \epsilon = -1 & \Rightarrow l_k + l_{k+1} \geq b_{k+1}. \end{cases}$$

Let $\mathcal{E} \in \mathrm{Eseg}(G)$ and write $\mathcal{E} = \cup_\rho \{([A_i, B_i]_\rho, l_i, \eta_i)\}_{i \in (I_\rho, >)}$. We say that $\mathcal{E}$ is non-negative if $B_i \geq 0$ for any $i \in I_\rho$ and for any ρ. We now state the non-vanishing theorem which was proven for $G = \mathrm{Sp}_{2n}(F)$ in [3, Theorems 3.6, 4.4] and for $G = \mathrm{O}_{2m}^{\pm}(F)$ in [19, Theorem 6.17].

Theorem 3.21. *Let $\mathcal{E} \in \mathrm{Eseg}(G)$ such that for any ρ, if there exists $i \in I_\rho$ with $B_i < 0$, then the admissible order on I_ρ satisfies (P').*

(i) Write $\mathcal{E} = \cup_\rho \{([A_i, B_i]_\rho, l_i, \eta_i)\}_{i \in (I_\rho, >)}$. We have $\pi(\mathcal{E}) \neq 0$ if and only if $\pi(sh^d(\mathcal{E})) \neq 0$ for any $d \gg 0$ such that $sh^d(\mathcal{E})$ is non-negative, and the following condition holds for all ρ and $i \in I_\rho$

$$(*) \qquad B_i + l_i \geq \begin{cases} 0 & \text{if } B_i \in \mathbb{Z}, \\ \frac{1}{2} & \text{if } B_i \notin \mathbb{Z} \text{ and } \eta_i = (-1)^{\alpha_i + 1}, \\ -\frac{1}{2} & \text{if } B_i \notin \mathbb{Z} \text{ and } \eta_i = (-1)^{\alpha_i}, \end{cases}$$

where

$$\alpha_i := \sum_{j<i} A_j + B_j + 1.$$

(ii) If $\mathcal{E}$ is non-negative, then $\pi(\mathcal{E}) \neq 0$ if and only if any adjacent pair $(i, j, \gg)$ satisfies Proposition 3.20(i).

Theorem 3.21 determines provides a method to determine when $\pi(\mathcal{E}) \neq 0$ purely combinatorially. We say that $\mathcal{E} \in \mathrm{Rep}(G)$ if $\pi(\mathcal{E}) \neq 0$. Similarly, we let $\mathrm{VRep}(G)$ denote the set of $\mathcal{F} = \cup_{\rho'} \mathcal{E}_{\rho'} \in \mathrm{Vseg}(G)$ such that $\mathcal{E} \in \mathrm{Rep}(G)$. Applying Theorem 3.21 to $\mathcal{E}$, we have that $\mathcal{F} \in \mathrm{VRep}(G)$ if and only if $\mathcal{F}$ formally satisfies the combinatorial nonvanishing conditions in Theorem 3.21. Furthermore, we set $\mathrm{Vseg}_\rho(G) = \{\mathcal{E}_\rho \mid \mathcal{E} \in \mathrm{Vseg}(G)\}$ and $\mathrm{VRep}_\rho(G) = \{\mathcal{E}_\rho \mid \mathcal{E} \in \mathrm{VRep}(G)\}$. Again, we have that $\mathcal{E} \in \mathrm{VRep}_\rho(G)$ if and only if $\mathcal{E}$ formally satisfies the combinatorial nonvanishing conditions in Theorem 3.21.

We provide an extension of Lemma 3.19(1).

Lemma 3.22 (commutativity of row exchanges)**.** *Let $\mathcal{E} = (r, r_1, r_2) \in \mathrm{VRep}(G)$. Assume further that the orders the supports of r, r_1, r_2 contain or are contained in each other. Then exchanging r to the third row and then exchanging r_1 and r_2 gives the same result as exchanging r_1 and r_2 first, and then exchanging r to the third row. Formally, $R_1(R_2(R_1(\mathcal{E}))) = R_2(R_1(R_2(\mathcal{E})))$.*

Proof. Since $\pi(\mathcal{E}) \neq 0$, by Theorem 3.21, we have $\pi(sh^d(\mathcal{E})) \neq 0$ for some $d \gg 0$ such that $sh^d(\mathcal{E})$ is non-negative. We observe from the definitions that the row exchange and shift operators commute. Thus, by Lemma 3.19(1), we obtain that

$$\begin{aligned} sh^d(R_1(R_2(R_1(\mathcal{E})))) = R_1(R_2(R_1(sh^d(\mathcal{E})))) &= R_2(R_1(R_2(sh^d(\mathcal{E})))) \\ &= sh^d(R_2(R_1(R_2(\mathcal{E})))). \end{aligned}$$

Therefore, we obtain that $R_1(R_2(R_1(\mathcal{E}))) = R_2(R_1(R_2(\mathcal{E})))$. □

We also verify that certain basic operators are "local" operators.

Lemma 3.23. *Suppose that $\mathcal{E} \in \mathrm{VRep}_\rho(G)$. Then the operations ui and $dual \circ ui \circ dual$ are "local" operations. More specifically, if a certain row is not involved in the union-intersection or any row exchanges, then it is fixed by these operations.*

Proof. For the operation ui the result follows immediately from Definition 3.10. In particular, the result is also true for the inverse of ui. Suppose now that we consider the operator $dual \circ ui \circ dual$, where the ui is not of type 3'. Then

$dual \circ ui \circ dual$ is the inverse of a ui operator (of the same type) by [18, Corollary 5.6] and thus the result follows. For $dual \circ ui \circ dual$ of type 3', the result follows from direct calculation (note that the rows to which the ui is applied must satisfy that $a_i + b_i \equiv a_j + b_j \mod 2$ since the ui is of type 3'). □

Next we give a lemma on a certain relation between the row exchange and union-intersection operators.

Lemma 3.24. *Let $\mathcal{E} = (r_1, r_2, r_3) \in \mathrm{VRep}_\rho(G)$ and suppose that $\mathrm{supp}(r_1) \supseteq \mathrm{supp}(r)$ for any $r \in \mathcal{E}$ and that ui_2 is applicable on $\mathcal{E}$. Let $\mathcal{F} = ui_2(\mathcal{E}) = (r_1, s_2, s_3)$ and consider the row exchange of r_1 to the bottom row in both $\mathcal{E}$ and $\mathcal{F}$. We write $\mathcal{E}' = (r_2', r_3', r_1')$ and $\mathcal{F}' = (s_2', s_3', r_1'')$ for the resulting virtual extended multi-segments. Then $r_1' = r_1''$.*

Proof. The proof is a straightforward consequence of Definitions 3.8 and 3.9. Alternatively, we can shift both $\mathcal{E}$ and $\mathcal{F}$ to be positive virtual extended multi-segments (i.e., the corresponding B_i's are shifted to be positive). Then (on the shifted virtual extended multi-segments) doing the row exchanges followed by a union-intersection (note the union-intersection remains applicable after the row exchange as we are in Case 1 of Definition 3.8) must result in the same virtual extended multi-segment as doing the union-intersection followed by the row exchanges by Theorem 3.3. Reversing the shifts then gives the claim. □

We end this subsection with the following result which will be used for certain reductions in several technical arguments later.

Corollary 3.25. *Suppose r is a row in $\mathcal{E} \in \mathrm{VRep}_\rho(G)$ immediately followed by some consecutive rows consisting a sub-multi-segment $\mathcal{E}_1 \subset \mathcal{E}$, $\mathcal{E}_2$ is an extended multi-segment equivalent to $\mathcal{E}_1$. We suppose further that $\mathrm{supp}(r) \supseteq \mathrm{supp}(s)$ for any $s \in \mathcal{E}_1 \cup \mathcal{E}_2$ and also that $A(r) \in \mathbb{Z}$. Then, the result after r is exchanged with $\mathcal{E}_1$ is the same as the result after r is exchanged with $\mathcal{E}_2$.*

Proof. Before proving the claim, we first recall an operator, phantom (dis)appearing, defined by Atobe ([4, Definition 3.4]). Let k be an integer and P_k denote the operator which formally attaches $([k-1, -k]_\rho, k, 1)$ (allowed only if $k > 0$) or $([k - \frac{1}{2}, -k - \frac{1}{2}]_\rho, k, 1)$ (allowed only if $k \geq 0$) as the first row of a virtual extended multi-segment. The choice is determined by the good parity condition. For split symplectic and odd special orthogonal groups, Atobe showed that any two equivalent extended multi-segments are related by a finite composition of P_k's, union-intersections, row exchanges, and their inverses([4, Theorem 1.4]). This result is equivalent with Theorem 3.16 (see [18, Remark 10.5]) and so the same result holds for quasi-split even orthogonal groups.

We return to the proof of the claim. For simplicity, we may assume that $\mathcal{E} = \{r\} \cup \mathcal{E}_1$. Let $\mathcal{E}' = \mathcal{E}_1' \cup \{r'\}$ denote the resulting virtual extended multi-segment obtained by row exchanging r with every segment in $\mathcal{E}_1$. Similarly, we consider $\{r\} \cup \mathcal{E}_2$ and let $\mathcal{E}_2' \cup \{r''\}$ denote the resulting virtual extended multi-segment obtained by row exchanging r with every segment in $\mathcal{E}_2$. We must show that $r' = r''$.

Furthermore, without loss of generality, we may assume that $\mathcal{E}_2$ is obtained from $\mathcal{E}_1$ by either a union-intersection or the composition of P_k and a union-intersection. The former case is taken care of by Lemma 3.24 and so we proceed with the latter case.

Let s denote the corresponding formally added element $([k-1,-k]_\rho, k, 1)$ (recall that $A(r) \in \mathbb{Z}$ and so this must be the case). We note that formally row exchanging r and s leaves both segments unchanged. Indeed, the row exchange of r with s is always in Case 1(b) or 1(c) of Definition 3.8 and formally $b(s)-2l(s)=0$. Consequently, the effect of row exchanging r to the last row of $\{r\} \cup \{s\} \cup \mathcal{E}_1$ is the same as row exchanging r to the last row of $\{r\} \cup \mathcal{E}_1$. The claim then follows from a similar argument as in the proof of Lemma 3.24. □

3.2. **The Adams conjecture revisited.** In this subsection, we reconsider the Adams conjecture (Conjecture 2.7) using extended multi-segments. We remark that the Adams conjecture was first understood using Mœglin's parameterization of local Arthur packets ([8, 17, 26]); however, these results are reinterpreted using extended multi-segments in [19]. We recall this reformulation here.

Definition 3.26. *Let $\mathcal{E} \in \mathrm{Eseg}(G_n)$ and write*

$$\mathcal{E} = \cup_\rho \{([A_i, B_i]_\rho, l_i, \eta_i)\}_{i \in (I_\rho >)}.$$

Let $\mathcal{E}'$ be the extended multisegment defined by replacing each ρ with $\chi_{V^\pm}^{-1}\rho$. Set $\frac{\alpha_0-1}{2} = \max\{A_i \mid i \in I_{\chi_{V^\pm}}\}+1$ and let $\alpha \geq \alpha_0$. In this case, we define

$$\mathcal{E}_\alpha^\pm = \mathcal{E}' \cup \left\{ \left(\left[\frac{\alpha-1}{2}, -\frac{\alpha-1}{2} \right]_{\chi_W}, \frac{\alpha-1}{2}, \pm 1 \right) \right\}.$$

We remark that the added extended segment should be inserted such that it is the first extended segment in the admissible order. The extended segments in $\mathcal{E}'$ should be ordered similarly to $\mathcal{E}$. Note that $\mathcal{E}_\alpha^\pm \in \mathrm{Eseg}(H_m^\pm)$.

The following is a reformulation of Mœglin's result on the Adams conjecture (Theorem 2.8).

Theorem 3.27 ([19, Proposition 6.2])**.** *Let $\mathcal{E} \in \mathrm{Eseg}(G_n)$. If $\pi = \pi(\mathcal{E}) \neq 0$ and $\alpha \gg 0$ ($\alpha \geq \alpha_0$ is sufficient), then*

$$\theta_{-\alpha}^\pm(\pi) = \pi(\mathcal{E}_\alpha^\pm).$$

This provides a simple way to compute the theta lift when α is large. For $\alpha < \alpha_0$, the calculation is done algorithmically. This is a reformulation of a construction of [8].

Algorithm 3.28. *Let $\mathcal{E} \in \mathrm{Rep}(G_n)$ with $\pi = \pi(\mathcal{E})$.*

(1) If $\alpha \geq \alpha_0$, then, by Theorem 3.27, we have $\theta_{-\alpha}^\pm(\pi) = \pi(\mathcal{E}_\alpha^\pm)$.
(2) If $\alpha < \alpha_0$, then we proceed recursively. From $\mathcal{E}_{\alpha_0}^\pm$, we construct $\mathcal{E}_{\alpha_0-2}^\pm$ as follows. By definition 3.26, we have

$$\mathcal{E}_\alpha^\pm = \mathcal{E}' \cup \left\{ \left(\left[\frac{\alpha-1}{2}, -\frac{\alpha-1}{2} \right]_{\chi_W}, \frac{\alpha-1}{2}, \pm 1 \right) \right\}.$$

We define $\mathcal{E}^{\pm}_{\alpha-2}$ *by replacing the added extended segment*

$$\left(\left[\frac{\alpha-1}{2},-\frac{\alpha-1}{2}\right]_{\chi_W},\frac{\alpha-1}{2},\pm 1\right)$$

with

$$\left(\left[\frac{\alpha-3}{2},-\frac{\alpha-3}{2}\right]_{\chi_W},\frac{\alpha-3}{2},\pm 1\right)$$

and then row exchanging this extended segment with any extended segment $([A_i,B_i]_{\chi_W},l_i,\eta_i)$ *(*$i\in I_{\chi_W}$*) where* $B_i=-\frac{\alpha-3}{2}$*. We proceed into Step 3 for the recursion.*

(3) *We define* $\mathcal{E}^{\pm}_{\alpha}$ *by repeatedly applying Step 2. That is, we construct a sequence* $\mathcal{E}^{\pm}_{\alpha_0}$*,* $\mathcal{E}^{\pm}_{\alpha_0-2}$*, ...,* $\mathcal{E}^{\pm}_{\alpha}$*, by repeatedly performing an* add^{-1} *operator on the added extended segment and row exchanging so that it is minimal in the order.*

This algorithm computes the theta lift if its is nonzero. From [19, Theorem 7.5], we directly obtain the following reformulation of [8, Theorem A].

Theorem 3.29. *Let* $\mathcal{E}\in\mathrm{Rep}(G_n)$*,* α *be any positive odd integer, and* $\pi=\pi(\mathcal{E})$*. If* $\pi(\mathcal{E}^{\pm}_{\alpha})\neq 0$*, then*

$$\theta^{\pm}_{-\alpha}(\pi)=\pi(\mathcal{E}^{\pm}_{\alpha}).$$

3.3. **Tempered representations.** In this subsection, we recall some results related to tempered representations. Let $\Pi_{temp}(G)$ denote the subset of $\Pi(G)$ consisting of tempered representations. We begin by recalling the parameterization of $\Pi_{temp}(G)$ via the local Langlands correspondence due to Arthur.

A local Arthur parameter $\psi\in\Psi^+(G)$ is called tempered if $\psi\in\Psi_{gp}(G)$ and ψ is trivial on the second $\mathrm{SL}_2(\mathbb{C})$, i.e., if $\psi=\bigoplus_{i=1}^r\rho_i\otimes S_{a_i}\otimes S_{b_i}$, then $b_i=1$ for any $i=1,\ldots,r$. We let $\Psi_{temp}(G)$ denote the subset of $\Psi^+(G)$ consisting of tempered local Arthur parameters.

Theorem 3.30 ([2, Theorem 1.5.1])**.** *We have that*

$$\Pi_{temp}(G)=\bigcup_{\psi\in\Psi_{temp}(G)}\Pi_{\psi}.$$

Moreover, if $\psi_1,\psi_2\in\Psi_{temp}(G)$ *and* $\psi_1\neq\psi_2$*, then* $\Pi_{\psi_1}\cap\Pi_{\psi_2}=\emptyset$*.*

In other words, local Arthur packets associated to tempered local Arthur packets partition $\Pi_{temp}(G)$. With Theorem 3.3 in mind, we say that $\mathcal{E}\in\mathrm{Rep}(G)$ is tempered if $\psi_{\mathcal{E}}$ is tempered. We remark that $\mathcal{E}\in\mathrm{Eseg}(G)$ is tempered if and only if

$$\mathcal{E}=\cup_\rho\{([A_i,A_i]_\rho,0,\eta_i)\}_{(i\in I_{\rho,>})}\in\mathrm{Eseg}(G),$$

where for any $i,j\in I_\rho$ with $A_i=A_j$, we have $\eta_i=\eta_j$. In this setting, for $i\in I_\rho$ we write $\eta_\rho(A_i):=\eta_i$ as a shorthand. Note that it is possible for $\pi(\mathcal{E})$ to be tempered even if $\mathcal{E}$ is not tempered.

We also recall how to compute the first occurrence and the going-up tower for tempered representations using extended multi-segments. The following result is a straightforward application of [5, Theorem 4.1] to our setting.

Theorem 3.31 ([5, Theorem 4.1])**.** *Let $\mathcal{E} \in \mathrm{Rep}(G_n)$ be tempered and write*

$$\mathcal{E}_{\chi_V} = \cup_{i=1}^{r} \{([A_i, A_i]_{\chi_V}, 0, \eta_i)\}.$$

We let $\mathcal{T}$ denote the set containing -1 and all odd positive integers l satisfying the following conditions:

- *(chain condition) The multi-set $\mathcal{A} = \cup_{i=1}^{r}\{A_i\}$ contains $\{0, 1, \ldots, \frac{l-1}{2}\}$;*
- *(oddness condition) the multiplicity of $\frac{i-1}{2}$ in $\mathcal{A}$ is odd for $i = 1, 3, \ldots, l$;*
- *(alternating condition) $\eta_{\chi_V}(A) = -\eta_{\chi_V}(A+1)$ for $A \in \{0, 1, \ldots, \frac{l-3}{2}\}$.*

Let $l(\pi(\mathcal{E})) = \max \mathcal{T}$. Then $m^{\mathrm{up}}(\pi(\mathcal{E})) = 2n + 3 + l(\pi)$. Moreover, if $l(\pi) = -1$, then $m^{\mathrm{up}}(\pi(\mathcal{E})) = m^{\mathrm{down}}(\pi(\mathcal{E})) = 2n + 2$. Otherwise, $\mathrm{up} = -\eta_{\chi_V}(0)$.

Note that $\chi_V \otimes S_a \otimes S_b$ (and also $\chi_W \otimes S_a \otimes S_b$) is of good parity if and only if $A = \frac{a+b}{2} - 1 \in \mathbb{Z}$ and $B = \frac{b-a}{2} \in \mathbb{Z}$. Consequently, with Definition 3.26 and the computation of the theta lift (Theorem 2.9) in mind, we often focus on the set $\mathrm{Vseg}^{\mathbb{Z}}(G)$ which is defined to be the set of elements $\mathcal{E} = \cup_\rho \{([A_i, B_i]_\rho, l_i, \eta_i)\}_{(i \in I_\rho, >)} \in \mathrm{Vseg}(G)$ such that $A_i, B_i \in \mathbb{Z}$ for any $i \in I_\rho$ and any ρ. We also set $\mathrm{VRep}^{\mathbb{Z}}(G) = \mathrm{VRep}(G) \cap \mathrm{Vseg}^{\mathbb{Z}}(G)$.

3.4. **Further Notation.** Throughout this subsection, we let ρ be an orthogonal supercuspidal representation of some $\mathrm{GL}_d(F)$. In this subsection, we provide a combinatorial extension of the results concerning the Adams conjecture, but replacing χ_V with ρ.

We let $\mathrm{Vseg}_\rho(G_n) = \{\mathcal{E}_\rho \mid \mathcal{E} \in \mathrm{Vseg}^{\mathbb{Z}}(G)\}$. Furthermore, we let $\mathrm{VRep}_\rho(G_n) = \mathrm{Vseg}_\rho(G_n) \cap \mathrm{VRep}(G)$. To $\mathcal{E} \in \mathrm{VRep}_\rho(G_n)$, we consider a formal local Arthur parameter $\psi_{\mathcal{E}}$ defined analogously to Definition 3.1(2).

As a direct consequence of Theorem 3.21, we have a map between $\mathrm{VRep}_\rho(G_n)$ and $\mathrm{VRep}_{\chi_V}(G_{n'})$ for a suitable n' given by simply replacing ρ by χ_V. For $\mathcal{E} \in \mathrm{VRep}_\rho(G_n)$, let $\mathcal{E}_{\rho \to \chi_V}$ denote its image in $\mathrm{VRep}_{\chi_V}(G_n)$ via this bijection.

Consider $\mathcal{E} \in \mathrm{Rep}(G_n)$ and recall the definition of $\mathcal{E}_\alpha^{\pm}$ from Definition 3.26. Formally, Definition 3.26 extends to $\mathcal{E} \in \mathrm{VRep}_{\chi_V}(G_n)$. Given $\mathcal{F} \in \mathrm{VRep}_\rho(G_n)$, we define $\mathcal{F}_\alpha^{\pm} = (\mathcal{F}_{\rho \to \chi_V})_\alpha^{\pm}$. Similarly, we extend Algorithm 3.28 to define $\mathcal{F}_\alpha^{\pm}$ generally.

Formally, Theorem 3.31 applies to any tempered $\mathcal{E} \in \mathrm{VRep}_{\chi_V}(G_n)$, i.e., the formal local Arthur parameter $\psi_{\mathcal{E}}$ is trivial on the second $\mathrm{SL}_2(\mathbb{C})$. Suppose now that $\mathcal{F} \in \mathrm{VRep}_\rho(G_n)$ is tempered. Then we can define the going-up tower (which we record its sign by up) and first occurrence of $\mathcal{F}$ by defining them to be those of $\mathcal{F}_{\rho \to \chi_V}$. We let up and $m^{\mathrm{up},\alpha}$ denote the corresponding sign and first occurrence. We further set

$$\Theta_1(\mathcal{F}) = \mathcal{F}^{\mathrm{up}}_{m^{\mathrm{up},\alpha}}.$$

Note that when $\rho = \chi_V$, $\Theta_1(\mathcal{F})$ is simply the χ_W-part of the extended multi-segment associated to the first occurrence on the going-up tower of some extended multi-segment in $\mathrm{Rep}(G_n)$. Moreover, in this case, $\Theta_1(\mathcal{F}) \in \mathrm{VRep}_{\chi_W}(H^{\mathrm{up}}_{m^{\mathrm{up}}})$ by the Adams conjecture for the going-up tower (Theorems 2.9 and 3.29).

Later, we shall see that there may be further operators applicable on $\Theta_1(\mathcal{F})$ which result in virtual extended multi-segments which are not predicted by the Adams conjecture. These arise in a systematic manner and will be denoted by $\Theta_i(\mathcal{F})$ for $i = 2, 3, 4$ (e.g., see §9).

4. Statement of results

Throughout this section, we let ρ be an orthogonal supercuspidal representation of some $\mathrm{GL}_d(F)$. Given $\mathcal{E} \in \mathrm{Vseg}_\rho(G)$, we often write $\mathcal{E} = \{([A_i, B_i], \ell_i, \eta_i)\}_{i\in I}$ where we implicitly identify $I = \{1, \ldots, k\}$ with the order $1 < 2 < \cdots < k$. That is, we omit ρ in the notation and identify the admissible order with the usual order.

For convenience we adopt the following definitions. First, we record the first and last columns of the symbols (see Example 3.2) attached to extended multi-segments as follows.

Definition 4.1. *Let $\mathcal{E} \in \mathrm{VRep}_\rho(G)$ and $c \in \mathbb{Z}$.*

- *We say that $\mathcal{E}$* starts at a column c *if there exists a row $r \in \mathcal{E}$ with $\rho|\cdot|^c \in \mathrm{supp}(r)$ and there does not exist a row $r' \in \mathcal{E}$ has $c' \in \mathrm{supp}(r')$ for any $c' < c$.*
- *Similarly, we say that $\mathcal{E}$* ends at a column c *if there exists a row $r \in \mathcal{E}$ with $c \in \mathrm{supp}(r)$ and there does not exist a row $r' \in \mathcal{E}$ such that $c' \in \mathrm{supp}(r')$ for any $c' > c$.*

If $\mathcal{E}$ starts at c, we also say c is the first column *of $\mathcal{E}$, and similarly if $\mathcal{E}$ ends at c, we say c is the* last column *of $\mathcal{E}$.*

Remark 4.2. *We often let c_{min}, resp. c_{max}, denote the first, resp. last, columns of $\mathcal{E} \in \mathrm{VRep}_\rho(G)$.*

We are particularly interested in the case that $\mathcal{E} \in \mathrm{VRep}_\rho(G)$ is tempered, i.e., for any $r \in \mathcal{E}$, we have that $\mathrm{supp}(r)$ is a singleton. In this case, we also keep track of the multiplicities of the columns as follows.

Definition 4.3. *Let $\mathcal{E} \in \mathrm{VRep}_\rho(G)$, and fix $c \in \mathbb{Z}$. We denote by m_c the* multiplicity of c in $\mathcal{E}$, *which is defined to be the number of times $([c, c], 0, \eta)$ appears in $\mathcal{E}$ for some η.*

In our later results and conjectures, we will make use of a decomposition of $\mathcal{E}$ into "blocks" which form the building blocks of our arguments.

Definition 4.4. *Let $\mathcal{E} = \{([A_i, B_i], \ell_i, \eta_i)\}_{i\in I} \in \mathrm{VRep}_\rho(G)$ be a tempered extended multi-segment (so that $A_i = B_i$ and $\ell_i = 0$). A* block *is a multi-subset $\mathcal{B} = \{([A_i, B_i], \ell_i, \eta_i)\}_{i\in J}$ with $J \subset I$ such that*

- *if $([A, A], 0, \eta_1) \in \mathcal{B}$ and $([A + 1, A + 1], 0, \eta_2) \in \mathcal{B}$ then $\eta_1 = -\eta_2$,*
- *if for some $\eta \in \{\pm 1\}$, we have $([A, A], 0, \eta) \in \mathcal{B}$ and $([A+2, A+2], 0, \eta) \in \mathcal{B}$, then $([A + 1, A + 1], 0, -\eta) \in \mathcal{B}$,*
- *if $([A, A], 0, \eta) \in \mathcal{B}$ then it does so with an odd multiplicity,*

and $\mathcal{B}$ is maximal for these properties (i.e. if $\mathcal{B}' = \{([A_i, B_i], \ell_i, \eta_i)\}_{i\in J'}$ with $J' \supset J$ also satisfies these conditions, then $J = J'$).

We say that $\mathcal{B}$ is an almost-block *if it satisfies the first two conditions and satisfies the third condition for all $([A, A], 0, \eta)$ except possibly when A is maximal among all $([A, A], 0, \eta)$ appearing in $\mathcal{B}$.*

More generally, we say that $\mathcal{B}$ is a block (or almost-block) if there exists some $\mathcal{E} \in \mathrm{VRep}_\rho(G)$ for which $\mathcal{B}$ is a block. We let $\mathrm{Block}_\rho(G)$, *resp.* $\mathrm{ABlock}_\rho(G)$ *denote the set of all blocks, resp. almost-blocks.*

It is straightforward to check the following fact.

Lemma 4.5. *Any tempered $\mathcal{E} \in \mathrm{VRep}_\rho(G)$ has a unique decomposition into disjoint blocks.*

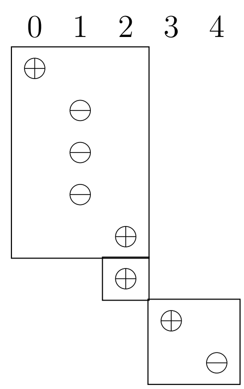
We will denote this decomposition $\mathcal{B}_1 \cup \cdots \cup \mathcal{B}_k$, where each of the $\mathcal{B}_i$ lie in $\mathrm{VRep}_\rho(G)$. Then $\mathcal{E} = \mathcal{B}_1 \cup \cdots \cup \mathcal{B}_k$ consists of the rows of $\mathcal{B}_1$, followed by the rows of $\mathcal{B}_2$, and so on. Note that the decomposition is ordered, in the sense that $\mathcal{B}_1 \cup \mathcal{B}_2 \neq \mathcal{B}_2 \cup \mathcal{B}_1$. As an example, if

$$\mathcal{E} = \begin{pmatrix} \overset{0}{\oplus} & \overset{1}{} & \overset{2}{} & \overset{3}{} & \overset{4}{} \\ & \ominus & & & \\ & \ominus & & & \\ & \ominus & & & \\ & & \oplus & & \\ & & \oplus & & \\ & & & \oplus & \\ & & & & \ominus \end{pmatrix}$$

then $\mathcal{E} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3$, where

$$\mathcal{B}_1 = \begin{pmatrix} \overset{0}{\oplus} & \overset{1}{} & \overset{2}{} \\ & \ominus & \\ & \ominus & \\ & \ominus & \\ & & \oplus \end{pmatrix}, \mathcal{B}_2 = \begin{pmatrix} \overset{2}{\oplus} \end{pmatrix}, \mathcal{B}_3 = \begin{pmatrix} \overset{3}{\oplus} & \overset{4}{} \\ & \ominus \end{pmatrix}.$$

In the future we will illustrate decompositions into blocks by putting boxes around each block, as in the following picture.



We define the remove column operator.

Definition 4.6. *Let $\mathcal{E} \in \mathrm{VRep}_\rho(G)$ be tempered. We define the* remove column operator rc_k *which removes the kth column of $\mathcal{E}$. More precisely, $\mathrm{rc}_k(\mathcal{E})$ consists of all the extended segments of $\mathcal{E}$ except those of the form $([k,k],0,\eta)$ for any η. If no such extended segments exist, then $\mathrm{rc}_k(\mathcal{E}) = \mathcal{E}$.*

Let $\mathcal{B} \in \mathrm{ABlock}(G)$. We let $\Psi(\pi(\mathcal{B}))$ denote the set of formal local Arthur parameters $\psi_{\mathcal{F}}$ where $\mathcal{F} \in \mathrm{VRep}_\rho(G)$ is such that $\mathcal{F}$ and $\mathcal{B}$ are related by a sequence of basic operators. We determine the cardinality of this set in the following theorem.

Theorem 4.7 (count for blocks)**.** *Let $\mathcal{B}$ be a block starting at c_{min}. Let c_{max} be the last column of $\mathcal{B}$, and let $m_{c_{max}-1}$ be the multiplicity of $c_{max}-1$ in $\mathcal{B}$. Let*

$$\mathcal{B}' := \mathrm{rc}_{c_{max}}(\mathcal{B})$$
$$\mathcal{B}'' := \mathrm{rc}_{c_{max}-1}(\mathcal{B}')$$

First suppose that $c_{min} = 0$. Then

$$|\Psi(\pi(\mathcal{B}))| = \begin{cases} 3|\Psi(\pi(\mathcal{B}'))| & \text{if } m_{c_{max}-1} = 1 \\ 4|\Psi(\pi(\mathcal{B}'))| - |\Psi(\pi(\mathcal{B}''))| & \text{if } m_{c_{max}-1} = 3, 5, \ldots. \end{cases} \tag{4.1}$$

On the other hand, if $c_{min} > 0$, then

$$|\Psi(\pi(\mathcal{B}))| = \begin{cases} 2|\Psi(\pi(\mathcal{B}'))| & \text{if } m_{c_{max}-1} = 1 \\ 3|\Psi(\pi(\mathcal{B}'))| - |\Psi(\pi(\mathcal{B}''))| & \text{if } m_{c_{max}-1} = 3, 5, \ldots. \end{cases} \tag{4.2}$$

We will discuss the motivation behind Theorem 4.7 in §5.1 and give a proof in §8. Next, we give a count for the "theta lift" of an almost-block.

Theorem 4.8 (count for theta lifts of almost-blocks)**.** *Let $\mathcal{B}$ be an almost-block starting at zero. Let c_{max} be the last column of $\mathcal{B}$, and let $m_{c_{max}}$ its multiplicity in $\mathcal{B}$. Let*

$$\mathcal{B}' := \mathrm{rc}_{c_{max}}(\mathcal{B}).$$

Then

$$|\Psi(\pi(\Theta_1(\mathcal{B})))| = \begin{cases} 3|\Psi(\pi(\mathcal{B}))| & \text{if } m_{c_{max}} = 1 \\ 3|\Psi(\pi(\mathcal{B}))| - |\Psi(\pi(\mathcal{B}'))| & \text{if } m_{c_{max}} = 2, 4, \ldots \\ 4|\Psi(\pi(\mathcal{B}))| - |\Psi(\pi(\mathcal{B}'))| & \text{if } m_{c_{max}} = 3, 5, \ldots. \end{cases}$$

We will discuss the reasoning behind the above theorem in §5.2 and prove it in §9.1.

Next we state how the numbered of local Arthur parameters containing a fixed tempered representation is determined by its block decomposition.

Theorem 4.9 (count for tempered representations)**.** *Let $\mathcal{E} \in \mathrm{VRep}_\rho(G)$ be a tempered extended multi-segment. Suppose that $\mathcal{E}$ decomposes into blocks $\mathcal{B}_1, \ldots, \mathcal{B}_k$ in that order. Then*

$$|\Psi(\pi(\mathcal{E}))| = |\Psi(\pi(\mathcal{B}_1))| \cdot \prod_{i=2}^{k} |\Psi(\pi(sh^1(\mathcal{B}_i)))|.$$

We will prove the above theorem in Theorem 10.14.

Remark 4.10. *Note that we can compute $|\Psi(\pi(\mathcal{B}_1))|$ using Equation (4.1) and $|\Psi(\pi(sh^1(\mathcal{B}_i)))|$ using Equation (4.2), so the above theorems together give a complete computation of $|\Psi(\pi(\mathcal{E}))|$ for any tempered $\mathcal{E} \in \mathrm{VRep}_\rho(G)$.*

Note that if $\mathcal{E} \in \mathrm{Rep}^{\mathbb{Z}}(G) = \mathrm{Rep}(G) \cap \mathrm{Vseg}^{\mathbb{Z}}(G)$, then $\mathcal{E}$ has a unique decomposition $\mathcal{E} = \mathcal{E}_{\rho_1} \cup \cdots \cup \mathcal{E}_{\rho_k}$ where $\rho_i \neq \rho_j$ for any $i \neq j$. Furthermore, we have $\Psi(\pi(\mathcal{E})) = \prod_{i=1}^{k} \Psi(\pi(\mathcal{E}_{\rho_i}))$. From this observation and Theorem 4.9, we obtain a formula for computing $\Psi(\pi(\mathcal{E}))$ for any tempered $\mathcal{E} \in \mathrm{Rep}^{\mathbb{Z}}(G) = \mathrm{Rep}(G) \cap \mathrm{Vseg}^{\mathbb{Z}}(G)$. That is, Theorem 4.9 implies Theorem 1.7.

Finally, we give a conjecture for counting the number of local Arthur packets containing a given theta lift of a tempered representation to its first occurrence in the going-up tower (in the sense of Remark 4.12).

Conjecture 4.11 (count for theta lifts of tempered representations)**.** *Let $\mathcal{E} = ([A_i, B_i], l_i, \eta_i) \in \mathrm{VRep}_{\rho}(G_n)$ be a tempered extended multi-segment. If $\mathcal{E}$ does not start at zero then $\Theta_1(\mathcal{E})$ is tempered, so $|\Psi(\pi(\Theta_1(\mathcal{E})))|$ can be computed by Theorem 4.9. Otherwise, suppose $\mathcal{E}$ has a decomposition into blocks $\mathcal{B}_1, \ldots, \mathcal{B}_k$. We split into cases depending on the columns near the end of the first block.*

Let H_{col} be the last column of $\mathcal{B}_1$ and let N_{col} be the first column of $\mathcal{B}_2$.

Case 1. $\underline{N_{col} > H_{col} + 1}$*:*

The representation $\Theta_1(\mathcal{E})$ is tempered.

Now suppose $N_{col} \leq H_{col} + 1$. Let η_H be such that $([H_{col}, H_{col}], 0, \eta_H)$ appears in $\mathcal{E}$, and let η_N be such that $([H_{col}+1, H_{col}+1], 0, \eta_N)$ appears in $\mathcal{E}$. (These are well-defined since $\mathcal{E}$ is tempered and $N_{col} \leq H_{col}+1$, so that an extended segment in column $H_{col}+1$ appears in $\mathcal{E}$.) Also let m_H be the multiplicity of H_{col} in $\mathcal{E}$ and let m_N be the multiplicity of $H_{col}+1$ in $\mathcal{E}$.

Case 2. $\underline{\eta_N \neq \eta_H}$*:*

Let $\mathcal{B}_1' := \mathrm{rc}_{H_{col}}(\mathcal{B}_1)$, $\mathcal{B}_2' := \mathrm{rc}_{H_{col}}(\mathcal{B}_2)$, and $\mathcal{B}_2'' := \mathrm{rc}_{H_{col}+1}(\mathcal{B}_2')$. Let

$$D = \begin{cases} |\Psi(\pi(\mathcal{B}_1))| \cdot |\Psi(\pi(\mathcal{B}_2))| + |\Psi(\pi(\mathcal{B}_1))| \cdot |\Psi(\pi(\mathcal{B}_2'))| \\ \qquad - |\Psi(\pi(\mathcal{B}_1'))| \cdot |\Psi(\pi(\mathcal{B}_2'))| & \text{if } m_H - 1 = 1 \\ 2|\Psi(\pi(\mathcal{B}_1))| \cdot |\Psi(\pi(\mathcal{B}_2))| - |\Psi(\pi(\mathcal{B}_1'))| \cdot |\Psi(\pi(\mathcal{B}_2))| & \text{if } m_H - 1 > 1. \end{cases}$$

If $\mathcal{B}_2''$ is empty then let $U = 0$, and otherwise let

$$U = |\Psi(\pi(\mathcal{B}_1))| \cdot (|\Psi(\pi(\mathcal{B}_2'))| - |\Psi(\pi(\mathcal{B}_2''))|).$$

Then

$$|\Psi(\pi(\Theta_1(\mathcal{E})))| = \big(|\Psi(\pi(\mathcal{B}_1))| \cdot |\Psi(\pi(\mathcal{B}_2))| + D + U\big) \cdot \prod_{i=3}^{k} |\Psi(\pi(\mathcal{B}_i))|.$$

In the remaining four cases, we suppose $\eta_N = \eta_H$.

Case 3. $\underline{m_H \equiv 0 \bmod 2,\ m_N \equiv 1 \bmod 2}$*:*

We have that

$$|\Psi(\pi(\Theta_1(\mathcal{E})))| = |\Psi(\pi(\Theta_1(\mathcal{B}_1 \cup \mathcal{B}_2)))| \cdot \prod_{i=3}^{k} |\Psi(\pi(\mathcal{B}_i))|.$$

Suppose $\mathcal{E}$ has at most two blocks in its block decomposition. Then we have the following recursive formulas:

Case 3.1. $\underline{\mathcal{E} \text{ ends at } H_{col}+1}$*:*

Let $\mathcal{E}' = \mathrm{rc}_{H_{col}+1}(\mathcal{E})$. Then

$$|\Psi(\pi(\Theta_1(\mathcal{E})))| = |\Psi(\pi(\Theta_1(\mathcal{E}')))|.$$

Case 3.2. *$\mathcal{E}$ ends at $H_{col}+2$:*
Let $\mathcal{E}' = \mathrm{rc}_{H_{col}+2}(\mathcal{E})$. Then

$$|\Psi(\pi(\Theta_1(\mathcal{E})))| = 4|\Psi(\pi(\Theta_1(\mathcal{E}')))|.$$

Case 3.3. *otherwise:*
We apply similar recursive formulas as Equation 4.1. More precisely, let c_{max} be the last column of $\mathcal{E}$, and let $m_{c_{max}-1}$ be the multiplicity of $c_{max}-1$. Let $\mathcal{E}' = \mathrm{rc}_{c_{max}}(\mathcal{E})$ and $\mathcal{E}'' = \mathrm{rc}_{c_{max}-1}(\mathcal{E}')$. Then

$$|\Psi(\pi(\Theta_1(\mathcal{E})))| = \begin{cases} 3|\Psi(\pi(\Theta_1(\mathcal{E}')))| & \text{if } m_{c_{max}-1} = 1 \\ 4|\Psi(\pi(\Theta_1(\mathcal{E}')))| - |\Psi(\pi(\Theta_1(\mathcal{E}'')))| & \text{if } m_{c_{max}-1} = 3, 5, \dots \end{cases}.$$

Case 4. *$m_H \equiv 1 \bmod 2$, $m_N \equiv 0 \bmod 2$:*
Then $\Theta_1(\mathcal{E})$ is tempered.

Case 5. *$m_H \equiv 0 \bmod 2$, $m_N \equiv 0 \bmod 2$:*

$$|\Psi(\pi(\Theta_1(\mathcal{E})))| = |\Psi(\theta^{\mathrm{up}}_{-m^{\mathrm{up},\alpha}}(\pi(\mathcal{B}_1 \cup \mathcal{B}_2)))| \cdot \prod_{i=3}^{k} |\Psi(\pi(\mathcal{B}_i))|.$$

Case 6. *$m_H \equiv 1 \bmod 2$, $m_N \equiv 1 \bmod 2$: We have $\Theta_1(\mathcal{E})$ is tempered, and moreover*

$$|\Psi(\pi(\Theta_1(\mathcal{E})))| = |\Psi(\pi(\Theta_1(\mathcal{B}_1)))| \cdot \prod_{i=2}^{k} |\Psi(\pi(\mathcal{B}_i))|.$$

We will prove Cases 1, 4, and 6 of the conjecture in §9.2, Case 3 in §9.3, and Case 5 in §11. We do not prove Case 2; however, we comment on its motivation in §5.3. The conjecture also allows us to compute $|\Psi(\theta^{\mathrm{up}}_{-m^{\alpha,\mathrm{up}}}(\pi(\mathcal{E})))|$ from $|\Psi(\pi(\mathcal{E}))|$ in the following sense.

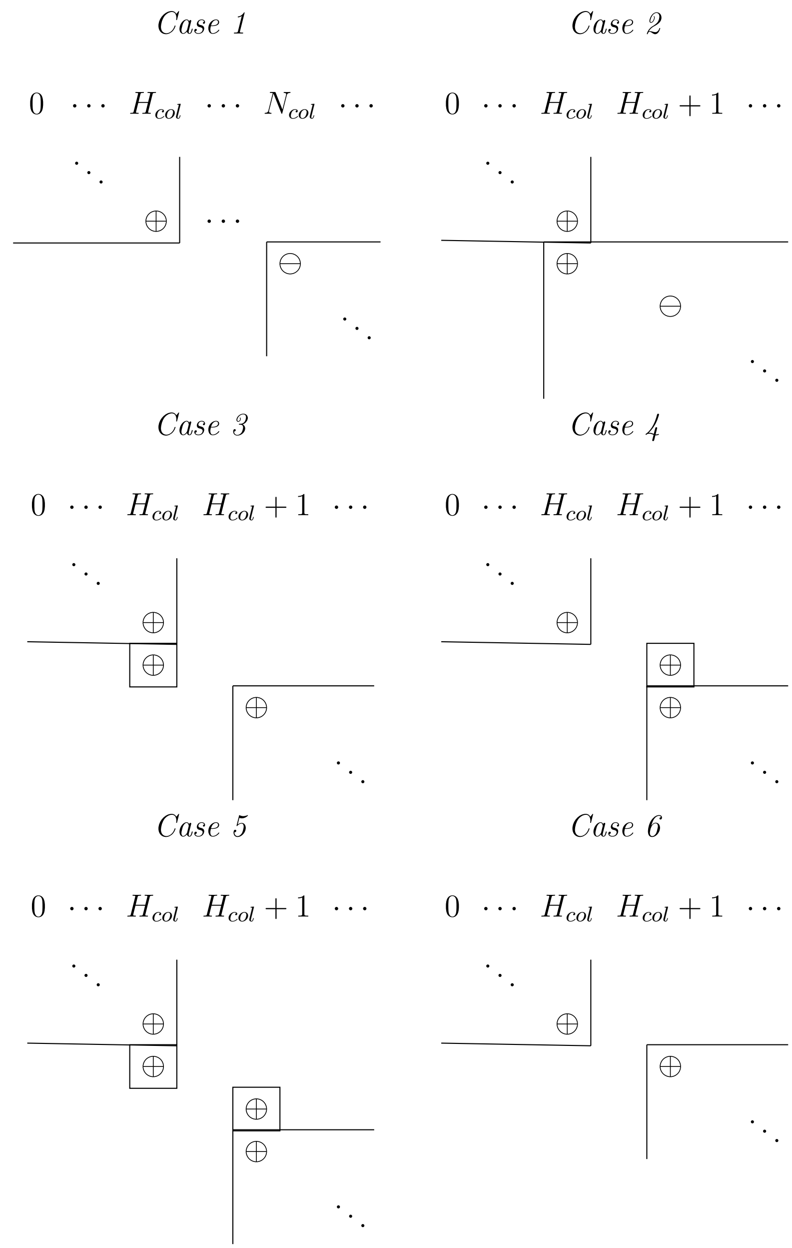
Remark 4.12. *Let $\mathcal{E} \in \mathrm{Rep}(G_n)$. Note that $\Psi(\pi(\mathcal{E})) = \prod_{\rho'} \Psi(\pi(\mathcal{E}_{\rho'}))$ and by Theorems 2.9 and 3.29, we have*

$$\Psi(\theta^{\mathrm{up}}_{-m^{\alpha,\mathrm{up}}}(\pi(\mathcal{E}))) = \Psi(\pi(\mathcal{E}^{\mathrm{up}}_{m^{\mathrm{up},\alpha}})) = \prod_{\rho''} \Psi(\pi((\mathcal{E}^{\mathrm{up}}_{m^{\mathrm{up},\alpha}})_{\rho''}).$$

Moreover, if $\rho \neq \chi_V$, then $|\Psi(\pi(\mathcal{E}_{\rho'}))| = |\Psi(\pi((\mathcal{E}^{\mathrm{up}}_{m^{\mathrm{up},\alpha}})_{\chi_W\chi_V^{-1}\rho'})|$. Thus, we understand that $|\Psi(\theta^{\mathrm{up}}_{-m^{\alpha,\mathrm{up}}}(\pi(\mathcal{E})))|$ can be computed from $|\Psi(\pi(\mathcal{E}))|$ provided that we understand $|\Psi(\pi((\mathcal{E}^{\mathrm{up}}_{m^{\mathrm{up},\alpha}})_{\chi_W})| = |\Psi(\pi(\Theta_1(\mathcal{E}_{\chi_V})))|$ which is described by Conjecture 4.11.

We give several remarks on the conjecture.

Remark 4.13. *The six cases in Conjecture 4.11 correspond to the various ways the first block could end, illustrated below. Case 1 corresponds to the situation where there is a gap between two columns. Case 2 corresponds to the situation where there is a column with an even number of circles, and the signs of the circles in the next column are opposite. Cases 3, 4, 5, and 6 correspond to the situation where the circles in two consecutive columns have the same sign.*



Remark 4.14. *In each of the six cases of Conjecture 4.11, each of the parts of the formulas can be computed using the previously stated results. We explicate this as follows.*

- *In Case 1, since $\Theta_1(\mathcal{E})$ is tempered, we can compute $|\Psi(\pi(\Theta_1(\mathcal{E})))|$ using Theorem 4.9.*
- *In Case 2, $\mathcal{B}_1'$, $\mathcal{B}_2'$, and $\mathcal{B}_2''$ are all blocks so we can compute $|\Psi(\pi(\mathcal{B}_1'))|$, $|\Psi(\pi(\mathcal{B}_2))|$, and $|\Psi(\pi(\mathcal{B}_2''))|$ using Theorem 4.7.*
- *Suppose that we are in Case 3.*
  - *In Case 3.1, since $\mathcal{E}$ contains circles in only one extra column after the first block, namely $H_{col}+1$, $\mathcal{E}'$ is almost-block, since it is $\mathcal{B}_1$ together with an extra circle in column H_{col}. So we can compute $|\Psi(\pi(\Theta_1(\mathcal{E}')))|$ using Theorem 4.8.*

  - *In Case 3.2, we note that the going-up tower and $m^{\mathrm{up},\alpha}$ are the same for $\mathcal{E}$ and $\mathcal{E}'$, since they only differ in columns after the first block, so in fact $\mathcal{E}'$ falls directly into Case 3.1.*
  - *In Case 3.3, we similarly note that $\mathcal{E}$, $\mathcal{E}'$, and $\mathcal{E}''$ all have the same up tower and value of $m^{\mathrm{up},\alpha}$, so again the recursive formulas eventually reduce to Case 3.1 or Case 3.2.*
- *In Case 4, as in Case 1, $\Theta_1(\mathcal{E})$ is tempered, so we can apply Theorem 4.9.*
- *In Case 5, note that since $m_H \equiv 0 \bmod 2$, the second block $\mathcal{B}_2$ consists of a single extended segment $([H_{col}, H_{col}], 0, \eta_H)$. So $\mathcal{B}_1 \cup \mathcal{B}_2$ is an almost-block, and hence $|\Psi(\pi(\Theta_1(\mathcal{B}_1 \cup \mathcal{B}_2)))|$ can be computed by Theorem 4.8. Note that the going-up tower and $m^{\mathrm{up},\alpha}$ are the same for $\mathcal{E}$ as for $\mathcal{B}_1 \cup \mathcal{B}_2$, so we can apply Theorem 4.8 directly. The other terms are just blocks and can be computed by Theorem 4.7.*
- *In Case 6, by similar reasoning, the first term can be computed by Theorem 4.8 and the other terms by Theorem 4.7.*

Remark 4.15. *In Cases 1 and 4 of Conjecture 4.11, while it is possible to compute the number of packets the theta lift lies in, there is no easy formula like in Case 6. This is because it is possible for the theta lift to cause two blocks to interact. For example, suppose*

$$\mathcal{E} = \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \left(\begin{array}{ccccc} \ominus & & & & \\ & \oplus & & & \\ & & & \ominus & \\ & & & \ominus & \\ & & & \ominus & \\ & & & & \oplus \end{array}\right)_{\chi_V} \end{array}.$$

The extended multi-segment corresponding to $\theta^{\mathrm{up}}_{-m^{\mathrm{up},\alpha}}(\pi(\mathcal{E}))$ is

$$\begin{array}{c} \begin{array}{ccccccc} -2 & -1 & 0 & 1 & 2 & 3 & 4 \end{array} \\ \left(\begin{array}{ccccccc} \lhd & \lhd & \oplus & \rhd & \rhd & & \\ & & \ominus & & & & \\ & & & \oplus & & & \\ & & & & & \ominus & \\ & & & & & \ominus & \\ & & & & & \ominus & \\ & & & & & & \oplus \end{array}\right)_{\chi_W} \end{array}.$$

This is equivalent to (via a $dual \circ ui \circ dual$ and a ui^{-1}) the extended multi-segment

$$\begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array}$$
$$\begin{pmatrix} \oplus & & & & \\ & \ominus & & & \\ & & \oplus & & \\ & & & \ominus & \\ & & & \ominus & \\ & & & \ominus & \\ & & & & \oplus \end{pmatrix}_{\chi_W}.$$

Note that this extended multi-segment now consists of a single block, not two. In particular, since the blocks $\mathcal{B}_1$ and $\mathcal{B}_2$ making up $\mathcal{E}$ have become one block, the number of local Arthur packets the theta lift lies in does not depend directly on $|\Psi(\pi(\mathcal{B}_1))|$ or $|\Psi(\pi(\mathcal{B}_2))|$ or other related quantities.

Finally, an analogous result follows from the above for the case where π is an anti-tempered representation.

Theorem 4.16. *Suppose $\mathcal{E}$ is an anti-tempered extended multi-segment, and let $([n, -n], n, \eta(\mathcal{E}))$ be its first row. Let m_n be the multiplicity of the nth column of $dual(\mathcal{E})$. Then:*

$$|\Psi(\theta^{\mathrm{up}}_{-m^{\mathrm{up},\alpha}}(\pi(\mathcal{E})))| = \begin{cases} 4|\Psi(\pi(\mathcal{E}))| - |\Psi(\pi(\mathcal{E}'))| & m_n > 1,\ dual(\mathcal{E}) \textit{ is a block}, \\ 3|\Psi(\pi(\mathcal{E}))| & m_n = 1,\ dual(\mathcal{E}) \textit{ is a block}, \\ 3|\Psi(\pi(\mathcal{E}))| - |\Psi(\pi(\mathcal{E}'))| & m_n > 1 \textit{ odd},\ dual(\mathcal{E}) \textit{ not a block}, \\ 2|\Psi(\pi(\mathcal{E}))| & \textit{otherwise}, \end{cases}$$

where $\mathcal{E}' = (dual \circ \mathrm{rc}_n \circ dual)(\mathcal{E})$.

We prove this theorem in §9.4.

5. Motivation

Throughout this section, we let ρ be an orthogonal supercuspidal representation of some $\mathrm{GL}_d(F)$ (often omitted in the notation). Let $([\frac{\alpha-1}{2}, -\frac{\alpha-1}{2}]_{\chi_W}, \frac{\alpha-1}{2}, \eta)$ denote the added extended segment in $\Theta_1(\mathcal{E})$ for some $\mathcal{E} \in \mathrm{VRep}_\rho(G_n)$. From the point of view of Definition 6.13, this extended segment is called a "hat". Throughout this section, we shall refer to it as the hat. Often, we apply a $dual \circ ui \circ dual$ which involves the hat with some other row in $\mathcal{E}$. We call refer to this operation as "dualizing the hat" (see Definition 7.25).

We give a sketch of the explanations for the main results and conjectures in Section 4.

5.1. **Theorem 4.7.** For simplicity, we only explain Equation (4.2) in this subsection. We remark that the motivation for Equation (4.1) arises from the theta lift which we discuss in the next subsection.

We observe that the only rows of $\mathcal{B}$ with support in column c_{max} are the circles in that column. After any number of row exchanges, union-intersections, and duals, the right endpoint of a row does not change. Furthermore, we claim that only the first circle in column c_{max} can interact with the other rows (see Theorem 7.17). Hence to count the number of extended multi-segments equivalent to $\mathcal{B}$, we can do casework based on what the row with support ending at c_{max} looks like. In each case, we perform some operation with the circle in column c_{max}, and then count the number of operations that are possible after that.

If the multiplicity $m_{c_{max}}$ is 1, then we just have two cases:

- either the circle in column c_{max} stays where it is, or
- we apply ui to the circle in column c_{max} with a circle in column $c_{max}-1$.

In either case we claim that the possible operations on the extended multi-segment that do not involve column c_{max} are exactly the same as those on $\mathcal{B}'$. Hence the count in either case is $|\Psi(\pi(\mathcal{B}'))|$, for a total of $2|\Psi(\pi(\mathcal{B}'))|$.

If the multiplicity $m_{c_{max}}$ is greater than 1, then we have three cases:

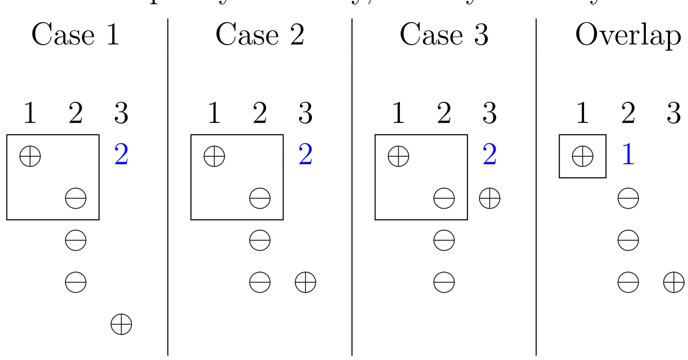
- either the circle in column c_{max} stays where it is,
- we apply the ui operator to the circle in column c_{max} with the last circle in column $c_{max}-1$, or
- we apply the ui operator on the circle in column c_{max} with the last circle in column $c_{max}-1$ and the resulting row is exchanged up until it is the first among rows with support starting at $c_{max}-1$.

Note that the second two cases differ by a row exchange, but in these cases we do not allow row operations involving the rows coming after the first row with support starting in $c_{max}-1$. This avoids double counting except in the case where the rows with support including $c_{max}-1$ are left completely unchanged, of which there are $|\Psi(\pi(\mathcal{B}''))|$. Hence our total count is $3|\Psi(\pi(\mathcal{B}'))|-|\Psi(\pi(\mathcal{B}''))|$.

Illustrated below are the three cases when

$$\mathcal{B}=(([0,0],0,1),([1,1],0,-1),([1,1],0,-1),([1,1],0,-1),([2,2],0,1)),$$

with a box in each case to highlight the row operations which are allowed. The number in blue below denotes the count of the number of equivalent virtual extended multi-segments of the adjacent box. The depiction of where the circle in column 3 is located is purely arbitrary, as they differ by a row exchange.



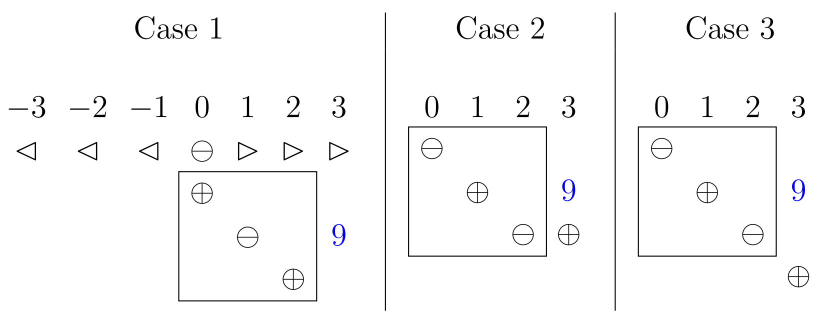
By direct computation, Theorem 4.7 is verified for the above example.

5.2. **Theorem 4.8.** Next we discuss Theorem 4.8. In the setting of Theorem 4.8, $\Theta_1(\mathcal{B})$ is the same as $\mathcal{B}$ but with an extra hat, and this hat is wider than all other rows. Since the right endpoint of this row (the hat) does not change after row operations, in any extended multi-segment equivalent to $\Theta_1(\mathcal{B})$, we can always uniquely identify this row. To count the number of equivalent extended multi-segments, we can therefore do casework on what this row looks like.

First suppose that the multiplicity of the last column is 1. Then we have three cases:

- either the hat stays where it is,
- the hat is dualized to the last row, or
- the hat is dualized to the last row, and then an extra circle is split off.

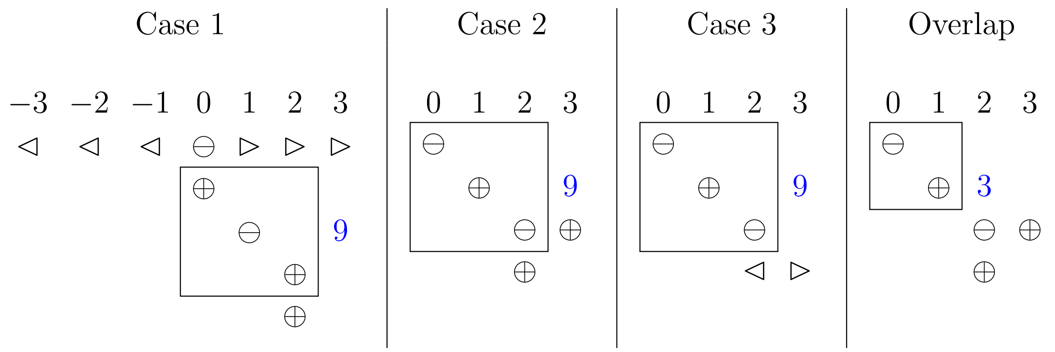
In the first and third cases it is clear that the operations on $\mathcal{B}$ are unaffected, so we obtain a count of $|\Psi(\pi(\mathcal{B}))|$. In the second case it is slightly less clear, but it turns out that exactly the same operations can be done on this extended multi-segment as on $\mathcal{B}$, since the extra circle is so far to the right that it does not interact. Hence this gives a count of $3|\Psi(\pi(\mathcal{B}))|$. Shown below are the three cases when $\mathcal{B} = (([0,0],0,1),([1,1],0,-1),([2,2],0,1))$.



Second suppose that the multiplicity of the last column is $2, 4, \ldots$. Then we consider three possibilities:

- the hat stays where it is,
- the hat is dualized to the second-to-last circle, or
- the hat is dualized to the last circle.

In the first case, $\mathcal{B}$ is unaffected so there are $|\Psi(\pi(\mathcal{B}))|$ possibilities. In the second case, we claim that the extra circle in the second-to-last row does not affect the operations that can be performed on $\mathcal{B}$. (This is clear if the multiplicity of the last column is > 2, but otherwise it requires some sort of argument.) Finally, in the third case it is clear that $\mathcal{B}$ is unaffected. However the second and third cases overlap in the case that the operations on $\mathcal{B}$ do not involve the second-to-last column, because then the extended multi-segments differ by a row exchange. There are $|\Psi(\pi(\mathcal{B}'))|$ of these. Hence we obtain the count $3|\Psi(\pi(\mathcal{B}))| - |\Psi(\pi(\mathcal{B}'))|$. Shown below are the three cases when $\mathcal{B} = ([0,0],0,1),([1,1],0,-1),([2,2],0,1),([2,2],0,1))$, as well as the overlap between the last two cases.



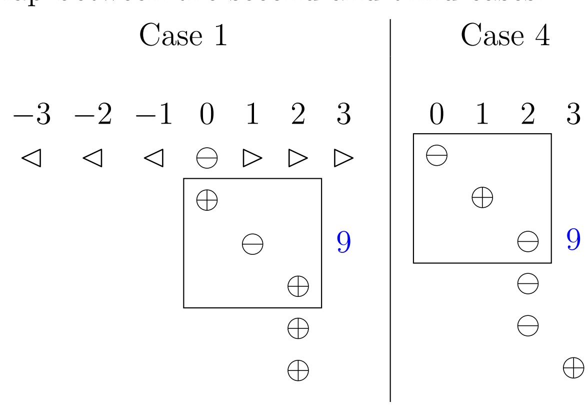
Finally suppose that the multiplicity of the last column is $3, 5, \dots$. Then the reasoning is much the same as before. Our four cases are:

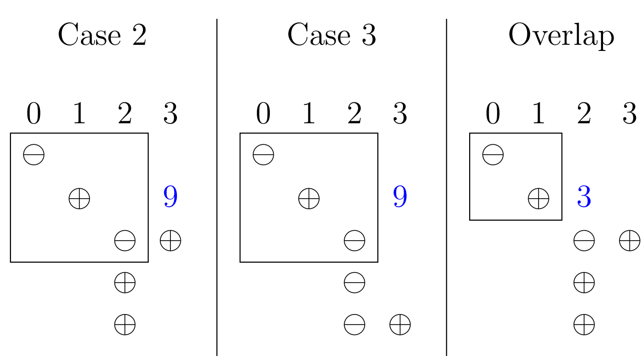
- the hat stays where it is,
- the hat is dualized to the first circle in the last column,
- the hat is dualized to some other circle in the last column,
- or the hat is dualized to the last circle in the last column, and then an extra circle is split off.

The second and third cases have the same overlap as before, while the first and fourth cases contribute $|\Psi(\pi(\mathcal{B}))|$. So we get a total count of $4|\Psi(\pi(\mathcal{B}))| - |\Psi(\pi(\mathcal{B}'))|$. Shown below are the cases when

$$\mathcal{B} = (([0,0],0,1),([1,1],0,-1),([2,2],0,1),([2,2],0,1),([2,2],0,1)),$$

as well as the overlap between the second and third cases.





5.3. **Conjecture 4.11.** Now we turn to Conjecture 4.11. Except for Case 2 of Conjecture 4.11, most of the cases will follow from the theory of blocks and almost blocks developed in the course of the proofs of Theorems 4.7 and 4.8 (albeit Case 5 requires further work, see §11).

Consequently, we will only explain the motivation for the formulas in Case 2 of Conjecture 4.11. By a similar argument as in the proof of Theorem 4.8, we can argue that the blocks after the first two are independent, in the sense that the count for the theta lift of $\mathcal{E}$ is the same as the count for the theta lift of $\mathcal{B}_1 \cup \mathcal{B}_2$ multiplied by $\prod_{i=3}^{k} |\Psi(\pi(\mathcal{B}_i))|$. So it suffices to consider the case when $\mathcal{E}$ decomposes into two blocks $\mathcal{B}_1$ and $\mathcal{B}_2$.

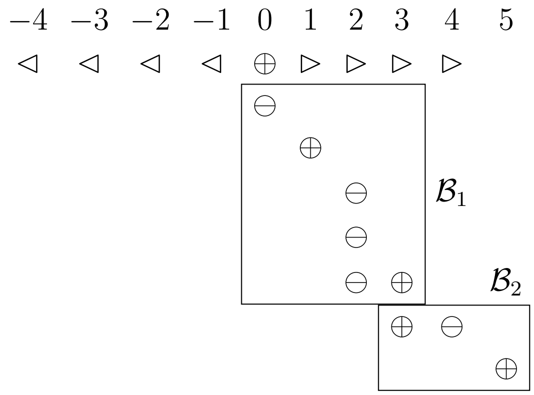
We take a slightly different perspective than in the previous formulas. Previously we started with a block $\mathcal{B}$, did casework on a certain row, and then performed arbitrary operations on a smaller extended multi-segment. Now we consider an arbitrary extended multi-segment $\mathcal{E}'$ equivalent to a tempered extended multi-segment $\mathcal{E}$, and ask what operations are possible on $\Theta_1(\mathcal{E}')$ that were not possible in $\mathcal{E}'$.

For each $\Theta_1(\mathcal{E}')$, the operations are of the following type. (It is not immediately clear that these are all the possible operations.) First we can perform some number of $dual \circ ui \circ dual$ operations between the hat in the first row and another row. Second we can perform additional ui operations between rows in extended multi-segments equivalent to $\mathcal{B}_1$ and $\mathcal{B}_2$. (In other words, the blocks are not "independent" as in Proposition 10.13.) The total number of extended multi-segments coming from the first type of operation is given by D, and the number coming from the second type of operation is given by U.

To give a count of D, we note that there are usually exactly two $dual \circ ui \circ dual$ operations involving the hat in the first row. This is for basically the same reasons as in the almost-block case (see Lemma 9.4): as long as the first circle in column H_{col} is combined with some row in $\mathcal{B}_1'$, either in a row of circles or in a hat, we can dualize the hat to that row; moreover, since m_H is even and hence greater than 1, there is always at least one additional circle in column H_{col} leftover to which the hat can dualize. There cannot be more than two dualizations because only the first circle in H_{col} can interact with $\mathcal{B}_1'$, and the dualizations to the remaining circles all differ by row exchanges.

The exceptions to the above reasoning are the following. In the case that $m_H - 1 > 1$ the only exception is that the first circle in H_{col} might not be combined with anything in $\mathcal{B}_1'$. In this case there are $|\Psi(\pi(\mathcal{B}_1'))| \cdot |\Psi(\pi(\mathcal{B}_2))|$ possibilities. This explains the second case of the formula for D, which is much the same as in the almost block-case (Theorem 4.8).

The case that $m_H - 1 = 1$ is more subtle. In these cases it is possible that there is only one $dual \circ ui \circ dual$ for another reason: the last circle in H_{col}, which belongs to $\mathcal{B}_2$, might be ui'd with another circle in $H_{col} + 1$, thereby "blocking" the $dual \circ ui \circ dual$ with the hat. For example, in the extended multi-segment below, there is only one $dual \circ ui \circ dual$ possible despite the fact that the first circle in column 3 has been combined with previous rows.



So, we count as follows:

- start with two $dual \circ ui \circ dual$ operations for each of the $\mathcal{E}'$, giving

$$2|\Psi(\pi(\mathcal{B}_1))| \cdot |\Psi(\pi(\mathcal{B}_2))|,$$

- subtract the case that there is only one $dual \circ ui \circ dual$ operation because the first circle in H_{col} is unchanged, of which there are

$$|\Psi(\pi(\mathcal{B}_1'))| \cdot |\Psi(\pi(\mathcal{B}_2))|,$$

- subtract the case that there is only one $dual \circ ui \circ dual$ operation because the second circle in H_{col} has been combined with other circles, of which there are

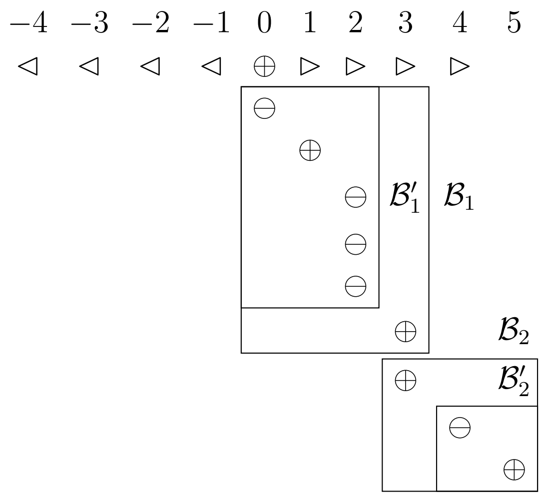
$$|\Psi(\pi(\mathcal{B}_1))| \cdot (|\Psi(\pi(\mathcal{B}_2))| - |\Psi(\pi(\mathcal{B}_2'))|),$$

- add back the overlap between the previous two cases, of which there are

$$|\Psi(\pi(\mathcal{B}_1'))| \cdot (|\Psi(\pi(\mathcal{B}_2))| - |\Psi(\pi(\mathcal{B}_2'))|).$$

The first two formulas are exactly the same as before. The third formula can be derived by complementary counting. We want to find the number of ways to form a $\mathcal{B}_2'$ equivalent to $\mathcal{B}_2$ such that the first row has more than one circle. The total number of such $\mathcal{B}_2'$ is $|\Psi(\pi(\mathcal{B}_2))|$, and the number of such $\mathcal{B}_2'$ where the first row has only one circle is just $|\Psi(\pi(\mathcal{B}_2'))|$, since we do not allow operations on the first row. Finally the fourth formula can be derived in a similar way. Combining the formulas and simplifying gives the claimed value of D for $m_H - 1 = 1$ in Case 2 of Conjecture 4.11.

Continuing in the above example, we have $\mathcal{B}_1'$ and $\mathcal{B}_2'$ are as follows.



We have the following counts:

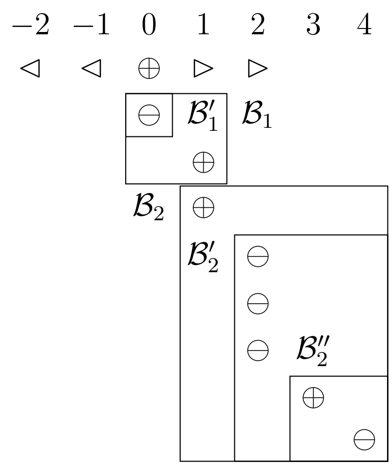
$$\begin{aligned}|\Psi(\pi(\mathcal{B}_1))| &= 33\\ |\Psi(\pi(\mathcal{B}_1'))| &= 9\\ |\Psi(\pi(\mathcal{B}_2))| &= 4\\ |\Psi(\pi(\mathcal{B}_2'))| &= 2.\end{aligned}$$

So the formula gives

$$D = 2(33\cdot 4) - (9\cdot 4 + 33\cdot(4-2) - 9\cdot(4-2)) = 180.$$

Finally we explain the count of U, the number of additional ui operations involving rows from extended multi-segments equivalent to $\mathcal{B}_1$ and $\mathcal{B}_2$. These occur in the following way: first, we perform a $dual\circ ui\circ dual$ operation between the hat and the last circle in column H_{col}, creating a row with a pair of triangles. (We claim this operation is always possible, even if the extended multi-segment is not tempered.) Then, if the row immediately following the pair of triangles has more than one circle, we can perform a union-intersection between the two rows. The number of new extended multi-segments created by this operation is possible is $|\Psi(\pi(\mathcal{B}_1))|\cdot(|\Psi(\pi(\mathcal{B}_2'))| - |\Psi(\pi(\mathcal{B}_2''))|)$. The term $|\Psi(\pi(\mathcal{B}_1))|$ comes from the fact that the first part of the extended multi-segment can be anything equivalent to $\mathcal{B}_1$, and the term $|\Psi(\pi(\mathcal{B}_2'))| - |\Psi(\pi(\mathcal{B}_2''))|$ comes from the fact that we need the first row with support in $H_{col}+1$ to have more than one circle.

As an example, consider the following theta lift of a tempered extended multi-segment.

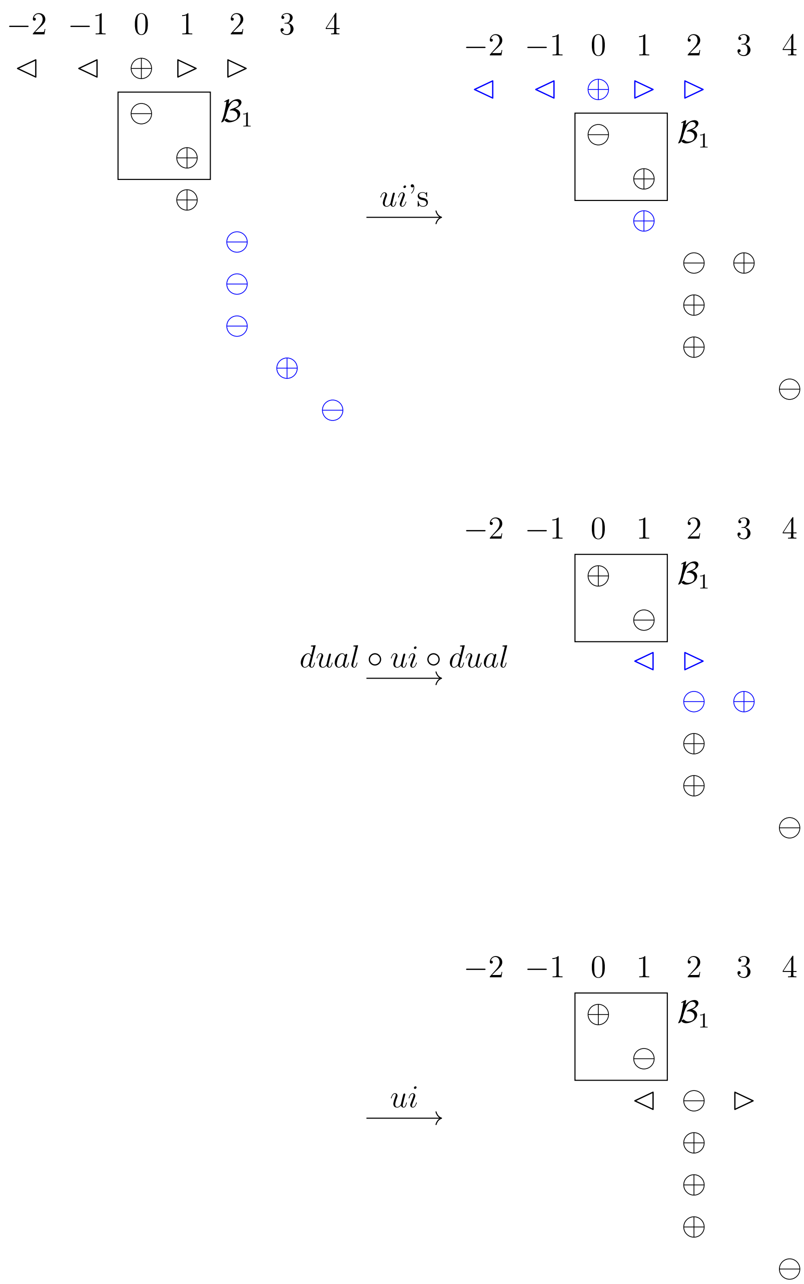


There are 6 additional extended multi-segments created by this *ui* operation, which are:

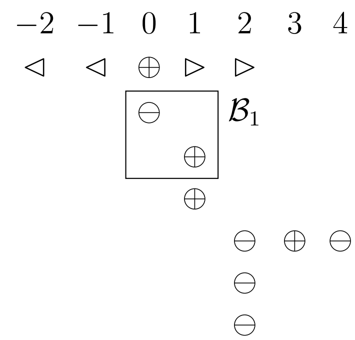
$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ \oplus & & & & \\ & \ominus & & & \\ & \lhd & \ominus & \rhd & \\ & & \oplus & & \\ & & \oplus & & \\ & & \oplus & & \\ & & & & \ominus \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ \oplus & \ominus & & & \\ & \lhd & \ominus & \rhd & \\ & & \oplus & & \\ & & \oplus & & \\ & & \oplus & & \\ & & & & \ominus \end{pmatrix} \quad \begin{pmatrix} -1 & 0 & 1 & 2 & 3 & 4 \\ \lhd & \oplus & \rhd & & & \\ & \ominus & & & & \\ & & \lhd & \ominus & \rhd & \\ & & & \oplus & & \\ & & & \oplus & & \\ & & & \oplus & & \\ & & & & & \ominus \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ \oplus & & & & \\ & \ominus & & & \\ & \lhd & \ominus & \oplus & \rhd \\ & & \ominus & & \\ & & \ominus & & \\ & & \ominus & & \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ \oplus & \ominus & & & \\ & \lhd & \ominus & \oplus & \rhd \\ & & \ominus & & \\ & & \ominus & & \\ & & \ominus & & \end{pmatrix} \quad \begin{pmatrix} -1 & 0 & 1 & 2 & 3 & 4 \\ \lhd & \oplus & \rhd & & & \\ & \ominus & & & & \\ & & \lhd & \ominus & \oplus & \rhd \\ & & & \ominus & & \\ & & & \ominus & & \\ & & & \ominus & & \end{pmatrix}$$

The first three are created by the following process, with an arbitrary equivalent extended multi-segment in place of $\mathcal{B}_1$. Note that the union-intersection is only valid in the third step because the 4th row has more than one circle.



The second three are created by an exactly analogous process, except that in the second step, the extended multi-segment takes the following form.



6. Basic notions and lemmas

In this section we introduce some basic notions and lemmas which will be helpful in the proofs of our results. First, we give a definition which counts the number of circles in the symbol associated to an extended segment (see Example 3.2) which we will also call a row in order to invoke the idea of the associated symbol.

Definition 6.1. *Let $r = ([A, B]_\rho, l, \eta)$ be a row. Then we denote the number of circles in r by $C(r)$, which is given by*

$$C(r) := b - 2l.$$

For $\mathcal{E}$ a virtual extended multi-segment, we let $C(\mathcal{E})$ be the total number of circles in $\mathcal{E}$, which is given by

$$C(\mathcal{E}) := \sum_{r \in \mathcal{E}} C(r).$$

We note that $C(r) + 1 \equiv A - B \mod 2$. Next, we attach a sign to $\mathcal{E}$.

Definition 6.2. *We let the* sign $\eta(\mathcal{E})$ *of an extended multi-segment $\mathcal{E}$ be*

$$\eta(\mathcal{E}) := \eta(r_1),$$

where r_1 is the first row in $\mathcal{E}$ in the admissible order.

6.1. **The alternating sign condition.** The alternating sign condition is an important condition on two rows which affects how they interact under the basic operations.

Definition 6.3. *Let r_1, r_2 be rows with $r_1 < r_2$ in some admissible order. We say that r_1 and r_2 (in that order) satisfy the* alternating sign condition *if*

$$\eta(r_2) = (-1)^{C(r_1)} \eta(r_1).$$

We say that an extended multi-segment $\mathcal{E}$ is alternating *if any two consecutive rows of $\mathcal{E}$ satisfy the alternating sign condition.*

In terms of symbols (see Example 3.2), the alternating sign condition says that the last circle of the first row r_1 and the first circle of the second row r_2 have opposite signs.

Definition 6.4. *Let r be a row. We call $(-1)^{C(r)-1}\eta(r)$ the* sign of the last circle of r.

Note that the above definition makes sense even if, in a picture, r does not have any circles. We now seek to prove some properties about rows that satisfy the alternating sign condition. First, we need the following lemma.

Lemma 6.5. *Let ρ be a unitary orthogonal supercuspidal representation of $\mathrm{GL}_d(F)$, $\mathcal{E} \in \mathrm{VRep}_\rho(G)$, and $r \in \mathcal{E}$ be a row. Let $\widehat{r}$ be the image of r under dual (see Definition 3.11). Then*

$$\eta(\widehat{r}) = (-1)^{C(\mathcal{E})-C(r)}\eta(r).$$

Proof. Suppose $r = r_i$ is the ith row of $\mathcal{E}$. Since $B \in \mathbb{Z}$ for every row, by Definition 3.11,

$$\eta(\widehat{r}) = (-1)^{\alpha_i+\beta_i}\eta(r),$$

where $\alpha_i = \sum_{j<i} a_j$ and $\beta_i = \sum_{j<i} b_j$. Since ρ is orthogonal, for any j we have $a_j \equiv b_j \bmod 2$, so

$$\alpha_i + \beta_i \equiv \sum_{j\neq i} b_j \equiv \sum_{j\neq i} C(r_j) \equiv C(\mathcal{E}) - C(r_i) \bmod 2. \qquad \square$$

We use the above lemma to verify that *dual* preserves the alternating sign condition.

Lemma 6.6. *Let ρ be a unitary orthogonal supercuspidal representation of $\mathrm{GL}_d(F)$ and $\mathcal{E} \in \mathrm{VRep}_\rho(G)$.*

(1) If $r_1 < r_2$ are two rows of $\mathcal{E}$ satisfying the alternating sign condition, then the rows $\widehat{r_2} < \widehat{r_1}$ in $dual(\mathcal{E})$ (see Definition 3.11) also satisfy the alternating sign condition. In particular, if $\mathcal{E}$ is alternating, then so is $dual(\mathcal{E})$.

(2) If $\mathcal{E}$ is alternating, then $\eta(\mathcal{E}) = \eta(dual(\mathcal{E}))$.

Proof. By Lemma 6.5 and the fact that r_1 and r_2 satisfy the alternating sign condition, we obtain

$$\begin{aligned}
\eta(\widehat{r_1}) &= (-1)^{C(\mathcal{E})-C(r_1)}\eta(r_1) \\
&= (-1)^{C(\mathcal{E})-C(r_1)}\left((-1)^{C(r_1)}\eta(r_2)\right) \\
&= (-1)^{C(\mathcal{E})}\eta(r_2) \\
&= (-1)^{C(\mathcal{E})}\left((-1)^{C(\mathcal{E})-C(r_2)}\eta(\widehat{r_2})\right) \\
&= (-1)^{C(r_2)}\eta(\widehat{r_2}).
\end{aligned}$$

Since $C(r_2) = C(\widehat{r_2})$, this shows $\widehat{r_2}$ and $\widehat{r_1}$ satisfy the alternating sign condition, in that order. This proves Part (1).

For Part (2), let r be the first row of $\mathcal{E}$ and let s be the last row of $\mathcal{E}$. Since $\mathcal{E}$ is alternating, by repeatedly applying the alternating sign condition, we obtain $\eta(s) = (-1)^{C(\mathcal{E})-C(s)}\eta(r)$. By Lemma 6.5, we have $\eta(\widehat{s}) = (-1)^{C(\mathcal{E})-C(s)}\eta(s)$. Combining these two gives that $\eta(\widehat{s}) = \eta(r)$. Since $\widehat{s}$ is the first row of $dual(\mathcal{E})$, this shows $\eta(\mathcal{E}) = \eta(dual(\mathcal{E}))$. $\square$

6.2. **Lemmas on row exchanges.** Much of the substance of the ensuing proofs will depend on understanding how a hat changes under certain row exchanges. The next few lemmas will be important in analyzing the behavior of hats in special cases of these types of operations.

Lemma 6.7. *Suppose $r_1 < r_2$ are consecutive rows satisfying the alternating sign condition and with $\mathrm{supp}(r_1) \supset \mathrm{supp}(r_2)$. Then, if $r_2' < r_1'$ are the images of r_1 and r_2 under a row exchange, we have:*

$$(l(r_1'), \eta(r_1')) = (l(r_1) - C(r_2), (-1)^{C(r_2)}\eta(r_1))$$
$$(l(r_2'), \eta(r_2')) = (l(r_2), (-1)^{C(r_1)+1}\eta(r_2)).$$

Moreover, $r_2' < r_1'$ fail the alternating sign condition.

Proof. First, we calculate that

$$\epsilon = (-1)^{A(r_1)-B(r_1)}\eta(r_1)\eta(r_2) = (-1)^{C(r_1)-1}\eta(r_1)\eta(r_1) \cdot (-1)^{C(r_1)} = -1.$$

So the row exchange falls into Case 1(c) of Definition 3.8. The formulas for $l(r_i'), \eta(r_i')$ follow immediately from the given formulas. To see that these rows fail the alternating sign condition, we note that because row exchange preserves supports, $b(r_2) = b(r_2')$, so $C(r_2) \equiv C(r_2') \bmod 2$. Applying this fact and using the above formulas,

$$\eta(r_2') = (-1)^{C(r_1)+1}\eta(r_2) = -\eta(r_1) = -(-1)^{C(r_2')}\eta(r_1'). \qquad \square$$

Lemma 6.8. *Suppose $r_1 < r_2$ are consecutive rows failing the alternating sign condition with $\mathrm{supp}(r_1) \subset \mathrm{supp}(r_2)$, and suppose that $C(r_2) \geq 2C(r_1)$. Then, if $r_2' < r_1'$ are the images of r_1 and r_2 under a row exchange, we have:*

$$(l(r_1'), \eta(r_1')) = (l(r_1), (-1)^{C(r_2)+1}\eta(r_1))$$
$$(l(r_2'), \eta(r_2')) = (l(r_2) + C(r_1), (-1)^{C(r_1)}\eta(r_2)).$$

Moreover, r_2' and r_1' satisfy the alternating sign condition.

Proof. Since r_1 and r_2 fail the alternating sign condition, we obtain that $\epsilon = 1$ (see Definition 3.8 for notation). Moreover, since $\mathrm{supp}(r_1) \subset \mathrm{supp}(r_2)$ and $C(r_2) \geq 2C(r_1)$, we fall into Case 2(b) of Definition 3.8. The formulas follow immediately from the given formulas. Finally, r_2' and r_1' satisfy the alternating sign condition because

$$\eta(r_1') = (-1)^{C(r_2)+1}\eta(r_1) = (-1)^{C(r_1)+C(r_2)}\eta(r_2) = (-1)^{C(r_2')}\eta(r_2'). \qquad \square$$

Lemma 6.9. *Suppose that $r_0 < r_1 < \cdots < r_k$ are consecutive rows satisfying the alternating sign condition with $\mathrm{supp}(r_0) \supset \mathrm{supp}(r_i)$ for $i > 0$. Let $r_1' < \cdots < r_k' < r_0^{(k)}$ be their images after r_0 is row exchanged k times. Then we have*

$$(l(r_0^{(k)}), \eta(r_0^{(k)})) = \left(l(r_0) - \sum_{i=1}^{k} C(r_i), (-1)^{\sum_{i=1}^{k} C(r_i)}\eta(r_0) \right),$$
$$(l(r_i'), \eta(r_i')) = (l(r_i'), (-1)^{C(r_0)+1}\eta(r_i)) \text{ for } i > 0.$$

Moreover, after the row exchanges every pair of adjacent rows satisfies the alternating sign condition except r_k' and $r_0^{(k)}$.

Proof. We proceed by induction on the number of times r_0 is exchanged. The base case is when r_0 is not exchanged, which is clear. Suppose the statement holds after r_0 has been exchanged i times, giving $r'_1 < \cdots < r'_i < r_0^{(i)} < r_{i+1} < \cdots < r_k$. Then since by inductive hypothesis $(r'_i, r_0^{(i)})$ fails the alternating sign condition and r_i and r_{i+1} satisfy the alternating sign condition, we have

$$\begin{aligned}\eta(r_0^{(i)}) &= (-1)^{C(r'_i)+1}\eta(r'_i)\\ &= (-1)^{C(r_i)+1}\left((-1)^{C(r_0)+1}\eta(r_i)\right)\\ &= (-1)^{C(r_0^{(i)})}\eta(r_{i+1}).\end{aligned}$$

So $r_0^{(i)}$ and r_{i+1} satisfy the alternating sign condition. Thus, we can apply Lemma 6.7. Exchanging these rows gives

$$\begin{aligned}l(r_0^{(i+1)}) &= l(r_0^{(i)}) - C(r_{i+1})\\ \eta(r_0^{(i+1)}) &= (-1)^{C(r_{i+1})}\eta(r_0^{(i)})\end{aligned}$$

which are the desired values (after applying the induction hypothesis). Meanwhile $l(r'_i) = l(r_i)$ and

$$\eta(r'_{i+1}) = (-1)^{C(r'_0)+1}\eta(r_{i+1}) = (-1)^{C(r_0)+1}\eta(r_{i+1})$$

as desired. Finally, to show the claims about the alternating sign condition, we observe that r'_i and r'_{i+1} are the same as r_i and r_{i+1} respectively except for their sign, which are both multiplied by the same quantity $(-1)^{C(r_0)+1}$. Since r_i and r_{i+1} satisfy the alternating sign condition, so do r'_i and r_{i+1}. Also, by Lemma 6.7, r'_{i+1} and $r_0^{(i+1)}$ do not alternate. This completes the inductive step. □

Lemma 6.10. *Suppose that $r_1 < \cdots < r_k < r_0$ are consecutive rows such that every pair of consecutive rows satisfies the alternating sign condition except r_k and r_0. Suppose further that* $\mathrm{supp}(r_0) \supset \mathrm{supp}(r_i)$ *for $i > 0$, and that $C(r_0) \geq 2\sum_{i=1}^{k} C(r_i)$. Let $r_0^{(k)} < r'_1 < \cdots < r'_k$ be their images after r_0 is exchanged up k times. Then:*

$$\begin{aligned}(l(r_0^{(k)}), \eta(r_0^{(k)})) &= \left(l(r_0) + \sum_{i=1}^{k} C(r_i), (-1)^{\sum_{i=1}^{k} C(r_i)}\eta(r_0)\right)\\ (l(r'_i), \eta(r'_i)) &= (l(r_i), (-1)^{C(r_0)+1}\eta(r_i)) \text{ for } i > 0.\end{aligned}$$

Moreover, after the row exchanges every pair of adjacent rows satisfies the alternating sign condition.

Proof. The proof is exactly analogous to the proof of Lemma 6.9, except that instead of applying Lemma 6.8, we apply Lemma 6.7. □

Lemma 6.11. *Suppose $h < h_1 < h_2$ are consecutive rows and that* $\mathrm{supp}(h) \supset \mathrm{supp}(h_1), \mathrm{supp}(h_2)$. *Further suppose $C(h) \neq 0$, and that $C(h_1) = C(h_2) = 1$ and $l(h_1) = l(h_2)$. Then h is unchanged after being exchanged with h_1 and h_2. Moreover, h_1 and h_2 are unchanged except that their sign is multiplied by* $(-1)^{C(h)+1}$.

Proof. Note that since $\mathrm{supp}(h) \supset \mathrm{supp}(h_1), \mathrm{supp}(h_2)$ and the support of a row is unchanged by row exchange, both row exchanges will be in Case 1 of Definition 3.8. We let $h'_1 < h' < h'_2$ and $h'_1 < h'_2 < h''$ denote the effects of the row exchanges. The second claim about the sign of h'_1 and h'_2 follows immediately from the definition. Note that since $C(h_1) = 1$, we have that $2l(h_1) + 1 = 2b(h_1)$ and similarly for h_2. For the first claim, we divide into three cases.

First suppose that the first row exchange is in Case 1(a) of Definition 3.8. Then $l(h') = b(h) - l(h) - 1$ and $\eta(h') = \eta(h)$. Since $A(h') = A(h)$, $B(h') = B(h)$, and $\eta(h) = \eta(h')$, and the first row exchange had $(-1)^{A(h)-B(h)}\eta(h)\eta(h_1) = 1$ by assumption, we obtain that the second row exchange also has $\epsilon = 1$. Now note that

$$C(h') = C(h) + 2(l(h) - l(h')) = C(h) + 2(2l(h) - b(h) + 1) = 2 - C(h).$$

Since the first row exchange was in Case 1(a), $C(h) < 2$. Since $C(h) \neq 0$, we must have $C(h) = 1$, and hence the second row exchange will be in Case 1(a). So

$$\begin{aligned} l(h'') &= b(h') - l(h') - 1 \\ &= b(h) - (b(h) - l(h) - 1) - 1 \\ &= l(h) \\ \eta(h'') &= \eta(h') = \eta(h). \end{aligned}$$

This completes the proof in this case.

Second suppose that the first row exchange is in Case 1(b). Then $l(h') = l(h) + 1$ and $\eta(h') = -\eta(h)$. Since ϵ for the first row exchange is 1, and $\eta(h') = -\eta(h)$, we conclude that ϵ for the second row exchange is -1, so it is in Case 1(c). Then $l(h'') = l(h') - 1 = l(h)$ and $\eta(h'') = -\eta(h') = \eta(h)$. This completes the proof in this case.

Finally suppose that the first row exchange is in Case 1(c). Then $l(h') = l(h) - 1$ and $\eta(h') = -\eta(h)$. So using similar reasoning as before, ϵ for the second row exchange is 1. However note that it is impossible for the second row exchange to be in Case 1(a), since $C(h') = C(h) + 2 \geq 2$, so we cannot have $C(h') < 2C(h_2) = 2$. So the second row exchange is in Case 1(b), where the formulas give $l(h'') = l(h') + 1 = l(h)$ and $\eta(h'') = -\eta(h') = \eta(h)$. This completes the proof in this case and hence proves the lemma. □

Lemma 6.12. *Suppose $h_1 < h_2 < h$ are consecutive rows and that* $\mathrm{supp}(h) \subset \mathrm{supp}(h_1), \mathrm{supp}(h_2)$. *Further suppose $C(h) \neq 0$, and that $C(h_1) = C(h_2) = 1$ and $l(h_1) = l(h_2)$. Then h is unchanged after being exchanged with h_1 and h_2. Moreover, h_1 and h_2 are unchanged except that their sign is multiplied by $(-1)^{C(h)}$.*

Proof. The proof is exactly analogous to the proof of Lemma 6.11, except that we are in Case 2 of Definition 3.8 instead of Case 1. □

6.3. **Lemmas on hats.** Recall that our goal is to classify all the extended multi-segments $\mathcal{E}'$ equivalent to a tempered extended multi-segment $\mathcal{E}$. One type of extended segment that appears in such extended multi-segments $\mathcal{E}'$ is of the following form.

Definition 6.13. *A* hat *is an extended segment of the form* $([A, B]_\rho, l, \eta)$ *where* $B = -l$.

The alternating sign condition helps provide an important criterion for whether consecutive hats can interact with each other.

Lemma 6.14. *Given two consecutive hats* $h_1 < h_2$ *in (P') order of the form*

$$h_1 = ([A_1, -B_1], B_1, \eta_1)$$
$$h_2 = ([A_2, -B_2], B_2, \eta_2)$$

it is possible to apply a nontrivial $dual \circ ui \circ dual$ *to* h_1 *and* h_2 *if and only if:*

- $A_2 = B_1 - 1$ *and*
- h_1 *and* h_2 *satisfy the alternating sign condition.*

The resulting row will be of the form

$$h = ([A_1, -B_2], B_2, \eta_1).$$

Proof. After applying *dual*, the corresponding dual rows will be of the form (and order)

$$\widehat{h_2} = ([A_2, B_2], 0, (-1)^{C(\mathcal{E})+C(h_2)} \cdot \eta_2)$$
$$\widehat{h_1} = ([A_1, B_1], 0, (-1)^{C(\mathcal{E})+C(h_1)} \cdot \eta_1)$$

Since these rows are composed entirely of circles, the only nontrivial *ui* is that of type 3'. Since h_1 and h_2 are in (P') order, $B_1 > B_2$. So such a *ui* can be applied if and only if $B_1 = A_2 + 1$ and if $\widehat{h_2}$ and $\widehat{h_1}$ satisfy the alternating sign condition. By Lemma 6.6, the second condition is equivalent to h_1 and h_2 satisfying the alternating sign condition. Applying *ui* gives a new row

$$\widehat{h} = ([A_1, B_2], 0, (-1)^{C(\mathcal{E})+C(h_2)}\eta_2).$$

Dualizing gives row gives a new row of the form

$$h = ([A_1, -B_2], B_2, \eta),$$

for some η.

In order to calculate the new value of η, we note that $C(h') = C(h_1') + C(h_2')$ and that dualization preserves the number of circles in each row. Therefore, $C(h') = C(h_1) + C(h_2)$ and $C(\mathcal{E}) = C(dual(\mathcal{E}))$. Now, we can calculate using Lemma 6.5:

$$\begin{aligned}\eta &= (-1)^{C(dual(\mathcal{E}))-C(\widehat{h})}\left((-1)^{C(\mathcal{E})+C(h_2)}\eta_2\right)\\ &= (-1)^{C(\mathcal{E})-C(h_1)-C(h_2)+C(\mathcal{E})+C(h_2)}\eta_2\\ &= (-1)^{C(h_1)}\eta_2\\ &= \eta_1.\end{aligned}$$

□

Definition 6.15. *We refer to the act of performing a* $dual \circ ui \circ dual$ *of type 3' on two consecutive hats* h_1, h_2 *as described above as* merging *the hats. We say two hats are* mergable *if they satisfy the conditions in Lemma 6.14. We denote the merged hat by* $h_1 * h_2$.

Remark 6.16. *The operation of merging two hats is dual to the operation of combining two rows r_1 and r_2 made entirely of circles through a ui of type $3'$.*

We observe from Lemma 6.14 that C is additive under mergings. Formally, if h_1 and h_2 are two hats, then $C(h_1 * h_2) = C(h_1) + C(h_2)$. It follows immediately from the formulas in Lemma 6.10 that row exchanges commute with the action of merging hats. We state this formally in the following corollary.

Corollary 6.17. *Let $r_1 < r_2 < r_3$ be three consecutive rows in a virtual extended multi-segment.*

- *Suppose r_2 and r_3 are mergable hats, and suppose $\mathrm{supp}(r_2), \mathrm{supp}(r_3) \subset \mathrm{supp}(r_1)$. Then the image of r_1 after being exchanged with r_2 and r_3 is the same as the image of r_1 after being exchanged with $r_2 * r_3$.*
- *Suppose r_1 and r_2 are mergable hats, and suppose $\mathrm{supp}(r_1), \mathrm{supp}(r_2) \subset \mathrm{supp}(r_3)$. Then the image of r_3 after being exchanged with r_2 and r_1 is the same as the image of r_3 after being exchanged with $r_1 * r_2$.*

7. Individual Blocks

Throughout this section, we let ρ be an orthogonal representation of $\mathrm{GL}_d(F)$ and $\mathcal{B} \in \mathrm{Block}_\rho(G)$. Recall that our goal is to determine the size of $\Psi(\pi(\mathcal{B}))$. This problem falls into two cases, namely, whether the block starts at zero or not (see Theorem 4.7). Throughout this section, we let ρ denote a unitary orthogonal supercuspidal representation of $\mathrm{GL}_d(F)$.

7.1. **Constructing Virtual Extended Multi-Segments from $\mathcal{S}$-data.** Let $\mathcal{B} \in \mathrm{Block}_\rho(G)$. We aim to give a classification of extended multi-segments that are equivalent to $\mathcal{B}$ (see Theorem 7.17). We begin by setting some notation. Let $c_{\min} = \min_{r \in \mathcal{B}} B(r)$ and $c_{\max} = \max_{r\in\mathcal{B}} A(r)$ be the integers that *supp*$(\mathcal{B})$ begins and ends at. For $c \in \mathbb{Z}$ we let

$$m_c = \#\{r \in \mathcal{B} \mid A(r) = B(r) = c\}$$

denote the number of rows in $\mathcal{B}$ with support $[c, c]$. We refer to m_c as the *multiplicity* of c. We let the tuple

$$\mathcal{M}_\mathcal{B} = (m_{c_{\min}}, m_{c_{\min}+1}, \ldots, m_{c_{\max}})$$

denote the multiplicities of all rows in $\mathcal{B}$. If $\mathcal{B}$ is fixed, we will simple write $\mathcal{M} = \mathcal{M}_\mathcal{B}$ for brevity. Note that all the numbers in $\mathcal{M}$ are positive odd integers by Definition 4.4. More generally, we fix two integers $c_{\min}, c_{\max} \in \mathbb{Z}_{\geq 0}$ with $c_{\max} \geq c_{\min}$. We say that a tuple

$$\mathcal{M}(c_{\min}, c_{\max}) = (m_{c_{\min}}, m_{c_{\min}+1}, \ldots, m_{c_{\max}}) \in \mathbb{Z}^{c_{\max}-c_{\min}+1}$$

is a *block-tuple* if each m_i is a positive odd integer. Note that if $\mathcal{B}$ is a block, then $\mathcal{M}_\mathcal{B}$ is a block-tuple.

Definition 7.1. *A* valid *tuple $\mathcal{S}$ for a block-tuple $\mathcal{M}(c_{\min}, c_{\max})$ is a tuple $(\mathcal{S}_1, \ldots, \mathcal{S}_k)$ of subsets of $\{c_{\min}, \ldots, c_{\max}\}$ satisfying the following conditions.*

(1) Each $\mathcal{S}_i \subset \{c_{\min}, \ldots, c_{\max}\}$ is a nonempty set of consecutive integers.
(2) $\bigcup_i \mathcal{S}_i = \{p, \ldots, q\}$.

(3) If $i < j$ and $s_i \in \mathcal{S}_i, s_j \in \mathcal{S}_j$, then $s_i \leq s_j$ (consequently, for $j \neq i$, we have $|\mathcal{S}_i \cap \mathcal{S}_j| \leq 1$).
(4) If $i < j$ and $c \in \mathcal{S}_i \cap \mathcal{S}_j$, then $j - i = 1$, $|\mathcal{S}_j| \geq 2$, and $m_c > 1$.
(5) If $i \geq 2$, $|\mathcal{S}_i| \geq 2$, and $c = \min \mathcal{S}_i$ has $m_c > 1$, then $c \in \mathcal{S}_{i-1}$.

Fix a block $\mathcal{B} \in \mathrm{Block}_\rho(G_n)$ and let $\mathcal{M} = \mathcal{M}_{\mathcal{B}}$. To any valid tuple $\mathcal{S}$ for $\mathcal{M}$, we associate a virtual extended multi-segment $\mathcal{E}(\mathcal{M}, \mathcal{S}, \eta)$ in the following manner.

Definition 7.2. *Let $\mathcal{M} = (m_{c_{\min}}, m_{c_{\min}+1}, \ldots, m_{c_{\max}})$ be a block-tuple. Given a sign $\eta \in \{\pm 1\}$ and a valid tuple $\mathcal{S} = (\mathcal{S}_1, \ldots, \mathcal{S}_k)$ for $\mathcal{M}$, we define a virtual extended multi-segment $\mathcal{E}(\mathcal{M}, \mathcal{S}, \eta)$ consisting of the following rows.*

(1) For each $\mathcal{S}_i$, we include a row of circles (i.e. a row with $l = 0$) with support

$$[\max \mathcal{S}_i, \min \mathcal{S}_i].$$

We refer to these rows as chains.

(2) For each $c \in \{c_{\min}, \ldots, c_{\max}\}$, we add $m_c - |\{i \mid c \in \mathcal{S}_i\}|$ copies of a row of circles with support

$$[c, c]$$

We refer to these rows as multiples.

The rows are ordered in a (P') ordering such that if $A(r) < A(r')$ then $r < r'$. In the case where r and r' both have $supp(r) = supp(r') = [c, c]$ but r is a chain and r' is a multiple, we choose the order so that $r < r'$.

The signs η are chosen so that the following conditions hold.

(1) $\eta(\mathcal{E}(\mathcal{M}, \mathcal{S}, \eta)) = \eta$.
(2) If r_i and r_{i+1} are consecutive rows and neither is a multiple, then r_i and r_{i+1}) satisfy the alternating sign condition (see Definition 6.3).
(3) If r_i and r_{i+1} are consecutive rows and r_{i+1} is a multiple, then r_i and r_{i+1}) fail the alternating sign condition.
(4) If r_i and r_{i+1} are consecutive rows and only r_i is a multiple, then r_i and r_{i+1} satisfy the alternating sign condition if and only if $B(r_{i+1}) > B(r_i)$.

Remark 7.3. *Note that the number of multiples $m_c - |\{\mathcal{S}_i \mid c \in \mathcal{S}_i\}|$ is chosen such that the number of circles in column c is always equal to m_c.*

We introduce some terminology related to the study of the virtual extended multi-segments $\mathcal{E}(\mathcal{M}, \mathcal{S}, \eta)$.

Definition 7.4. *If two chains r and r' are associated to $\mathcal{S}_i$ and $\mathcal{S}_{i+1}$ such that $\mathcal{S}_i \cap \mathcal{S}_j \neq \emptyset$, then we say that r' is a* z-chain. *An example is given in Figure 1.*

Definition 7.5. *We say that two chains r_1 and r_2 are* consecutive chains *if they are associated (in the sense of Case (1) of Definition 7.2) with consecutive sets $\mathcal{S}_i$ and $\mathcal{S}_{i+1}$.*

Definition 7.6. *Let r_1 and r_2 be consecutive chains. If r_3 is a multiple such that $r_1 < r_3 < r_2$, then we say that r_3* belongs *to r_1.*

Remark 7.7. *It follows from the sign condition in Definition 7.2 that all multiples belonging to some chain r have the same sign $\eta(r) \cdot (-1)^{C(r)-1}$.*

$$\begin{array}{cccc} 0 & 1 & 2 & 3 \end{array}$$
$$\begin{pmatrix} \oplus & \ominus & & \\ & \ominus & & \\ & \ominus & \oplus & \ominus \end{pmatrix}$$

FIGURE 1. The $\mathcal{S}$ data for the above multi-segment is $(\{0,1\},\{1,2,3\})$. Since $\mathcal{S}_1 \cap \mathcal{S}_2 = \{1\}$, the chain with support $[3,1]$ is a z-chain.

Remark 7.8. *Suppose $\mathcal{E} = \mathcal{E}(\mathcal{M}, \mathcal{S}, \eta)$ as in Definition 7.2. Suppose the circles of $\mathcal{E}$ are read out in order, row by row, then the symbols $\oplus$ or $\ominus$ always appear an odd number of times consecutively. For example, in Figure 1, there is one $\oplus$, then three $\ominus$s, then one $\oplus$, and then one $\ominus$.*

It turns out that for blocks $\mathcal{B}$ starting at zero (i.e., $c_{\min} = 0$), we will need to specify more data to describe all the extended multi-segments equivalent to $\mathcal{B}$. In particular, some extended multi-segments equivalent to $\mathcal{B}$ have rows that are hats, but the construction in Definition 7.2 only produces extended multi-segments with rows having $l = 0$. We therefore make the following modification to the construction $\mathcal{E}(\mathcal{M}, \mathcal{S}, \eta)$ by defining a new parameter $\mathcal{T}$.

Definition 7.9. *Suppose $\mathcal{B}$ starts at zero and let $\mathcal{M} := \mathcal{M}_{\mathcal{B}}$ be the corresponding block-tuple. Let $\mathcal{S} = (\mathcal{S}_1, \ldots, \mathcal{S}_k)$ be a valid tuple for $\mathcal{M}$. Then we say a partition $(\mathcal{T}_i^0, \mathcal{T}_i^1, \ldots, \mathcal{T}_i^{\ell_i})$ of each $\mathcal{S}_i$ is* valid *if*

(1) each $\mathcal{T}_i^j$ is a nonempty set of consecutive integers;
(2) if $j < j'$ and $t_j \in \mathcal{T}_i^j$, $t_{j'} \in \mathcal{T}_i^{j'}$, then $t_j \leq t'_j$;
(3) if $|\mathcal{S}_i \cap \mathcal{S}_{i+1}| \geq 1$, then $|\mathcal{T}_{i+1}^0| \geq 2$.

A valid *tuple $(\mathcal{M}, \mathcal{S}, \mathcal{T})$ is one such that $\mathcal{S}$ and $\mathcal{T}$ are both valid.*

Remark 7.10. *As a means of shorthand, we will represent the subsets $\mathcal{T}_i^j \subset \mathcal{S}_i$ by overlining each $\mathcal{T}_i^j$ for $j > 0$. As an example, suppose $\mathcal{S}_i = \{2,3,4,5,6,7,8,9\}$, and $\mathcal{T}_i^0 = \{2,3,4\}, \mathcal{T}_i^1 = \{5,6\}, \mathcal{T}_i^2 = \{7\}, \mathcal{T}_i^3 = \{8,9\}$. As shorthand, we write*

$$\mathcal{S}_i = \{2,3,4,\overline{5,6},\overline{7},\overline{8,9}\}.$$

Definition 7.11. *Let $\mathcal{M} = (m_{c_{min}}, \ldots, m_{c_{max}})$ be a block-tuple with $c_{min} = 0$. Given valid $(\mathcal{M}, \mathcal{S}, \mathcal{T})$ along with a sign $\eta \in \{\pm 1\}$, we associate an extended multi-segment $\mathcal{E}(\mathcal{M}, \mathcal{S}, \mathcal{T}, \eta)$ as follows.*

(1) For each $\mathcal{T}_i^j$ with $j \geq 1$, we include a hat with support
$$[\max \mathcal{T}_i^j, -\min \mathcal{T}_i^j].$$
(2) For each $\mathcal{T}_i^0 \subset \mathcal{S}_i$, we include a row of circles with support
$$[\max \mathcal{T}_i^0, \min \mathcal{T}_i^0].$$
(3) For each $c \in \{0, \ldots, c_{max}\}$, we add $m_c - |\{i \mid c \in \mathcal{S}_i\}|$ copies of a row of circles with support
$$[c, c]$$
for some $\eta_i \in \{\pm 1\}$.

The order of the rows and the signs of each row follow exactly the same rules as in Definition 7.2.

As before, we call rows falling into case (2) *chains* and rows falling into case (3) *multiples*. We also have exactly the same notions of z-chains (Definition 7.4) and a multiple belonging to a chain (Definition 7.6) as before.

Definition 7.12. *We say that $\mathcal{E}$ is of* type $Y_{\mathcal{M}}$ *if it is of the form $\mathcal{E}(\mathcal{M}, \mathcal{S}, \mathcal{T}, \eta)$ for valid $(\mathcal{M}, \mathcal{S}, \mathcal{T})$ or $\mathcal{E}(\mathcal{M}, \mathcal{S}, \eta)$ for valid $(\mathcal{M}, \mathcal{S})$.*

Definition 7.13. *If $\mathcal{E}$ is of type $Y_{\mathcal{M}}$, then we refer to the associated $\mathcal{S}$ (or $\mathcal{S}$ and $\mathcal{T}$, if applicable) as the* $\mathcal{S}$-data *of $\mathcal{E}$.*

Since the construction above is quite elaborate, we provide an example of a type $Y_{\mathcal{M}}$ virtual extended multi-segment.

Example 7.14. *Let $n = 9$, $\mathcal{M} = (1, 1, 3, 1, 1, 3, 1, 3, 1)$, and $\mathcal{S} = \{\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3, \mathcal{S}_4, \mathcal{S}_5\}$, where*

$$\begin{aligned} \mathcal{S}_1 &= \{0, \overline{1, 2}\} \\ \mathcal{S}_2 &= \{2, 3, \overline{4}\} \\ \mathcal{S}_3 &= \{5\} \\ \mathcal{S}_4 &= \{5, 6, 7\} \\ \mathcal{S}_5 &= \{8, 9\}. \end{aligned}$$

Then the associated multi-segment $\mathcal{E}(\mathcal{M}, \mathcal{S}, \mathcal{T})$ with $\eta(\mathcal{E}) = 1$ is:

$$\begin{array}{cccccccccccccc} -4 & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array}$$
$$\left(\begin{array}{cccccccccccccc} \lhd & \lhd & \lhd & \lhd & \oplus & \rhd & \rhd & \rhd & \rhd & & & & & \\ & & & \lhd & \ominus & \oplus & \rhd & & & & & & & \\ & & & & \ominus & & & & & & & & & \\ & & & & & & \ominus & & & & & & & \\ & & & & & & \ominus & \oplus & & & & & & \\ & & & & & & & & & \ominus & & & & \\ & & & & & & & & & \ominus & & & & \\ & & & & & & & & & \ominus & \oplus & \ominus & & \\ & & & & & & & & & & & & \oplus & \ominus \\ & & & & & & & & & & & & \ominus & \\ & & & & & & & & & & & & \ominus & \end{array}\right)$$

Here, the row with support $[2, 2]$ is a multiple belonging to the chain with support $[0, 0]$ and the multiples with support $[8, 8]$ belong to the chain with support $[8, 9]$. Meanwhile, the first row with support $[5, 5]$ is a chain, and the second such row is a multiple belonging to it.

Example 7.15. *Suppose $\mathcal{B}$ is a block. Then $\mathcal{B}$ is of type $Y_{\mathcal{M}_{\mathcal{B}}}$, where $\mathcal{S}_i = \{i\}$ (and $\mathcal{T}_i^0 = \{i\}$, if $\mathcal{B}$ starts at zero).*

One important property of extended multi-segments of type $Y_{\mathcal{M}}$ is the following lemma.

Lemma 7.16. *If $\mathcal{E}$ is of type $Y_{\mathcal{M}}$, then $\eta(\mathcal{E}) = \eta(dual(\mathcal{E}))$.*

Proof. Let r be the last row of $\mathcal{E}$. Then Lemma 6.5 implies that

$$\eta(dual(\mathcal{E})) = \eta(\widehat{r}) = (-1)^{C(\mathcal{E})-C(r)}\eta(r),$$

where $\widehat{r}$ is the image of r under $dual$. Note that Remark 7.8 holds for all multi-segments of type $Y_{\mathcal{M}}$, even those where $\mathcal{M}$ starts after zero. Therefore, listing the symbols $\oplus$ and $\ominus$ in order produces streaks of odd multiplicity. Therefore, an even number of circles can be removed from $\mathcal{E}$ so that each of these streaks has one circle; i.e., so that $\mathcal{E}$ is alternating. Doing this does not change the sign $(-1)^{C(\mathcal{E})-C(r)}\eta(r)$. Therefore, the result follows from Lemma 6.6. □

This definition of type $Y_{\mathcal{M}}$ has been set up with the intention of proving the following theorem.

Theorem 7.17. *Let $\mathcal{B} \in \mathrm{VRep}_\rho(G_n)$ be a block with multiplicities $\mathcal{M} := \mathcal{M}_{\mathcal{B}}$. Then, the set $\Psi(\pi(\mathcal{B}))$ is precisely the set of $\psi_{\mathcal{E}}$ where $\mathcal{E} \in \mathrm{VRep}_\rho(G_n)$ is of type $Y_{\mathcal{M}}$ with $\eta(\mathcal{E}) = \eta(\mathcal{B})$.*

We will prove this theorem in Section 7.4.

7.2. **Type X_k.** In this section, we define a special case of type $Y_{\mathcal{M}}$, which we call type X_k. We will use this simple case to prove the existence of four basic combinations of operators on multi-segments virtual extended multi-segments $\mathcal{E}$ of type X_k, all of which preserve such a type.

Definition 7.18. *We say that a virtual extended multi-segment $\mathcal{E}$ is of* type X_k *if $\mathcal{E}$ is of type $Y_{\mathcal{M}}$ for a block-tuple $\mathcal{M} = (m_0, \dots, m_k) = (1, \dots, 1)$.*

Here, we note a few properties about extended multi-segments of type X_k.

Lemma 7.19. *Let $\mathcal{E}$ be of type X_k. Then,*

(1) $\mathcal{E}$ is alternating,
(2) $C(\mathcal{E}) = k+1$, and
(3) $dual(\mathcal{E})$ is also of type X_k.

Proof. Suppose that $\mathcal{E} = \mathcal{E}(\mathcal{M}, \mathcal{S}, \mathcal{T}, \eta)$. Since all the multiplicities of $\mathcal{E}$ are 1, $\mathcal{E}$ has no multiples. Therefore, Part (1) of the lemma follows from the sign condition of Definition 7.2.

Since all the multiplicities of $\mathcal{E}$ are 1, by Condition (4) of Definition 7.1, none of the $\mathcal{S}_i$ overlap. Each $\mathcal{S}_i$ contributes rows with a total of $|\mathcal{S}_i|$ circles, so

$$C(\mathcal{E}) = \sum_{i=1}^{\ell} |\mathcal{S}_i| = \left|\bigcup_{i=1}^{\ell} \mathcal{S}_i\right| = |\{0, 1, \dots, k\}|,$$

which proves Part (2) of the lemma.

For Part (3), we first show $dual(\mathcal{E})$ has the supports of an extended multi-segment of type X_k. The dual of a hat with support $[\max \mathcal{T}_i^j, -\min \mathcal{T}_i^j]$ is a row of circles with support $[\max \mathcal{T}_i^j, \min \mathcal{T}_i^j]$. The dual of a row of circles with support $[\max \mathcal{T}_i^0, \min \mathcal{T}_i^0]$ is a hat with support $[\max \mathcal{T}_i^0, -\min \mathcal{T}_i^0]$, except when $\min \mathcal{T}_i^0 = 0$, in which case it is still a row of circles. So the supports of $dual(\mathcal{E})$ are the same as the supports of $\mathcal{E}(\mathcal{M}, \mathcal{S}', \mathcal{T}', \eta)$, where $\mathcal{S}'$ and $\mathcal{T}'$ is created by the following procedure. The elements of $\mathcal{S}'$ are

- for all i and for all $0 < j < \ell_i$, the set $\mathcal{T}_i^j$, with the corresponding partition consisting of just the set $\mathcal{T}_i^j$,
- for all i with $\ell_i > 0$, the union of $\mathcal{T}_i^{\ell_i}$ and every $\mathcal{T}_j^0$ such that $\ell_k = 0$ for all $i < k \leq j$, with the corresponding partition consisting of the sets $\mathcal{T}_i^{\ell_i}, \mathcal{T}_{i+1}^0, \ldots$, in that order, or
- $\mathcal{T}_1^0$, with the corresponding partition consisting of just the set $\mathcal{T}_1^0$.

It is clear that the union of all of these sets is the same as the union of the $\mathcal{S}_i$, and that they do not overlap. We simply order them such that condition (3) of Definition 7.1 is satisfied. Conditions (4) and (5) are not applicable. We simply order the partitions such that condition (2) of Definition 7.9 is satisfied, and condition (3) is not applicable. So this is indeed valid. Moreover, we see that, except for $\mathcal{T}_1^0$, every $\mathcal{T}_i^j$ with $j > 0$, corresponding to a hat, is some $\mathcal{T}'^{j'}_{i'}$ for $j' = 0$, corresponding to a row of circles, and similarly every $\mathcal{T}_i^j$ with $j = 0$, corresponding to a row of circles, is some $\mathcal{T}'^{j'}_{i'}$ for $j' > 0$. So $\mathcal{E}(\mathcal{M}, \mathcal{S}', \mathcal{T}', \eta)$ does indeed have the correct supports.

Next we check that $\mathcal{E}(\mathcal{M}, \mathcal{S}', \mathcal{T}', \eta)$ has the same order as $dual(\mathcal{E})$. Note that applying $dual$ to $\mathcal{E}$ replaces each row with its dual and reverses their order. So if $\mathcal{E}$ is in (P') order then so is $dual(\mathcal{E})$. Moreover, since the $\mathcal{S}_i$ do not overlap, each of the rows of $\mathcal{E}$ starts in a distinct column, and the same is true for $dual(\mathcal{E})$. So the (P') condition is enough to specify the order exactly. But $\mathcal{E}(\mathcal{M}, \mathcal{S}', \mathcal{T}', \eta)$ also satisfies (P'), so they have the same order.

Finally, we check the signs. By Lemma 6.6, $dual(\mathcal{E})$ is alternating, and $\eta(dual(\mathcal{E})) = \eta$. Since $\mathcal{E}(\mathcal{M}, \mathcal{S}', \mathcal{T}', \eta)$ is also alternating and has sign η, the signs match up, so we are done. □

We provide an example of the proof of Part (3) of Lemma 7.19.

Example 7.20. *Let $\mathcal{E}$ correspond to the $\mathcal{S}$-data*

$$\{0, \overline{1}, \overline{2, 3}\}, \{4, 5\}, \{6\}, \{7, \overline{8}\}.$$

Applying the dual operator gives an extended multi-segment with $\mathcal{S}$-data

$$\{0\}, \{1\}, \{2, 3, \overline{4, 5}, \overline{6}, \overline{7}\}, \{8\}.$$

We aim to prove the following result as a precursor to the more general Theorem 7.17.

Theorem 7.21. *Suppose that $\mathcal{E} \in \mathrm{VRep}_\rho(G)$ is tempered of type X_k with sign $\eta(\mathcal{E})$. Then, the set $\Psi(\pi(\mathcal{E}))$ is precisely the set of $\psi_{\mathcal{F}}$ where $\mathcal{F} \in \mathrm{VRep}_\rho(G)$ is of type X_k with $\eta(\mathcal{F}) = \eta(\mathcal{E})$.*

We will prove this theorem at the end of this subsection. In order to prove this theorem, we will completely classify all raising operators that are possible on virtual extended multi-segments of type X_k. First, we prove the existence of two very important operations for virtual extended multi-segments of type X_k.

Lemma 7.22. *Suppose that $\mathcal{E}$ is of type X_k and that $h = ([A, -B], B, \eta)$ is a hat in $\mathcal{E}$ with $C(h) > c$. Then there exists a series of operations on $\mathcal{E}$ resulting in a new virtual extended multi-segment $\mathcal{E}'$ satisfying the following conditions:*

- *$\mathcal{E}'$ has a hat of the form $h' = ([A - c, -B], B, \eta)$,*

- *$\mathcal{E}'$ has a row of c circles with support $[A, A-c+1]$,*
- *the other segments of $\mathcal{E}'$ are precisely the same as the segments of $\mathcal{E}-\{h\}$, except with possibly different signs,*
- *$\eta(\mathcal{E}') = \eta(\mathcal{E})$, and*
- *$\mathcal{E}'$ is of type X_k.*

Proof. We first prove the statement in the case where $A = k$. The operation whose existence we shall establish consists of the following combination:

(1) Row exchange h until it is the last row $\mathcal{E}$.
(2) Perform a ui^{-1} of type 3' to separate c circles from the bottom row.
(3) Row exchange the second to last row all the way to the top.

In light of Corollary 6.17 and Lemma 3.19(3), we know that these row exchanges commute with merging hats and ui's of type 3'. Therefore, to determine the image of h under the row exchanges, we may assume that $\mathcal{E}$ is completely unmerged: i.e., each row $r \in \mathcal{E}$ has $C(r) = 1$.

Thus, the row h is followed by $k+1-C(h)$ rows $r_1, \dots, r_{k+1-C(h)}$, all satisfying the alternating sign condition. It is clear by analyzing $\mathcal{S}$-data that $\mathcal{E} - \{h\}$ is of type $X_{k-C(h)}$; therefore, supp(h) contains the supports of all other rows. Therefore, according to Lemma 6.9, exchanging h down to the bottom gives us a row h' such that

$$l(h') = l(h) - (k + 1 - C(h)) = B - k - 1 + (k - B + 1) = 0,$$

$$\eta(h') = (-1)^{k+1-C(h)}\eta(h).$$

Therefore, h' is a row of circles. Because $c < C(h) = k - B + 1$, we have that $B \leq k - c$, which means that it is possible to use a ui^{-1} to separate off the last c circles into a new row. We denote the two resulting rows by h_1, h_2.

Lemma 6.9 says that the images $r'_1 < \cdots < r'_{k+1-C(h)}$ are alternating but $(r'_{k+1-C(h)}, h')$ fails the alternating sign condition, This means that $(r'_{k+1-C(h)}, h_1)$ also fails the alternating sign condition. Again, $supp(h_1) = [k-c, B]$ still contains the supports of all the rows r'_i. Therefore, Lemma 6.10 tells us that exchanging h_1 to the top of the multi-segment gives us new rows $h'_1 < r''_1 < \cdots < r''_{k+1-C(h)}$ satisfying the alternating sign condition. In particular:

$$l(h'_1) = \eta(h_1) + k + 1 - C(h) = 0 + l(h) = B,$$

$$\begin{aligned}\eta(h'_1) &= (-1)^{k+1-C(h)}\eta(h_1) \\ &= (-1)^{k+1-C(h)}\eta(h') \\ &= (-1)^{k+1-C(h)}(-1)^{k+1-C(h)}\eta(h) \\ &= \eta(h).\end{aligned}$$

These equations tell us that the resulting row h'_1 is a hat of the form $([A - c, -B], B, \eta)$. In particular, if we call the resulting extended multi-segment $\mathcal{E}'$, then $\eta(\mathcal{E}) = \eta(\mathcal{E}')$. Finally, we need to check that the pair $(r''_{k+1-C(h)}, h_2)$ satisfies the alternating sign condition:

$$\begin{aligned}\eta(h_2) &= (-1)^{C(h_1)+c}\eta(h_1)\\ &= (-1)^{C(h)+c}(-1)^{k+1-C(h)}\eta(h')\\ &= (-1)^{k+1+c}\eta(h)\end{aligned}$$

Meanwhile:

$$\begin{aligned}\eta(r''_{k+1-C(h)}) &= (-1)^{C(h)-c}(-1)^{C(h)}\eta(r_{k+1-C(h)})\\ &= (-1)^{c}\eta(r_{k+1-C(h)}).\end{aligned}$$

But since the original rows $h < r_1 < \cdots < r_{k+1-C(h)}$ were alternating, we obtain

$$(-1)^c\eta(r_{k+1-C(h)}) = (-1)^c(-1)^{k-C(h)}(-1)^{C(h)}\eta(h) = -\eta(h_2),$$

and since $C(r_{k+1-C(h)}) = 1$, this suffices to show that the pair is alternating. Now, it is clear that $\mathcal{E}' - \{h'_1, h_2\}$ has the same segments as $\mathcal{E} - \{h\}$ except, possibly, for the signs. Since it is alternating, we see that $\mathcal{E}' - \{h'_1, h_2\}$ is of type $X_{k-C(h)}$. We conclude that $\mathcal{E}'$ is itself X_k.

Now, we progress to the more general case where $k \geq A$. Since $\mathcal{E}$ is of type X_k, there exists a sub-virtual extended multi-segment $\mathcal{E}_A$ of type X_A, of which h is the first row. As we have seen, performing the aforementioned operations on $\mathcal{E}_A$ gives a segment $\mathcal{E}'_A$ that has the same sign and is of type X_A. Just as $\mathcal{E}$ is built from $\mathcal{E}_A$ by including alternating merged hats and rows of circles, $\mathcal{E}'$ is likewise built the same way from $\mathcal{E}'_A$. Thus, to show that $\mathcal{E}'$ is of type X_k, it suffices to show that it is alternating.

In particular, it suffices to show that h'_1 and the row that comes before it satisfy the alternating sign condition, and that h_2 and any row following it satisfies the alternating sign condition. The first of these follows from the fact that h and h'_1 have the same sign. For the second of these, we note that $C(\mathcal{E}_A) = C(\mathcal{E}'_A)$. Because $\mathcal{E}$ satisfies the alternating sign condition, the row following the block $\mathcal{E}'_A$ should have sign $\eta(\mathcal{E}_A)(-1)^{C(\mathcal{E}_A)}$. This is equal to $\eta(\mathcal{E}'_A)(-1)^{C(\mathcal{E}'_A)}$, so it has the correct sign in $\mathcal{E}'$ as well. Thus, $\mathcal{E}'$ is X_k. That $\eta(\mathcal{E}') = \eta(\mathcal{E})$ is clear, since any rows before h are unaffected. ☐

Next, we show that it is sometimes possible to perform exactly one $dual \circ ui \circ dual$ on certain virtual extended multi-segments of type X_k.

Lemma 7.23. *Suppose $\mathcal{E}$ is of type X_k and $h = ([k, -B], B, \eta)$ is the first row. Then it is possible to perform exactly one dual ∘ ui ∘ dual involving the row h. This operation preserves the property of being type X_k and the sign $\eta(\mathcal{E})$.*

Proof. The image of h in $dual(\mathcal{E})$ is a row of circles $\widehat{h} = ([k, B], 0, \eta')$. Since $\mathcal{E} - \{h\}$ is of type $X_{k-C(h)}$, none of the other rows of $dual(\mathcal{E})$ have supports intersecting $[k, B]$. Therefore, the only possible ui in $dual(\mathcal{E})$ involving $\widehat{h}$ is one of type 3'. Therefore, a ui must involve the image of the unique row $r \in \mathcal{E}$ with support ending at $A(r) = B - 1$. The existence of a row ending at $B - 1$ is guaranteed because $\mathcal{E} \setminus \{h\}$ is of type $X_{k-C(h)} = X_{B-1}$.

There are two possible cases. The first case is that the row $r \in \mathcal{E}$ ending at $B - 1$ is the hat directly after h, in which case the unique $dual \circ ui \circ dual$

involving $\widehat{h}$ is the operator merging the hats. The second case is that the row $r \in \mathcal{E}$ ending at $B-1$ is a row of circles on the bottom of $\mathcal{E}$. In $dual(\mathcal{E})$, the image $\widehat{r}$ is the first hat, and its support still ends at $B-1$. The assertion that it is possible to perform a ui between $\widehat{r}$ and $\widehat{h}$ is equivalent to the assertion that the ui^{-1} operation in Lemma 7.22 can be undone. The fact that such an operation preserves the sign and the property of being of type X_k follows from Lemmas 6.5 and 7.22. □

In the more general case where $h = ([A, -B], B, \eta)$ is not the first row of $\mathcal{E}$, we still have some sub-multi-segment $\mathcal{E}_A$ of type X_A, with h as the first hat. This means that it is always possible to perform a $dual \circ ui \circ dual$ on any hat, though this operation may not be unique. To keep the uniqueness property, we restate the result as such:

Lemma 7.24. *Suppose $\mathcal{E}$ is of type X_k and $h = ([A, B], -B, \eta) \in \mathcal{E}$ is any hat. Then there exists a unique way to apply $dual \circ ui \circ dual$ to a pair of rows (h, r) where $A(r) < A$.*

Again, this operation will fall into one of two cases: (1) r is a hat and the operation is merging, and (2) r is a row of circles. For the sake of brevity, we will refer to the latter operation as *dualizing* the hat h to r. Note that a hat with support $[A, B]$ can only be dualized to a row of circles r whose support ends at $-B-1$.

The previous two operations are two of four possible raising operators that can be performed for virtual extended multi-segments $\mathcal{E}$ of type X_k. Here are the four possible raising operators.

Definition 7.25.

- *Separation of a row of circles via ui^{-1} of type 3' which we denote by S,*
- *Merging consecutive hats which we denote by M,*
- *ui^{-1} a la Corollary 7.22 (again of type 3') which we denote by U, and*
- *Dualizing a hat h which we denote by D.*

Note that operators S and M are in a sense inverse-dual to each other; specifically, M can be obtained by dualizing $\mathcal{E}$ and performing the inverse of S. These operations are different from the other two in that they involve the interaction of consecutive rows. U and D are similarly inverse-duals, and both of them involve non-consecutive interactions: applying U or D to a hat near the top of $\mathcal{E}$ results in the appearance of circles near the bottom of $\mathcal{E}$. These operations are summarized in the following table.

	Consecutive	Non-Consecutive
ui^{-1}	S	U
$dual \circ ui \circ dual$	M	D

In fact, these operators S, M, U and D completely classify equivalence for virtual extended multi-segments of type X_k.

Lemma 7.26. *Let $\mathcal{E}$ be of type X_k. Then the only raising or lowering operators that can be applied to $\mathcal{E}$ are those of the forms S, M, U, D and their inverses.*

Proof. All raising operators are of the form either $dual \circ ui \circ dual$ or ui^{-1}. We have already seen from Lemma 7.24 that the only possible $dual \circ ui \circ dual$s are M or D. Applying ui^{-1} of type 3' to a row of circles is necessarily an S operation. Applying ui^{-1} of type 3' to a hat involves conducting row exchanges until that hat is a row of circles, which is necessarily a U operation (recall that a ui^{-1} not of type 3' must be of the form $dual \circ ui \circ dual$ by Lemma 3.13). Meanwhile, if T is a lowering operator on $\mathcal{E}$, then $dual \circ T \circ dual$ is a raising operator on $dual(E)$ and must therefore be one of S, M, U, D. Since these operations come in inverse dual pairs, T must be one of their inverses. □

Corollary 7.27. *If $\mathcal{E} \sim \mathcal{E}'$ and $\mathcal{E}$ is of type X_k, then $\mathcal{E}'$ is also of type X_k. Furthermore, $\eta(\mathcal{E}) = \eta(\mathcal{E}')$.*

Proof. It suffices to check that the operations S, M, U, D, and their inverses all preserve the property of being of type X_k and also preserve the sign. For S and M, and their inverses, this fact is obvious. For U and D, it follows from Lemmas 7.22 and 7.24 respectively. For the inverses of U and D, it follows from the fact that U and D are inverse-duals of each other, combined with Lemmas 7.16 and 7.19. □

Now, we are finally ready to prove the classification formulated in Theorem 7.21.

Proof of Theorem 7.21. Suppose that $\mathcal{E} \in \mathrm{VRep}_\rho(G)$ is tempered of type X_k with sign $\eta(\mathcal{E})$. By Lemma 7.27 implies that all $\mathcal{E}' \in \mathrm{VRep}_\rho(G)$ which are equivalent to $\mathcal{E}$ are also X_k with the same sign. Conversely, if $\mathcal{E}'$ is of type X_k, applying D_{h_i} for each hat of $\mathcal{E}'$ gives an equivalent virtual extended multi-segment made up only of circles. Finally, applying S successively leaves a tempered segment with the same multiplicities as $\mathcal{E}$. Sign preservation guarantees that this multi-segment is exactly $\mathcal{E}$. □

Going forward, we introduce more elaborate notation for the operators above to make the ensuing proofs more precise.

Definition 7.28.
- *Let $S_{r,c}$ be the operator separating out c circles from a row r.*
- *Let $U_{h,c}$ be the ui^{-1} operator that removes c circles from the hat h.*
- *Let M_{h_1,h_2} be the operator merging the hats h_2, h_2.*
- *Let $D_{h,r}$ be the operator that dualizes the hat h to the row r.*

7.3. **Operations for type $Y_{\mathcal{M}}$.** For the duration of this section, let $\mathcal{E}$ be of type $Y_{\mathcal{M}}$ for some block-tuple $\mathcal{M}$ of odd-integers. The goal of this section is to describe and classify the row operations that can be implemented on $\mathcal{E}$. These are analogous to the operations S, M, U, and D seen for type X_k, but they are generally more complicated. Furthermore, these operators may be understood nicely through their effect on the $\mathcal{S}$-data of $\mathcal{E}$. Note in advance that the discussion of the S operator will apply to all multi-segments of type $Y_{\mathcal{M}}$, while the other operators only concern virtual extended multi-segments starting at zero.

First, we consider the operation S. For multi-segments of type X_k, applying operations ui^{-1} to rows of circles is simple: all operations $S_{r,c}$ are possible, and no row exchanges are necessary. However, this is not always the case with general multi-segments of type $Y_{\mathcal{M}}$.

$$\begin{array}{cccc} 0 & 1 & 2 & 3 \end{array}$$
$$\begin{pmatrix} \oplus & \ominus & \oplus & \\ & \oplus & & \\ & \oplus & & \\ & \oplus & & \\ & \oplus & & \\ & & & \ominus \end{pmatrix}$$

FIGURE 2. An extended multi-segment where not all ui^{-1} are admissible without row exchanges.

Let r be the row $([2,0],0,1)$ in Figure 2. It is not possible to apply $S_{r,1}$ because then the resulting multi-segment would not be in admissible order. However, we can still perform this operation if we first exchange r down four rows until it is below the multiples $([1,1],0,1)$, then separate one circles, and finally exchange the row back up to its original position. If we did this, we would end up with the following multisegment

$$\begin{array}{cccc} 0 & 1 & 2 & 3 \end{array}$$
$$\begin{pmatrix} \oplus & \ominus & & \\ & \ominus & & \\ & \ominus & & \\ & \ominus & & \\ & \ominus & & \\ & & \oplus & \\ & & & \ominus \end{pmatrix}.$$

We note that the resulting segment is odd-alternating. In fact, the ui^{-1} that produced this multi-segment can be understood through its effect on the $\mathcal{S}$-data of the original multi-segment:

$$(\{0,1,2\},\{3\}) \longrightarrow (\{0,1\},\{2\},\{3\}).$$

Meanwhile the multiplicities $\mathcal{M} = (1,5,1,1)$ are preserved through these operations. Such an operation can be conducted more generally, and we prove this in the following lemma.

Lemma 7.29 (Existence of S)**.** *Suppose $\mathcal{E}$ is of type $Y_{\mathcal{M}}$, $r = ([A,B],0,\eta)$ is a chain in $\mathcal{E}$, and $k < C(r)$ is an integer. Then there exists a series of operations preserving the type $Y_{\mathcal{M}}$ that splits $\mathcal{S}_i = \{B,\dots,A,\dots\}$ into two sets*

$$\mathcal{S}_i^1 = \{B,\dots,A-c\} \text{ and } \mathcal{S}_i^2 = \{A-c+1,\dots,A,\dots\}.$$

Proof. We will conduct the following series of operations.

- Exchange r downwards past all rows whose supports begin at $A-c$ or earlier. The resulting row r' will be equal to r.
- Apply $S_{r,c}$.
- Exchange the remaining circles back up to the original position of r.

We assert that the only rows whose supports begin at $A-c$ or earlier are multiples. This is equivalent to the assertion that there are no chains beginning at $B-c$ or earlier, which must be true because two chains can only overlap for at most one circle. Now, all of these multiplies must belong to the chain r, and since they do not overlap with any other chains, there must be evenly many of each of them. Therefore, by Lemma 6.11, the row r stays the same after exchanging with these rows. Meanwhile, each multiple's sign is multiplied by $(-1)^{C(r)-1}$.

Now, the rows after r are precisely the rows whose supports begin at least at $A-c+1$, so we can separate r into two rows:

$$r_1 = ([A-c], 0, \eta),\ r_2 = ([A, A-c+1], 0, \eta^{C(r_1)}).$$

Then exchanging r_1 up with all the multiplicities once again preserves r_1 by Lemma 6.11, while all the multiplicities have their signs multiplied by $(-1)^{C(r_1)} = (-1)^{C(r)-1-c}$. It is clear that the resulting multi-segment has the same $\mathcal{S}$-data as $\mathcal{E}$, except that $\mathcal{S}_i = \{B, \ldots, A, \ldots\}$ has been split into $\{B, \ldots, A-c\}$ and $\{B-c+1, \ldots, A, \ldots\}$. It remains to be checked that the multi-segment is odd-alternating.

In $\mathcal{E}$, all the multiplicities belonged to r and therefore had sign $\eta(-1)^{C(r)-1}$. These row exchanges have changed their sign by $(-1)^k$, so the new sign is $\eta(-1)^{C(r)-c-1} = \eta(-1)^{C(r_1)-1}$. This obeys the odd-alternating condition. Before r_1 was exchanged up, the pair (r_1, r_2) obeyed the alternating sign condition, so r_2 must alternate with the multiplicity right before it after the row exchange: since there are an even number of multiplicities, this is alternating. Finally, the multiplicities belonging to r_2 are precisely those that belonged to r but do not now belong to r_1. These still have sign

$$\eta(-1)^{C(r)-1} = \eta(-1)^{C(r_1)}(-1)^{C(r_2)-1} = \eta(r_2)(-1)^{C(r_2)-1},$$

so they obey the alternating sign condition. □

We continue to refer to the operation described in Lemma 7.29 as $S_{r,c}$.

Remark 7.30. *There is a possible problem that could occur with our notation. Suppose we apply $S_{r,k}$ to a chain $r = ([A, B], 0, \eta)$ such that $A - B = k$ and r is a z-chain. Then separating k circles leaves behind a one circle row $([B, B], 0, \eta)$ associated to $\mathcal{S}_i^1$. However, the conditions for being of type $Y_{\mathcal{M}}$ require that $|\mathcal{S}_i^1| \geq 2$. That is, we would like to count it as a multiple rather than a chain. To fix this notational problem, instead of saying that we have separated $\mathcal{S}_i$ into two sets $\mathcal{S}_i^1 = \{B\}$ $\mathcal{S}_i^1 = \{B+1, \cdots, A\}$, we instead throw out $\mathcal{S}_i^1$ and replace $\mathcal{S}_i$ with $\mathcal{S}_i^2$ solely. If $A \geq B+2$, the chain corresponding to $\{B+1, \cdots, A\}$ must be row exchanged below the (evenly many) multiples with support $[B+1, B+1]$.*

To summarize, we can describe the impact of S on the $\mathcal{S}$-data of $\mathcal{E}$ as follows:

Remark 7.31. *Let $r = ([A, B], 0, \eta)$. We have two cases as follows.*

- *If $c = A - B$ and r is a z-chain, then applying $S_{r,c}$ replaces:*
$$\{B, \ldots, A, \ldots\} \longrightarrow \{B+1, \ldots, A, \ldots\}.$$
- *Otherwise, applying $S_{r,c}$ replaces:*
$$\{B, \ldots, A, \ldots\} \longrightarrow \{B, \ldots, A-c\}, \{A-c+1, \ldots, A, \ldots\}.$$

Definition 7.32. *We shall refer to the case where $k = A - B$ and r is a z-chain as S^2, and the other case as S^1.*

We note that this S operation still commutes with row exchanges. More precisely, the following corollary follows immediately from Parts (1) and (2) of Lemma 3.19.

Corollary 7.33. *Let R denote the operation of exchanging some row r with a series of rows $r_i < r_{i+1} < \cdots < r_j$. Let S be some ui^{-1} of type S involving the rows $r_i < r_{i+1} < \cdots < r_j$. Then $R \circ S = S \circ R$.*

For the remaining operations M, U, and D discussed in this section, we assume that $\mathcal{E} = \mathcal{E}(\mathcal{M}, \mathcal{S}, \mathcal{T}, \eta)$ begins at zero. The most simple operation that can be implemented on $\mathcal{E}$ is M, the merging of hats.

Lemma 7.34. *Suppose $\mathcal{E} = \mathcal{E}(\mathcal{M}, \mathcal{S}, \mathcal{T}, \eta)$ begins at zero and has hats $h_1 < h_2$ obtained from $\mathcal{T}_i^j$ and $\mathcal{T}_{i'}^{j'}$. Then h_1 and h_2 can be merged if and only if $i = i'$ and $j = j' + 1$.*

Proof. Suppose that $h_1 = ([A_1, -B_1], B_1, \eta_1)$ and $h_2 = ([A_2, -B_2], -B_2, \eta_2)$. Then we must have $\mathcal{T}_i^j = \{B_1, \ldots, A_1\}$ and $\mathcal{T}_{i'}^{j'} = \{B_2, \ldots, A_2\}$, and these sets must be disjoint for $(\mathcal{S}, \mathcal{T})$ to be valid. Since $h_1 < h_2$, we must have $B_1 > A_2$. Because $\mathcal{E}$ is odd-alternating and in (P') order, h_1 and h_2 can be merged if and only if $B_1 = A_2 + 1$ by Lemma 6.14.

We prove that if $B_1 = A_2 + 1$, then $i = i'$ and $j = j' + 1$, as the converse is clear. Suppose for the sake of contradiction that $i \neq i'$, in which case $i > i'$ since $B_1 > A_2$. But because $A_2 = B_1 - 1 \in \mathcal{S}_{i'}$ and $|\mathcal{S}_i \cap \mathcal{S}_{i'}| \leq 1$, we must have that $\min \mathcal{S}_i = B_1$ or A_2. But since $j \geq 1$, this would imply that $\mathcal{T}_i^0 \subset \{A_1\}$, contradicting the condition that $|\mathcal{T}_i^0| \geq 2$. Therefore, we conclude that $i = i'$. The fact that $j = j' + 1$ follows from the fact that $B_1 = A_2 + 1$. ∎

Remark 7.35. *With this lemma in mind, we observe that the operation M has a nice description in terms of the $\mathcal{S}$ data of $\mathcal{E}$. If $\mathcal{T}_i^j$ and $\mathcal{T}_i^j$ are the sets associated with h_1 and h_2, then merging $h_1 * h_2$ corresponds with replacing these sets with $\mathcal{T}_i^j \cup \mathcal{T}_i^{j+1}$.*

In other words, applying M_{h_1,h_2} to $h_1 = ([A, -B], B, \eta_1)$ and $h_2 = ([-B + 1, C], -C, \eta_2)$ gives the replacement

$$\{\ldots \overline{C, \ldots, B-1}, \overline{B, \ldots, A}, \ldots\} \longrightarrow \{\ldots \overline{C, \ldots, B-1, B, \ldots, A}, \ldots\}.$$

Next, we show that it is possible to dualize hats.

Lemma 7.36 (Existence of D)**.** *Suppose $\mathcal{E} = \mathcal{E}(\mathcal{M}, \mathcal{S}, \mathcal{T}, \eta)$ with $\mathcal{M}$ beginning at zero and that $h = ([A, -B], B, \eta)$ is a hat in $\mathcal{E}$. Suppose that $r_1, r_2, \ldots, r_k$ are the rows of circles whose support contains $B - 1$. The following hold.*

1. *The row r_k ends at $B - 1$, and it is possible to perform a dual $\circ$ ui $\circ$ dual between h and r_k, which we notate D_{h,r_k}. We have that $D_{h,r_k}(\mathcal{E})$ is also of type $Y_{\mathcal{M}}$ and has the same sign.*
2. *Suppose $r = r_i$ is a row of circles and $k - i \in 2\mathbb{Z}$. Then it is possible to perform a dual $\circ$ ui $\circ$ dual between h and r, which we notate $D_{h,r}$.*
3. *If $\mathcal{E}'$ is the image of $\mathcal{E}$ under such a dual$\circ$ui$\circ$dual in either Part (1) or (3), then $\mathcal{E}$ and $\mathcal{E}'$ are of type $Y_{\mathcal{M}}$ for the same $\mathcal{M}$. Moreover, $\eta(\mathcal{E}) = \eta(\mathcal{E})$.*

(4) $\mathcal{E}$ is equivalent to a tempered virtual extended multi-segment of type $Y_{\mathcal{M}}$.

Before proving this lemma, we note that the operation $D_{h,r}$ as it is described above commutes with row exchanges.

Lemma 7.37. *Suppose $\mathcal{E} = \mathcal{E}(\mathcal{M}, \mathcal{S}, \mathcal{T}, \eta)$ begins at zero. Let $r \in \mathcal{E}$ be a row, and let r_1 and r_2 be rows such that the operation D_{r_1,r_2} is valid. Suppose that $supp(r)$ contains the supports of the rows between r_1 and r_2 inclusively. Then exchanging r down with the rows $r_1 < \cdots < r_2$ results in the same r' as exchanging r with the images of these rows under D_{r_1,r_2}.*

Proof. We prove the result by inducting on the number of hats below r_1. As part of the inductive assumption, we can assume that all hats after r_1 can be dualized without affecting the image r' under row exchanges. Therefore, the rows between r_1 and r_2 (excluding r_1 but including r_2) are all rows of circles. In light of Lemma 7.33, we can presume that all chains between r_1 and r_2 are completely unmerged so that the rows from r_1 to r_2 (not inclusive) form a tempered multi-segment. Then, by Lemmas 6.11 and 6.12, we can presume that there are no multiples, so that the rows $r_1 < \cdots < r_2$ consists of a multi-segment of type X_k for some k. Then, these rows must alternate, so Lemma 6.9 indicates that r' depends only on the total number of circles in these rows, which is preserved under D_{r_1,r_2}. □

With the above lemma in hand, we prove Lemma 7.36.

Proof of Lemma 7.36. We first verify that r_k ends at $B-1$. Assume otherwise for the sake of contradiction. Then r_k must have support containing B, so $B \in \mathcal{T}_i^0$ for some i. This is impossible because B is in the hat h, so $B \in \mathcal{T}_{i'}^{j'}$ for some $j \geq 1$. Then we must have that $|\mathcal{S}_i \cap \mathcal{S}_{i'}| \geq 2$, which is a contradiction.

We prove Parts (1) and (4) together within the same inductive argument. In Part (1), we are asserting that it is possible to conduct the following series of operations:

- Dualize $\mathcal{E}$, in which $\widehat{r_k}$ will be a hat and $\widehat{h}$ a row of $C(h)$ circles.
- Exchange down $\widehat{r_\ell'}$ in $dual(\mathcal{E})$ until its image $\widehat{r_\ell}'$ is the row right before $\widehat{h}$.
- Perform a ui of type 3' between $dual(r_\ell)'$ and $\widehat{h}$.
- Row exchange the resulting row until it is in the original position of $\widehat{r_\ell}$ the first row again.
- Dualize the resulting extended multi-segment.

Firstly, it is clear that none of the rows except for those between h and r_k are relevant to the existence of these operations, so it suffices to prove the lemma in the case where h is the first row of $\mathcal{E}$ and r_k is the last. We will induct on the number of hats after h, so as an inductive step we may suppose that all the hats after h have already been dualized. We reduce to the case where these hats are dualized by Lemma 7.37. By Corollary 7.33, we can also reduce to the case where all chains that come before r_k have exactly one circle. Thus, we have supposed inductively that $\mathcal{E} \setminus \{h, r_k\}$ is a tempered virtual extended multi-segment. Now, we separately consider each of the two following cases.

The first case is that $m_{B-1} \geq 3$, in which case r_k is a multiple of the form $([B-1, B-1], 0, \eta')$. Then $\mathcal{E} \setminus \{h\}$ is tempered as well since all its rows have exactly one circle. So we have

$$\mathcal{E} \setminus \{h\} = \bigcup_{i=1}^{B-1} \bigcup_{j=1}^{m_i} ([i,i], 0, (-1)^i \eta(\mathcal{E})).$$

This means that

$$dual(\mathcal{E}) \setminus \{\widehat{h}\} = \bigcup_{i=1}^{B-1} \bigcup_{j=1}^{m_i} ([i,-i], i, \eta_i').$$

However, in light of Lemma 6.11, it is clear that exchanging a row with

$$\bigcup_{j=1}^{m_i'} ([i,-i], i, \eta_i')$$

is equivalent to exchanging it with

$$\bigcup_{j=1}^{m_i' \mod 2} ([i,-i], i, \eta_i').$$

Therefore, proving this result for $\mathcal{E}$ is equivalent to proving it for an extended multi-segment where all multiples have been removed. We then presume

$$\mathcal{E} \setminus \{h\} = \bigcup_{i=1}^{B-1} ([i,i], 0, (-1)^i \eta(\mathcal{E})),$$

which is precisely the case where $\mathcal{E}$ is of type X_k, so Part (1) follows from Lemma 7.24. If $\mathcal{E} - \{h\}$ is tempered, then employing this $dual \circ ui \circ dual$ to combine h and r_k into a single row r' and then using successive operations $S_{r_k',1}$ shows that $\mathcal{E}$ itself is equivalent to a tempered multi-segment which proves Part (4) in this case.

The second case is that $m_{B-1} = 1$, in which case $r_k = ([C,B], 0, \eta')$ is a chain. Depending on whether r_k overlaps with another chain at C, we have that

$$\mathcal{E} \setminus \{h, r_k\} = \bigcup_{i=1}^{C} \bigcup_{j=1}^{m_i'} ([i,i], 0, (-1)^i \eta(\mathcal{E})),$$

where

$$m_i' := \begin{cases} m_i - 2 & \text{if } i = C \text{ and } r_k \text{ overlaps with another chain,} \\ m_i - 1 & \text{if } i = C \text{ and } r_k \text{ does not overlap with another chain,} \\ m_i & \text{otherwise.} \end{cases}$$

Again, when row exchanging $\widehat{r_k}$ with the other rows of $dual(\mathcal{E})$, the multiplicities will not impact the image $\widehat{r_k}'$ due to Lemma 6.11. Therefore, we can presume that each m_i is equal to 0 or 1, and we have reduced to the case where $\mathcal{E}$ is of type X_k. Again, this suffices for Part (1). Dualizing h and applying S suffices to show Part (4).

We note that Part (2) can be reduced to Part (1) by similarly only considering rows between h and r_i. Since r_i and r_k have the same parity, such a multi-segment is still of type $Y_{\mathcal{M}}$ and the proof still holds.

Similarly, it suffices to prove Part (3) for the case where $i = k$. The fact that the sign $\eta(\mathcal{E})$ is preserved under the $dual \circ ui \circ dual$ operation is inherited from the fact that the sign is preserved when $\mathcal{E}$ is of type X_k. This is because the presence of multiplicities does not cause a cumulative change in sign under row operations. To show that type $Y_{\mathcal{M}}$ is preserved, we note that none of the multiplicities are affected under the operation; it suffices to show that the odd-alternating condition is preserved.

We note that exchanging $\widehat{r}$ down until it is the row before $dual(h)$ changes the signs of all the intermediary rows by $(-1)^{C(r)-1}$, and exchanging the merged row back up changes all the signs by $(-1)^{C(r)+C(h)-1}$ (Lemmas 6.9 and 6.10). Therefore, all the intermediate rows have a net sign change of $(-1)^{C(h)}$. Consecutive intermediate rows all satisfy the same sign conditions as before. Meanwhile, it follows from the proof of Lemma 7.24 that the signs of the first and last circles of the block of rows $h < \cdots < r$ stays the same under the $D_{h,r}$ operation. Then r' must satisfy the sign condition with the row after it prescribed by Definition 7.2. Meanwhile, the sign of the row before r is changed by $(-1)^{C(h)-1}$, as is the last circle of $\widehat{r}'$, so r' must also satisfy the proper sign condition with the row before it. □

The above lemma is enough to deduce the following result.

Corollary 7.38. *Any two $Y_{\mathcal{M}}$ multi-segments with the same η are equivalent up to the operators S, D, and their inverses.*

Proof. First, we suppose $\mathcal{M}$ starts at zero. Part (4) of the Lemma 7.36 implies that all $Y_{\mathcal{M}}$ multi-segments with the same $\mathcal{M}$ and η are equivalent to a tempered multi-segment of type $Y_{\mathcal{M}}$. Part (1) guarantees that this tempered multi-segment always has the same $\mathcal{M}$ and η.

If $\mathcal{M}$ starts after zero, then any type $Y_{\mathcal{M}}$ multi-segment has no hats and can be separated via the S operation into a tempered multi-segment. By Lemma 7.29, this multi-segment still must have the same $\mathcal{M}$ and η. □

By reducing to the case where $\mathcal{E}$ is of type X_k, we have shown that the operation D has the same structural properties as it did for type X_k. That is to say, if $\mathcal{E}' = D_{h,r}(\mathcal{E})$ and r' is the resulting row, then:

- $\mathcal{E} \setminus \{h, r\}$ and $\mathcal{E}' \setminus \{r'\}$ have the same rows up to a change in sign.
- If r is a row of circles with support $[A, B]$, then r' is a row of circles with support $[A + C(h), B]$.

Remark 7.39. *Lemma 7.36 and the above description concern only the case where $r = r_i$, when i and k have the same parity. One may wonder what would happen if we dualize h to some multiple r_i where i and k have different parity. Indeed, it is possible to do this, and Lemma 3.19 implies that if $r_i = r_j$, then D_{h,r_i} and D_{h,r_j} are equivalent up to row exchanges, We illustrate this idea in the following figure.*

Indeed, dualizing a hat h to a multiple r_i with different parity from k will always produce a row with $l \geq 1$. To see this, suppose $r_1 = ([B, B], 0, \eta)$ and $r_2 = ([A, B], 0, \eta)$ are two consecutive rows with the same sign. The formulae indicate that r_1 stays the same under a row exchange except that its sign is

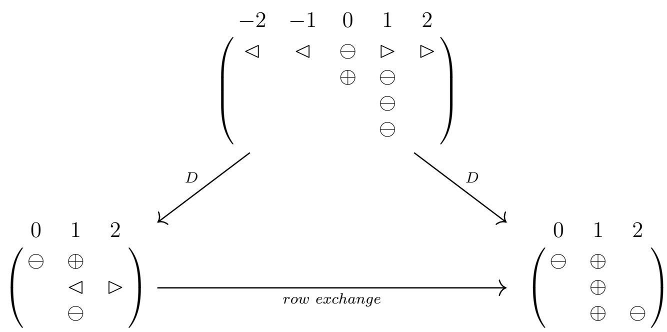


FIGURE 3. These applications of D to consecutive multiples are equivalent via a row exchange.

changed by $(-1)^{C(r_2)}$*, whereas* $l(r_2)$ *is increased by 1. This is the case regardless of the order of* r_1 *and* r_2*.*

Since it does not matter which multiple a hat is dualized to, the convention will always be dualize to the last multiple, r_k*.*

The next lemma uses the observation that dualizing to each of the multiples produces the same multi-segment up to row exchanges in order to determine how many ways there are to dualize a given hat.

Lemma 7.40. *Let* $h = ([A, -B], B, \eta)$ *be a hat in* $\mathcal{E} = \mathcal{E}(\mathcal{M}, \mathcal{S}, \mathcal{T}, \eta')$*.*

(1) *If* $m_{B-1} = 1$*, then there is a unique* $dual \circ ui \circ dual$ *involving* h *and rows lower than* h*.*
(2) *If* $m_{B-1} \geq 3$ *and* $\mathcal{T}_i^0 = \{B-1\} \subset \mathcal{S}_i$ *for some* i*, then there is exactly one* $dual \circ ui \circ dual$*, unique up to row exchange, involving* h *and rows lower than* h*.*
(3) *Otherwise, there are exactly two* $dual \circ ui \circ duals$ *involving* h *and rows lower than* h*, up to row exchanges.*

Proof. For Part (1), if $m_{B-1} = 1$, then there is exactly one row r with support ending at $B-1$. If r is a hat, it must be the row right after h, so the unique $dual \circ ui \circ dual$ is $M_{h,r}$. If r is a row of circles, then the operation is $D_{h,r}$ by Lemma 7.36.

For Part (2), if $m_{B-1} \geq 3$, then there are m_{B-1} rows with support ending at $B-1$. Since there exists some $\mathcal{T}_i^0 = \{B-1\}$, none of these rows can be a hat. Instead, one of them is a chain with support $[B-1, B-1]$, and the others are all multiples. Therefore, all these m_{B-1} rows are consecutive and equal to each other. By Lemma 7.36, h can be dualized to each of them, but per Remark 7.39 all of these are equivalent up to row exchange.

For Part (3), if $m_{B-1} \geq 3$ and there does not exist any $\mathcal{T}_i^0 = \{B-1\}$, then B is contained in some $\mathcal{S}_i = \{\dots, B-2, B-1, \overline{B, \dots, A}, \dots\}$. Since two elements of $\mathcal{S}$ have intersection of size at most one, $B-1 \notin \mathcal{S}_j$ for any $j \neq i$. Now, one of the following must be true.

- If $B-1$ is contained in a hat, then this hat h' and m_{B-1} multiples of support $[B-1, B-1]$ are precisely the m_{B-1} rows with support ending at $B-1$. One $dual \circ ui \circ dual$ is obtained by merging $h * h'$. The other is gotten by dualizing h to any of the multiples. Per Remark 7.39, these dualizations are the same up to row exchange since the multiples are all equal to each other.

- If $B-1$ is not contained in a hat, then it must be contained in a chain of length ≥ 2, lest there be some $\mathcal{T}_i^0 = \{B-1\}$. Then one $dual \circ ui \circ dual$ is gotten by dualizing h to this chain, and the other is gotten by dualizing it to any of the multiples.

From here, it is clear that the two $dual \circ ui \circ dual$s do not produce virtual extended multi-segments associated with the same Arthur parameters. This can be seen because the supports of the multi-segments are different. Indeed, the one obtained from the second case contains a row of support $[B, B-1]$ and the one obtained from the first case cannot contain such a row. □

In summary, when $m_{B-1} \geq 3$, a hat can sometimes be dualized in two ways: either to a chain or to a multiple. Although these operations are quite similar, they impact the $\mathcal{S}$-data of $\mathcal{E}$ very differently. Therefore, we will distinguish between them by notating a dualization to a chain as D^1 and a dualization to a multiple as D^2.

Remark 7.41. *These dualizations change the $\mathcal{S}$-data as follows.*

- *Applying $D^1_{h,r}$ to $h = ([A, -B], B, \eta_1)$ and $r = ([B-1, C], 0, \eta_2)$ replaces*
$$\{C, \ldots, B-1, \overline{B, \ldots, A}, \ldots\} \longrightarrow \{C, \ldots, B-1, B, \ldots, A, \ldots\}.$$
- *Applying $D^2_{h,r}$ to $h = ([A, -B], B, \eta_1)$ and a multiple $r = ([B-1, B-1], 0, \eta_2)$ replaces*
$$\{\ldots, B-1, \overline{B, \ldots, A}, \ldots\} \longrightarrow \{\ldots, B-1\}, \{B-1, \ldots, A, \ldots\},$$
thereby resulting in the creation of a z-chain.

Finally, we prove that the operation U can be applied to $\mathcal{E}$ in the same way structurally as if $\mathcal{E}$ were of type X_k.

Lemma 7.42 (Existence of U)**.** *Suppose $\mathcal{E} = \mathcal{E}(\mathcal{M}, \mathcal{S}, \mathcal{T}, \eta)$ begins at zero and that $h = ([A, -B], B, \eta)$ is a hat in $\mathcal{E}$ with $C(h) > c$. Then there exists a series of operations on $\mathcal{E}$ resulting in a new extended multi-segment $\mathcal{E}'$ of type $Y_{\mathcal{M}}$ satisfying the following*

- *$\mathcal{E}'$ has a hat of the form $h' = ([A-c, -B], B, \eta)$,*
- *$\mathcal{E}'$ has a row of k circles with support $[A, A-c+1]$,*
- *the other segments of $\mathcal{E}'$ are precisely the same as the segments of $\mathcal{E} - \{h\}$, except with possibly different signs, and*
- *$\eta(\mathcal{E}') = \eta(\mathcal{E})$.*

Proof. Again, we will first prove the statement in the case where $h = ([k, -B], B, \eta)$, where $k = c_{\max}$. This consists of showing that the following combination of operations is valid.

- Row exchange h until the image h' is at the bottom of the multi-segment. h' should be a row of circles.
- Apply $S_{h',c}$.

- Row exchange the remaining circles back to the top.

Due to Lemma 7.36, we know that $\mathcal{E} \setminus \{h\}$ is equivalent to a tempered representation through some series of row operations, all of which commute with row exchange by Corollary 7.33 and Lemma 7.37. Therefore, we can presume that $\mathcal{E} \setminus \{h\}$ is tempered. Furthermore, we can presume that $\mathcal{E} \setminus \{h\}$ has no multiples due to Lemma 6.11. Thus, we have reduced to the type X_k case, and the result follows from Lemma 7.22.

The fact that $\eta(\mathcal{E})$ is preserved is clear from the fact that the sign of h is unchanged throughout the row exchanges: the sign changes from exchanging down and exchanging up cancel each other by Lemmas 6.9 and 6.10. These lemmas also show that the rows that h is exchanged with all have their signs changed by $(-1)^c$, so they satisfy the same sign conditions that they did before; therefore, the multi-segment is still odd alternating. That h alternates with the row below it after the U operation if and only if it does before the row operation is clear since $C(h)$ is also decreased by c.

Finally, the fact that the broken-off row of c circles alternates with the row before it can be seen in two cases. If the row before it is a chain, then the proof is exactly the same as in the type X_k case (see the proof of Lemma 7.22). If the preceding row is a multiple with support $[C, C]$, it has the same sign as the last circle of the chain that it belongs to. Since $C \leq A - c < A$, it is impossible for C to be in more than one $\mathcal{S}_i$, so there are evenly many such multiples. Therefore, if we reduce to the case where the multiples are not present via Lemma 6.11, it is clear that the row of c circles alternates with the chain, so it must also alternate with the multiple. Thus, we observe that the image of this U operation is odd-alternating and therefore of type $Y_{\mathcal{M}}$. □

Remark 7.43. *As for the $\mathcal{S}$-data, applying $U_{h,c}$ to $h = ([A, -B], B, \eta)$ replaces*

$$\{\dots \overline{B, \dots, A}, \dots\} \longrightarrow \{\dots \overline{B, \dots, A - c}\}, \{\overline{A - c + 1, \dots, A}, \dots\}.$$

As with the other operators, the U operator commutes with row exchange. Specifically, it follows from a combination of Parts (1) and (3) of Lemma 3.19 that the following corollary holds.

Corollary 7.44. *Let R denote the operation of exchanging some row r with a series of rows $r_i < r_{i+1} < \cdots < r_j$. Let T be some U operation involving the rows $r_i < r_{i+1} < \cdots < r_j$. Then $R \circ T = T \circ R$.*

7.4. **Exhaustion of Operators on $Y_{\mathcal{M}}$ Segments.** We now aim to prove Theorem 7.17. Due to Corollary 7.38, we know that all type $Y_{\mathcal{M}}$ multi-segments with the same sign are equivalent. Then, it suffices to show that if $\mathcal{E}$ is a virtual extended multi-segment of type $Y_{\mathcal{M}}$, and sign η, then any virtual extended multi-segment equivalent to $\mathcal{E}$ is also $Y_{\mathcal{M}}$ with sign η. First we will prove that $\mathcal{E}^{min}$ is of type $Y_{\mathcal{M}}$ with sign η. Then we will show that all possible raising operators on type $Y_{\mathcal{M}}$ virtual extended multi-segments preserve type $Y_{\mathcal{M}}$ and sign. Since all equivalent virtual extended multi-segments can be obtained from $\mathcal{E}^{min}$ through raising operators, this will suffice for our proof. We begin with the following simple technical lemma about row exchange.

Lemma 7.45. *Let h_1 and h_2 be hats with order $h_1 < h_2$, and suppose $C(h_1) = C(h_2) = 1$ and $\eta(h_1) = \eta(h_2)$. Then after row exchanging h_1 and h_2, we have $h_1' = h_1$ and $h_2' = h_2$.*

Proof. Note that since $C(h_i) = 1$, we have that $A(h_i) \equiv B(h_i) \mod 2$. Thus, the row exchange has $\epsilon = (-1)^{A(h_1)-B(h_1)}\eta(h_1)\eta(h_2) = 1$, $b(h_1) - 2l(h_1) = 1$, and $2(b(h_2) - 2l(h_2)) = 2$, so it falls into Case 1(a) of Definition 3.8. So $l(h_2') = l(h_2)$, $\eta(h_2') = \eta(h_2)$, and

$$\begin{aligned} l(h_1') &= A(h_1) - B(h_1) + 1 - (l(h_1) + 1) \\ &= l(h_1) \\ \eta(h_1') &= (-1)^{A(h_2)-B(h_2)}\eta(h_1) = \eta(h_1). \end{aligned}$$

So the rows are unchanged. □

Now, we classify $\mathcal{E}^{\min}$ according to $\mathcal{S}$-data, breaking into cases depending on whether $\mathcal{M}$ starts at zero.

Lemma 7.46. *Suppose $\mathcal{E}$ is of type $Y_{\mathcal{M}}$ with $\mathcal{M}$ beginning after zero. Suppose that the $\mathcal{S}$-data is such that $\mathcal{S}$ contains some $\mathcal{S}_i = \{c_{\min}, \ldots, c_{\max}\}$. Then $\mathcal{E} = \mathcal{E}^{\min}$.*

Proof. We check that no lowering operators are possible on $\mathcal{E}$. There are two cases: either $\mathcal{S} = (S_i)$ or $\mathcal{S} = (\{c_{\min}\}, \mathcal{S}_i)$. In the first case, $\mathcal{E}$ consists of one chain and multiples contained within the support of this chain. Clearly, no *ui* operators involving the chain are possible. All the multiples have circles of the same sign, so it is impossible to apply a *ui* of type 3'. In the second case, there are two chains, and the second chain contains all the other rows, which each have one circle. Again, it is evident that there are no *ui* operators.

Less obvious is the fact that there are no $dual \circ ui^{-1} \circ dual$ operators on $\mathcal{E}$. To see this, note that $dual(\mathcal{E})$ consists of three different kinds of rows

- one hat h with support $[c_{\max}, -c_{\min}]$,
- evenly many (possibly zero) hats below h with support $[c_{\min}, -c_{\min}]$, or
- numerous hats with one circle each above h corresponding to multiples belonging to the chain with support $[c_{\max}, c_{\min}]$.

None of these rows have $l = 0$. In order to perform a ui^{-1} of type 3', we must obtain a row with $l = 0$ through row exchanges. However, the rows above h cannot be exchanged with h or the rows below h since no such pair of rows has one row's support containing the other. By Lemma 7.45, none of the rows above h are changed by any row exchanges with each other. Also, no row exchange involving only rows below h changes $dual(\mathcal{E})$. Thus, the only row exchanges that could possibly give an $l = 0$ are those involving h. But h can only be exchanged with the hats below, and since each of these hats has one circle of the same sign, by Lemma 6.11, we need only consider what happens after swapping h with any one of these rows, as doing two row exchanges leaves h unchanged. Since h fails the alternating sign condition with the row under it, this row exchange increases $l(h)$ by one and leaves the second row the same except for a possible sign change. Thus, no row exchanges create a row of circles, so no ui^{-1} of type 3' is possible. □

Lemma 7.47. *Suppose $\mathcal{E}$ is of type $Y_{\mathcal{M}}$ with $\mathcal{S}$-data $(\{0, \overline{1}, \overline{2}, \dots, \overline{k}\})$. Then $\mathcal{E} = \mathcal{E}^{min}$.*

Proof. Note that $\mathcal{E}$ consists of hats $([A, -A], A, \eta_A)$ for $A \in \{1, \dots, k\}$, a single chain $([0, 0], 0, \eta_0)$, and multiples. We check that there are no possible lowering operators on $\mathcal{E}$.

First, we check lowering operators of type 3'; i.e., lowering operators involving ui or ui^{-1} of type 3'. It is clear that the only uis of type 3' are S^{-1} and U^{-1}: S^{-1} cannot be applied because all the multiples have the same sign as the single chain, and U^{-1} cannot be applied because there is only one chain. Meanwhile, there are no $dual \circ ui^{-1} \circ dual$ operations of type 3' possible on $\mathcal{E}$ because every row r in $\mathcal{E}$ has $C(r) = 1$: therefore, the same is true of $dual(\mathcal{E})$, so no ui^{-1} can be performed.

The only lowering operator not of type 3' is a ui. We observe that there are no ui operators not of the type 3', since there are no rows whose supports overlap and do not contain each other. This can be seen because all the hats have supports containing each other, and all other rows have supports of one circle. □

Clearly, any $\mathcal{E}$ of type $Y_{\mathcal{M}}$ is equivalent to a virtual extended multi-segment of the form described in Lemma 7.46 or Lemma 7.47, since all $Y_{\mathcal{M}}$ multi-segments of the same sign are equivalent to the same tempered block via applications of S. Therefore, given $\mathcal{E}$ of type $Y_{\mathcal{M}}$, $\mathcal{E}^{\min}$ is always of type $Y_{\mathcal{M}}$ and sign $\eta(\mathcal{E})$.

Now, it suffices to check that all raising operators on $\mathcal{E}$ preserve sign and type $Y_{\mathcal{M}}$ form. Such raising operators must be of one of the following three forms

- ui^{-1} of type 3',
- $dual \circ ui \circ dual$ where the ui is of type 3', or
- $dual \circ ui \circ dual$ where the ui is not of type 3'.

For the first of these, it is clear that S and U are the only ui^{-1} operations of type 3'. For the second, when $\mathcal{M}$ starts at zero, any $dual \circ ui \circ dual$ of type 3' must have one of the rows it involves be a hat (lest the rows would have intersecting duals). If the other row is also a hat, then the operation is M, and if the other operation is a row of circles, then the operation is D. When $\mathcal{M}$ starts after zero, it is clear that there are no $dual \circ ui \circ dual$ operations of type 3' because the rows of $dual(\mathcal{E})$ are all hats and therefore have intersecting supports because they all contain zero.

Since the operations S, M, U, and D all preserve type $Y_{\mathcal{M}}$ and η, it is clear we only need to account for $dual \circ ui \circ dual$s not of type 3'. We will prove that such operations preserve type $Y_{\mathcal{M}}$ and η through the following two lemmas. Lemma 7.48 will account for all such $dual \circ ui \circ dual$ operations between two rows which are not hats. Lemma 7.49 will account for $dual \circ ui \circ dual$ operations (when $\mathcal{M}$ starts at zero) between two rows, at least one of which is a hat.

Lemma 7.48. *Suppose T is a $dual \circ ui \circ dual$ not of type 3'. Suppose that T involves two rows $r_1 < r_2$, neither of which are hats. Then T preserves type $Y_{\mathcal{M}}$ and $\eta(\mathcal{E})$.*

Proof. Suppose $supp(r_1) = [A_1, B_1]$ and $supp(r_2) = [A_2, B_2]$. Since $r_1 < r_2$, we presume that $B_1 \leq B_2$. Then, $\widehat{r_1}$ and $\widehat{r_2}$ have supports $[A_1, -B_1]$ and $[A_2, -B_2]$, respectively. A ui not of type 3' between these rows is possible only if $-B_2 <$

$-B_1 \leq A_2 < A_1$. This implies that $B_1 < B_2 \leq A_2 < A_1$, so $supp(r_1) \supsetneq supp(r_2)$. Therefore, r_2 must be a multiple belonging to r_1 and we have $B_2 = A_2 < A_1$. Since $A_2 \neq A_1$, r_2 cannot be contained in the support of any chain other than r_1. There are evenly many such multiples, and using $dual \circ ui \circ dual$ to combine r_1 with any of these multiples gives multi-segments that are equivalent up to row exchange. Therefore, we assume that r_2 is the last of these multiples.

Applying such a ui to $\widehat{r_1}$ and $\widehat{r_2}$ replaces them with rows that have support $[A_1, -A_2]$ and $[A_2, -B_1]$, in that order. Applying dual gives rows with supports $[A_2, B_1]$ and $[A_1, A_2]$, in that order. The resulting rows have the same order and support as the multi-segment obtained by implementing the following change on the $\mathcal{S}$-data of $\mathcal{E}$

$$\{B_1, \ldots, A_1, \ldots\} \longrightarrow \{B_1, \ldots, A_2\}, \{A_2, \ldots, A_1, \ldots\}.$$

The multi-segment resulting from this change in $\mathcal{S}$-data is equivalent to $\mathcal{E}$, which is equivalent to $T \circ \mathcal{E}$. Therefore, Corollary 3.4 shows that $T(\mathcal{E})$ is equal to this multi-segment and therefore has type $Y_{\mathcal{M}}$ and satisfies $\eta(T(\mathcal{E})) = \eta(\mathcal{E})$. □

Lemma 7.49. *Suppose T is a $dual \circ ui \circ dual$ not of type 3'. Suppose that T involves two rows $h < r$, where h is a hat. Then T preserves type $Y_{\mathcal{M}}$ and $\eta(\mathcal{E})$.*

Proof. If $h = ([A, -B], 0, \eta)$, then $\widehat{h}$ is a row of circles with support $[A, B]$. Note that if h is associated to the set $\mathcal{T}_i^j$, then $\mathcal{T}_i^j = \{B, \ldots, A\}$. Since all $\mathcal{T}_i^j$ for $j \geq 1$ are disjoint, we conclude that if r is a hat, then $\mathrm{supp}(\widehat{h}) \cap \mathrm{supp}(\widehat{r}) = \emptyset$, rendering the operation T impossible. Therefore, r has to be some row of circles with support $[C, D]$, which dualizes to a hat with support $[C, -D]$. In order to apply T, we need to have $B \leq C < A$. Given the intersection conditions on the sets $\mathcal{S}_i$, this is only possible if r is a multiple with support $[C, C]$. We note that since $C \in \{B, \ldots, A\}$, there must be an even number of multiples equal to r. Since performing $dual \circ ui \circ dual$ with each of these multiples is equivalent up to row exchanges, we presume that r is the last row.

We claim that implementing T will replace h with a hat of the same sign $([C, -B], B, \eta)$ and change r into a chain with support $[A, C]$. This guarantees that $\eta(\mathcal{E})$ is preserved. All the while, the segment will remain odd-alternating. Given this claim, we see that T implements the following change in the $\mathcal{S}$-data:

$$\{\ldots, \overline{B, \ldots, A}, \ldots\} \longrightarrow \{\ldots, \overline{B, \ldots, C}\}, \{C, \ldots, A, \ldots\},$$

which is indeed valid. Thus, the claim implies T preserves type $Y_{\mathcal{M}}$.

To prove the claim, let $\widehat{r} = ([C, -C], C, \eta_1)$ and $\widehat{h} = ([A, B], 0, \eta_2)$. Should a ui between these rows be possible, they would be replaced with rows of support $[A, -C]$ and $[C, B]$, which would exist in the same positions as $\widehat{r}$ and $\widehat{h}$ respectively in order to satisfy P' order. Dualizing back, we obtain a multi-segment with the same order and support as $\mathcal{E}$ except:

- The row r is replaced by a row with support $[A, C]$.
- The hat h is replaced by a row of support $[C, -B]$.

Such an extended multi-segment has precisely the same support and order as the multi-segment resulting from the change in $\mathcal{S}$-data described in the claim.

These virtual extended multi-segments correspond to the same non-zero tempered blocks, so Corollary 3.4 suffices to show that they are equal. □

Having seen that all possible *dual* ∘ *ui* ∘ *dual*s not of case 3' preserve $Y_{\mathcal{M}}$ and $\eta(\mathcal{E})$, we conclude Theorem 7.17.

8. The Count $|\Psi(\pi(\mathcal{B}))|$

The goal of this section is to prove Theorem 4.7. We will do this by counting extended multi-segments of type $Y_{\mathcal{M}}$. In order to reduce to this case, we must establish a correspondence between $\Psi(\pi(\mathcal{B}))$ and extended multi-segments $\mathcal{E}$ of type $Y_{\mathcal{M}_{\mathcal{B}}}$. In particular, different type $Y_{\mathcal{M}}$ multi-segments with different $\mathcal{S}$-data should be associated with different virtual extended multi-segments. This is guaranteed by the following lemma.

Lemma 8.1. *If $\mathcal{E}_1$ and $\mathcal{E}_2$ are both of type $Y_{\mathcal{M}}$ and have $\eta(\mathcal{E}_1) = \eta(\mathcal{E}_2)$, then $\psi(\mathcal{E}_1) = \psi(\mathcal{E}_2)$ if and only if $\mathcal{E}_1 = \mathcal{E}_2$.*

Proof. It is obvious that $\mathcal{E}_1 = \mathcal{E}_2$ implies $\psi(\mathcal{E}_1) = \psi(\mathcal{E}_2)$. Meanwhile, if $\psi(\mathcal{E}_1) = \psi(\mathcal{E}_2)$, then these extended multi-segments must have the same support. The orders of their rows, furthermore, have already been prescribed and therefore must be the same. Since $\mathcal{E}_1$ and $\mathcal{E}_2$ are both $Y_{\mathcal{M}}$ and have $\eta(\mathcal{E}_1) = \eta(\mathcal{E}_2)$, we must have $\mathcal{E}_1 \sim \mathcal{E}_2$. Corollary 3.4 now suffices to prove the desired result. □

8.1. Blocks Starting at Zero.

Motivated by Definition 3.26 and Theorem 3.31 (which is the case for $\rho = \chi_V$), we consider a "lift" for arbitrary ρ in the case $\mathcal{E}$ is of type $Y_{\mathcal{M}}$, with $\mathcal{M}$ starting at zero, as follows.

Definition 8.2. *Suppose that $\mathcal{E} \in \mathrm{VRep}_\rho(G_n)$ is of type $Y_{\mathcal{M}}$, with $\mathcal{M}$ starting at zero, and write*

$$\mathcal{E} = \cup_{i=0}^{k}\{([A_i, B_i]_\rho, l_i, \eta_i)\}.$$

Let

$$\mathcal{E}' = \cup_{i=0}^{k}\{([A_i, B_i]_{\chi_W}, l_i, \eta_i)\}.$$

Recall from Theorem 7.17 that there exists a tempered virtual extended multi-segment $\mathcal{E}_{temp}$ which is equivalent to $\mathcal{E}$. We assume further that $\mathcal{E}_{temp}$ starts at 0.

We define

$$\Theta_1(\mathcal{E}) = \left\{\left([c_{\max}+1, -c_{\max}-1]_{\chi_W}, c_{\max}+1, -\eta(\mathcal{E})\right)\right\} \cup \mathcal{E}'.$$

We remark that the added extended segment should be inserted such that it is the first extended segment in the admissible order. Note that $\Theta_1(\mathcal{E}) \in \mathrm{Vseg}(H_m^{\pm})$ for some $m \in \mathbb{Z}_{\geq 0}$ and some choice of sign $\pm$.

When $\rho = \chi_V$ and $\mathcal{E} = \mathcal{E}_{\chi_V}$, this definition coincides with the local theta lift of $\pi(\mathcal{E})$ to its first occurrence in the going-up tower via Definition 3.26 and Theorems 3.29 and 3.31. We verify the $\Theta_1(\mathcal{E})$ is also an element of $\mathrm{VRep}_{\chi_W}(H_m^{\pm})$.

Lemma 8.3. *Suppose that $\mathcal{E} \in \mathrm{VRep}_\rho(G_n)$ of type $Y_{\mathcal{M}}$. Then $\Theta_1(\mathcal{E}) \in \mathrm{VRep}_{\chi_W}(H_m^{\pm})$.*

Proof. The claim follows directly from Theorem 3.21. □

Suppose that $\mathcal{E} \in \mathrm{VRep}_\rho(G_n)$ is tempered of type $Y_{\mathcal{M}}$ with sign $\eta(\mathcal{E})$. It is apparent from the definition that $\Theta_1(\mathcal{E})$ is of type $Y_{\mathcal{M}'}$ with $\eta(\Theta_1(\mathcal{E})) = -\eta(\mathcal{E})$, where $\mathcal{M}' = (m_0, \ldots, m_k, 1)$.

We will prove the following theorems, which will help us prove Theorem 4.7.

Theorem 8.4. *Let $\mathcal{E}$ be of type $Y_{\mathcal{M}}$ where $\mathcal{M} = (m_0, \ldots, m_{c_{\max}-1}, 1)$. Then the following holds.*

(1) We can perform a $dual \circ ui \circ dual$ involving the first row of $\Theta_1(\mathcal{E})$ to get a virtual extended multi-segment which we denote by $\Theta_2(\mathcal{E})$. We can further perform an additional ui^{-1} involving the $(c_{\max}+1)$-st column of $\Theta_2(\mathcal{E})$ to obtain another virtual extended multi-segment which we denote by $\Theta_3(\mathcal{E})$.

(2) We have the relation

$$\Psi(\Theta_1(\mathcal{E})) = \{\psi_{\Theta_i(\mathcal{E}')} \mid \mathcal{E}' \sim \mathcal{E}; i = 1, 2, 3\}.$$

Theorem 8.5. *Let $\mathcal{E}$ be of type $Y_{\mathcal{M}}$ where $\mathcal{M} = (m_0, \ldots, m_{c_{\max}})$ with $m_{c_{\max}} > 1$. Then the following holds.*

(1) We can perform two $dual \circ ui \circ dual$s of type 3' involving the first row of $\Theta_1(\mathcal{E})$: one with the first row whose support ends at $c_{\max}$, and one with the last row whose support ends at $c_{\max}$. We call the resulting segments $\Theta_2(\mathcal{E})$ and $\Theta_4(\mathcal{E})$, and we have $\psi_{\Theta_2(\mathcal{E})} = \psi_{\Theta_4(\mathcal{E})}$ if and only if the $\mathcal{S}$-data of $\mathcal{E}$ has some $\mathcal{S}_i = \{c_{\max}\}$. We can perform an additional ui^{-1} to both Θ_2 and Θ_4 to remove a row of one circle from the row with support ending at $c_{\max}+1$ to obtain the same $\Theta_3(\mathcal{E})$.

(2) We have the relation

$$\Psi(\pi(\Theta_1(\mathcal{E}))) = \{\psi_{\Theta_i(\mathcal{E}')} \mid \mathcal{E}' \sim \mathcal{E}; i = 1, 2, 3, 4\}.$$

We begin by proving Part (1) of Theorem 8.4 from the existence of the operators D, U, and S.

Proof of Theorem 8.4 Part (1). Suppose that $\mathcal{E}$ is of type $Y_{\mathcal{M}}$ with final multiplicity $m_{c_{\max}} = 1$. We know that $\Theta_1(\mathcal{E}) := h \cup \mathcal{E}$, where

$$h := ([c_{\max}+1, -c_{\max}-1], c_{\max}+1, -\eta(\mathcal{E})).$$

Lemma 7.40 states that there is exactly one $dual \circ ui \circ dual$ involving h. There are two cases.

- First, $\Theta_2(\mathcal{E}) = D_{h,r} \circ \Theta_1(\mathcal{E})$, for some chain r. In this case, we let
$$\Theta_3(\mathcal{E}) := S_{r',1} \circ \Theta_2(\mathcal{E}),$$
where r' is the image of r under $D_{h,r}$.
- Second, $\Theta_2(\mathcal{E}) = M_{h,h'} \circ \Theta_1(\mathcal{E})$, where h' is the first hat of $\mathcal{E}$. Here, we let
$$\Theta_3(\mathcal{E}) := U_{h*h',1} \circ \Theta_2(\mathcal{E}).$$ □

The proof of Part (1) of Theorem 8.5 is similar. To complete it we will utilize the following lemma, which considers how the two different operations D^1 and D^2 can be related to each other.

Lemma 8.6. *Suppose that $\mathcal{E}$ is of type $Y_{\mathcal{M}}$ and has a hat $h = ([A, B], -B, \eta)$. Suppose $m_{B-1} \geq 3$, and let r_1, r_2 be the first and last rows ending at $B-1$.*

(1) If r_1 is a hat, then

$$U_{h*r_1,C(h)} \circ M_{h,r_1} = S_{r_2',C(h)} \circ D^2_{h,r_2}.$$

(2) If r_1 is a chain, then

$$S_{r_1',C(h)} \circ D^1_{h,r_1} = S_{r_2',C(h)} \circ D^2_{h,r_2}.$$

Here, r_1' and r_2' indicate the images of r_1 and r_2 under $dual \circ ui \circ duals$ with h.

Proof. For Part (1), let $r_1 = ([C, -B+1], B-1, \eta_1)$ and let $r_2 = ([B-1, B-1], 0, \eta_2)$. Then conducting $U_{h*r_1,C(h)} \circ M_{h,r_1}$ implements the following changes to the $\mathcal{S}$-data of $\mathcal{E}$ according to Remarks 7.35 and 7.43.

$$\begin{aligned}\{\ldots\overline{C,\ldots,B-1},\overline{B,\ldots,A},\ldots\} &\longrightarrow \{\ldots\overline{C,\ldots,B-1,B,\ldots,A},\ldots\}\\ &\longrightarrow \{\ldots\overline{C,\ldots,B-1}\},\{B,\ldots,A,\ldots\}.\end{aligned}$$

Meanwhile, applying $S_{r_2',C(h)} \circ D^2_{h,r_2}$ gives the changes according to Remarks 7.31 and 7.41:

$$\begin{aligned}\{\ldots\overline{C,\ldots,B-1},\overline{B,\ldots,A},\ldots\} &\longrightarrow \{\ldots\overline{C,\ldots,B-1}\},\{B-1,B,\ldots,A,\ldots\}\\ &\longrightarrow \{\ldots\overline{C,\ldots,B-1}\},\{B,\ldots,A,\ldots\},\end{aligned}$$

where the last arrow takes place due to the exception discussed in Remark 7.30.

For Part (2), let $r_1 = ([C, B-1], 0, \eta_1)$. Then conducting $S_{r_1',C(h)} \circ D^1_{h,r_1}$ gives:

$$\begin{aligned}\{C,\ldots,B-1,\overline{B,\ldots,A},\ldots\} &\longrightarrow \{C,\ldots,B-1,B,\ldots,A,\ldots\}\\ &\longrightarrow \{C,\ldots,B-1\},\{B,\ldots,A,\ldots\}.\end{aligned}$$

Meanwhile, applying $S_{r_2',C(h)} \circ D^2_{h,r_2}$ gives the changes:

$$\begin{aligned}\{C,\ldots,B-1,\overline{B,\ldots,A},\ldots\} &\longrightarrow \{C,\ldots,B-1\},\{B-1,B,\ldots,A,\ldots\}\\ &\longrightarrow \{C,\ldots,B-1\},\{B,\ldots,A,\ldots\},\end{aligned}$$

which suffices to complete the proof. □

Proof of Theorem 8.5 Part (1). By Lemma 7.40, given any $\Theta_1(\mathcal{E})$ where $m_{c_{\max}} \geq 3$, it is possible to perform two different $dual \circ ui \circ dual$'s of type 3' involving the hat $h = ([c_{\max}+1, -c_{\max}-1], c_{\max}+1, -\eta(\mathcal{E}))$, resulting in virtual extended multi-segments $\Theta_2(\mathcal{E})$ and $\Theta_4(\mathcal{E})$ respectively. Lemma 7.40 states that $\Theta_2(\mathcal{E})$ and $\Theta_4(\mathcal{E})$ are equivalent via row exchanges if and only if the $\mathcal{S}$-data of $\mathcal{E}$ contains $\{c_{\max}\}$.

Now, we consider two cases. If the multi-segment $\mathcal{E}$ has a hat h' with support containing n, then the two $dual \circ ui \circ dual$'s associated with h that produce $\Theta_2(\mathcal{E})$ and $\Theta_4(\mathcal{E})$ are $M_{h,h'}$ and $D^2_{h,r}$ respectively, where r is a multiple. We can produce another virtual extended multi-segment from $\Theta_2(\mathcal{E})$ by applying $U_{h*h',1}$, and we can obtain another virtual extended multi-segment from $\Theta_4(\mathcal{E})$ by applying $S_{r',1}$, where r' is the image of r under $D^2_{h,r}$. Part (1) of Lemma 8.6 implies that both of ui^{-1}'s produce the same extended multi-segment which we call $\Theta_3(\mathcal{E})$.

In the case where the two operators that produce $\Theta_2(\mathcal{E})$ and $\Theta_4(\mathcal{E})$ are D^1 and D^2, then we can perform a ui^{-1} of the form S to separate one circle at $c_{\max}+1$ from each. Part (2) of Lemma 8.6 guarantees that both resulting multi-segments are the same, and again we let this be $\Theta_3(\mathcal{E})$. □

Next we want to establish that, whenever $\mathcal{E}$ is of type $Y_{\mathcal{M}}$ and final multiplicity $m_{c_{\max}} = 1$, no two elements of the set $\{\Theta_i(\mathcal{E}') \mid i \in \{1,2,3\}; \mathcal{E}' \sim \mathcal{E}\}$ have the same local Arthur parameter. To do this, we must simply argue that these extended multi-segments have different supports. In particular, we want to prove the following.

Lemma 8.7. *Let $\mathcal{E}_1 \sim \mathcal{E}_2$ be of type $Y_{\mathcal{M}}$, with $\mathcal{M}$ starting after zero. Then*

$$supp(\Theta_i(\mathcal{E}_1)) = supp(\Theta_j(\mathcal{E}_2)) \Rightarrow i = j, \mathcal{E}_1 = \mathcal{E}_2.$$

Proof. Starting with the case where $m_{c_{\max}} = 1$, let $\mathcal{E}$ be of type $Y_{\mathcal{M}}$ and consider some multi-segment $\Theta_i(\mathcal{E})$ for $i \in \{1,2,3\}$. Let $r \in \Theta_i(\mathcal{E})$ be the unique row whose support contains $c_{\max}+1$. Then we must be in one of the following cases:

- If $r = ([c_{\max}+1, -c_{\max}-1], c_{\max}+1, \eta)$ for some $\eta \in \{\pm 1\}$, then $i = 1$.
- If $C(r) > 1$ (i.e., r is a merged hat or merged chain), then $i = 2$.
- If $r = ([c_{\max}+1, c_{\max}+1], 0, \eta)$, for some $\eta \in \{\pm 1\}$, then $i = 3$.

This confirms that $supp(\Theta_i(\mathcal{E}_1)) = supp(\Theta_j(\mathcal{E}_2)) \Rightarrow i = j$, so it suffices to show that the multi-segment $\mathcal{E}$ can be uniquely determined from $\Theta_i(\mathcal{E})$. This is obvious in the case where $i = 1$ since $supp(\mathcal{E}) = supp(\Theta_1(\mathcal{E}) \setminus \{r\})$, as implied by Definition 8.2; note that the support suffices to identify $\mathcal{E}$ by Lemma 8.1. The statement $supp(\mathcal{E}) = supp(\Theta_1(\mathcal{E}) \setminus \{r\})$ is also clearly true for $i = 3$. In the case $i = 2$, $\mathcal{E}$ can be determined because $\Theta_3(\mathcal{E})$ can be determined: the operation from $\Theta_2(\mathcal{E})$ to $\Theta_3(\mathcal{E})$ is unique.

Meanwhile, in the case where $c_{\max} > 1$, let $\mathcal{E}$ be of type $Y_{\mathcal{M}}$. Any multi-segment $\Theta_i(\mathcal{E})$ must fall into one of the following slightly different cases:

- If $r = ([c_{\max}+1, -c_{\max}], k, \eta)$ for some $\eta \in \{\pm 1\}$, then $i = 1$.
- If $C(r) > 1$ and $|\{\mathcal{S}_i \mid c_{\max} \in \mathcal{S}_i\}| = 1$, then $i(\mathcal{E}) = 2$.
- If $C(r) > 1$ and $|\{\mathcal{S}_i \mid c_{\max} \in \mathcal{S}_i\}| = 2$, then $i(\mathcal{E}) = 4$.
- If $r = ([c_{\max}+1, c_{\max}+1], 0, \eta)$, then $i = 3$.

Again, this shows that the support of $\Theta_i(\mathcal{E})$ determines i, except in the case when the unique chain containing $c_{\max}+1$ has support $[c_{\max}+1, c_{\max}]$. This case could occur both for $\theta_2(\mathcal{E})$ and $\Theta_4(\mathcal{E})$, but only when the unique chain in $\mathcal{E}$ containing $c_{\max}$ has support $[c_{\max}, c_{\max}]$. This however, is precisely the case where $\Theta_2(\mathcal{E}) = \Theta_4(\mathcal{E})$. Meanwhile, $\mathcal{E}$ can be determined from $\Theta_i(\mathcal{E})$ again using the same methods as the case where $m_{c_{\max}} > 1$. This suffices for the proof. ☐

Now, we prove Part (2) of Theorem 8.4. Lemma 8.7 shows that the elements $\{\psi_{\Theta_i(\mathcal{E}')} \mid i = 1,2,3;\ \mathcal{E}' \sim \mathcal{E}\}$ are all distinct, and so now it suffices to show that any $\psi_{\mathcal{F}}$, where $\mathcal{F} \sim \Theta_1(\mathcal{E})$, belongs to this set. To do this, we examine the $\mathcal{S}$-data.

Proof of Theorem 8.4 Part (2). Let $\mathcal{F} \sim \Theta_1(\mathcal{E})$, where $\mathcal{E}$ is of type $Y_{\mathcal{M}}$ and $\mathcal{M} = (m_0, \ldots, m_{c_{\max}})$. Then Theorem 7.17 and Part (1) of Theorem 8.4 imply that $\mathcal{F}$ must be type $Y_{\mathcal{M}'}$, for $\mathcal{M}' = (m_0, \ldots, c_{\max}, 1)$. Consider the $\mathcal{S}$-data of $\mathcal{F}$. Since $m_{c_{\max}} = m_{c_{\max}+1} = 1$, we note $c_{\max}$ can only belong to one set $\mathcal{S}_i$, and likewise with $c_{\max}+1$.

Let $\mathcal{E}' := \mathcal{E}(\mathcal{M}, \mathcal{S}', \mathcal{T}', \eta(\mathcal{E})$, where $(\mathcal{S}', \mathcal{T}')$ is obtained by removing $k+1$ from $(\mathcal{S}, \mathcal{T})$. From Theorem 7.17, we have $\mathcal{E}' \sim \mathcal{E}$. We have the following cases.

- If both k and $k+1$ belong to the same set $\mathcal{S}_\ell = \{\dots, k, k+1\}$, then clearly $\mathcal{F} = \Theta_2(\mathcal{E}')$.
- If both k and $k+1$ belong to $\mathcal{S}_\ell = \{\dots, \overline{k, k+1}\}$, then $\mathcal{F} = \Theta_2(\mathcal{E}')$.
- If both k and $k+1$ belong to $\mathcal{S}_\ell = \{\dots, \overline{k}, \overline{k+1}\}$, then $\mathcal{F} = \Theta_1(\mathcal{E}')$.
- If both k and $k+1$ belong to $\mathcal{S}_\ell = \{\dots, k, \overline{k+1}\}$, then $\mathcal{F} = \Theta_2(\mathcal{E}')$.
- If $k+1$ belongs to its own set $\mathcal{S}_\ell = \{k+1\}$, then $\mathcal{F} = \Theta_3(\mathcal{E}')$. □

The proof of Part (2) of Theorem 8.5 is quite similar, since again, in light of Lemma 8.7, we only need to show that any $\mathcal{F} \sim \Theta_1(\mathcal{E})$ is of the form $\Theta_i(\mathcal{E}')$ for some $i \in \{1, 2, 3, 4\}$ and $\mathcal{E}' \sim \mathcal{E}$.

Proof of Theorem 8.5 Part (2). Let $\mathcal{F} \sim \Theta_1(\mathcal{E})$. Then again Theorem 7.17 and Part (1) of Theorem 8.4 imply that $\mathcal{F}$ must be of type $Y_{\mathcal{M}'}$, where $\mathcal{M}' = (m_0, \dots, c_{\max}, 1)$. Consider the $\mathcal{S}$-data of $\mathcal{F}$. Since $m_{k+1} = 1$, $k+1$ can only belong to one set $\mathcal{S}_i$. However, k can belong to at most two $\mathcal{S}_i$, but only when the latter set has $\mathcal{T}_i^0 = \{k, k+1\}$. In such a case, we clearly have $\mathcal{F} = \Theta_4(\mathcal{E}')$, where $\mathcal{E}'$ is defined as in the proof of Part (2) of Theorem 8.4. The cases where k only belongs to one $\mathcal{S}_i$ can be identified with various $\Theta_i(\mathcal{E}')$ in exactly the same manner as the proof for Part (2) of Theorem 8.4. □

Having proved Theorems 8.4 and 8.5, we can finally prove Theorem 4.7 for blocks $\mathcal{B}$ that start at zero. To do this, we need the following lemma.

Lemma 8.8. *Suppose $\mathcal{E}$ is of type $Y_{\mathcal{M}}$, where $\mathcal{M} = (m_0, \dots, m_{c_{\max}})$. Let $k \in \mathbb{Z}_{>0}$ and*

$$\mathcal{E}' := \mathcal{E} \cup \bigcup_{i=1}^{c} ([c_{\max}, c_{\max}], 0, \eta')$$

be the extended multi-segment obtained by k circles of support $[c_{\max}, c_{\max}]$ to $\mathcal{E}$. Here, the sign η' is chosen to match the sign on the last circle of $\mathcal{E}$. Then, a multi-segment $\mathcal{E}_1'$ is equivalent to $\mathcal{E}'$ if and only if $\mathcal{E}_1' = \mathcal{E}_1 \cup \bigcup_{i=1}^{k}([c_{\max}, c_{\max}], 0, \eta')$ for $\mathcal{E}_1 \sim \mathcal{E}$. In particular, we obtain that

$$|\Psi(\pi(\mathcal{E}'))| = |\Psi(\pi(\mathcal{E}))|.$$

Proof. Per the proof of Theorem 7.17, we have seen that the raising operators on $\mathcal{E}$ are precisely S, M, U, D, and certain $dual \circ ui \circ dual$s not of type 3' that we have shown to be equivalent to some combination of S, M, U, D, and their inverses (Lemmas 7.48 and 7.49). It is clear from the definitions of S, M, U, D, that all of these operators and their inverses can still be performed on the $\mathcal{E}$ when considered as a sub-virtual extended multi-segment of $\mathcal{E}'$. This proves the forward direction of the desired statement. To prove the other direction, we claim that these are the only operators that can be applied to $\mathcal{E}.'$

We assume otherwise for the sake of contradiction, i.e, we assume that there exists an operator T applicable on $\mathcal{E}'$ which is not applicable on $\mathcal{E}$. Since none of the multiples $r_i = ([c_{\max}, c_{\max}], 0, \eta')$ allow the other rows of $\mathcal{E}$ to interact with each other in new ways, such an operator T would have to involve one of these multiples r_i. We consider all possible operators. T cannot be a $dual \circ ui \circ dual$ because $dual(r_i)$ has support $[c_{\max}, -c_{\max}]$, and all other rows in $dual(\mathcal{E}')$ would be contained in this support. T also cannot be a ui^{-1} of type 3'. since these rows have only one circle. Meanwhile, T obviously cannot be a $dual \circ ui^{-1} \circ dual$ of type 3' since each of these multiples has only one circle.

The only remaining possibility is that T is a ui, in which case it must be one of type 3' since the rows r_i have only one circle. T would have to merge r_i with r, a row ending at $c_{\max} - 1$. Consider the sets $\mathcal{S}_i$ containing $c_{\max} - 1$ or k. If we have a set $\{\ldots, c_{\max} - 1, c_{\max}\}$, then T is impossible because the only rows r ending at $k - 1$ are multiples, which must have the same sign as the rows with support $[c_{\max}, c_{\max}]$. If we have a set $\{\ldots, k-1, \overline{c_{\max}}\}$, then the sign of r_i has been chosen so that it can be treated as a multiple belonging to r and therefore has the same sign as the circle at $c_{\max} - 1$, rendering T impossible. If we have $\{c_{\max} - 1\}, \{c_{\max}\}$, then applying T to merge r and r_i is the same as applying the operation to merge $c_{\max}$ with the chain of support $[c_{\max}, c_{\max}]$, which is already defined in $\mathcal{E}$.

We conclude that the virtual extended multi-segments equivalent of $\mathcal{E}'$, up to row exchange, are in one-to-one correspondence with the virtual extended-multi-segments which are equivalent to $\mathcal{E}$, up to row exchange. In particular, this means that there is a one-to-one correspondence between $\Psi(\pi(\mathcal{E}))$ and $\Psi(\pi(\mathcal{E}'))$. □

Now, we are ready to prove Theorem 4.7 for blocks that start at 0. We shall restate the result here.

Theorem 8.9. *Let $\mathcal{E}$ be a tempered almost-block with columns 0 through $c_{\max} \geq 2$. Let $\mathcal{E}_k$ denote the extended multi-segment consisting of only the rows with supports contained in $[k, 0]$. Then,*

$$|\Psi(\mathcal{E})| = \begin{cases} 3 \cdot |\Psi(\pi(\mathcal{E}_{c_{\max}-1}))| & \text{if } m_{c_{\max}-1} = 1, \\ 4 \cdot |\Psi(\pi(\mathcal{E}_{c_{\max}-1}))| - |\Psi(\pi(\mathcal{E}_{c_{\max}-1}))| & \text{if } m_{c_{\max}-2} \geq 3. \end{cases}$$

Proof. Let $\mathcal{E}'$ be the extended multi-segment obtained by removing all but one of the rows in $\mathcal{E}$ of support $[c_{\max}, c_{\max}]$. Then $\mathcal{E}'$ is a block of type $Y_{\mathcal{M}}$ for some $\mathcal{M}$ ending at $c_{\max}$. From Lemma 8.8, we have $|\Psi(\pi(\mathcal{E}))| = |\Psi(\pi(\mathcal{E}'))|$. Therefore, we may assume for the sake of the proof that $\mathcal{E}$ is a block with $m_{c_{\max}} = 1$. In this case, we have $\mathcal{E} = \Theta_3(\mathcal{E}_{c_{\max}-1})$ up to twisting by the appropriate supercuspidals. If $m_{c_{\max}-1} = 1$, the fact that $|\Psi(\pi(\mathcal{E}))| = 3 \cdot |\Psi(\pi(\mathcal{E}_{c_{\max}-1}))|$ follows immediately from Part (2) of Theorem 8.4.

Meanwhile, Part (2) of Theorem 8.5 gives us that if $m_{c_{\max}-1} > 1$, then

$$|\Psi(\pi(\mathcal{E}))| = 4 \cdot |\Psi(\pi(\mathcal{E}_{c_{\max}-1}))| - R,$$

where R is the cardinality of the set $\{\mathcal{E} \mid \mathcal{E} \sim \mathcal{E}_{c_{\max}-1}, \Theta_2(\mathcal{E}) = \Theta_4(\mathcal{E})\}$. According to part (2) of Theorem 8.5, these are precisley the multi-segments with $\{k-1\}$ in their $\mathcal{S}$-data.

Applying Lemma 8.8 again allows us to assume each of the $\mathcal{E} \in \{\mathcal{E} \mid \mathcal{E} \sim \mathcal{E}_{c_{\max}-1}, \Theta_2(\mathcal{E}) = \Theta_4(\mathcal{E})\}$ has $m_{c_{\max}-1} = 1$. In that case, these are precisely the multi-segments equal to $\Theta_3(\mathcal{E}'')$ for $\mathcal{E}'' \sim \mathcal{E}_{k-2}$. The cardinality of this set is precisely $|\Psi(\pi(\mathcal{E}_{c_{\max}-2}))|$. □

8.2. **Blocks not starting at zero.** In this subsection, we use Theorem 7.17 to derive Equation (4.2) in Theorem 4.7, which we restate below for convenience.

Theorem 8.10 (count for blocks not starting at zero)**.** *Let $\mathcal{B}$ be a block starting at c_{min} and ending at c_{max}, and suppose that $c_{min} > 0$. Let*

$$\mathcal{B}' := \mathrm{rc}_{c_{max}}(\mathcal{B})$$
$$\mathcal{B}'' := \mathrm{rc}_{c_{max}-1}(\mathcal{B}').$$

Then

$$|\Psi(\pi(\mathcal{B}))| = \begin{cases} 2|\Psi(\pi(\mathcal{B}'))| & \text{if } m_{c_{max}-1} = 1, \\ 3|\Psi(\pi(\mathcal{B}'))| - |\Psi(\pi(\mathcal{B}''))| & \text{if } m_{c_{max}-1} = 3, 5, \ldots. \end{cases}$$

Proof. First we consider the case that $m_{c_{max}-1} = 1$. From Theorem 7.17, the set $\Psi(\pi(\mathcal{B}))$ is in bijection with the set of valid $\mathcal{S}$ for the block-tuple $\mathcal{M}_{\mathcal{B}}$, and analogously for the set $\Psi(\pi(\mathcal{B}'))$. To prove the formula, we will partition the set of valid $\mathcal{S}$ for $\mathcal{M}_{\mathcal{B}}$ into two sets, Ψ_1 and Ψ_2, each of which is in bijection with the set of valid $\mathcal{S}'$ for $\mathcal{M}_{\mathcal{B}'}$. The set of valid $\mathcal{S}'$ for $\mathcal{M}_{\mathcal{B}'}$ is in turn in bijection with $\Psi(\pi(\mathcal{B}'))$ by Theorem 7.17, so this suffices to prove the formula.

The first set Ψ_1 consists of the valid $\mathcal{S} = (\mathcal{S}_1, \ldots, \mathcal{S}_k)$ with $\mathcal{S}_k = \{c_{max}\}$. In this case, if we let

$$\mathcal{S}' = (\mathcal{S}'_1, \ldots, \mathcal{S}'_{k-1}) := (\mathcal{S}_1, \ldots, \mathcal{S}_{k-1}),$$

then we claim $\mathcal{S}'$ is valid for $\mathcal{M}_{\mathcal{B}'}$. To see this, we simply check each of the conditions in Definition 7.1. Condition (1) is clear. Since $\mathcal{M}_{\mathcal{B}'} = (m_{c_{min}}, \ldots, m_{c_{max}-1})$, Condition (2) holds. Finally, Conditions (3), (4), and (5) are strictly weaker for $\mathcal{S}'$ compared to $\mathcal{S}$.

Moreover, we claim that given $\mathcal{S}' = (\mathcal{S}'_1, \ldots, \mathcal{S}'_{k-1})$ valid for $\mathcal{M}_{\mathcal{B}'}$,

$$\mathcal{S} := (\mathcal{S}'_1, \ldots, \mathcal{S}'_{k-1}, \{c_{max}\})$$

is valid for $\mathcal{M}_{\mathcal{B}}$. Conditions (1), (2) of Definition 7.1 are clear. Condition (3) holds because every element of any $\mathcal{S}'_i$ is at most $c_{max} - 1$, since $\mathcal{B}'$ ends at $c_{max} - 1$. Condition (4) is clear because we only need to check it in the case that $j = k$, but $\mathcal{S}_k$ does not intersect with any other $\mathcal{S}_i$. Finally, Condition (5) is clear because $|\mathcal{S}_k| < 2$, and it holds for $\mathcal{S}'$.

The second set Ψ_2 consists of all other valid $\mathcal{S}$. In this case due to Condition (3) of Definition 7.1, it must be that $c_{max} \in \mathcal{S}_k$. Since $\mathcal{S} \notin \Psi_1$, $\mathcal{S}_k$ does not consist solely of c_{max}, and so we have $|\mathcal{S}_k| \geq 2$. In this case, if we let

$$\mathcal{S}' = (\mathcal{S}'_1, \ldots, \mathcal{S}'_k) := (\mathcal{S}_1, \ldots, \mathcal{S}_{k-1}, \mathcal{S}_k \setminus \{c_{max}\}),$$

then we claim $\mathcal{S}'$ is valid for $\mathcal{M}_{\mathcal{B}'}$. Condition (1) holds because $|\mathcal{S}_k| \geq 2$, so $\mathcal{S}'_k \neq \emptyset$. Conditions (2) and (3) are clear, and Conditions (4) and (5) hold for the same reason as before: they are strictly weaker for $\mathcal{S}'$ than for $\mathcal{S}$.

Moreover, we claim that given $\mathcal{S}' = (\mathcal{S}'_1, \ldots, \mathcal{S}'_k)$ valid for $\mathcal{M}_{\mathcal{B}'}$,

$$\mathcal{S} = (\mathcal{S}'_1, \ldots, \mathcal{S}'_{k-1}, \mathcal{S}'_k \cup \{c_{max}\})$$

is valid for $\mathcal{M}_{\mathcal{B}}$. Conditions (1), (2), (3) are clear. Condition (4) is clear because we only need to check the case of $\mathcal{S}'_k \cup \{c_{max}\}$ intersecting with other $\mathcal{S}'_i$, but since $c_{max} \notin \mathcal{S}'_i$, it must be that $\mathcal{S}'_i$ and $\mathcal{S}'_k$ have nontrivial intersection, so it follows that $k - i = 1$, $|\mathcal{S}'_k| \geq 2$, and $m_c > 1$, where $c \in \mathcal{S}'_i \cap \mathcal{S}'_k$. So certainly $|\mathcal{S}'_k \cup \{c_{max}\}| \geq 2$ as well. For Condition (5), it suffices to check the case where $\mathcal{S}'_k$ does not satisfy the hypotheses of Condition (5) in $\mathcal{S}'$ but $\mathcal{S}'_k \cup \{c_{max}\}$ does satisfy the hypotheses. Since $|\mathcal{S}'_k \cup \{c_{max}\}| \geq 2$ while $|\mathcal{S}'_k| < 2$, it must be that $\mathcal{S}'_k = \{c_{max}-1\}$. But $m_{c_{max}-1} = 1$ by assumption, so the hypotheses of Condition (5) are not satisfied, so the condition is satisfied. This completes the bijections for $m_{c_{max}-1} = 1$ case of Theorem 8.10.

Second we consider the case that $m_{c_{max}-1} > 1$. We will partition the set of valid $\mathcal{S}$ for $\mathcal{M}_{\mathcal{B}}$ into three sets Ψ_1, Ψ_2, and Ψ_2. The first two sets will both be in bijection with the set of valid $\mathcal{S}'$ for $\mathcal{M}_{\mathcal{B}'}$. The third set Ψ_2 will be in bijection with a subset of the set of valid $\mathcal{S}'$ for $\mathcal{M}_{\mathcal{B}'}$, with the complement of this subset being in bijection with the set of valid $\mathcal{S}''$ for $\mathcal{M}_{\mathcal{B}''}$. Together with Theorem 7.17, this proves the formula.

The three sets Ψ_1, Ψ_2, Ψ_3 are as follows. Let $\mathcal{S} = (\mathcal{S}_1, \ldots, \mathcal{S}_k)$ be valid for $\mathcal{M}_{\mathcal{B}}$. We have that $\mathcal{S}$

- lies in Ψ_1 if $\mathcal{S}_k = \{c_{max}\}$,
- lies in Ψ_2 if $c_{max} - 1$ appears in more than one $\mathcal{S}_i$, or
- lies in Ψ_3 if $c_{max} - 1$ appears in exactly one $\mathcal{S}_i$, say $\mathcal{S}_{i_0}$, and $c_{max} \in \mathcal{S}_{i_0}$.

Observe that these three sets partition the set of valid $\mathcal{S}$. Certainly $\Psi_2 \cap \Psi_3 = \emptyset$. If $\mathcal{S} \notin \Psi_2 \cup \Psi_3$, then $c_{max} - 1$ appears in exactly one $\mathcal{S}_i$, say $\mathcal{S}_{i_0}$, and $c_{max} \notin \mathcal{S}_{i_0}$. By Condition (3) of Definition 7.1, the set $\mathcal{S}_{i_0+1}$ can contain only c_{max}. By Condition (4), there can be no further sets $\mathcal{S}_j$ with $j > i_0 + 1$. Hence $k = i_0 + 1$, and we have $\mathcal{S}_k = \{c_{max}\}$, so $\mathcal{S}$ lies in the first set.

Now we construct our bijections. The bijection for Ψ_1 is exactly the same as before, when $m_{c_{max}-1} = 1$, since we did not use this fact.

For Ψ_2, let $\mathcal{S}$ be valid for $\mathcal{M}_{\mathcal{B}}$. If $c_{max} - 1$ appears in more than on $\mathcal{S}_i$, then by Condition (3) it appears in exactly two, say $\mathcal{S}_{i_0}$ and $\mathcal{S}_{i_0+1}$. By Condition (4), we have $|\mathcal{S}_{i_0+1}| > 1$, so $\mathcal{S}_{i_0+1} = \{c_{max} - 1, c_{max}\}$. By the same reasoning as before, there can be no further sets $\mathcal{S}_j$ with $j > i_0 + 1$, or in other words $k = i_0 + 1$. Then we claim

$$\mathcal{S}' = (\mathcal{S}'_1, \ldots, \mathcal{S}'_{k-1}) := (\mathcal{S}_1, \ldots, \mathcal{S}_{k-1})$$

is valid for $\mathcal{M}_{\mathcal{B}'}$. Condition (1) is clear. Condition (2) follows from the fact that $c_{max} - 1 \in \mathcal{S}'_{k-1}$, and the corresponding condition holds for $\mathcal{S}$. Conditions (3), (4), (5) all follow directly from the corresponding conditions for $\mathcal{S}$.

Moreover, we claim that given $\mathcal{S}' = (\mathcal{S}'_1, \ldots, \mathcal{S}'_{k-1})$ valid for $\mathcal{M}_{\mathcal{B}'}$, then

$$\mathcal{S} = (\mathcal{S}'_1, \ldots, \mathcal{S}'_{k-1}, \{c_{max} - 1, c_{max}\})$$

is valid for $\mathcal{M}_{\mathcal{B}}$. Conditions (1), (2), (3) are clear. For Condition (4), it suffices to check overlaps between $\{c_{max} - 1, c_{max}\}$ and some other $\mathcal{S}'_i$. Since the condition holds for $\mathcal{S}'$, the only possible such overlap is with $\mathcal{S}'_{k-1}$, with intersection $\{c_{max} - 1\}$. Then it is indeed the case that $|\{c_{max} - 1, c_{max}\}| \geq 2$ and $m_{c_{max}-1} > 1$. Finally, for Condition (5) it again suffices to check the condition for $\{c_{max} - 1, c_{max}\}$, which does indeed satisfy the hypotheses. We have $c_{max} - 1 \in \mathcal{S}'_{k-1}$ since $\mathcal{S}'$ is valid for $\mathcal{M}_{\mathcal{B}'}$ and therefore satisfies Condition (2), (3).

Finally, we construct our bijection between Ψ_3 and a subset of the valid $\mathcal{S}' = (\mathcal{S}'_1, \ldots, \mathcal{S}'_k)$ for $\mathcal{M}_{\mathcal{B}'}$. The subset consists of those $\mathcal{S}'$ satisfying the following property.

$$\text{(Q)} \qquad \text{If } \min(\mathcal{S}'_k) = c_{max} - 1, \text{then } c_{max} - 1 \in \mathcal{S}'_{k-1}.$$

First suppose we have a valid $\mathcal{S}$ for $\mathcal{M}_{\mathcal{B}}$ lying in Ψ_3. If $c_{max}-1$ appears in only $\mathcal{S}_{i_0}$ and $c_{max} \in \mathcal{S}_{i_0}$, then let

$$\mathcal{S}' = (\mathcal{S}'_1, \ldots, \mathcal{S}'_k) := (\mathcal{S}_1, \ldots, \mathcal{S}_{k-1}, \mathcal{S}_k \setminus \{c_{max}\}).$$

Again it is clear that the conditions for $\mathcal{S}'$ to be valid for $\mathcal{M}_{\mathcal{B}'}$ are strictly weaker than the conditions for $\mathcal{S}$ to be valid for $\mathcal{M}_{\mathcal{B}}$. We also observe that $\mathcal{S}'$ always has property (Q), because if $\min(\mathcal{S}_k \setminus \{c_{max}\}) = c_{max}-1$, then since $m_{c_{max}-1} > 1$, by Condition (5) we have $c_{max}-1 \in \mathcal{S}'_{k-1}$.

Moreover, suppose we have $\mathcal{S}'$ valid for $\mathcal{M}_{\mathcal{B}'}$ satisfying property (Q). Then we claim

$$\mathcal{S} = (\mathcal{S}'_1, \ldots, \mathcal{S}'_k \cup \{c_{max}\})$$

is valid for $\mathcal{M}_{\mathcal{B}}$. Conditions (1), (2), (3) are clear. For Condition (4), it suffices to check the case where one of the sets is $\mathcal{S}'_k \cup \{c_{max}\}$. Since $\mathcal{S}'$ is valid, this can only have nontrivial intersection with $\mathcal{S}'_{k-1}$. If c is this intersection, then the fact that $m_c > 1$ follows from the fact that Condition (4) holds for $\mathcal{S}'$. Also, $|\mathcal{S}'_k \cup \{c_{max}\}| \geq 2$. Finally, for Condition (5) it suffices to check the set $\mathcal{S}'_k \cup \{c_{max}\}$. If $\min(\mathcal{S}'_k) < c_{max}-1$, then the hypotheses of Condition (5) for this set are equivalent to those for $\mathcal{S}'_k$ in $\mathcal{S}'$. So the condition for $\mathcal{S}$ is also satisfied. If $\min(\mathcal{S}'_k) = c_{max}-1$, then Condition (5) follows from the fact that $\mathcal{S}'$ has property (Q).

To complete the proof, we construct a bijection between the $\mathcal{S}'$ not satisfying property (Q) and the $\mathcal{S}''$ valid for $\mathcal{M}_{\mathcal{B}''}$. Given $\mathcal{S}' = (\mathcal{S}'_1, \ldots, \mathcal{S}'_k)$ not satisfying property (Q), we claim

$$\mathcal{S}'' = (\mathcal{S}'_1, \ldots, \mathcal{S}'_{k-1})$$

is valid for $\mathcal{M}_{\mathcal{B}''}$. Since $\mathcal{S}'$ does not satisfy property (Q), we have $\min(\mathcal{S}'_k) = c_{max}-1$ and $c_{max}-1 \notin \mathcal{S}'_{k-1}$. Hence Condition (2) follows. Conditions (1), (3), (4), (5) are clear from the fact that $\mathcal{S}'$ is valid.

Moreover, given $\mathcal{S}'' = (\mathcal{S}''_1, \ldots, \mathcal{S}''_{k-1})$ valid for $\mathcal{M}_{\mathcal{B}''}$, we claim

$$\mathcal{S}' = (\mathcal{S}''_1, \ldots, \mathcal{S}''_{k-1}, \{c_{max}-1\})$$

is valid for $\mathcal{M}_{\mathcal{B}'}$. Conditions (1), (2), (3) are clear. Note that no additional constraints are imposed by Conditions (4) and (5) applied to $\mathcal{S}'$ since $\{c_{max}-1\}$ cannot intersect any other $\mathcal{S}''_i$, and its size is not at least 2, respectively. Finally, observe that this map is the inverse of the previous one because $\mathcal{S}'$ not satisfying property (Q) implies that $\mathcal{S}'_k = \{c_{max}-1\}$. $\square$

9. Proof of Results Involving the Theta Correspondence

Here, we use our previous results about tempered representations to provide proofs for Theorem 4.8, Cases 1, 3, 4, and 6 of Conjecture 4.11, and Theorem 4.16.

9.1. Theta Correspondence for Almost-Blocks.

Suppose that $\mathcal{E} \in \mathrm{Rep}(G_n)$ is tempered and let $\pi = \pi(\mathcal{E})$. For brevity, we write $m^{\mathrm{up},\alpha} = m^{\mathrm{up},\alpha}(\pi)$. We are interested in determining $\Psi(\theta^{\mathrm{up}}_{-m^{\mathrm{up},\alpha}}(\pi(\mathcal{E}))$ using $\Psi(\pi)$. We decompose $\mathcal{E} =$

$\cup_{i=1}^{p}\mathcal{E}_{\rho_i}$ where $\rho_i \neq \rho_j$ for any $i \neq j$. First, we note by Theorem 3.16 that

$$\Psi(\pi(\mathcal{E})) = \prod_{i=1}^{p} \Psi(\pi(\mathcal{E}_{\rho_i})).$$

We begin by remarking on a trivial case.

Remark 9.1. *First suppose that $\rho_i \neq \chi_V$ for any $i = 1, \ldots, p$. By Theorem 3.31, we have that $\mathrm{up} = \mathrm{down}$ and $m^{\mathrm{up},\alpha}(\pi) = m^{\mathrm{down},\alpha}(\pi) = 1$ and so the distinction between the going-up and going-down towers does not matter. From Theorem 3.29, we find that*

$$\Psi(\theta_{-1}^{\pm}(\pi(\mathcal{E})) = \{\psi_1 \mid \psi \in \Psi(\pi)\},$$

where $\psi_1 = \psi_\alpha$ where $\alpha = 1$ (see Conjecture 2.7).

Suppose that some $\rho_i = \chi_v$, say $\rho_1 = \chi_V$. By Theorems 3.16 and 3.29, we have that

$$|\Psi(\theta_{-m^{\mathrm{up},\alpha}}^{\mathrm{up}}(\pi(\mathcal{E})))| = |\Psi(\pi((\mathcal{E}_{\chi_V})_{m^{\mathrm{up},\alpha}}^{\mathrm{up}})))| \prod_{i=2}^{p} |\Psi(\pi(\mathcal{E}_{\rho_i}))|,$$

where $\pi((\mathcal{E}_{\chi_V})_{m^{\mathrm{up},\alpha}}^{\mathrm{up}})$ is defined by Algorithm 3.28. Thus, it suffices to determine $|\Psi(\pi((\mathcal{E}_{\chi_V})_{m^{\mathrm{up},\alpha}}^{\mathrm{up}})))|$.

We proceed in two cases based on whether $\mathcal{E}_{\chi_V}$ starts at 0 or not. We begin with the latter case. Note that, by Theorem 3.31, $m^{\mathrm{up},\alpha}(\pi) = m^{\mathrm{down},\alpha}(\pi) = 1$ in this case and so again the distinction between the going-up and going-down towers does not matter.

Lemma 9.2. *If $\mathcal{E}_{\chi_V}$ does not start at 0, then $\theta_{-1}^{\pm}(\pi(\mathcal{E}))$ is tempered. In particular, $(\mathcal{E}_{\chi_V})_1^{\pm}$ is tempered and so $|\Psi(\pi((\mathcal{E}_{\chi_V})_1^{\pm}))|$ can be computed by Theorem 4.7.*

Proof. First, we note that [8, Theorem B] implies that $(\mathcal{E}_{\chi_V})_1^{\pm} \in \mathrm{Rep}(H_{n+1}^{\pm})$. By Algorithm 3.28 and the fact that $\mathcal{E}_{\chi_V}$ does not start at 0, it follows that $\mathcal{E}_1^{\pm}$ is tempered which further implies that $\theta_{-1}^{\pm}(\pi(\mathcal{E}))$ is tempered. □

Hereinafter, we treat the case that $\mathcal{E}_{\chi_V}$ starts at 0. By Theorem 3.31, we have that $\mathrm{up} = -\eta(\mathcal{E}_{\chi_V})$ and $\mathrm{up} \neq \mathrm{down}$. Furthermore, we have that $(\mathcal{E}_{\chi_V})_{m^{\mathrm{up},\alpha}}^{\mathrm{up}} = \Theta_1(\mathcal{E}_{\chi_V})$ in this setting (see Definition 8.2). Thus, it suffices to study $|\Psi(\Theta_1(\mathcal{E}_{\chi_V}))|$.

This case naturally breaks into two cases based on the block decomposition of $\mathcal{E}_{\chi_V}$.

Lemma 9.3. *Suppose that $\mathcal{E}_{\chi_V}$ consists of a single block starting at 0. Then $\Theta_1(\mathcal{E}_{\chi_V})$ is equivalent to an almost block and hence $|\Psi(\Theta_1(\mathcal{E}_{\chi_V}))|$ is determined by Theorem 4.7.*

Proof. This is a direct consequence of Theorem 7.17. □

In the case where $\mathcal{E}$ is a single block starting at zero, then it follows from Theorem 7.17 that the theta lift $\theta_{-m^{\mathrm{up},\alpha}}^{\mathrm{up}}(\pi(\mathcal{E}))$ is itself a tempered representation, since it is of type $Y_{\mathcal{M}}$ for some $\mathcal{M}$, so the count $|\Psi(\theta_{-m^{\mathrm{up},\alpha}}^{\mathrm{up}}(\pi(\mathcal{E})))|$ follows immediately from Theorem 4.7.

Next we consider consider the case where $\mathcal{E}_{\chi_V} = \mathcal{E}' \cup r$, where $\mathcal{E}'$ is of type $Y_{\mathcal{M}}$ and $r = ([c_{\max}, c_{\max}]_{\chi_V}, 0, \eta)$ for η equaling the sign of the last circle of $\mathcal{E}'$. It should be noted that the proof of Lemma 7.40 holds even when the multiplicities

m_i are not odd. In fact, this argument works if we replace χ_V by any orthogonal representation ρ of $\mathrm{GL}_d(F)$ Therefore, we obtain the following lemma.

Lemma 9.4. *Suppose $\mathcal{E}_{\chi_V} = \mathcal{E}' \cup ([c_{\max}, c_{\max}]_{\chi_V}, 0, \eta)$ as described above. Let $\Theta_1(\mathcal{E}_{\chi_V}) = h \cup \mathcal{E}''$, where $h = ([c_{\max}+1, -c_{\max}-1]_{\chi_W}, n+1, -\eta(\mathcal{E}))$ and $\mathcal{E}''$ is obtained from $\mathcal{E}$ by replacing each χ_V by χ_W. Then we can perform precisely two dual $\circ$ ui $\circ$ duals (up to row exchange) combining the first row of $\Theta_1(\mathcal{E})$: one with the first row whose support ends at $c_{\max}$, and one with the last row whose support ends at $c_{\max}$. We call the resulting segments $\Theta_2(\mathcal{E})$ and $\Theta_4(\mathcal{E})$, and we have $\psi_{\Theta_2(\mathcal{E}_{\chi_V})} = \psi_{\Theta_4(\mathcal{E}_{\chi_V})}$ if and only if $\mathcal{E}'$ has $\{c_{\max}\}$ in its $\mathcal{S}$-data.*

By the same proof as for Lemma 8.7, we have that all the elements of the set $\{\psi_{\Theta_i(\mathcal{E})} \mid \psi_{\mathcal{E}} \in \Psi(\pi), i \in \{1,2,4\}\}$ are distinct, except for the case where $\psi_{\Theta_2(\mathcal{E})} = \psi_{\Theta_4(\mathcal{E})}$, as specified in Lemma 9.4. In order to prove Theorem 4.8, it suffices to prove that the elements in this set are the only Arthur packets that $\pi(\Theta_1(\mathcal{E}))$ belongs to.

Lemma 9.5. *If $\mathcal{E}$ is an almost block with $c_{\max}$ columns, then*

$$\Psi(\theta^{\mathrm{up}}_{-m^{\mathrm{up},\alpha}}(\pi(\mathcal{E}))) = \{\psi_{\Theta_i(\mathcal{E}')} \mid \psi_{\mathcal{E}'} \in \Psi(\pi(\mathcal{E})), i \in \{1,2,4\}\}.$$

Proof. As shorthand, let P denote the set $P = \{\psi_{\Theta_i(\mathcal{E}')} \mid \psi_{\mathcal{E}'} \in \Psi(\pi(\mathcal{E})), i \in \{1,2,4\}\}$. Lemma 9.4 gives us that $\Psi(\theta^{\mathrm{up}}_{-m^{\mathrm{up},\alpha}}(\pi(\mathcal{E}))) \subset P$, so we need only prove the reverse direction. It suffices to show that P is closed under the basic operators, i.e., if $\psi_{\Theta_i(\mathcal{E}')} \in P$, then we claim that there is no raising or lowering operator from $\Theta_i(\mathcal{E}')$ that does not induce another element of P. To observe this, we split into several cases.

If $i=1$, then $\Theta_i(\mathcal{E}')$ has the hat $h = ([c_{\max}+1, -c_{\max}-1], c_{\max}+1, -\eta(\mathcal{E}))$ as its first row. It is clear that any operator not involving h produces a multi-segment in P with $i=1$. Meanwhile, any operator involving h must be a *dual* $\circ$ *ui* $\circ$ *dual* and end up producing either $\Theta_2(\mathcal{E}')$ or $\Theta_4(\mathcal{E}')$ by Lemma 9.4.

If $i=2$ or $i=4$, it is impossible to execute a ui^{-1} to $\Theta_3(\mathcal{E}')$ analogous to the one given in Theorem 8.5. This is because exchanging the row whose support ends at $c_{\max}+1$ to the bottom of the extended multi-segment produces a row with two triangles rather than a row of circles, due to the extra multiple with support $[c_{\max}, c_{\max}]$, as discussed in Remark 7.39.

To see that any operator not yet considered is closed under P, consider the extended multi-segment $\mathcal{E}' \cup r$, where r is an extra multiple of support $[c_{\max}, c_{\max}]$ (with $\eta(r)$ chosen so that $\mathcal{E}' \cup r$ is nonvanishing). This multi-segment must be of the form $\Theta_i(\mathcal{E}'')$, where $i \in \{2,4\}$ and $\mathcal{E}''$ is of type $Y_{\mathcal{M}}$ for $\mathcal{M}$ ending at $c_{\max}$ with odd multiplicities. Any operator on the sub-virtual extended multi-segment $\mathcal{E}' \subset \Theta_i(\mathcal{E}'')$ must induce an operator on $\Theta_i(\mathcal{E}'')$. This is because the only operator that could interact with the added row r is a ui^{-1} separating one circle of support $[c_{\max}+1, c_{\max}+1]$, and we have seen that this operator does not exist. Since $i \in \{2,4\}$ and any operator on $\mathcal{E}'$ cannot ascend to an operator from Θ_i to Θ_3, it is clear that any operator on $\mathcal{E}'$ must produce some other Θ_i with $i \neq 3$, which belongs to the set P. ☐

9.2. **Cases 1, 4, and 6 of Conjecture 4.11.** First, suppose that $\mathcal{E} \in \mathrm{VRep}_{\chi_V}(G_n)$ is a multi-segment as described in Cases 1, 4, and 6 of Conjecture 4.11. We will prove that $\mathcal{E}^{\mathrm{up}}_{m^{\mathrm{up},\alpha}}$ is tempered, and this along with the counts described in Theorem 4.9 will imply the desired results.

If $\mathcal{B}_1, \ldots, \mathcal{B}_k$ is the block decomposition of $\mathcal{E}$, we have that $\mathcal{B}_2$ begins at least one column after $\mathcal{B}_1$ ends. It follows then from Theorem 3.31 that

$$(\mathcal{B}_1 \cup \mathcal{B}_2 \cup \cdots \cup \mathcal{B}_k)^{\mathrm{up}}_{m^{\mathrm{up},\alpha}} = (\mathcal{B}_1)^{\mathrm{up}}_{m^{\mathrm{up},\alpha}} \cup \mathcal{B}_2 \cup \cdots \cup \mathcal{B}_k.$$

Yet, according to Theorems 8.4 and 8.5, $(\mathcal{B}_1)^{\mathrm{up}}_{m^{\mathrm{up},\alpha}}$ can be changed into a new block $\mathcal{B}_1'$ with one more column than $\mathcal{B}$ via a $dual \circ ui \circ dual$ and then a ui^{-1}. These operations can be implemented on the first component of $(\mathcal{B}_1)^{\mathrm{up}}_{m^{\mathrm{up},\alpha}} \cup \mathcal{B}_2 \cup \cdots \cup \mathcal{B}_k$, giving

$$(\mathcal{B}_1 \cup \mathcal{B}_2 \cup \cdots \cup \mathcal{B}_k)^{\mathrm{up}}_{m^{\mathrm{up},\alpha}} = \mathcal{B}_1' \cup \mathcal{B}_2 \cup \cdots \cup \mathcal{B}_k.$$

In the event where $\mathcal{B}_1'$ and $\mathcal{B}_2$ share a column, we must have that the circles in this column have the same sign by Theorem 3.21(ii). Since each component $\mathcal{B}_1'$ is tempered, we conclude that $\pi(\Theta_3(\mathcal{E}))$ is a tempered representation, from which we obtain Cases 1, 4, and 6 of Conjecture 4.11.

Remark 9.6. *Note that $\mathcal{B}_1' \cup \mathcal{B}_2 \cup \cdots \cup \mathcal{B}_k$ is not necessarily itself the block decomposition of the theta lift. In the case where $\mathcal{B}_1'$ and $\mathcal{B}_2$ have an overlapping column, all but one of the circles in the first row of $\mathcal{B}_2$ need to be transferred to the end of $\mathcal{B}_1'$, thereby replacing them with two new blocks $\mathcal{B}_1''$ and $\mathcal{B}_2$. Yet, Lemma 8.8 gives $\Psi(\pi(B_1'')) = \Psi(\pi(B_1'))$, while the formula in Theorem 4.7 guarantees that $\Psi(\pi(B_2')) = \Psi(\pi(B_2))$. This, along with 10.13, implies the formula*

$$|\Psi(\theta^{\mathrm{up}}_{-m^{\mathrm{up},\alpha}}(\pi(\mathcal{E})))| = |\Psi(\theta^{\mathrm{up}}_{-m^{\mathrm{up},\alpha}}(\pi(\mathcal{B}_1)))| \cdot \prod_{i=2}^{k} |\Psi(\pi(\mathcal{B}_i))|.$$

Such a formula is not generally possible for Cases 1 and 4, as described in Remark 4.15.

9.3. **Case 3 of Conjecture 4.11.** Throughout this subsection, we often omit the orthogonal supercuspidal representations, e.g., χ_V and χ_W.

In order to prove Case 3 of Conjecture 4.11, we will first specify a modification of type $Y_{\mathcal{M}}$, which we will call type $Z_{\mathcal{M}}$. Let k_1 and k_2 be integers where $0 < k_1 < k_2$. Let $\mathcal{M} = (m_0, m_1, \ldots, m_{k_2})$ be a tuple of positive integers. Here, we require that $m_i \in 2\mathbb{Z}$ if and only if $i \in \{k_1, k_1+1\}$. Again, we set a collection of sets $\mathcal{S} = \{\mathcal{S}_1, \ldots, \mathcal{S}_\ell\}$ and subsets $\mathcal{T}$ with the same rules as before as in Definitions 7.1 and 7.9, with the additional requirement that there exists no i such that $\mathcal{T}_i^0 = \{k_1\}$. We construct an associated $\mathcal{E}(\mathcal{M}, \mathcal{S}, \mathcal{T}, \eta)$ exactly as before as in Definition 7.11, except with the following differences:

- If $\min \mathcal{T}_i^0 = k_1$, we associate the row $([\max \mathcal{T}_i^0, \min \mathcal{T}_i^0], 1, \eta)$ rather than $([\max \mathcal{T}_i^0, \min \mathcal{T}_i^0], 0, \eta)$. This chain should alternate in sign with the previous row.
- If $r = ([A, k_1+1], 0, \eta)$ is a chain beginning at k_1+1, then r should not alternate with the previous row.

Remark 9.7. *Note that the first bullet point makes sense since there exists no $\mathcal{T}_i^0 = \{k_1\}$, so any row corresponding to a $\mathcal{T}_i^0$ with $\min \mathcal{T}_i^0 = k_1$ has a support of length at least two.*

Remark 9.8. *As a means of shorthand, we will refer to a multi-segment of Type $Z_{\mathcal{M}}$ as being of Type Z_{k_1,k_2}.*

Example 9.9. *Let $k_1 = 1$ and $k_2 = 4$. Then the virtual extended multi-segment associated to $\eta = 1, \mathcal{M} = (3,2,2,1,3,1)$ and $\mathcal{S} = (\{0\},\{1,2,3,\overline{4}\},\{5\})$ is*

$$\begin{array}{cccccccccc} -4 & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 & 5 \end{array}$$
$$\left(\begin{array}{cccccccccc}
\lhd & \lhd & \lhd & \lhd & \oplus & \rhd & \rhd & \rhd & \rhd & \\
 & & & & \ominus & & & & & \\
 & & & & \ominus & & & & & \\
 & & & & \ominus & & & & & \\
 & & & & & \oplus & & & & \\
 & & & & & \lhd & \ominus & \rhd & & \\
 & & & & & & \ominus & & & \\
 & & & & & & & \ominus & & \\
 & & & & & & & & \ominus & \\
 & & & & & & & & \ominus & \\
 & & & & & & & & \ominus & \\
 & & & & & & & & & \oplus
\end{array}\right)$$

Here, the chain with support $[3,1]$ has $l = 1$ and alternates with the row above it, in contrast to what would happen in a multi-segment of type $Y_{\mathcal{M}}$.

Example 9.10. *Suppose $\mathcal{E}$ is a tempered multi-segment satisfying $m_H \equiv 0 \bmod 2$, $m_N \equiv 1 \bmod 2$ as in Case 3 of Conjecture 4.11. Then $\mathcal{E}^{\mathrm{up}}_{m^{\mathrm{up},\alpha}}$ is of type Z_{k_1,k_2} with $\mathcal{S}$-data $(\{0\},\{1\},\ldots,\{k-1\},\{k,\overline{k+1}\},\{k+2\},\ldots,\{n\})$. We provide an example below.*

$$\mathcal{E} = \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \left(\begin{array}{cccc}
\ominus & & & \\
 & \oplus & & \\
 & \oplus & & \\
 & & \oplus & \\
 & & & \ominus
\end{array}\right)\end{array} \longrightarrow \mathcal{E}^{\mathrm{up}}_{m^{\mathrm{up},\alpha}} = \begin{array}{c} \begin{array}{cccccc} -2 & -1 & 0 & 1 & 2 & 3 \end{array} \\ \left(\begin{array}{cccccc}
\lhd & \lhd & \oplus & \rhd & \rhd & \\
 & & \ominus & & & \\
 & & & \oplus & & \\
 & & & \oplus & & \\
 & & & & \oplus & \\
 & & & & & \ominus
\end{array}\right)\end{array}$$

Note that in this example, the chain with support $[2,2]$ satisfies the necessary sign condition by not alternating with the previous row.

We now briefly classify all operators on extended multi-segments $\mathcal{E}$ of type Z_{k_1,k_2}. In short, we will verify that the operators S, M, U, D (as defined for type $Y_{\mathcal{M}}$ in Lemmas 7.29, 6.14, 7.42, and 7.36) are still well defined, preserve type Z_{k_1,k_2}, $\eta(\mathcal{E})$, and have the same effect on the $\mathcal{S}$-data of type Z_{k_1,k_2} multi-segments. We will then verify that all other operators preserve type Z_{k_1,k_2}. Much of the proof will be exactly the same as the analogous proofs for type $Y_{\mathcal{M}}$, so some details will be omitted.

The proof that the operator M behaves as prescribed for Z_{k_1,k_2} extended multi-segments carries over immediately for the proof in the case of type $Y_{\mathcal{M}}$ (see Lemma 7.34. Therefore, we will start with the operator S.

Lemma 9.11. *Suppose $\mathcal{E}$ is of type Z_{k_1,k_2}, $r = ([A,B],0,\eta)$ is a chain, and $c < C(r)$ is a positive integer. Then there exists an operator, notated $S_{r,c}$, preserving type Z_{k_1,k_2} and $\eta(\mathcal{E})$ that splits $\mathcal{S}_i = \{B,\dots,A,\dots\}$ into two sets*

$$\mathcal{S}_i^1 = \{B,\dots,A-c\} \text{ and } \mathcal{S}_i^2 = \{A-c+1,\dots,A,\dots\}.$$

Moreover, if $l(r) = 1$, then $S_{r,c}$ is defined even when $c = C(r)$.

Remark 9.12. *An exception to this effect on the $\mathcal{S}$-data still holds in specific cases where r is a z-chain analogous to the one detailed in Remark 7.30.*

Proof. The proof that $S_{r,c}$ is possible carries over from Lemma 7.29 except when the row r has to be exchanged with rows whose supports contain k_1 or k_1+1. Such a step is only a part of the S operation whenever one of these multiples belongs to r. We must fall into one of the following cases.

- Both the multiples with support $[k_1,k_1]$ and $[k_1+1,k_1+1]$ belong to r and the operation S involves row exchanging r with all of these multiples. There is an odd number of each of these, and they have the same sign, so exchanging r with these multiples does not change r by Lemma 6.11. We must necessarily have that $l(r) = 0$, so a ui^{-1} of type 3' may be applied after r is exchanged down. Row exchanging up again has no effect by Lemma 6.12.
- Both the multiples with support $[k_1,k_1]$ and $[k_1+1,k_1+1]$ belong to r but the operation S involves row exchanging r with only the multiples with support $[k_1,k_1]$. Here, we apply ui^{-1} of type 3' to separate some circles from r to obtain a row of circles r' with support $[A,k_1]$. We then row exchange r' until it is right below all the multiples with support $[k_1,k_1]$. Either $A = k_1$ and the resulting row is then another such multiple, or $A > k_1$, in which case exchanging with an odd number of multiples produces a chain with $l = 1$ as required.
- Only the multiples with support k_1+1 belong to r, in which case r must be a row with support $[A,k_1]$ for some $A \geq k_1+1$. Such a row must have $l = 1$, so we must actually have $A \geq k_1+2$, lest $C(r) = 0$. Then, exchanging r with the odd number of multiples of support $[k_1+1,k_1+1]$ yields a row with $l = 0$, so a ui^{-1} may be applied.

The proof that sign conditions are conserved proceeds analogously to the proof of Lemma 7.29. Note that none of these separations can produce some set $\mathcal{T}_0^i = \{k+1\}$, so the resulting virtual extended multi-segment is still of type Z_{k_1,k_2}. □

Next, we verify the existence of the D operator.

Lemma 9.13. *Given $\mathcal{E}$ of type Z_{k_1,k_2}, the operator D is well defined and affects the $\mathcal{S}$-data the same as in the $Y_{\mathcal{M}}$ case (see Lemma 7.36).*

Proof. Suppose $h = ([A,-B],B,\eta)$ be the hat to be dualized by the D operator. Firstly, if $A \leq k_1$, then all possible *dual* $\circ$ *ui* $\circ$ *dual* operations of type 3' are contained in a sub-virtual extended multi-segment of type $Y_{\mathcal{M}}$ with $c_{\max} = A$, from which the result follows immediately from Lemma 7.36.

If $A = k_1+1$, then h can either be dualized to a chain r ending at k_1 or a multiple with support $[k_1,k_1]$ (these virtual extended multi-segments being equivalent by row-exchange if and only if $C(r) = 1$). By Remark 7.39, dualizing to the bottom multiple with support $[k_1, k_1]$ will produce a row $([k_1 + 1, k_1], 1, \eta)$ as desired.

If $A \geq k_1 + 2$, let r_1 be the chain ending at $A - 1$ (if such a chain exists) and let r_2 be the final multiple with support $[A - 1, A - 1]$. If we want to dualize h to r_1, we may suppose by Corollary 7.33 and Lemma 7.37 that all rows between h and r_1 are unmerged and none of them are hats. We have the following four cases.

- Suppose r_1 has support $[A-1, B]$ with $B > k_1+1$. If there exists a chain beginning at k with $l = 1$, then we row exchange the chain up so that it is the first row beginning at k and it has $l = 0$. Now, there are an odd number of rows of circles with support $[k_1, k_1]$ and an odd number with support $[k_1 + 1, k_1 + 1]$, all above r_1. By Lemmas 6.11 and 6.12, we can suppose that these evenly many multiples are not present since they do not affect row exchanges. This reduces to the $Y_{\mathcal{M}}$ case from which the claim follows by Lemma 7.36.
- If r_1 has support $[A - 1, k_1 + 1]$, then suppose $\mathcal{S}_i = \{k_1 + 1, \dots, \overline{A - 1, A}, \dots, \dots\}$ be the associated set. But $\mathcal{S}_{i-1}$ must also contain $k_1 + 1$ because $m_{k_1+1} > 1$. Since all rows between h and r_1 are as separated as possible, we must have $\mathcal{S}_{i-1} = \{k_1, k_1 + 1\}$ (since we cannot have $\mathcal{T}^0_{i-1} = \{k_1 + 1\}$. The associated chain is of the form $([k_1 + 1, k_1], 1, \eta')$. Removing this chain does not impact row exchanges and reduces to the $Y_{\mathcal{M}}$ case.
- If r_1 has support $[A - 1, k_1]$, then $r_1 = ([A - 1, k_1], 1, \eta)$ for some η. In this case, row exchange r_1 up with one of the rows with support $[k_1, k_1]$ so it has $l = 0$. Now the rows between k_1 and r_1, inclusive, constitute a sub-virtual extended multi-segment of type $Y_{\mathcal{M}}$. Perform the dualization and exchange the chain back down, restoring the value $l = 1$.
- Otherwise, r_1 must be above all the multiples of support $[k_1, k_1]$ or $[k_1 + 1, k_1 + 1]$, in which case the multi-segment of rows from h to r_1 is of type $Y_{\mathcal{M}}$ for $\mathcal{M}$ ending at A, and the existence of the dualization operator follows immediately.

If we want to dualize h to r_2, we again suppose that all rows in between are completely unmerged and all intermediate hats are dualized away. In this case, dualizing to r_2 is equivalent up to dualizing to the chain that r_2 belongs to, and we have already proved that this is possible. □

We show the existence of the U operator.

Lemma 9.14. *Given $\mathcal{E}$ of type Z_{k_1,k_2}, the operator U is well-defined and affects the $\mathcal{S}$-data the same as in the $Y_{\mathcal{M}}$ case.*

Proof. Let $h = ([A, -B], B, \eta)$ for some η and that we wish to perform $U_{h,c}$, the ui^{-1} operator separating c circles from the hat h. We use the following relation

$$D_{h',r} \circ U_{h,c} = S_{r',c} \circ D_{h,r},$$

where r is the chain obtained from the same $\mathcal{S}_i$ as h. Here, r' is the image of r under D and h' is the remaining hat after U. This relation is proven as Part

(9) of Theorem 12.3 for the $Y_{\mathcal{M}}$ case, but its proof relies solely on $\mathcal{S}$-data. The existence of a U operation that impacts the $\mathcal{S}$-data in the same way as the U-operation for type $Y_{\mathcal{M}}$ follows since the other operators and their inverses are already known to exist. Any ui^{-1} applied to a hat will produce an extended multi-segment of the same support and order as U and therefore must produce the same multi-segment by Corollary 3.4. □

Now we aim to prove the following classification for extended multi-segments of type $Z_{\mathcal{M}}$.

Theorem 9.15. *Suppose $\mathcal{E}$ is of type $Z_{\mathcal{M}}$. Then $\mathcal{E}' \sim \mathcal{E}$ if and only if $\mathcal{E}'$ is of type $Z_{\mathcal{M}}$ and $\eta(\mathcal{E}) = \eta(\mathcal{E}')$.*

In order to do this, we will use the following description of $\mathcal{E}^{\min}$ for extended multi-segments $\mathcal{E}$ of type $Z_{\mathcal{M}}$. The proof of this lemma is exactly the same as the proof of Lemma 7.47 and will therefore be omitted.

Lemma 9.16. *Suppose $\mathcal{E}$ is of type $Z_{\mathcal{M}}$ with $\mathcal{S}$-data $(\{0, \overline{1}, \overline{2}, \ldots, \overline{n}\})$. Then $\mathcal{E} = \mathcal{E}^{min}$.*

Any $\mathcal{E}$ of type Z_{k_1,k_2} can clearly be turned into one of the form specified in Lemma 9.16 with the same $\mathcal{M}$ and η through a series of $-S, -D$ and $-M$ operators. This suffices to show that two extended multi-segments of type $Z_{\mathcal{M}}$ with the same η are equivalent. Now, to prove Theorem 9.15, it suffices to show that if $\mathcal{E}$ is of type $Z_{\mathcal{M}}$ then any equivalent extended multi-segment is of type $Z_{\mathcal{M}}$ with the same η. Since we know that $\mathcal{E}^{\min}$ has the same $\mathcal{E}$ and η, it suffices to show that these properties are preserved under raising operators.

Lemma 9.17. *Let $\mathcal{E}$ be of type $Z_{\mathcal{M}}$ and T be a raising operator. Then T preserves $Z_{\mathcal{M}}$ form, including $\mathcal{M}$, as well as $\eta(\mathcal{E})$.*

Proof. It is apparent that the only raising operators of type 3' applicable to multi-segments of form Z_{k_1,k_2} are S, M, U, and D, and these are known to conserve $\mathcal{M}$ and η. Otherwise, T must be a $dual \circ ui \circ dual$ not of type 3'. In this case, T consists of some combination of the raising operators of type 3' and their inverses; the proof of this fact is exactly the same as the proofs of Lemmas 7.49 and 7.48. □

Next, we need to ensure that two unequal but equivalent $Z_{\mathcal{M}}$ multi-segments are associated with different Arthur packets. The proof of the following result is exactly the same as the proof of Lemma 8.1.

Lemma 9.18. *If $\mathcal{E}_1$ and $\mathcal{E}_2$ are both of type $Z_{\mathcal{M}}$ and $\eta(\mathcal{E}_1) = \eta(\mathcal{E}_2)$, then $\psi(\mathcal{E}_1) = \psi(\mathcal{E}_2)$ if and only if $\mathcal{E}_1 = \mathcal{E}_2$.*

Let $\mathcal{E}$ be an extended multi-segment of type $Z_{k,k+1}$ and let $\Theta_1'(\mathcal{E}) = r \cup \mathcal{E}$, where $r = ([k+2, -k-2], k+2, -\eta(\mathcal{E}))$. We remark that $\Theta_1'(\mathcal{E})$ only differs definition of $\Theta_1(\mathcal{E})$ by twisting by the appropriate supercuspidal representations. Note that $\Theta_1'(\mathcal{E})$ is of type $Z_{k,k+2}$. Then we have the following result.

Lemma 9.19. *We can perform exactly two $dual \circ ui \circ duals$ (up to row exchange) involving the first row of $\Theta_1'(\mathcal{E})$: one with the first row whose support ends at*

$k+1$, and one with the last row whose support ends at $k+1$. The resulting multi-segments are not equivalent, and we call them $\Theta_2'(\mathcal{E})$ and $\Theta_4'(\mathcal{E})$. We can perform an additional ui^{-1} to both Θ_2' and Θ_4' separating one circle from the row with support ending at $k+2$ to obtain the same $\Theta_3'(\mathcal{E})$.

Proof. It is immediately obvious that there are only two $dual \circ ui \circ dual$s involving the first row of $\mathcal{E}$, up to row exchange. This is because all but one of the rows ending at $k+1$ must be repeats. The first row ending at $k+1$ cannot have support $[k+1, k+1]$ because we cannot have some $\mathcal{T}_i^0 = \{k+1\}$. Therefore, we can be sure that $\psi_{\Theta_2'(\mathcal{E})} \neq \psi_{\Theta_4'(\mathcal{E})}$ by comparing supports. Lastly, we can be sure that applying an additional ui^{-1} to separate one circle with support $[k+2, k+2]$ produces the same $\Theta_4'(\mathcal{E})$ by means of Lemma 8.6, which still holds for type $Z_{\mathcal{M}}$. □

Lemma 9.20. *All of the elements of the set $\{\psi_{\Theta_i'(\mathcal{E}')} \mid \mathcal{E}' \sim \mathcal{E}; i = 1, 2, 3, 4\}$ are distinct.*

Proof. The proof is analogous to that of Lemma 8.7. In light of Lemma 9.18, it suffices to show that these extended multi-segments are not equal to each other. Suppose $\psi_{\Theta_i'(\mathcal{E}_1)} = \psi_{\Theta_i'(\Theta_j'(\mathcal{E}_2))}$. First, we show that $i = j$ by comparing the supports of these multi-segments. Specifically, let $r \in \Theta_i(\mathcal{E}_1)$ be the row whose support ends at $k+2$.

- If $supp(r) = [k+2, -k-2]$, then $i = 1$.
- If $supp(r) = [k+2, k+2]$, then $i = 4$.
- If $supp(r) = [k+2, k+1]$, then $i = 3$.
- Otherwise, $i = 2$.

Therefore, it is clear that $\psi_{\Theta_i'(\mathcal{E}_1)} = \psi_{\Theta_i'(\Theta_j'(\mathcal{E}_2)} \Rightarrow i = j$. Now, we aim to show that $\mathcal{E}_1 = \mathcal{E}_2$. If $i \in \{1, 3\}$ or $i = 3$, then $\mathrm{supp}(\mathcal{E}) = \mathrm{supp}(\Theta_i'(\mathcal{E}) \setminus \{r\})$, so $\mathcal{E}$ can be uniquely determined from $\Theta_i'(\mathcal{E})$. If $i \in \{2, 4\}$, then $\Theta_i'(\mathcal{E})$ uniquely determines $\Theta_3'(\mathcal{E})$, which determines $\mathcal{E}$. □

Lemma 9.21. *We have the relation*

$$\Psi(\pi(\Theta_1'(\mathcal{E}))) = \{\psi_{\Theta_i'(\mathcal{E}')} \mid \mathcal{E}' \sim \mathcal{E}; i = 1, 2, 3, 4\}.$$

Moreover, Lemma 9.20 then implies that $|\Psi(\pi(\Theta_1'(\mathcal{E})))| = 4|\Psi(\pi(\mathcal{E}))|$.

Proof. Suppose that $\psi_{\mathcal{E}_1} \in \Psi(\pi(\Theta_1(\mathcal{E})))$.Since $\Theta_1(\mathcal{E})$ is of type $Z_{k,k+2}$, then $\mathcal{E}_1$ must also be of type $Z_{k,k+2}$ with the same $\mathcal{M}$ and η. We examine the $\mathcal{S}$-data of $\mathcal{E}_1$. In particular, we study the unique set containing $k+2$. Let $\mathcal{E}_2$ be the virtual extended multi-segment $\mathcal{E}(\mathcal{M}, \mathcal{S}', \mathcal{T}', \eta(\mathcal{E})$, where $(\mathcal{S}', \mathcal{T}')$ is obtained by removing $k+2$ from the $\mathcal{S}$ data of $\mathcal{E}_1$. We have the following cases.

- $\mathcal{S} = (\ldots, \{\ldots, k+1\}, \{k+2\})$ in which case $\mathcal{E}_1 = \Theta_4'(\mathcal{E}_2)$.
- $\mathcal{S} = (\ldots, \{\overline{\ldots\ldots, k+1, k+2}\})$, in which case $\mathcal{E}_1 = \Theta_2'(\mathcal{E}_2)$.
- $\mathcal{S} = (\ldots, \{\ldots k, k+1, k+2\})$, in which case $\mathcal{E}_1 = \Theta_2'(\mathcal{E}_2)$.
- $\mathcal{S} = (\ldots, \{k+1, \overline{k+2}\})$, in which case $\mathcal{E}_1 = \Theta_4'(\mathcal{E}_2)$.
- $\mathcal{S} = (\ldots, \{\ldots, k+1, \overline{k+2}\})$, in which case $\mathcal{E}_1 = \Theta_3'(\mathcal{E}_2)$. □

Now, we prove Case 3 of Conjecture 4.11, starting with Case 3.1. Let $\mathcal{E}$ be an extended multi-segment with $m_H \equiv 0 \bmod 2$, $m_N \equiv 1 \bmod 2$ such that $\mathcal{E}$ ends at

$H_{col}+1$ and let $\mathcal{E}' = \mathrm{rc}_{H_{col}+1}(\mathcal{E})$. Then $\mathcal{E}^{\mathrm{up}}_{m^{\mathrm{up},\alpha}}$ is of type $Z_{k,k+1}$, while $(\mathcal{E}^{\mathrm{up}}_{m^{\mathrm{up},\alpha}})' = \mathcal{E}^{\mathrm{up}}_{m^{\mathrm{up},\alpha}} \setminus \{r_2,\dots,r_{m_{k+1}}\}$, where $\{r_2,\dots,r_{m_{k+1}}\}$ are the rows with support $[k+1,k+1]$. The statement of Part 3.1 is equivalent to the assertion that

$$|\Psi(\pi(\mathcal{E}^{\mathrm{up}}_{m^{\mathrm{up},\alpha}}))| = |\Psi(\pi((\mathcal{E}^{\mathrm{up}}_{m^{\mathrm{up},\alpha}})'))|.$$

This, however, follows from the fact that all the operators on $\pi(\mathcal{E}^{\mathrm{up}}_{m^{\mathrm{up},\alpha}})$ are some composition of S, M, U, D, and their inverses. None of these operators affect the rows $\{r_2,\dots,r_{m_{k+1}}\}$, and they are all still well defined even if these rows are removed. Thus, we have a one-to-one correspondence

$$\{\mathcal{E}_1 \mid \mathcal{E}_1 \sim \mathcal{E}_{m^{\mathrm{up},\alpha}}\} = \{\mathcal{E}_1' \cup \{r_1,\dots,r_k\} \mid \mathcal{E}_1' \sim \mathcal{E}'_{m^{\mathrm{up},\alpha}}\}.$$

We thus have a similar one-to-one correspondence between the sets $\Psi(\pi(\mathcal{E}_{m^{\mathrm{up},\alpha}}))$ and $\Psi(\pi(\mathcal{E}'_{m^{\mathrm{up},\alpha}}))$, which suffices for the proof.

Next, we prove Case 3.2. Let $\mathcal{E}$ be a multi-segment with $m_H \equiv 0 \bmod 2$, $m_N \equiv 1 \bmod 2$ such that $\mathcal{E}$ ends at $H_{col}+2$, and let $\mathcal{E}' = \mathrm{rc}_{H_{col}+2}(\mathcal{E})$. An argument equivalent to the proof of Lemma 8.8 allows us to reduce to the case where $m_{k+2} = 1$. Again, $\mathcal{E}'_\alpha$ is an extended multi-segment of type $Z_{k,k+1}$, and our reduction means that $\mathcal{E}_\alpha$ is an extended multi-segment of type $Z_{k,k+2}$ obtained by taking the union $\mathcal{E}'_\alpha \cup r$. Then it follows from Lemma 9.21 that

$$|\Psi(\pi(\mathcal{E}_\alpha))| = 4|\Psi(\pi(\mathcal{E}'_\alpha))|.$$

The proof of Case 3.3 is almost exactly the same as the proof of the recursive formula in Theorem 4.7, so we omit it here.

Remark 9.22. *The more general statement of Case 3 is:*

$$|\Psi(\theta^{\mathrm{up}}_{-m^{\mathrm{up},\alpha}}(\pi(\mathcal{E})))| = |\Psi(\theta^{\mathrm{up}}_{-m^{\mathrm{up},\alpha}}(\pi(\mathcal{B}_1 \cup \mathcal{B}_2)))| \cdot \prod_{i=3}^{k} |\Psi(\pi(\mathcal{B}_i))|,$$

which essentially states that taking the theta correspondence only impacts the first and second block, while multi-segments on the remaining blocks act completely independently of these blocks a la the statement of Proposition 10.13. The details of the proof of this proposition still apply if a pair of blocks of Type $Y_{\mathcal{M}}$ is replaced by a single block of type $Z_{\mathcal{M}}$, which is exactly what happens under the theta correspondence.

9.4. **Anti-Tempered Extended Multi-Segments.** Calculating $\theta^{\mathrm{up}}_{-m^{\mathrm{up},\alpha}}(\pi)$ when π is an anti-tempered representation is simpler than the tempered case because $\theta^{\mathrm{up}}_{-m^{\mathrm{up},\alpha}}(\pi)$ itself must be an anti-tempered representation. Moreover, given an anti-tempered extended multi-segment $\mathcal{E}$, we can easily identify a multi-segment associated to the theta lift $\theta^{\mathrm{up}}_{-m^{\mathrm{up},\alpha}}(\pi(\mathcal{E}))$.

Lemma 9.23. *Let $\mathcal{E}$ be an anti-tempered extended multi-segment in (P') order. Then:*

(1) The first row of $\mathcal{E}$ is of the form $([k,-k],k,\eta(\mathcal{E}))$.
(2) $\mathcal{E}_{m^{\mathrm{up},\alpha}}$ is of the form $([k+1,-k-1],k+1,-\eta(\mathcal{E})) \cup \mathcal{E}$.
(3) $\mathcal{E}_{m^{\mathrm{up},\alpha}}$ is itself an anti-tempered multi-segment.

Proof. Part (1) follows immediately from the fact that $dual(\mathcal{E})$ must be tempered. Since the first row of $\mathcal{E}$ must dualize to a row with a single circle, this indeed must be a row of the form $([k,-k],n,\eta(\mathcal{E}))$. Furthermore, since we impose a (P') order, we note that no row of $\mathcal{E}$ is supported $\geq k$.

Therefore, from Algorithm 3.28, it follows that for α such that $\frac{\alpha-1}{2}\geq n$, $\theta^{\eta}_{-\alpha}(\pi(\mathcal{E})$ is associated to an extended multi-segment:

$$\mathcal{E}' := \left(\left[\frac{\alpha-1}{2},\frac{-\alpha+1}{2}\right],\frac{\alpha-1}{2},\eta\right)\cup\mathcal{E}.$$

Forming such an extended multi-segment for the case where $\frac{\alpha-1}{2}=n$, we note that $\pi(\mathcal{E}')\neq 0$ if and only if $\eta=\eta(\mathcal{E})$. To see this, via Lemma 3.12, it suffices to show that $\pi(dual(\mathcal{E}'))\neq 0$ if and only if $\eta=\eta(\mathcal{E}')$. This is true because the hat $\left(\left[\frac{\alpha-1}{2},\frac{-\alpha+1}{2}\right],\frac{\alpha-1}{2},\eta\right)=([k,-k],k,\eta)$ is equal to the first row of $\mathcal{E}$, except possibly for the sign. If $\eta=\eta(\mathcal{E})$, then $dual(\mathcal{E})$ will be a tempered extended multi-segment with an extra circle in the nth column and therefore will correspond to a non-zero tempered representation. If $\eta=-\eta(\mathcal{E})$, then $dual(\mathcal{E})$ will have two circles in the same column with opposite signs and therefore correspond with the 0 representation.

Therefore, it is clear that the sign $\eta(\mathcal{E})$ corresponds with the down-tower, and that $m^{\mathrm{up},\alpha}$ must satisfy $\frac{m^{\mathrm{up},\alpha}-1}{2}\geq k+1$. In fact, we have $\frac{m^{\mathrm{up},\alpha}-1}{2}=k+1$, since the extended multi-segment $([k+1,-k-1],k+1,-\eta(\mathcal{E}))\cup\mathcal{E}$ clearly has a tempered dual. This suffices to prove (2), and also (3). ☐

The proof of Theorem 4.16 now follows immediately from a combination of Lemma 9.23 and Theorem 4.11.

10. Interactions of Blocks

Throughout this section, we focus primarily on the case that ρ is an orthogonal representation of $\mathrm{GL}_d(F)$. In this section, we consider the cases where a tempered $\mathcal{E}\in\mathrm{VRep}_\rho(G)$ may consist of multiple blocks (see Lemma 4.5).

10.1. **Existence of operations on blocks.** In this subsection, we study the effects operators on blocks. Intuitively, the valid operations on a block in a virtual extended multi-segment are the same as if the block was its own virtual extended multi-segment. (A similar statement holds if the extended multi-segment is not tempered and we are considering one of the pieces of its decomposition.) The only exception to this is that, for the blocks after the first block, they behave as if they do not start at zero, even if they do. This is what Lemma 10.2 amounts to. Lemmas 10.6 and 10.7 show the validity of all of the operations, as the intuitive picture might suggest (with this additional adjustment).

The following definition formalizes the idea of a block which starts at zero behaving as if it does not start at zero.

Definition 10.1. *Let $\mathcal{E}$ be a block starting at zero of type $Y_{\mathcal{M}}$, where $\mathcal{M}=(m_0,\ldots,m_{c_{\max}})$. We say $\mathcal{E}$ is of type $Y^{>0}_{\mathcal{M}}$ if $\mathcal{E}=sh^{-1}(\mathcal{E}')$ for some $\mathcal{E}'$ of type $Y_{\mathcal{M}'}$, where $\mathcal{M}'=(m'_1,\ldots,m'_{c_{\max}+1})$ with $m'_i=m_{i-1}$.*

Lemma 10.2. *Let $\mathcal{E}\in\mathrm{VRep}_\rho(G)$ be equivalent to a tempered virtual extended multi-segment $\mathcal{E}_{temp}$. Suppose that $\mathcal{E}_{temp}$ has a block decomposition $\mathcal{B}_{1,temp}\cup\cdots\cup\mathcal{B}_{k,temp}$ with $k>1$ and that $\mathcal{E}$ has decomposition $\mathcal{B}_1\cup\cdots\cup\mathcal{B}_k$ with $\mathcal{B}_i$ equivalent to $\mathcal{B}_{i,temp}$. Suppose further that $\mathcal{B}_k$ is of type $Y^{>0}_{\mathcal{M}}$. We claim the following.*

(1) *Every operation on $\mathcal{E}$ which is an operation on $\mathcal{B}_k$ (in the sense that it only involves rows from $\mathcal{B}_k$) is analogous to an operation on the extended multi-segment $sh^1(\mathcal{B}_k)$, in the following way:*
   - *the operation ui on rows r_1 and r_2 in $\mathcal{E}$ corresponds to the operation ui on rows $sh^1(r_1)$ and $sh^1(r_2)$ in $sh^1(\mathcal{B}_k)$;*
   - *the operation dual on $\mathcal{E}$ corresponds to the operation dual on $sh^1(\mathcal{B}_k)$;*
   - *and the operation ui^{-1} of type 3' on $\mathcal{E}$ which splits off k circles from row r corresponds to the operation ui^{-1} of type 3' in $sh^1(\mathcal{B}_k)$ which splits off k circles from $sh^1(r)$.*

(2) *After any operation on $\mathcal{E}$ which is an operation on $\mathcal{B}_k$, $\mathcal{B}_k$ is still of type $Y_{\mathcal{M}}^{>0}$.*

Proof. Except for the case of $dual \circ ui^{-1} \circ dual$ operations, the proof is similar to the proof of exhaustion in §7.4. The main difference is that we prove a slightly stronger claim here, which is a characterization of the operators possible at each step, rather than a characterization of all equivalent extended multi-segments. Therefore we do not find the minimal extended multi-segment with respect to the admissible order, so in our exhaustion step we must also consider lowering operators.

For operations of the form $dual \circ ui \circ dual$, the same support argument as before (see discussion before Lemma 7.48) holds to show that no operations are possible except between a chain and a multiple belonging to it. In this case the $dual \circ ui \circ dual$ operation only involves rows between the chain and the multiple in question. In particular, it only involves rows in $\mathcal{B}_k$. So applying the same reasoning as in the proof of Lemma 7.48, the operation preserves type $Y_{\mathcal{M}}^{>0}$.

For operations of the form $dual \circ ui^{-1} \circ dual$ where the ui^{-1} is of type 3', we claim that no such operations are possible. First suppose a row $\widehat{r}$ in $dual(\mathcal{E})$ is exchanged up. Then any such operation only involves rows in $dual(\mathcal{B}_k)$, so it would be a valid operation on the extended multi-segment $\mathcal{B}_k$. But no such operations are possible, by a similar reasoning as in Lemma 7.46, applied to each chain and multiples contained in the support of the chain.

So we only consider a row r in $\mathcal{E}$ which is exchanged down in $dual(\mathcal{E})$. If $\mathcal{B}_k$ does not start at zero, then again we can imitate the proof of Lemma 7.46: in short, after exchanging a row in the dual, it will always have $l > 0$, so no ui^{-1} of type 3' are possible.

On the other hand, if $\mathcal{B}_k$ starts at zero, then no $dual \circ ui^{-1} \circ dual$ operations are possible, but the reason is slightly different. Let r be the row in $\mathcal{E}$ to which ui^{-1} is eventually applied. Note that since $\mathcal{B}_k$ starts at zero, we must have $k = 2$ and $\mathcal{B}_1$ must consist of some number of circles in column 0.

We first claim that if $\widehat{r}$ is not exchanged to the last row in $dual(\mathcal{E})$, no ui^{-1} is possible. Suppose $\widehat{r}$ is split into two rows, $\widehat{r_1}$ and $\widehat{r_2}$, with $\widehat{r_1} < \widehat{r_2}$ in the resulting admissible order. If ρ is in the support of $\widehat{r_1}$, then ρ is not in the support of $\widehat{r_2}$, and so $B(\widehat{r_2}) > 0$. But since $\mathcal{B}_1$ has a circle in column 0, the last row of $dual(\mathcal{E})$ is a single circle in column 0 and it comes after $\widehat{r_2}$. So the order it not admissible. If on the other hand ρ is not in the support of $\widehat{r_1}$, then its $A(\widehat{r_1}) < 0$ which is of course impossible. Hence the ui^{-1} operation was not valid.

However, if $\widehat{r}$ is exchanged to the last row in $dual(\mathcal{E})$, we claim ui^{-1} is still not possible. Since a $dual \circ ui^{-1} \circ dual$ would be inverse to the D operation, by similar reasoning as the existence of the D operation (Lemma 7.36) but in reverse, since $\mathcal{B}_2$ is of type $Y_{\mathcal{M}}$, when $\widehat{r}$ is exchanged all the way to the last row of $dual(\mathcal{B}_2)$, it has only circles. Since $\mathcal{B}_1$ has an odd number of circles, by Lemma 6.11, we can assume without loss of generality that it has exactly one circle. But the row exchange is of Case 1(b) and therefore after the last row exchange we must have $l \geq 1$ and so no ui^{-1} is possible.

For operations of the form ui^{-1} of type 3', these must be of type S, which have already been addressed above.

Finally, for operations of the form ui, since they do not depend on the column, the valid ui operations on $\mathcal{B}_k$ are precisely the valid ui operations on $sh^1(\mathcal{B}_k)$. These all preserve type $Y_{\mathcal{M}}^{>0}$ by definition. □

Remark 10.3. *The intuitive reason why the second block $\mathcal{B}_2$ behaves the same as a block not starting at zero is that, in both cases, it is impossible to form hats via a $dual \circ ui^{-1} \circ dual$ operation. Hence the only extended multi-segments are ones that can be obtained by doing ui's of type 3' together with row exchanges.*

As an example, consider $\mathcal{E} = (([0,0],0,1),([0,0],0,1),([1,1],0,-1),([2,2],0,1))$ which has a block decomposition

$$\mathcal{B}_1 = (([0,0],0,1))$$
$$\mathcal{B}_2 = (([0,0],0,1),([1,1],0,-1),([2,2],0,1)).$$

Shown below are all the extended multi-segments created by row exchanging the first row of $dual(\mathcal{E})$ down.

$$dual(\mathcal{E}) = \begin{array}{c} \begin{matrix} -2 & -1 & 0 & 1 & 2 \end{matrix} \\ \begin{pmatrix} \lhd & \lhd & \ominus & \rhd & \rhd \\ & \lhd & \oplus & \rhd & \\ & & \ominus & & \\ & & \ominus & & \end{pmatrix} \end{array}$$

$$R_1(dual(\mathcal{E})) = \begin{array}{c} \begin{matrix} -2 & -1 & 0 & 1 & 2 \end{matrix} \\ \begin{pmatrix} & \lhd & \oplus & \rhd & \\ \lhd & \oplus & \ominus & \oplus & \rhd \\ & & \ominus & & \\ & & \ominus & & \end{pmatrix} \end{array}$$

$$R_2(R_1(dual(\mathcal{E}))) = \begin{array}{c} \begin{matrix} -2 & -1 & 0 & 1 & 2 \end{matrix} \\ \begin{pmatrix} & \lhd & \oplus & \rhd & \\ & & \ominus & & \\ \ominus & \oplus & \ominus & \oplus & \ominus \\ & & \ominus & & \end{pmatrix} \end{array}$$

$$R_3(R_2(R_1(dual(\mathcal{E})))) = \begin{array}{c} \begin{matrix} -2 & -1 & 0 & 1 & 2 \end{matrix} \\ \begin{pmatrix} & \lhd & \oplus & \rhd & \\ & & \ominus & & \\ & & \ominus & & \\ \lhd & \oplus & \ominus & \oplus & \rhd \end{pmatrix} \end{array}$$

We see that only after two exchanges does the row have $l = 0$, but since there is another row with a circle in column 0 following it, no ui^{-1} is possible. In the case of a block not starting at zero, no ui^{-1} is possible because no sequence of row exchanges gives $l = 0$.

The previous lemma allows us to make the following very important observation about the supports of each part of the decomposition.

Lemma 10.4. *Let $\mathcal{E}_{temp}$ have a block decomposition $\mathcal{B}_{1,temp} \cup \cdots \cup \mathcal{B}_{k,temp}$, and let $\mathcal{E}$ be equivalent to $\mathcal{E}_{temp}$. Suppose $\mathcal{E}$ has a decomposition $\mathcal{B}_1 \cup \cdots \cup \mathcal{B}_k$ with $\mathcal{B}_i$ equivalent to $\mathcal{B}_{i,temp}$. Then this decomposition has the property that for $i_1 < i_2$, and rows $r_1 \in \mathcal{B}_{i_1}$, $r_2 \in \mathcal{B}_{i_2}$, we have*

$$A(r_1) \leq B(r_2).$$

Proof. Note that in $\mathcal{E}_{temp}$, for $i_1 < i_2$ and any row $r_1 \in \mathcal{B}_{i_1,temp}$ and $r_2 \in \mathcal{B}_{i_2,temp}$, we have

$$0 \leq B(r_1) \leq A(r_1) \leq B(r_2) \leq A(r_2).$$

Since $\mathcal{E}$ is equivalent to $\mathcal{E}_{temp}$ and each of the $\mathcal{B}_i$ are equivalent to $\mathcal{B}_{i,temp}$, we can get $\mathcal{E}$ by applying basic operations to each of the $\mathcal{B}_{i,temp}$ taking them to $\mathcal{B}_i$. Note that basic operations on $\mathcal{B}_{i_1}$ do not change the maximum of $A(r)$ across rows $r \in \mathcal{B}_{i_1}$. Moreover, the only basic operation which might change the minimum of support is $dual \circ ui^{-1} \circ dual$ of type 3', but from Lemma 10.2 we know these are not possible. So for $r_1 \in \mathcal{B}_{i_1}$ and $r_2 \in \mathcal{B}_{i_2}$ with $i_1 < i_2$, we have that $A(r_1) \leq B(r_2)$. □

Definition 10.5. *We call the above property the* staircase *property of the decomposition. In future sections, we will also consider other sorts of decompositions arising in different ways, and refer to the same property as the* staircase *property.*

Next we show that any operator on a block induces an operator on the virtual extended multi-segment.

Lemma 10.6. *Let $\mathcal{E} \in \mathrm{VRep}_\rho(G)$ be an extended multi-segment equivalent to a tempered extended multi-segment, and let $\mathcal{E}$ have decomposition $\mathcal{B}_1 \cup \cdots \cup \mathcal{B}_k$, with the same conditions as before. Then any operation on $\mathcal{B}_1$ as its own extended multi-segment is also a valid operation on $\mathcal{E}$.*

Proof. Let T be a basic operation on $\mathcal{B}_1$. Any basic operation is one of $dual \circ ui \circ dual$, ui^{-1} of type 3', $dual \circ ui^{-1} \circ dual$ of type 3', or ui. Since the conditions for ui are purely local (see Lemma 3.23), if T is a union-intersection, then it is clearly also valid on $\mathcal{E}$. Similarly, observe that $dual(\mathcal{E}) = dual(\mathcal{B}_k) \cup \cdots \cup dual(\mathcal{B}_1)$ possibly up to a global sign change on each part $dual(\mathcal{B}_i)$. Hence if T is an operation of the form $dual \circ ui \circ dual$, then it is also valid on $\mathcal{E}$, since a global sign change on $dual(\mathcal{B}_1)$ does not affect whether a union-intersection is valid.

Now we consider the case of ui^{-1} of type 3'. Let $\mathcal{B}_1'$ be such that $ui(\mathcal{B}_1') = \mathcal{B}_1$, where the union-intersection is of type 3'. Since union-intersection is local, it is always the case that $ui(\mathcal{B}_1' \cup \cdots \cup \mathcal{B}_k) = \mathcal{B}_1 \cup \cdots \cup \mathcal{B}_k = \mathcal{E}$, unless $\mathcal{B}_1' \cup \cdots \cup \mathcal{B}_k$ is not in admissible order. Since ui^{-1} is a valid operation on $\mathcal{B}_1$, we see that $\mathcal{B}_1'$ is in admissible order. Also, each $\mathcal{B}_i$ for $i > 1$ is in admissible order. So it suffices to check that for a row r_1 from $\mathcal{B}_1'$ and a row r_2 from $\mathcal{B}_{i_2}$ for $i_2 > 1$, r_1 and r_2 are in admissible order. Since r_2 comes after r_1, this can only fail if $A(r_1) > A(r_2)$ and $B(r_1) > B(r_2)$. But since the decomposition has the staircase property, we always have that $B(r_2) \geq A(r_1)$. So $B(r_2) \geq B(r_1)$, so the order is admissible. Hence we conclude ui^{-1} is a valid operation on $\mathcal{E}$.

Finally we consider the case of $dual \circ ui^{-1} \circ dual$ where the ui^{-1} is of type 3'. Here the exact same argument applies, but to $dual(\mathcal{E})$. The only possible obstruction to ui^{-1} being a valid operation is that the result is not in admissible order. Again it suffices to check rows r_1 from $dual(\mathcal{B}_1)$ and r_2 from $dual(\mathcal{B}_{i_2})$ for $i_2 > 1$. However, every row in $dual(\mathcal{B}_{i_2})$ has support containing the support of every row in $dual(\mathcal{B}_1)$, since the decomposition has the staircase property. Since this is still true even after a ui^{-1}, it is impossible for these rows to fail to satisfy the admissibility condition. □

Lemma 10.7. *Let $\mathcal{E}_{temp}$ be a tempered extended multi-segment with a block decomposition $\mathcal{B}_{1,temp} \cup \cdots \cup \mathcal{B}_{k,temp}$, and let $\mathcal{E}$ be any equivalent extended multi-segment, with decomposition $\mathcal{B}_1 \cup \cdots \cup \mathcal{B}_k$. Suppose that $\mathcal{B}_k = \mathcal{B}_{k,temp}$, and let $\mathcal{B}_k'$ be an extended multi-segment equivalent to $\mathcal{B}_k$ of type $Y_{\mathcal{M}}^{>0}$. Then there exists a sequence of operations taking $\mathcal{E}$ to $\mathcal{E}' = \mathcal{B}_1 \cup \cdots \cup \mathcal{B}_{k-1} \cup \mathcal{B}_k'$.*

Proof. Note that of the S, M, U, D^1, and D^2 operators, the only one which applies to extended multi-segments of type $Y_{\mathcal{M}}^{>0}$ is S, since such extended multi-segments have no hats. By Corollary 7.38, it suffices to show that the S and S^{-1} operations on $\mathcal{B}_k$ are valid as an operation on $\mathcal{E}$, where we only assume $\mathcal{B}_k$ is of type $Y_{\mathcal{M}}^{>0}$. We can then repeatedly apply S or S^{-1} operations to take $\mathcal{E}$ to $\mathcal{E}'$.

Since S and S^{-1} are local, it just remains to check that they do not create rows violating the admissible order condition. This is clear for S^{-1} since it is a ui operation.

For S operations, since $\mathcal{B}_k$ is the last piece of the decomposition, the only way this could happen is if it created a row with A smaller than a row above it. But a ui^{-1} operation does not change the minimum value of A among the rows of $\mathcal{B}_k$, so it cannot create a non-admissible order.

Also, the S operation preserves type $Y_{\mathcal{M}}^{>0}$ since it is local, and the S operation is also valid on $sh^1(\mathcal{B}_k)$, since it does not depend on the column. So this follows from the existence of S on a single block (Lemma 7.29). □

10.2. **Independence of blocks.** In this subsection, we prove that the blocks which occur in the block decomposition a tempered $\mathcal{E} \in \mathrm{VRep}_\rho(G)$ (see Lemma 4.5) determine $\Psi(\pi(\mathcal{E}))$ (see Proposition 10.13). We begin by giving some technical lemmas. The first lemma allows us to only consider certain sequences of row exchanges.

Lemma 10.8. *Let $\mathcal{E} \in \mathrm{VRep}_\rho(G)$ and let $\mathcal{E}'$ be the result after any number of row exchanges on $\mathcal{E}$. Then there exists a row of $\mathcal{E}$ so that, when it is exchanged monotonically (i.e. either all up or all down) until it is the i-th row, it is the same as the i-th row of $\mathcal{E}'$.*

Proof. Let r be the row of $\mathcal{E}$ which, after the row exchanges, becomes the ith row of $\mathcal{E}'$. By commutativity of row exchanges (Lemma 3.22), when performing the row exchanges taking $\mathcal{E}$ to $\mathcal{E}'$, we can equivalently perform all the row exchanges involving r first, and then perform the other row exchanges. Since the other row exchanges do not involve r, and are performed last, they do not affect the ith row, so without loss of generality we can omit them. So we are left with a sequence of row exchanges only involving r. Since row exchanging is its own inverse, after removing pairs of row exchanges which are their own inverses, we can assume that the sequence of row exchanges either moves r monotonically up or monotonically down. Hence the ith row of $\mathcal{E}'$ is the same as the ith row if we only performed row exchanges monotonically on r. □

Our second lemma concerns row exchanges within blocks of type $Y_{\mathcal{M}}$, and follows from the theorems in Section 7.

Lemma 10.9. *Let $\mathcal{B}$ be any virtual extended multi-segment equivalent to a block $\mathcal{B}_{temp}$.*

(1) *Suppose $\mathcal{B}$ starts at 0. Suppose that there is a row r such that it is possible to row exchange it until it is the last row, and let r' be its image after those row swaps. Then $l(r') = 0$. Moreover, the sign of the last circle in $\mathcal{B}$ is the same as the sign of the last circle in r'.*

(2) *Suppose $\mathcal{B}$ does not start at* 0. *Suppose that there is a row r such that it is possible to row exchange it until it is the first row, and let r' be its image. Then $l(r') = 0$, and the sign of the first circle in $\mathcal{B}$ is the same as the sign of the first circle in r'.*

Proof. For Part (1), first suppose $\mathcal{B}_{temp}$ starts at 0 and ends at k. Then $\mathcal{B}$ is of type $Y_{\mathcal{M}}$ by Theorem 7.17.

If r is a hat, then we use similar reasoning as in the proof of Lemma 7.22. In particular, we can assume that all rows following r have $C = 1$. Moreover, by Lemma 6.11 we can assume that the multiplicity of every column in $\mathcal{B}_{temp}$ is 1. Hence without loss of generality $\mathcal{B}_{temp}$ is therefore of type X_k. If r' is the image of r after it is exchanged to the bottom, the fact that $l(r') = 0$ follows from the proof of Lemma 7.22. By the alternating sign condition,

$$\eta(r) = \eta(\mathcal{B}) \cdot (-1)^{\sum_{s<r} C(s)}.$$

Lemma 6.9 implies that the sign of the last circle in r' is

$$\eta(r) \cdot (-1)^{\sum_{s>r} C(s)} \cdot (-1)^{C(r')-1}.$$

Excluding r, there are $k + 1 - C(r)$ rows s in $\mathcal{B}$ each with $C(s) = 1$, so the sign of the last circle is

$$\eta(\mathcal{B})(-1)^{\sum_{s\in\mathcal{B}-\{r\}} C(s)}(-1)^{C(r')-1} = (-1)^{k+C(r')-C(r)}\eta(\mathcal{B}).$$

But $C(r') - C(r)$ is even since r and r' have the same support, so the sign is $(-1)^k\eta(\mathcal{B})$, which is precisely the sign of the last circle of $\mathcal{B}$.

If r is not a hat, then r is a chain or a multiple. Again it is only possible to exchange r with the rows following it if $\mathrm{supp}(r)$ contains their supports, which means that all rows $s > r$ must be multiples. We presume r is a chain, since otherwise r is trivially unchanged by row exchanges with identical multiples. Since there is no chain after r, there must be an even number of each multiple; thus, by Lemma 6.11, exchanging r with these rows leaves r unchanged, so $l(r') = l(r) = 0$ as desired. The sign of the last circle of r' is the same as the last circle of r, which is the same as the sign of the last circle in $\mathcal{B}_{temp}$ (and $\mathcal{B}$) due to the odd-alternating condition.

For part (2), if $\mathcal{B}_{temp}$ is a block not starting at zero, we break into two cases. First suppose $\mathrm{supp}(r)$ has length at least 2. Then it must be a chain. Moreover, its support cannot be contained in the support of a row before it, since by the axioms of $\mathcal{S}$-data (Definition 7.2) the intersection of any two supports has size at most 1. Since we can exchange r all the way to the top, $\mathrm{supp}(r)$ must contain the support of every row before it. For the same reason, this implies that every row before r must be a single circle. Since r is a chain, by the axioms of type $Y_{\mathcal{M}}$, all the circles must lie in column $B(r)$ and have the same sign as $\eta(r)$, and there must be an even number of circles. So by Lemma 6.11, r is unchanged when it is exchanged to the top. So it has the same sign as the sign of $\mathcal{B}$.

Second suppose $\mathrm{supp}(r)$ has length 1. If $\mathrm{supp}(r)$ contains the support of all the rows above it, then it must be a multiple of a chain of length 1. Since these all have the same sign, row exchanges have no affect. Otherwise, $\mathrm{supp}(r)$ is contained in the support of all the rows above it, which means it belongs to a chain s of length at least 2. Since $\mathrm{supp}(r)$ is contained in the support of all the rows above it, every row coming before r other than s must be another multiple in the same column as r. So without loss of generality r immediately follows the chain s. Then we can compute from Definition 3.8 that $l(r') = 0$ and $\eta(r') = \eta(s)$, using the fact that $\mathcal{B}$ is odd-alternating. □

Finally, we need the following lemmas, which will be helpful when we consider operations which involve row exchanges. This lemma allows us to treat the row exchange as if the relevant row was exchanged all the way to the top/bottom, by truncating the extended multi-segment.

Lemma 10.10. *Let $\mathcal{E}$ be of type $Y_{\mathcal{M}}$, and let $\mathcal{E}^{\mathrm{tc}}$ be a truncated virtual extended multi-segment containing all but the first i rows of $\mathcal{E}$. Then regardless of i, the extended multi-segment $\mathcal{E}^{\mathrm{tc}}$ is equivalent to a tempered virtual extended multi-segment where the multiplicity of each column is at most the corresponding multiplicity in $\mathcal{E}_{temp}$. Moreover, the operations realizing this equivalence can be performed on $\mathcal{E}$.*

Proof. Observe that $\mathcal{E}$ can be transformed to $\mathcal{E}_{temp}$ using two types of operations: undualizing hats, and splitting rows of circles (Corollary 7.38). Undualizing a hat only involves rows between the hat and the row to which it is undualized. In particular, for any hat in $\mathcal{E}^{\mathrm{tc}}$, since the truncation removes rows from the top, the corresponding row is also in $\mathcal{E}^{\mathrm{tc}}$. So the undualization is a valid operation on $\mathcal{E}^{\mathrm{tc}}$. Similarly, we see that splitting rows of circles can only involve a row of circles and the rows following it, so it is again a valid operation on $\mathcal{E}^{\mathrm{tc}}$. So we can undualize all the hats in $\mathcal{E}^{\mathrm{tc}}$, if there are any, and then split rows of circles until the result is tempered.

After performing all of these operations, we are left with the truncated rows followed by a tempered virtual extended multi-segment equivalent to $\mathcal{E}^{\mathrm{tc}}$. The operations which take this virtual extended multi-segment to $\mathcal{E}_{temp}$ only increase the multiplicity of each column, since undualizing hats increases multiplicities, while splitting rows of circles leaves them unchanged. Hence the multiplicity of each column in the extended multi-segment equivalent to $\mathcal{E}^{\mathrm{tc}}$ must be at most the multiplicity of the corresponding column in $\mathcal{E}_{temp}$. □

We are now read to begin the proof of independence of blocks. A key observation is that due to the maximality of blocks, there are only a few configurations that can occur between adjacent blocks. The following definition of type 1, type 2, and type 3 boundaries formalizes this.

Definition 10.11. *Suppose $\mathcal{E} \in \mathrm{VRep}_\rho(G)$ is equivalent to a tempered virtual extended multi-segment $\mathcal{E}_{temp}$, and suppose $\mathcal{E}_{temp}$ has a block decomposition $\mathcal{B}_{1,temp} \cup \cdots \cup \mathcal{B}_{k,temp}$. Suppose that $\mathcal{E}$ has a decomposition $\mathcal{B}_1 \cup \cdots \cup \mathcal{B}_k$ with $\mathcal{B}_i$ equivalent to $\mathcal{B}_{i,temp}$. In this setup we give a classification of the ways in which $\mathcal{B}_k$ could start, which depend on how it support overlaps with the support of $\mathcal{B}_{k-1}$, which we call the* boundary *between $\mathcal{B}_k$ and $\mathcal{B}_{k-1}$. Let H_{col} be the last nonempty column in $\mathcal{B}_{k-1}$ and let N_{col} be the first nonempty column in $\mathcal{B}_k$.*

- *A* type 1 boundary *occurs when there is a gap between $\mathcal{B}_{k-1}$ and $\mathcal{B}_k$, or more formally $N_{col} > H_{col} + 1$.*
- *A* type 2 boundary *occurs when $N_{col} = H_{col}$.*

- *A* type 3 boundary *occurs when* $N_{col} = H_{col} + 1$.

Note that in the case of a type 2 boundary, the sign of the last circle in $\mathcal{B}_{k-1,temp}$ must be the same as the sign of the first circle in $\mathcal{B}_{k,temp}$, since $\mathcal{B}_{k,temp}$ started a new block. Similarly in case of a type 3 boundary, the sign of the last circle in $\mathcal{B}_{k-1,temp}$ must be the same as the sign of the first circle in $\mathcal{B}_{k,temp}$. See Figure 4 below.

Type 1

$$\begin{matrix} 0 & \cdots & H_{col} & \cdots & N_{col} & \cdots \end{matrix}$$
$$\begin{pmatrix} & \ddots & & & & \\ & & \oplus & \cdots & & \\ & & & & \ominus & \\ & & & & & \ddots \end{pmatrix}$$

Type 2

$$\begin{matrix} 0 & \cdots & H_{col} & & \cdots \end{matrix}$$
$$\begin{pmatrix} & \ddots & & & \\ & & \oplus & & \\ & & \oplus & & \\ & & & \ominus & \\ & & & & \ddots \end{pmatrix}$$

Type 3

$$\begin{matrix} 0 & \cdots & H_{col} & N_{col} & \cdots \end{matrix}$$
$$\begin{pmatrix} & \ddots & & & \\ & & \oplus & & \\ & & & \oplus & \\ & & & & \ddots \end{pmatrix}$$

FIGURE 4. Boundary Types

We also make a few observations about row exchanges across a boundary. A row exchange cannot occur across a boundary (i.e. between rows of the virtual extended multi-segments in question) of type 1 or type 3, since in those cases the supports of the two extended multi-segments are disjoint.

Moreover, observe that in the case of a type 2 boundary, say between $\mathcal{E}_1$ and $\mathcal{E}_2$, at most one row exchange can occur across the boundary. Let H_{col} be the last column of $\mathcal{E}_1$, which is the same as the first column of $\mathcal{E}_2$, and let $\mathcal{E}_{1,temp}$ and $\mathcal{E}_{2,temp}$ be the tempered extended multi-segments equivalent to $\mathcal{E}_1$ and $\mathcal{E}_2$. Note that $\mathcal{E}_{2,temp}$ has exactly one circle in the first column of $\mathcal{E}_{2,temp}$, since otherwise the circles would be included in $\mathcal{E}_{1,temp}$. Since none of the operations can increase the support of a row, and none of the operations can create two rows with support in a column from one row with support in a column, $\mathcal{E}_2$ can only have one row with support including the first column of $\mathcal{E}_{2,temp}$, which must be H_{col}. By similar reasoning, the support of a row in $\mathcal{E}_1$ cannot extend beyond H_{col}. So the only way for a row exchange to occur is with a single circle in H_{col} from $\mathcal{E}_2$, after which no other row exchanges can occur because every other row in $\mathcal{E}_2$ has support not including H_{col}.

Now we are ready to state the main lemma of this subsection.

Lemma 10.12. *Suppose that* $\mathcal{E} \in \mathrm{VRep}_\rho(G)$ *is equivalent to a tempered virtual extended multi-segment* $\mathcal{E}_{temp}$. *Let* $\mathcal{E}_{temp}$ *have block decomposition* $\mathcal{B}_{1,temp} \cup \cdots \cup \mathcal{B}_{k,temp}$. *Suppose that* $\mathcal{E}$ *has a decomposition* $\mathcal{B}_1 \cup \cdots \cup \mathcal{B}_k$ *with* $\mathcal{B}_i$ *equivalent to* $\mathcal{B}_{i,temp}$. *Then any basic operation* T *on* $\mathcal{E}$ *cannot involve (even after row exchanges) rows from both* $\mathcal{B}_{i_1}$ *and* $\mathcal{B}_{i_2}$ *for* $i_1 < i_2$.

Proof. Recall from Theorem 7.17 that any virtual extended multi-segment equivalent to a block is of type $Y_{\mathcal{M}}$, up to row exchanges. Since these row exchanges do not involve rows from different parts of the decomposition, we can assume without loss of generality that each $\mathcal{B}_i$ is of type $Y_{\mathcal{M}}$.

We have four possible operations: ui^{-1} of type 3', $dual \circ ui^{-1} \circ dual$ where the ui^{-1} is of type 3', $dual \circ ui \circ dual$, and ui.

Operations of the form ui^{-1} of type 3'

First we show that no ui^{-1} operation of type 3' can occur involving a row from $\mathcal{B}_{i_1}$ and a row from $\mathcal{B}_{i_2}$ for $i_1 \neq i_2$. Note that a general ui^{-1} operation of type 3' consists of some number of row exchanges, followed by ui^{-1}, followed by the inverse row exchanges. Consequently, we need to rule out the row exchanges affecting another block. Suppose ui^{-1} is applied to a row r. By Lemma 10.8, it suffices to consider only monotonic row exchanges involving r.

The only way this operation could involve a row from $\mathcal{B}_{i_1}$ and a row from $\mathcal{B}_{i_2}$ is if r lies in $\mathcal{B}_{i_1}$ and it is exchanged into a position in $\mathcal{B}_{i_2}$, or vice versa. However we know that exchanges across a boundary of type 1 and type 3 are impossible, and the only exchange across a boundary of type 2 is a row exchange with exactly one circle in H_{col}. So suppose we have one of these row exchanges. By Lemma 10.9, a row becomes a row of circles after it is exchanged to the end of one of the $\mathcal{B}_i$. In order for the row exchange to be nontrivial, we need r to have at least two circles, since it is being exchanged with a row with exactly one circle. (Otherwise, both rows would have to be supported in the same column, which would mean they are identical.) Therefore we lie in Case 1(b) or Case 2(b) of Definition 3.8, which means that after the row exchange, $l(r') = 1$. In particular, no ui^{-1} of type 3' on r' is possible.

Operations of the form $dual \circ ui^{-1} \circ dual$ of type 3'

Second we show that no $dual \circ ui^{-1} \circ dual$ operation involving a row from $\mathcal{B}_{i_1}$ and a row from $\mathcal{B}_{i_2}$ is possible. Again by Lemma 10.8 it suffices to consider two possibilities: either a row $\widehat{r}$ from $dual(\mathcal{B}_{i_1})$ is exchanged up into $dual(\mathcal{B}_{i_2})$ (recall from Definition 3.11 that the order is reversed), or a row $\widehat{r}$ from $dual(\mathcal{B}_{i_2})$ is exchanged down into $dual(\mathcal{B}_{i_1})$.

When r lies in $\mathcal{B}_{i_1}$ and we exchange $\widehat{r}$ up, we claim no $dual \circ ui^{-1} \circ dual$ operation is possible for a much more general reason. Observe that $\mathcal{E}$ is in (P') order. This is because each $\mathcal{B}_i$ is of type $Y_{\mathcal{M}}$, so it is in (P') order. Moreover, the block decomposition has the property that for a row r_1 lying in $\mathcal{B}_{i_1}$ and a row r_2 lying in $\mathcal{B}_{i_2}$ for $i_1 < i_2$, we have $A(r_1) \leq B(r_2) \leq A(r_2)$.

Since $\mathcal{E}$ is in (P') order, so is $dual(\mathcal{E})$. Now suppose $\widehat{r}$ is exchanged with some row, say $\widehat{s}$, before the ui^{-1} occurs. Since the order is (P'), and $\widehat{r}$ is being exchanged up (by assumption), we must have $B(\widehat{s}) \leq B(\widehat{r})$. Let $\widehat{r}'$ be the image of $\widehat{r}$ after the row exchanges, and let $\widehat{r_1}$ and $\widehat{r_2}$ be the result of applying the ui^{-1} operation to $\widehat{r}'$, with $B(\widehat{r_1}) < B(\widehat{r_2})$. Since the operation only row exchanges $\widehat{r_2}$, after the ui^{-1} operation, the row $\widehat{r_1}$ comes before $\widehat{s}$. Since $dual$ requires the extended multi-segment to have (P') order, we must have $B(\widehat{r_1}) \leq B(\widehat{s})$. But $B(\widehat{r_1}) = B(\widehat{r})$, so we conclude $B(\widehat{s}) = B(\widehat{r})$. However we know the row exchange between $\widehat{r_2}$ and $\widehat{s}$ is possible. By the above reasoning the first column of $\widehat{r}$ lies in the support of $\widehat{s}$, but it clearly does not lie in the support of $\widehat{r_2}$. So we have a contradiction.

When r lies in $\mathcal{B}_{i_2}$ and we exchange $\widehat{r}$ down, we can apply a similar analysis as in the proof of Lemma 7.22, except that this time the operation is not possible. By truncating the extended multi-segment, we can assume that $\widehat{r}$ is exchanged to the last row. By Lemma 10.10 the truncation is equivalent to a tempered extended multi-segment. Since the decomposition satisfies the staircase property (Lemma 10.4) and $\mathcal{B}_{i_2}$ is (P') order (as it is type $Y_{\mathcal{M}}$), the support of $\widehat{r}$ must contain the support of all rows below it. So by Corollary 3.25 we can assume the truncation is indeed tempered, or in other words that $\widehat{r}$ is swapped with rows with $C = 1$. Let $\widehat{r}'$ be the image of $\widehat{r}$ after all row exchanges, or all but one row exchange in the case that the multiplicity in the first column of the reduction is even. By examining the cases of row exchange, we see that in these cases l cannot decrease by more than 1 for every row exchange. Moreover, the only cases in which l decreases by exactly 1 is in Case 1(c), or Case 1(a) and $C(\widehat{r}) = 0$. Since a row exchange with two hats with identical widths leaves $\widehat{r}$ unchanged (Lemma 6.11), in order for $l(\widehat{r}')$ to be 0, every row exchange (ignoring multiplicities) must decrease l by exactly 1. Moreover, there must be a row exchange with a row with every width less than $|B(\widehat{r})|$. This is impossible if $\widehat{r}$ encounters a boundary of type 1.

In the case of a boundary of type 2, we note that the row exchanges with the duals of the circles in H_{col} leave $\widehat{r}$ unchanged. In particular, the net effect is that there are no row exchanges with hats with width H_{col}. So once again we have a contradiction.

In the case of a boundary of type 3, we see that since the circles in H_{col} and N_{col} fail the alternating sign condition, so do their corresponding hats in the dual (Lemma 6.6). In particular, the image of $\widehat{r}$ after row exchanges will fail the alternating sign condition with the hat of width H_{col}. In order for $l(\widehat{r}')$ to be 0, by the reasoning above this forces the row exchange to be Case 1(a) and have $C(\widehat{r}) = 0$. However the latter is impossible since $C(\widehat{r})$ starts nonzero, and only increases at each step. Hence in any case $\widehat{r}$ cannot be exchanged across a boundary and still have $l(\widehat{r}') = 0$. This shows no $dual \circ ui^{-1} \circ dual$ of type 3' is possible.

<u>Operations of the form $dual \circ ui \circ dual$</u>

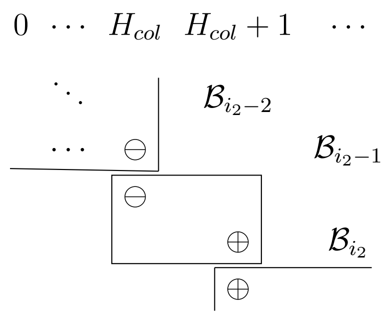
Third we show that no operations $dual \circ ui \circ dual$ can occur between a rows in $\mathcal{B}_{i_1}$ and $\mathcal{B}_{i_2}$. Since the decomposition has the staircase property (Lemma 10.4), for any $r_1 \in \mathcal{B}_{i_1,temp}$ and $r_2 \in \mathcal{B}_{i_2,temp}$, $\mathrm{supp}(\widehat{r_2}) \supset \mathrm{supp}(\widehat{r_1})$. So there is no nontrivial union-intersection possible between different blocks.

<u>Operations of the form ui</u>

Fourth we show that no ui operations can occur between a row r_1 in $\mathcal{B}_{i_1}$ and a row r_2 in $\mathcal{B}_{i_2}$. We have two cases based on i_1.

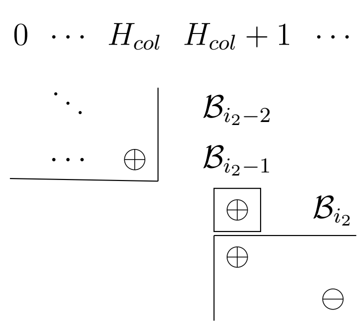
Case 1. First suppose that the row r_1 lies in some $\mathcal{B}_i$ for $i < i_2 - 1$. Note that in the case of a type 1 or type 3 boundary, the maximum of the supports of all rows in the first extended multi-segment is at least 1 less than the minimum of the supports of all rows in the second extended multi-segment. Also, in the case of two consecutive type 2 boundaries, the middle extended multi-segment must have at least two columns, since blocks are maximal. So the maximum of the supports of the rows in the first extended multi-segment is at least 1 less than the minimum of the supports of the rows in the third extended multi-segment. In order for r_1 and r_2 to have a valid union-intersection operation, we must have that $A(r_1) \geq B(r_2) - 1$. So r_1 cannot lie in $\mathcal{B}_i$ for $i < i_2 - 2$, and if r_1 lies in $\mathcal{B}_{i_2-2}$, then the boundaries between $\mathcal{B}_{i_2-2}$, $\mathcal{B}_{i_2-1}$ and $\mathcal{B}_{i_2-1}$, $\mathcal{B}_{i_2}$ must be either both type 2, or one of them is type 2 and one of them is type 3.

In the case that both boundaries are type 2, it must be that $\mathcal{B}_{i_2-1}$ has exactly two columns, say H_{col} and $H_{col}+1$, as shown below.



For support reasons, the only way for the union-intersection to occur is if r_1 is exchanged to the first position in $\mathcal{B}_{i_2-1}$, and r_2 is exchanged to the position immediately after. If either r_1 or r_2 has only one circle, then the first row exchange for r_1 or r_2 is trivial, so the ui is equivalent to a ui between adjacent parts of the decomposition, which is covered in the cases below. But if r_1 has more than one circle, the row exchange with the first row of $\mathcal{B}_{i_2-1}$ is Case 1(b) of Definition 3.8, since by Lemma 10.9 the sign of the last circle of r_1 is the same as the sign of the last circle of $\mathcal{B}_{i_2-2,temp}$, which is the same as the sign of the first circle of $\mathcal{B}_{i_2-1,temp}$, which is the same as the sign of the first circle of $\mathcal{B}_{i_2-1}$. So the result after the row exchange, denoted r_1', has $l(r_1') = 1$, so no ui is possible.

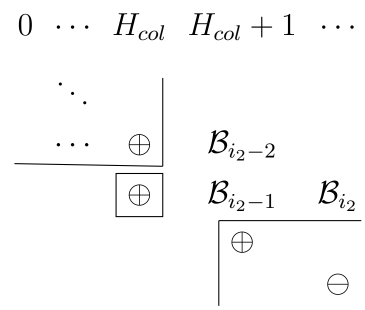
In the case that one of the boundaries is type 2 and one of the boundaries is type 3, first suppose that the boundary between $\mathcal{B}_{i_2-2}$ and $\mathcal{B}_{i_2-1}$ is of type 3, and the boundary between $\mathcal{B}_{i_2-1}$ and $\mathcal{B}_{i_2}$ is of type 2. Let the last column of $\mathcal{B}_{i_2-2}$ be H_{col}. In order for the ui to be possible, it must be that $\mathcal{B}_{i_2-1}$ is only supported in one column, namely $H_{col}+1$, as depicted below.



The union-intersection must be performed by exchanging r_1 to the last row of $\mathcal{B}_{i_2-2}$ and exchanging r_2 to the first row of $\mathcal{B}_{i_2-1}$. By Lemma 6.11, without loss of generality we can assume $\mathcal{B}_{i_2-1}$ has only one circle. Then the row exchange for

r_2 is in Case 2(b) of Definition 3.8 by Lemma 10.9, so the result r_2' has $l(r_2') = 1$ and hence no ui is possible.

Now suppose that the boundary between $\mathcal{B}_{i_2-2}$ and $\mathcal{B}_{i_2-1}$ is of type 2, and the boundary between $\mathcal{B}_{i_2-1}$ and $\mathcal{B}_{i_2}$ is of type 3. In this case, in order for the support of r_1 to include $H_{col} - 1$, it must be that $\mathcal{B}_{i_2-1}$ lies only in the column $H_{col} - 1$. Since it follows a boundary of type 2, it must be a single circle. In summary, we have the following situation:



In this case, by Lemma 10.9, when row r_1 is exchanged so that it takes the place of the last row in $\mathcal{B}_{i_2-2}$, it must be a row of only circles ending in the same sign as $\mathcal{B}_{i_2-2}$. After another row exchange with the only row of $\mathcal{B}_{i_2-1}$, it contains a pair of triangles. So no union-intersection is possible with row r_2, which after row exchanges, by Lemma 10.9, is a row of circles starting at $H_{col} + 1$.

Case 2. Second suppose that r_1 lies in $\mathcal{B}_{i_2-1}$. If the boundary between $\mathcal{B}_{i_2-1}$ and $\mathcal{B}_{i_2}$ is type 1 or type 3, since no row exchanges can occur across the boundary, in order for a union-intersection to occur between a row of $\mathcal{B}_{i_2-1}$ and a row of $\mathcal{B}_{i_2}$, it must be that a row of $\mathcal{B}_{i_2-1}$ is exchanged to the last row of $\mathcal{B}_{i_2-1}$, and a row of $\mathcal{B}_{i_2}$ is exchanged to the first row of $\mathcal{B}_{i_2}$. Let r_1' be the image of the last row of $\mathcal{B}_{i_2-1}$ after the row exchanges, and let r_2' be the image of the first row of $\mathcal{B}_{i_2}$ after row exchanges. By Lemma 10.9, $l(r_1') = 0$ and $l(r_2') = 0$. However for a type 1 boundary, no ui is possible since $B(r_2') > A(r_1') + 1$. In a type 3 boundary, for support reasons the ui would have to be of type 3', since $A(r_1') = H_{col}$ and $B(r_2') = H_{col} + 1$. By Lemma 10.9, since the sign of the last circle of $\mathcal{B}_{i_2-1,temp}$ is the same as the sign of the first circle of $\mathcal{B}_{i_2,temp}$, the sign of the last circle of r_1' is the same as the sign of the first circle of r_2'. Hence no ui of type 3' is possible.

If the boundary between $\mathcal{B}_{i_2-1}$ and $\mathcal{B}_{i_2}$ is type 2, then there are two ways a union-intersection might occur. First, r_1 is exchanged across the boundary, which can only happen if it is exchanged with the first circle of $\mathcal{B}_{i_2}$. In this case, since by Lemma 10.9 the row contains only circles when it is the last row of $\mathcal{B}_{i_2-1}$, it contains a pair of triangles when it is the first row of $\mathcal{B}_{i_2}$. However r_2 after row exchanges, by Lemma 10.9, is a row of circles starting at $H_{col} + 1$, so no union-intersection is possible. Second, r_1 is not exchanged across the boundary. After row exchanges r_1 and r_2 are both only circles, say r_1' and r_2', so the only possible ui is of type 3'. However the sign of the last circle of $\mathcal{B}_{i_2-1}$ is the same as the sign of the first circle of $\mathcal{B}_{i_2}$, so by Lemma 10.9, the sign of the last circle of r_1' is the same as the sign of the first circle of r_2'. This means that union-intersection is not possible. □

We now prove the main result of this subsection, namely, that the blocks occurring in the block decomposition control the equivalence of tempered virtual extended multi-segments.

Proposition 10.13 (Independence of blocks)**.** *Let $\mathcal{E} \in \mathrm{VRep}_\rho(G)$ be equivalent to a tempered virtual extended multi-segment $\mathcal{E}_{temp}$. If $\mathcal{E}_{temp}$ has a block decomposition $\mathcal{B}_{1,temp} \cup \cdots \cup \mathcal{B}_{k,temp}$, then there exists a decomposition $\mathcal{E} = \mathcal{B}_1 \cup \cdots \cup \mathcal{B}_k$ such that $\mathcal{B}_i$ is equivalent to $\mathcal{B}_{i,temp}$ for all i.*

Proof. Since $\mathcal{E}$ is equivalent to $\mathcal{E}_{temp}$, there exists a sequence of basic operations taking $\mathcal{E}_{temp}$ to $\mathcal{E}$. Suppose the first operation sends $\mathcal{E}_{temp}$ to $\mathcal{E}^{(1)}$, then the second operation sends $\mathcal{E}^{(1)}$ to $\mathcal{E}^{(2)}$, and so on. Note that a decomposition of the desired form exists on $\mathcal{E}_{temp}$, namely the decomposition into blocks. So it suffices to show that if $\mathcal{E}^{(s)}$ has a decomposition of the desired form, after applying a basic operation, we can still find a suitable decomposition. Then by repeating this procedure, we find a suitable decomposition of $\mathcal{E}$.

So suppose we have some basic operation T on $\mathcal{E}^{(s)}$, assuming that $\mathcal{E}^{(s)}$ has some decomposition $\mathcal{B}_1 \cup \cdots \cup \mathcal{B}_k$ with $\mathcal{B}_i$ equivalent to $\mathcal{B}_{i,temp}$. By Lemma 10.12, T cannot involve (either by union-intersection or by row exchanges) a row from $\mathcal{B}_{i_1}$ and a row from $\mathcal{B}_{i_2}$ for $i_1 \neq i_2$. So T involves only rows from $\mathcal{B}_{i_0}$ for some i_0. By Lemma 3.23, this means T does not affect rows from $\mathcal{B}_i$ for $i \neq i_0$. Then $T(\mathcal{E}^{(s)}) = \mathcal{B}_i \cup \cdots \cup T'(\mathcal{B}_{i_0}) \cup \cdots \cup \mathcal{B}_k$, where T' denotes the corresponding operation. Therefore, we have found a suitable decomposition of $T(\mathcal{E}^{(s)})$. ☐

We end this subsection with a proof Theorem 4.9 which we recall below (with slightly different notation).

Theorem 10.14 (count for tempered representations)**.** *Let $\mathcal{E} \in \mathrm{VRep}_\rho(G)$ be tempered and suppose that $\mathcal{E}_{temp}$ decomposes into blocks $\mathcal{B}_{1,temp} \cup \cdots \cup \mathcal{B}_{k,temp}$ in that order. Then*

$$|\Psi(\pi(\mathcal{E}_{temp}))| = |\Psi(\pi(\mathcal{B}_{1,temp}))| \cdot \prod_{i=2}^{k} |\Psi(\pi(sh^1(\mathcal{B}_{i,temp})))|.$$

Proof. Let $\mathcal{E} \in \mathrm{VRep}_\rho(G)$ be an equivalent to $\mathcal{E}_{temp}$. By Proposition 10.13, $\mathcal{E}$ has a decomposition $\mathcal{B}_1 \cup \cdots \cup \mathcal{B}_k$ with each $\mathcal{B}_i$ equivalent to $\mathcal{B}_{i,temp}$. We claim that each $\mathcal{B}_i$ for $i > 1$ must be of type $Y_{\mathcal{M}}^{>0}$. Conversely, we claim that for every $\mathcal{B}_1$ equivalent to $\mathcal{B}_{1,temp}$ and for every $\mathcal{B}_i$ of type $Y_{\mathcal{M}}^{>0}$ equivalent to $\mathcal{B}_{i,temp}$, there exists $\mathcal{E}$ equivalent to $\mathcal{E}_{temp}$ with the decomposition $\mathcal{B}_1 \cup \cdots \cup \mathcal{B}_k$.

For the first claim, since $\mathcal{E}_{temp}$ satisfies these conditions, it suffices to show that after applying any basic operation, the extended multi-segment still satisfies these conditions. Let $\mathcal{E}$ be some extended multi-segment equivalent to $\mathcal{E}_{temp}$ such that each $\mathcal{B}_i$ for $i > 1$ is of type $Y_{\mathcal{M}}^{>0}$. Let T be an operation on $\mathcal{E}$. By Lemma 10.12, we can view T as an operation on one of the parts of the decomposition $\mathcal{B}_i$. If T is an operation on $\mathcal{B}_1$, then there is nothing to prove. Otherwise, suppose T is an operation on $\mathcal{B}_i$. Then applying Lemma 10.2 to the truncated extended multi-segment $\mathcal{B}_1 \cup \cdots \cup \mathcal{B}_i$, we see that after T is applied, $\mathcal{B}_i'$ is still of type $Y_{\mathcal{M}}^{>0}$. Note that since T was a valid operation on $\mathcal{E}$, it is still a valid operation on the truncated extended multi-segment.

For the second claim, by Lemma 10.6 we can apply the operations taking $\mathcal{B}_{1,temp}$ to $\mathcal{B}_1$ to the first block of $\mathcal{E}_{temp}$. Since these operations are local (see Lemma 3.23), they do not affect the other rows and so we obtain an equivalent extended multi-segment with decomposition $\mathcal{B}_1 \cup \mathcal{B}_{2,temp} \cup \cdots \cup \mathcal{B}_{k,temp}$. By Lemma 10.7, we can repeat the same logic for each of the other blocks by considering successive truncations and applying the relevant operations to the last part of each truncation. □

11. Case 5 of Conjecture 4.11

In this section we prove the formula in Case 5 of Conjecture 4.11. This has two parts. First, we prove an independence-type result analogous to Proposition 10.13. Second, we prove an existence-type result analogous to Lemmas 10.6 and 10.7.

In all the statements that follow, we assume that $\mathcal{E}_{temp}$ lies in Case 5 of Conjecture 4.11.

11.1. **Independence-type result.** In the (near) future we will consider decompositions of the following form. If $\mathcal{E}_{temp}$ is a tempered extended multi-segment with block decomposition $\mathcal{B}_{1,temp} \cup \cdots \cup \mathcal{B}_{k,temp}$, and $\mathcal{E}$ is equivalent to $\Theta_1(\mathcal{E}_{temp})$, then we want to consider a decomposition $\mathcal{C}_1 \cup \cdots \cup \mathcal{C}_{k-1}$ such that

$$\begin{aligned} \mathcal{C}_1 &\sim \Theta_1(\mathcal{B}_{1,temp} \cup \mathcal{B}_{2,temp}) \\ \mathcal{C}_i &\sim \mathcal{B}_{i+1,temp} \text{ for } i = 2, \ldots, k-1. \end{aligned}$$

Like the decomposition into parts equivalent to blocks from the previous section, this decomposition has the staircase property, which will be important.

Lemma 11.1. *The decomposition $\mathcal{C}_1 \cup \cdots \cup \mathcal{C}_{k-1}$ described above, if it exists, has the property that for $i_1 < i_2$, and rows $r_1 \in \mathcal{C}_{i_1}$, $r_2 \in \mathcal{C}_{i_2}$, we have $A(r_1) \leq B(r_2)$.*

Proof. The proof is much the same as Lemma 10.4, except this time we know $\mathcal{C}_i$ for $i > 1$ is type $Y_{\mathcal{M}}$ and does not begin at zero, so the minimum of $B(r)$ for $r \in \mathcal{B}_{i+1,temp}$ is the same as the minimum of $B(r)$ for $r \in \mathcal{C}_i$. Hence we need not rely on Lemma 10.2.

More precisely, this follows from the fact that

- the analogous statement is true for $\Theta_1(\mathcal{B}_{1,temp}, \mathcal{B}_{2,temp}), \mathcal{B}_{3,temp}, \ldots, \mathcal{B}_{k,temp}$,
- equivalent extended multi-segments have the same max of support, and
- the minimum of the support of $\mathcal{C}_{i+1}$ is the same as the minimum of the support of $\mathcal{B}_i$. □

We also need the following fact, which follows from the results on lifts of almost-blocks.

Lemma 11.2. *Let $\mathcal{B}$ be an almost-block ending at column k with the multiplicity of the last column even, and let $\mathcal{C}_1$ be any virtual extended multi-segment equivalent to $\Theta_1(\mathcal{B})$. Pick any row $r \in \mathcal{C}_1$ for which we can row exchange r until it becomes the last row. We denote its image under these row exchanges by r'. Then either*

- *r consists of a single circle in column k, with the same sign as the last sign of $\mathcal{B}$ or*

- *$l(r')=1$, and the sign of the last circle of r' is the opposite of the sign of the last circle in $\mathcal{B}$.*

Proof. Let $\mathcal{B}$ have block decomposition $\mathcal{B}_{1,temp}\cup\mathcal{B}_{2,temp}$. Note that $\mathcal{B}_{2,temp}$ is just a single circle. First recall from Lemma 8.8 that every extended multi-segment equivalent to $\mathcal{B}_{1,temp}\cup\mathcal{B}_{2,temp}$ takes the form $\mathcal{B}_1\cup\mathcal{B}_{2,temp}$ for some $\mathcal{B}_1\sim\mathcal{B}_{1,temp}$. By Lemma 9.4, the only operations on $\Theta_1(\mathcal{B}_1\cup\mathcal{B}_{2,temp})$ which involve the hat are $dual\circ ui\circ dual$ operations between the hat and the first and last rows ending at column k. So we split into cases depending on whether $\mathcal{C}_1$ is of the form $\Theta_1(\mathcal{B})$, $\Theta_2(\mathcal{B})$, or $\Theta_4(\mathcal{B})$.

First consider the virtual extended multi-segments of the form $\Theta_1(\mathcal{B}_1\cup\mathcal{B}_{2,temp})$ or $\Theta_2(\mathcal{B}_1\cup\mathcal{B}_{2,temp})$. Observe that both of these have a decomposition of the form $\mathcal{B}_1'\cup\mathcal{B}_{2,temp}$ where $\mathcal{B}_1'\sim\Theta_1(\mathcal{B}_{1,temp})$. If the row r lies in $\mathcal{B}_{2,temp}$, then it is already the last row. Its sign must be the same as the sign of the last circle in $\mathcal{B}_{1,temp}$, and it lies in column k. So this satisfies the conditions of the first case.

Otherwise, since $\Theta_1(\mathcal{B}_{1,temp})$ is a block, when the row r is exchanged to the last position in $\mathcal{B}_1'$, giving r', we have $l(r')=0$ by Lemma 10.9. Furthermore, the last circle of r' has the same sign as the last circle of $\Theta_1(\mathcal{B}_{1,temp})$. Since the first circle of $\Theta_1(\mathcal{B}_{1,temp})$ has the opposite sign as the first circle of $\mathcal{B}_{1,temp}$, but $\Theta_1(\mathcal{B}_{1,temp})$ ends in column $k+1$ while $\mathcal{B}_{1,temp}$ ends in column k, the sign of the last circle is the same. So we conclude that the last circle of r' has the same sign as the last circle of $\mathcal{B}_{1,temp}$, which is the same as the sign of $\mathcal{B}_{2,temp}$. So the row exchange between r' and the (only) row of $\mathcal{B}_{2,temp}$ lies in Case 1(b). If r'' is the result, then $l(r'')=1$ and $\eta(r'')=-\eta(r)$. So the sign of the last circle of r'' is opposite the sign of the last circle of r', which is the same as the sign of the last circle of $\mathcal{B}$.

Now we consider extended multi-segments of the form $\Theta_4(\mathcal{B}_1\cup\mathcal{B}_{2,temp})$. In this case since the last row of $\mathcal{B}_1\cup\mathcal{B}_{2,temp}$ ending at k is just $\mathcal{B}_{2,temp}$, the $dual\circ ui\circ dual$ occurs between the hat and $\mathcal{B}_{2,temp}$. By reducing to the case where all rows have one circle by Corollary 3.25 and then performing a similar calculation as in Lemma 7.24, we see that the resulting row is a pair of triangles with the same sign as the last circle of $\mathcal{B}$. So by Definition 6.4 the sign of the last circle of this row is the opposite of the sign of the last circle of $\mathcal{B}$. ☐

Lemma 11.3. *Let $\mathcal{E}$ be equivalent to the theta lift of an almost-block. Suppose the almost-block has block decomposition $\mathcal{B}_{1,temp}\cup\mathcal{B}_{2,temp}$, with $\mathcal{B}_{1,temp}$ ending at column H_{col}. Let $\mathcal{E}^{\mathrm{tc}}$ be a truncated extended multi-segment containing all but the first i rows of $\mathcal{E}$. Then $\mathcal{E}^{\mathrm{tc}}$ is equivalent to one of the following. Moreover, in each case the operations realizing this equivalence can be performed on $\mathcal{E}$.*

(1) If $\mathcal{E}=\Theta_1(\mathcal{B}_1\cup\mathcal{B}_{2,temp})$ for some $\mathcal{B}_1\sim\mathcal{B}_{1,temp}$, then
  - *$\mathcal{E}^{\mathrm{tc}}$ is equivalent to $\Theta_1(\mathcal{B}_{1,temp}\cup\mathcal{B}_{2,temp})$,*
  - *or is equivalent to $\mathcal{B}_1'\cup\mathcal{B}_{2,temp}$ where $\mathcal{B}_1'$ is tempered, and the multiplicity of each column is at most the corresponding multiplicity in $\mathcal{B}_{1,temp}$.*

(2) If $\mathcal{E}=\Theta_2(\mathcal{B}_1\cup\mathcal{B}_{2,temp})$ for some $\mathcal{B}_1\sim\mathcal{B}_{1,temp}$, then
  - *$\mathcal{E}^{\mathrm{tc}}$ is equivalent to the second form in the previous case,*

- *or is equivalent to $\mathcal{B}_1' \cup \mathcal{B}_{2,temp}$ where $\mathcal{B}_1'$ is the result after applying a certain ui to a tempered extended multi-segment whose multiplicities are at most the multiplicities in $\Theta_3(\mathcal{B}_{1,temp})$.*

(3) If $\mathcal{E} = \Theta_4(\mathcal{B}_1 \cup \mathcal{B}_{2,temp})$ for some $\mathcal{B}_1 \sim \mathcal{B}_{1,temp}$, then $\mathcal{E}^{\mathrm{tc}}$ is equivalent to $\mathcal{B}_1' \cup \mathcal{B}_2'$, where $\mathcal{B}_1'$ is tempered with the multiplicities of each column at most the multiplicities in $\mathcal{B}_{1,temp}$, and $\mathcal{B}_2'$ is the same as the last row of $\mathcal{E}$.

Proof. The proof is similar to the proof of Lemma 10.10.

Assume first that was are in Case (1). if $\mathcal{E}^{\mathrm{tc}} \neq \mathcal{E}$, then the hat added by Θ_1 must be truncated. Hence a nontrivial truncation of $\mathcal{E}$ is just a (possibly trivial) truncation of a virtual extended multi-segment equivalent to an almost-block. In this case, the result follows from Lemma 10.10.

Assume now that was are in Case (2). Further, we first suppose that the row to which the *dual* $\circ$ *ui* $\circ$ *dual* is applied is truncated. Then whether or not the *dual* $\circ$ *ui* $\circ$ *dual* was applied is irrelevant, so this reduces to the second form of Case (1). Now suppose that the row was not truncated. Recall that $\mathcal{E}$ takes the form $\mathcal{E}' \cup \mathcal{B}_{2,temp}$ where $\mathcal{E}' \sim \Theta_1(\mathcal{B}_{1,temp})$, since $\mathcal{B}_{2,temp}$ cannot be the last row ending at H_{col}. By the same reasoning as in the proof of Lemma 10.10, we can undualize hats and split rows of circles in $\mathcal{E}'$, with one exception. Since $\mathcal{B}_{2,temp}$ is a single circle in H_{col}, we cannot perform a ui^{-1} operation to split off circles in $H_{col}+1$. Hence $\mathcal{E}$ is equivalent to $\mathcal{B}_1' \cup \mathcal{B}_{2,temp}$, where $\mathcal{B}_1'$ is "almost tempered", except that there cannot be a single circle in $H_{col}+1$. In fact, observe that since the *dual* $\circ$ *ui* $\circ$ *dual* is applied to the first row ending at H_{col}, and this row was not truncated, in $\mathcal{B}_1'$ we must have at least one circle in H_{col}, since the dual of this row (if it is a hat, or the row itself if it is a row of circles) has a circle in H_{col}. So in $\mathcal{B}_1'$ there exists a row with support $[H_{col}+1, H_{col}]$ consisting of two circles, which is the result of applying a certain *ui*.

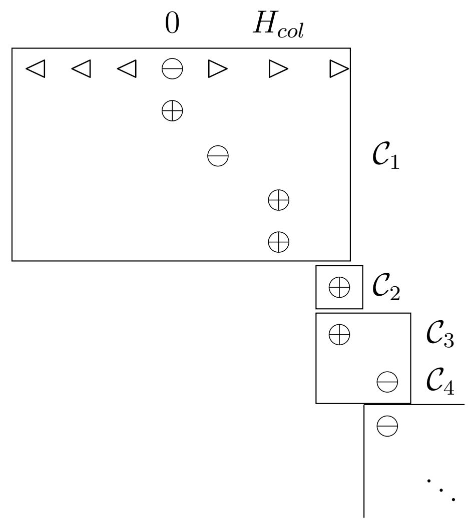
Finally, we assume that we are in Case (3). We note that the last row of $\mathcal{B}_1 \cup \mathcal{B}_{2,temp}$ with support ending in H_{col} is just the last row $\mathcal{B}_{2,temp}$, so the truncation does not affect this row. Hence a truncation of $\mathcal{E}$ is just a truncation of $\mathcal{B}_1$ followed by the last row of $\mathcal{E}$. Since this last row has support $[H_{col}+1, H_{col}]$, all undualizations and all ui^{-1} operations of type 3' on $\mathcal{B}_1$ are valid. Thus by Lemma 10.10, the truncation of $\mathcal{B}_1$ is equivalent to an extended multi-segment taking the desired form $\mathcal{B}_1'$. □

Lemma 11.4. *Let $\mathcal{E}_{temp}$ be a tempered extended multi-segment with block decomposition $\mathcal{B}_{1,temp} \cup \cdots \cup \mathcal{B}_{k,temp}$. Let $\mathcal{E}$ be any extended multi-segment equivalent to $\Theta_1(\mathcal{E}_{temp})$, and suppose $\mathcal{E}$ has a decomposition $\mathcal{C}_1 \cup \cdots \cup \mathcal{C}_{k-1}$ such that $\mathcal{C}_1 \sim \Theta_1(\mathcal{B}_{1,temp} \cup \mathcal{B}_{2,temp})$ and $\mathcal{C}_i \sim \mathcal{B}_{i+1,temp}$ for $i = 2, \ldots, k-1$. Then any operation on $\mathcal{E}$ cannot involve (even by row exchanges) a row from $\mathcal{C}_{i_1}$ and a row from $\mathcal{C}_{i_2}$ for $i_1 < i_2$.*

Proof. The proof is similar in structure to the proof of Lemma 10.12, although specific the cases are different.

Without loss of generality suppose that each of the $\mathcal{C}_i$ for $i > 1$ are of type $Y_{\mathcal{M}}$, since any extended multi-segment equivalent to a block is of type $Y_{\mathcal{M}}$ up to row exchanges. For $\mathcal{C}_1$, we can assume that it takes the form $\Theta_1(\mathcal{B}_1 \cup \mathcal{B}_{2,temp})$, $\Theta_2(\mathcal{B}_1 \cup \mathcal{B}_{2,temp})$, or $\Theta_4(\mathcal{B}_1 \cup \mathcal{B}_{2,temp})$ for some $\mathcal{B}_1$ of type $Y_{\mathcal{M}}$ equivalent to $\mathcal{B}_{1,temp}$.

Throughout this proof, let H_{col} be the last column of $\mathcal{B}_1$. In the picture below we illustrate the decomposition into parts $\mathcal{C}_i$ for a prototypical example of Case 5 of Conjecture 4.11.



Operations of the form ui^{-1} of type 3'

As before, it suffices to consider a ui^{-1} operation consisting of a sequence of monotonic row exchanges, followed by a ui^{-1}, followed by the inverse sequence. If $i_1, i_2 \neq 1$ then we can view the ui^{-1} operation as occurring on $\mathcal{C}_2 \cup \cdots \cup \mathcal{C}_{k-1}$, which are all equivalent to blocks. So by Lemma 10.12, no such operation is possible. So suppose $i_1 = 1$.

First suppose that the ui^{-1} operation is applied to a row r that was originally from $\mathcal{C}_{i_1} = \mathcal{C}_1$, before row exchanges. Note that each $\mathcal{C}_i$ for $i > 1$ starts at $H_{col}+1$ or later, while r ends at $H_{col}+1$ or earlier. So the only rows r can row exchange with that do not lie in $\mathcal{C}_1$ are single circles in column $H_{col}+1$, and r must end at $H_{col}+1$. Let r' be the image of r after the row exchanges. Since r lies in $\mathcal{C}_1$, after the row exchanges r' comes after a row which is a single circle in column $H_{col}+1$. Since r' ends at $H_{col}+1$, this means that if any number of circles are split off, this will result in a row ending before $H_{col}+1$. So the resulting extended multi-segment is not in admissible order, so no ui^{-1} is possible.

Second suppose that the ui^{-1} operation is applied to a row r that was originally from $\mathcal{C}_{i_2}$. We claim that it is impossible for r to be exchanged into $\mathcal{C}_1$ if $i_2 \geq 4$. If the boundary between $\mathcal{C}_3$ and $\mathcal{C}_4$ is type 1 or type 3[1], then $\mathcal{C}_4$ starts at $H_{col}+3$ or later. If the boundary is type 2, then since the boundary between $\mathcal{C}_2$ and $\mathcal{C}_3$ is also type 2, and blocks are maximal, it must be that $\mathcal{C}_3$ has at least two columns. So $\mathcal{C}_4$ starts at $H_{col}+2$ or later. In either case, since $\mathcal{C}_1$ ends at $H_{col}+1$, no row exchanges are possible with rows of $\mathcal{C}_1$.

Now suppose $i_2 = 3$. Since the supports of $\mathcal{C}_3$ and $\mathcal{C}_1$ only overlap in column $H_{col}+1$, if r is exchanged with any rows of $\mathcal{C}_1$, they must have support only in $H_{col}+1$, so they must be single circles. Moreover, r must have support starting at

[1]Note that $\mathcal{C}_i$ for $i > 1$ are equivalent to blocks, so we can speak of the boundary in the same way as before.

$H_{col}+1$. But by the same reasoning as before (in the case that r lies in $\mathcal{C}_1$ and is exchanged down), the image after row exchanges r' has no valid ui^{-1} operations which leave the order admissible, since it comes before the single circles in column $H_{col}+1$.

Finally, if $i_2 = 2$ then r has support of length 1, so certainly no ui^{-1} is possible.

Operations of the form $dual \circ ui^{-1} \circ dual$ of type 3'

There are two cases for a possible $dual \circ ui^{-1} \circ dual$ operation. Either a row r from $\mathcal{E}$ corresponding to $\widehat{r}$ in $dual(\mathcal{E})$ is exchanged up in $dual(\mathcal{E})$, then a ui^{-1} operation is applied, or $\widehat{r}$ is exchanged down in $dual(\mathcal{E})$.

In the first case, the proof is very similar to the proof in Lemma 10.12. It suffices to show that $\mathcal{E}$ is in (P') order. The decomposition $\mathcal{C}_1 \cup \cdots \cup \mathcal{C}_{k-1}$ satisfies the staircase property, so it suffices to show that each $\mathcal{C}_i$ is in (P') order. For $\mathcal{C}_i$ for $i > 1$, this follows from the fact that $\mathcal{C}_i$ is of the type $Y_{\mathcal{M}}$. For $\mathcal{C}_1$, if it takes the form $\Theta_1(\mathcal{B}_1 \cup \mathcal{B}_{2,temp})$ then since the hat is wider than all the rows, adding it preserves the (P') order. If $\mathcal{C}_1$ takes the form $\Theta_2(\mathcal{B}_1 \cup \mathcal{B}_{2,temp})$ or $\Theta_4(\mathcal{B}_1 \cup \mathcal{B}_{2,temp})$ then we see that the order is still (P') since none of the left endpoints have changed compared to $\mathcal{B}_1 \cup \mathcal{B}_{2,temp}$.

In the second case, we once again imitate Lemma 10.12, but we make use of Lemma 11.3 instead of Lemma 10.10, which makes it slightly more involved. If $\widehat{r}$ is not exchanged with any rows of $dual(\mathcal{C}_1)$, then we can apply the exact same reasoning as in Lemma 10.12, since the later parts $\mathcal{C}_2, \mathcal{C}_3, \ldots$ of the decomposition are equivalent to blocks. Otherwise, we suppose $\widehat{r}$ is exchanged with rows of $dual(\mathcal{C}_1)$. We get a truncation of $\mathcal{C}_1$ by considering the rows in $\mathcal{C}_1$ corresponding to the rows in $dual(\mathcal{C}_1)$ which are exchanged with $\widehat{r}$. By Corollary 3.25, we can assume that $\widehat{r}$ is exchanged with the dual of extended multi-segments of the form specified in Lemma 11.3.

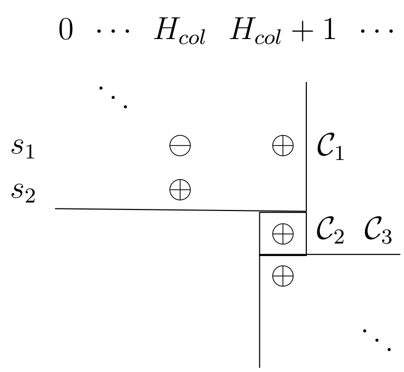
We observe that r cannot lie in $\mathcal{C}_2$. This is because r is only supported in one column, so $B(\widehat{r}) = -A(\widehat{r})$, so any ui^{-1} of type 3' would create we a row s satisfying $|B(s)| > A(s)$, which is impossible.

Now recall from the proof of Lemma 10.12 that any row exchange with a row with with one circle cannot decrease $l(\widehat{r})$ by more than 1. So in order for l to be 0 after all the row exchanges, it must decrease once for every column. Also, if $\widehat{r}$ is exchanged across the boundary of two parts of the decomposition equivalent to blocks, then the net result is that l decreases less than once for every column. Since r cannot lie in $\mathcal{C}_2$, necessarily $\widehat{r}$ is exchanged across such a boundary. Hence it remains to show that when $\widehat{r}$ is exchanged with the rows of $dual(\mathcal{C}_1)$ corresponding to the truncation of $\mathcal{C}_1$, l does not decrease more than the number of columns of the truncation, excluding the column $H_{col}+1$. (The reason for the caveat is that the supports of $\mathcal{C}_1$ and $\mathcal{C}_2$ have an overlap of one column, namely $H_{col}+1$.)

First suppose that the truncation of $\mathcal{C}_1$ is trivial, or in other words $\widehat{r}$ is exchanged all the way to the last row of $dual(\mathcal{E})$. Let $\widehat{r}'$ be the image of $\widehat{r}$ after all row exchanges except the last one (i.e. the one with the dual of the hat). We note that $l(\widehat{r}') \geq 2$, since columns H_{col} and $H_{col}+1$ both have an even multiplicity in $\mathcal{E}_{temp}$. (This is similar to the reasoning in the case of a type 2 boundary in the proof of Lemma 10.12.) So for the last row exchange, between $\widehat{r}'$ and the dual of the hat, if it is Case 1(b) or Case 1(c) then $l(\widehat{r}'') = l(\widehat{r}') \pm 1$, which in either case is still positive. If it is Case 1(a), then $l(\widehat{r}'') = C(\widehat{r}') + l(\widehat{r}') - 1$, but $l(\widehat{r}') - 1 > 0$, so this is still positive. So in any case no ui^{-1} of type 3' is possible.

Second suppose that the truncation is equivalent to something of the form $\mathcal{B}_1' \cup \mathcal{B}_{2,temp}$, where $\mathcal{B}_1'$ is tempered and the multiplicities are at most the multiplicities of $\mathcal{B}_{1,temp}$. In this case since the truncation is not supported in $H_{col}+1$, we just want to show that l does not decrease by more than the number of columns of the truncation. This follows directly from the aforementioned fact that any row exchange does not decrease l by more than 1, and Lemma 6.11.

Third suppose that the truncation is equivalent to something of the form $\mathcal{B}_1' \cup \mathcal{B}_{2,temp}$, where $\mathcal{B}_1'$ is the result after applying a ui involving circles in H_{col} and $H_{col}+1$ to a tempered extended multi-segment whose multiplicities are at most the multiplicities in $\Theta_3(\mathcal{B}_{1,temp})$. This is depicted in the diagram below. By Lemma 6.11, we can assume that there are no rows of $\mathcal{B}_1'$ supported in column H_{col} except for the row with support $[H_{col}+1, H_{col}]$, because the multiplicity of H_{col} is even. By the same reasoning as before, for every column before than H_{col}, exchanging with the hats corresponding to the circles in that column does not decrease l by more than 1. So it suffices to show that exchanging with the duals of the row with support $[H_{col}+1, H_{col}]$ and $\mathcal{B}_{2,temp}$ does not decrease l by more than 1. Call the first row s_1 and the second row s_2.

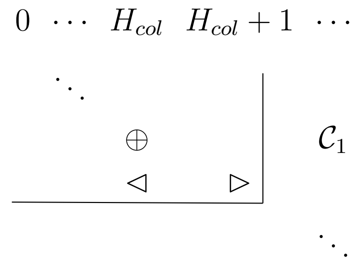


To show this, we first observe that by examining the cases of Definition 3.8, exchanging $\widehat{r}$ with $\widehat{s_2}$ cannot decrease l by more than 1, and exchanging it with $\widehat{s_1}$ cannot decrease l by more than 2. Hence the row exchange with $\widehat{s_2}$ must either keep l unchanged, or decrease it by 1. In the former case, the row exchange with $\widehat{s_1}$ must decrease l by 2, and in the latter case the row exchange with $\widehat{s_1}$ must decrease l by 1.

In the former case, if l is unchanged, then it must have been Case 1(a) of Definition 3.8 and $C(\widehat{r}) = 1$. But note that s_1 and s_2 fail the alternating sign condition, since the last circle of both of them has the same sign as the last circle of $\mathcal{B}_1$. So $\widehat{s_2}$ and $\widehat{s_1}$ also fail the alternating sign condition, and since $C(\widehat{s_2}) = 1$, this means $\eta(\widehat{s_2}) = \eta(\widehat{s_1})$. Since the row exchange of $\widehat{r}$ with $\widehat{s_1}$ is Case 1(a), they fail the alternating sign condition. Moreover, $\eta(\widehat{r}) = \eta(\widehat{r}')$, where $\widehat{r}'$ is the image of $\widehat{r}$ after the first row exchange. So the row exchange of $\widehat{r}$ with $\widehat{s_2}$ is either Case 1(a) or Case 1(b). But the only way for l to decrease by 2 is if it is Case 1(a) and $C(\widehat{r}') = 0$, which is impossible, as $\widehat{r}$ always has at least one circle.

In the latter case, if l decreases by 1 in the first row exchange, then it must have been Case 1(c). By similar reasoning as before, since $\eta(\widehat{r}') = -\eta(\widehat{r})$, we have that $\widehat{r}'$ fails the alternating sign condition with $\widehat{s_2}$. So the row exchange must be Case 1(a) or Case 1(b). But in order for l to decrease by 1 in the second row exchange, it must be Case 1(a) and $C(\widehat{r}') = 1$. But this is clearly impossible since $l(\widehat{r}')$ is not maximal, as it just decreased by 1.

Fourth and finally suppose that the truncation is equivalent to something of the form $\mathcal{B}_1' \cup \mathcal{B}_2'$, where $\mathcal{B}_1'$ is tempered and the multiplicities are at most the multiplicities in $\mathcal{B}_{1,temp}$, and $\mathcal{B}_2'$ is a pair of triangles, as shown below. Since $\mathcal{B}_2'$ is the only row of the truncation supported in $H_{col}+1$, it suffices to show that row exchanges with the dual of this row do not decrease l. This is clear by examining the cases of Definition 3.8, since the row has no circles.



Operations of the form $dual \circ ui \circ dual$

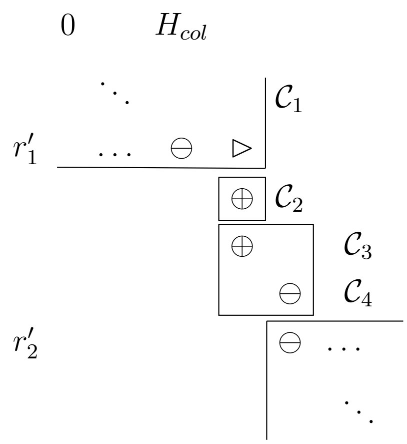
As in the proof of Lemma 10.12, these are impossible because the decomposition has the staircase property (Lemma 11.1).

Operations of the form ui

Observe that if $i_1 > 1$, then the ui operation between r_1 and r_2 can be viewed as a ui on the truncation $\mathcal{C}_2 \cup \cdots \cup \mathcal{C}_{k-1}$. All of the parts of this decomposition are equivalent to blocks. So by Lemma 10.12, no ui is possible. Hence we only have to consider the case where $i_1 = 1$.

Now observe that for $i_2 > 4$, no ui is possible for support reasons. By the same reasoning as when we considered ui^{-1} operations, the earliest column that $\mathcal{C}_4$ could start at is $H_{col}+2$, which is achieved when the boundaries between $\mathcal{C}_2$ and $\mathcal{C}_3$, and $\mathcal{C}_3$ and $\mathcal{C}_4$ are both type 2. If $\mathcal{C}_5$ started at $H_{col}+2$, then $\mathcal{C}_4$ would only be supported in one column, which means that its circles should be part of $\mathcal{C}_3$ or $\mathcal{C}_4$, by maximality of blocks. So $\mathcal{C}_5$ must start at least at $H_{col}+3$, which means no ui is possible with a row from $\mathcal{C}_1$, which ends at $H_{col}+1$. So it remains to consider the cases $i_2 = 4$, 3, 2.

First suppose $i_2 = 4$. Then by the above reasoning it must be that the boundaries between $\mathcal{C}_2$ and $\mathcal{C}_3$ and $\mathcal{C}_3$ and $\mathcal{C}_4$ are both type 2, and that $\mathcal{C}_3$ has two columns. It also follows that r_1 must end at $H_{col}+1$ and r_2 must begin at $H_{col}+2$. Moreover, it must be that every row of $\mathcal{C}_3$ has exactly one circle. Otherwise, row with support $[H_{col}+2, H_{col}+1]$ cannot be exchanged with r_1 or r_2, so they can never be exchanged to adjacent positions. By Lemma 6.11 and 6.12, we can assume the multiplicities of $H_{col}+1$ and $H_{col}+2$ in $\mathcal{C}_2$ and $\mathcal{C}_3$ are all 1. Let r_1' be the image of r_1 after it is exchanged to the last row of $\mathcal{C}_1$, and let r_2' be the image of r_2 after it is exchanged to the first row of $\mathcal{C}_4$. In summary, we have the following situation.



To see that no ui is possible, we note that for support reasons the ui must be performed by exchanging r'_1 with the circles in column $H_{col}+1$, exchanging r'_2 with the circles in column $H_{col}+2$, and then applying ui to adjacent rows, and exchanging back. By Lemma 6.11, the image of r'_1 after these row exchanges, which we call r''_1, is identical to r'_1. Let r''_2 be the image of r'_2 after the row exchange with the circle in column $H_{col}+2$. Since the supports of r_1 and r_2 do not overlap, the union-intersection cannot be Cases 1 or 2 of Definition 3.8. Also, since r''_1 has $l(r''_1)=1$,

$$A(r''_1) - l(r''_1) + 1 = H_{col} + 1 < H_{col} + 2 \leq B(r''_2) + l(r''_2).$$

So no ui of Case 3 is possible either.

Second suppose $i_2 = 3$. We have the following three cases:

(1) $A(r_1) = H_{col} + 1$ and $B(r_2) = H_{col} + 1$
(2) $A(r_1) = H_{col} + 1$ and $B(r_2) = H_{col} + 2$
(3) $A(r_1) = H_{col}$ and $B(r_2) = H_{col} + 1$

In both the first and the second case, since $A(r_1) = H_{col} + 1$, by Lemma 11.2, we have that if r'_1 is the image of r_1 after it is exchanged to the last row of $\mathcal{C}_1$, then $l(r'_1) = 1$ and the sign of the last circle of r'_1 is opposite the sign of the last circle of $\mathcal{B}_{1,temp}$.

In the first case, by Lemma 6.11, we can assume without loss of generality that $\mathcal{C}_2$ has multiplicity 1 in $H_{col}+1$. First suppose that r_2 is supported only in $H_{col}+1$. Then the ui could occur in one of two ways: either r'_1 is exchanged with the row of $\mathcal{C}_2$, and then the ui occurs, or r_2 is exchanged with the row of $\mathcal{C}_2$, and then the ui occurs.

For the first way, since the sign of the last circle of r'_1 is opposite the sign of the last circle of $\mathcal{B}_{1,temp}$, and this is the same as the sign of the circle in $\mathcal{C}_2$, the row exchange with the row of $\mathcal{C}_2$ is Case 1(c) of Definition 3.8. If r''_1 is the image of r'_1 after the row exchange, then $l(r''_1) = 0$ and $\eta(r''_1) = -\eta(r'_1)$. Since $\eta(r_2)$ is the same as the sign of the circle in $\mathcal{C}_2$, it follows that r''_1 and r_2 fail the alternating sign condition. So the only possible union-intersection would be Case 1 or Case 2 of Definition 3.10, but $A(r_2) - l(r_2) \neq A(r''_1) - l(r''_1)$ and $B(r_2) + l(r_2) \neq B(r''_1) + l(r''_1)$.

For the second way the row exchange between r_2 and the row in $\mathcal{C}_2$ is trivial, giving r_2' with $l(r_2') = l(r_2)$ and $\eta(r_2') = \eta(r_2)$. In this case a union-intersection between r_1' and r_2' of Case 3 of Definition 3.10 is possible, but it leaves the rows (and the order) unchanged, so it is trivial.

Next suppose that r_2 is supported in more than one column. We once again have to ways for the ui to occur: either r_1' is exchanged with the circle in $\mathcal{C}_2$, or r_2 is exchanged with the circle in $\mathcal{C}_2$. The first way is impossible for exactly the same reason as before. For the second way, the row exchange between r_2 and the circle in $\mathcal{C}_2$ is Case 2(b) of Definition 3.8. So the image r_2' after row exchange has $l(r_2') = 1$ and $\eta(r_2') = -\eta(r_2)$. So r_1' and r_2' fail the alternating sign condition, so the union-intersection would have to be Case 1 or Case 2. But $A(r_1') - l(r_1') \neq A(r_2') - l(r_2')$ and $B(r_1') + l(r_1') \neq B(r_2') + l(r_2')$, so union-intersection is impossible.

For the second case, that $A(r_1) = H_{col} + 1$ and $B(r_2) = H_{col} + 2$, the only way the union-intersection can occur is if r_1' is exchanged with all the circles in column $H_{col} + 1$. Since the multiplicity of $H_{col} + 1$ is even, by Lemma 6.11, the result r_1'' is the same as r_1'. So r_1'' and r_2 fail the alternating sign condition, but again neither Case 1 nor Case 2 of Definition 3.10 applies.

For the third case, that $A(r_1) = H_{col}$ and $B(r_2) = H_{col} + 1$, for support reasons the only way the union-intersection could occur is if r_1 is exchanged to the last row of $\mathcal{C}_1$, giving r_1', and r_2 is exchanged to the first row of $\mathcal{C}_2$, giving r_2'. For support reasons the ui would have to be Case 3 of Definition 3.10. Moreover, since $B(r_2') + l(r_2') \geq H_{col} + 1$, and $A(r_1') - l(r_1') + 1 \leq H_{col} + 1$, it follows that $l(r_1') = 0$ and $l(r_2') = 0$. By Lemma 10.9, this implies r_1' is a single circle in column H_{col}, with the same sign as the last circle of $\mathcal{B}_{1,temp}$.

If r_2 has support only in $H_{col} + 1$, then the row exchanges with the circles in $\mathcal{C}_2$ leave it unchanged. So it fails the alternating sign condition with r_1', so no ui is possible.

If r_2 has support in multiple columns, then after row exchanging with circles in $\mathcal{C}_2$, the result r_2' has $l(r_2') = 1$ using much the same reasoning as before. So again no ui is possible.

Third suppose $i_2 = 2$. Since r_2 is then a single circle with support in H_{col}, this case has already been considered when $i_2 = 3$ and r_2 is a single circle with support in H_{col}, since row exchanges leave it unchanged. □

Proposition 11.5. *Let $\mathcal{E}_{temp}$ be a tempered extended multi-segment with block decomposition $\mathcal{B}_{1,temp} \cup \cdots \cup \mathcal{B}_{k,temp}$. Let $\mathcal{E}$ be any extended multi-segment equivalent to $\Theta_1(\mathcal{E}_{temp})$. Then there exists a decomposition $\mathcal{C}_1 \cup \cdots \cup \mathcal{C}_{k-1}$ of $\mathcal{E}$ such that*

$$\mathcal{C}_1 \sim \Theta_1(\mathcal{B}_{1,temp} \cup \mathcal{B}_{2,temp})$$
$$\mathcal{C}_i \sim \mathcal{B}_{i+1,temp} \text{ for } i = 2, \ldots, k-1.$$

Proof. The proof is analogous to the proof of Proposition 10.13. Observe that $\Theta_1(\mathcal{E}_{temp})$ has such a decomposition, namely $\mathcal{C}_1 = \Theta_1(\mathcal{B}_{1,temp} \cup \mathcal{B}_{2,temp})$ and $\mathcal{C}_i = \mathcal{B}_{i+1,temp}$ for $i > 1$. Since $\mathcal{E}$ is equivalent to $\Theta_1(\mathcal{E}_{temp})$, there exists a sequence of basic operations taking $\mathcal{E}$ to $\Theta_1(\mathcal{E}_{temp})$. So it suffices to show that each one of these operations preserves the decomposition. By Lemma 11.4, each operation only involves rows from a single part $\mathcal{C}_i$. Since operations are local (Lemma 3.23), they do not affect the other parts of the decomposition. So we can just apply the operation to the relevant $\mathcal{C}_i$, which gives a new decomposition. $\square$

11.2. **Existence-type result.** We prove the following existence type result.

Lemma 11.6. *Let $\mathcal{E}_{temp}$ be a tempered extended multi-segment with block decomposition $\mathcal{B}_{1,temp} \cup \cdots \cup \mathcal{B}_{k,temp}$. Let $\mathcal{E}$ be any extended multi-segment equivalent to $\Theta_1(\mathcal{E}_{temp})$ with decomposition $\mathcal{C}_1 \cup \cdots \cup \mathcal{C}_{k-1}$ such that $\mathcal{C}_1 \sim \Theta_1(\mathcal{B}_{1,temp} \cup \mathcal{B}_{2,temp})$ and $\mathcal{C}_i \sim \mathcal{B}_{i+1,temp}$ for $i = 2, \ldots, k-1$, which exists by Proposition 11.5. Then any operation on $\mathcal{C}_i$ as its own extended multi-segment is also a valid operation on $\mathcal{E}$.*

Proof. The proof is analogous to the proof of Lemma 10.6, but we sketch it again for clarity, since the context is slightly more general here.

Any operation T on $\mathcal{C}_i$ is either a ui, a $dual \circ ui \circ dual$, a ui^{-1} of type 3', or a $dual \circ ui^{-1} \circ dual$ where the ui^{-1} is of type 3'. In the case that T is a ui, it is clear that whether the conditions of union-intersection are satisfied is purely local. In the case that T is a $dual \circ ui \circ dual$, we note that $dual(\mathcal{E}) = dual(\mathcal{C}_{k-1}) \cup \cdots \cup dual(\mathcal{C}_1)$ possibly up to a global sign change on each $dual(\mathcal{C}_i)$. Hence this does not affect whether the union-intersection is valid.

In the case that T is a ui^{-1} of type 3', the only thing we need to check is that the order is still admissible after the ui^{-1}. Since T is a valid operation on $\mathcal{C}_i$, the rows created satisfy the admissible order condition with other rows from $\mathcal{C}_i$. Also since the decomposition has the staircase property (Lemma 11.1), the rows created by applying a ui^{-1} to $\mathcal{C}_i$ cannot fail the admissible order condition with a row from $\mathcal{C}_j$ for $j \neq i$.

Finally, in the case that T is a $dual \circ ui^{-1} \circ dual$, again the only thing we need to check is that the order is still admissible. Note that

$$\max_{r \in \mathcal{C}_i} |B(r)| \leq \max_{r \in \mathcal{C}_i} A(r) \leq \min_{r \in \mathcal{C}_{i+1}} B(r),$$

where the first inequality follows from the definition of A and B and the second inequality follows from the fact that the decomposition has the staircase property. This implies that for any $i < j$, the support of every row in $dual(\mathcal{C}_j)$ contains the support of every row in $dual(\mathcal{C}_i)$. Hence the new rows created by applying a ui^{-1} to $dual(\mathcal{C}_i)$ cannot fail the admissible order condition with a row from $\mathcal{C}_j$ for $j \neq i$, since it does not even apply. $\square$

Proof (of Case 5 of Conjecture 4.11). By Proposition 11.5, every $\mathcal{E}$ equivalent to $\mathcal{E}_{temp}$ has a decomposition $\mathcal{C}_1 \cup \cdots \cup \mathcal{C}_{k-1}$ with $\mathcal{C}_1 \sim \Theta_1(\mathcal{B}_{1,temp} \cup \mathcal{B}_{2,temp})$ and $\mathcal{C}_i \sim \mathcal{B}_{i+1,temp}$ for $i > 1$. By repeatedly applying Lemma 11.6, all such combinations of $\mathcal{C}_i$ are achievable. $\square$

12. Commutativity Results

Throughout this section, we assume that ρ is an orthogonal representation of $\mathrm{GL}_d(F)$ and often suppress it in the notation. In the language of extended multi-segments, the spirit of the Adams conjecture in the case of the first occurrence of the up-tower is that for some $\mathcal{E} \in \mathrm{Rep}_\rho(G_n)$, we have

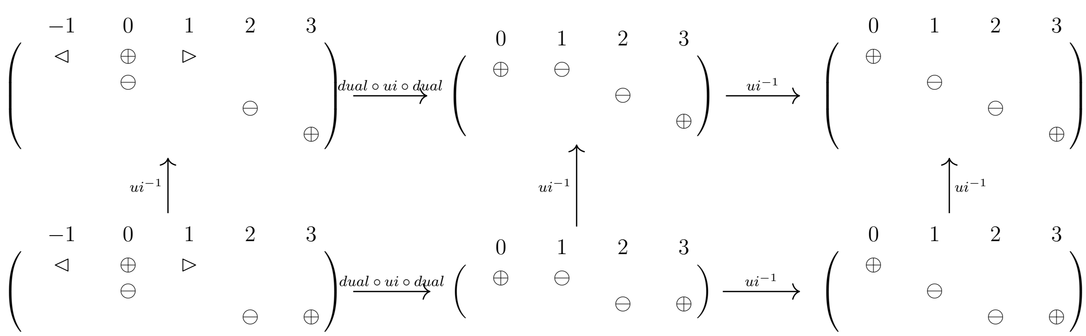
$$\Psi(\theta^{\mathrm{up}}_{-m^{\mathrm{up},\alpha}}(\pi(\mathcal{E}))) \supseteq \{\psi_{\mathcal{E}'_{m^{\mathrm{up},\alpha}}} \mid \mathcal{E}' \sim \mathcal{E}\}.$$

Our previous results show that it is too much to hopeful for an equality between these sets generally, however in many cases they offer a meaningful resolution for tempered $\mathcal{E}$. In particular, Theorem 8.4 describes a case in which $\Psi(\theta^{\mathrm{up}}_{-m^{\mathrm{up},\alpha}}\pi(\mathcal{E}))$ is equal not to $\{\psi_{\mathcal{E}'_{m^{\mathrm{up},\alpha}}} \mid \mathcal{E}' \sim \mathcal{E}\}$, but to three "copies" of this set. Theorem 8.5 offer a bit more complexity. In this case, $\Psi(\theta^{\mathrm{up}}_{-m^{\mathrm{up},\alpha}}\pi(\mathcal{E}))$ is equal to the union of four sets like $\{\psi_{\mathcal{E}'_{m^{\mathrm{up},\alpha}}} \mid \mathcal{E}' \sim \mathcal{E}\}$, but two of these sets may have some overlap. Here, we offer an elaboration of these results that better elucidates the connection between these sets, rather than simply counting their sizes.

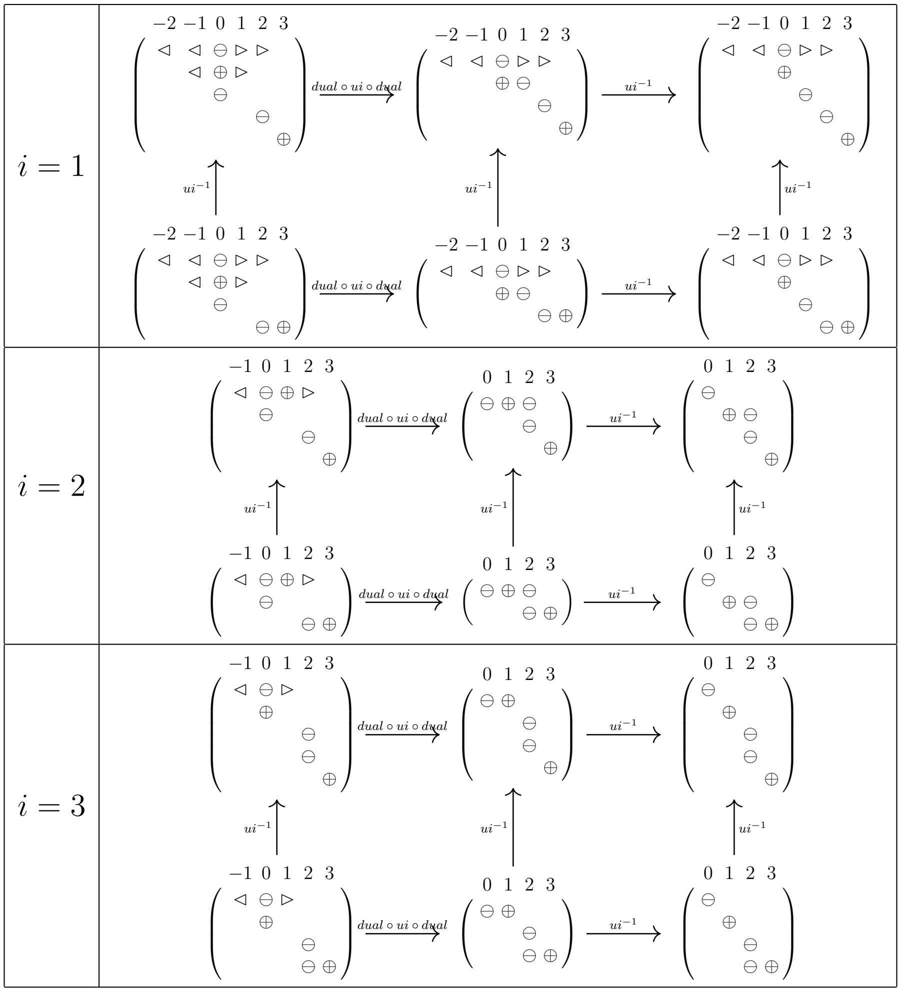
12.1. **A Motivating Example.** Consider the following.

$$\mathcal{E} = \begin{pmatrix} \overset{0}{\oplus} & \overset{1}{} & \overset{2}{} & \overset{3}{} \\ & \ominus & & \\ & & \ominus & \\ & & & \oplus \end{pmatrix}, \quad \mathcal{E}_{m^{\mathrm{up},\alpha}} = \begin{pmatrix} \overset{-2}{\lhd} & \overset{-1}{\lhd} & \overset{0}{\ominus} & \overset{1}{\rhd} & \overset{2}{\rhd} & \overset{3}{} \\ & & \oplus & & & \\ & & & \ominus & & \\ & & & & \ominus & \\ & & & & & \oplus \end{pmatrix}.$$

The multi-segment $\mathcal{E} \in \mathrm{Rep}(G_n)$ can be written in the block decomposition $\mathcal{B}_1 \cup \mathcal{B}_2$, where $|\Psi(\pi(\mathcal{B}_1))| = 3$ and $|\Psi(\pi(\mathcal{B}_2))| = 2$, from which it follows that $|\Psi(\pi(\mathcal{E}))| = 6$ via Theorem 4.9. These six extended multi-segments and all connecting raising operators are depicted below.



The block $\mathcal{B}_1$ is of type $Y_{\mathcal{M}_\mathcal{B}}$ with $\mathcal{M}_\mathcal{B} = (m_0, m_1) = (1, 1)$, so Theorem 8.4 guarantees that $\Psi(\Theta(\mathcal{B}_1)) = \{\psi_{\Theta_i(\mathcal{B}')} \mid \mathcal{B}' \sim \mathcal{B}_1, i \in \{1, 2, 3\}\}$. Independence of blocks, as stated in Proposition 10.13, guarantees a similar 3-to-1 correspondence between extended multi-segments equivalent to $\mathcal{E}_{m^{\mathrm{up},\alpha}}$ and $\mathcal{E}$ respectively. The following three diagrams show the layers of multi-segments $\{\Theta_i(\mathcal{E}') \mid \mathcal{E}' \sim \mathcal{E}\}$ for $i = 1, 2$, and 3 respectively.



Not only does each of these layers have the same number of multi-segments as $\Psi(\pi(\mathcal{E}))$, the arrows between the corresponding multi-segments is exactly the same in each diagram. In other words, raising operators between multi-segments equivalent to $\mathcal{E}$ seem to "commute" with the $dual \circ ui \circ dual$ and ui^{-1} operators which go between Θ_1, Θ_2, and Θ_3. In a sense, then, each set $\{\Theta_i(\mathcal{E}') \mid \mathcal{E}' \sim \mathcal{E}\}$ is a precise copy of the set $\{\mathcal{E}' \mid \mathcal{E}' \sim \mathcal{E}\}$. This leads us to the main theorem we aim to prove regarding the structure of row operations on $\mathcal{E}_{m^{\mathrm{up},\alpha}}$, which we aim to prove at the end of this section.

Theorem 12.1. *Let $\mathcal{E}$ be of type $Y_{\mathcal{M}}$ with $\mathcal{M}$ starting at zero, such that $\Psi(\pi(\mathcal{E})) = \{\psi(\Theta_i(\mathcal{E})) \mid \mathcal{E}' \sim \mathcal{E}, i \in I_{\mathcal{E}}\}$, for some index set $I_{\mathcal{E}} = \{1, 2, 3, 4\}$ or $\{1, 2, 3\}$ depending on $\mathcal{E}$. Then, if $\mathcal{E}' \sim \mathcal{E}'' \sim \mathcal{E}$ but $\mathcal{E}' \neq \mathcal{E}''$, there exists a raising operator $T \in \{dual \circ ui \circ dual, ui^{-1}\}$ from $\Theta_i(\mathcal{E}')$ to $\Theta_j(\mathcal{E}'')$ if and only if $i = j$ and the same raising operator T can be applied from $\mathcal{E}'$ to $\mathcal{E}''$.*

Remark 12.2. *We have stated this theorem for the case where $\mathcal{E}$ is of type $Y_{\mathcal{M}}$. Similar results can be deduced for other cases, such as*

- *$\mathcal{E}$ is an almost-block, given the structure detailed in Lemma 9.5,*
- *$\mathcal{E}$ if of type $Y_{\mathcal{M}}$ with $\mathcal{M}$ starting after zero; i.e., $\mathcal{E}$ is equivalent to a tempered multi-segment not starting at zero, or*
- *$\mathcal{E}$ has a block decomposition similar to the one in the above example.*

For the sake of brevity, we only prove the stated theorem.

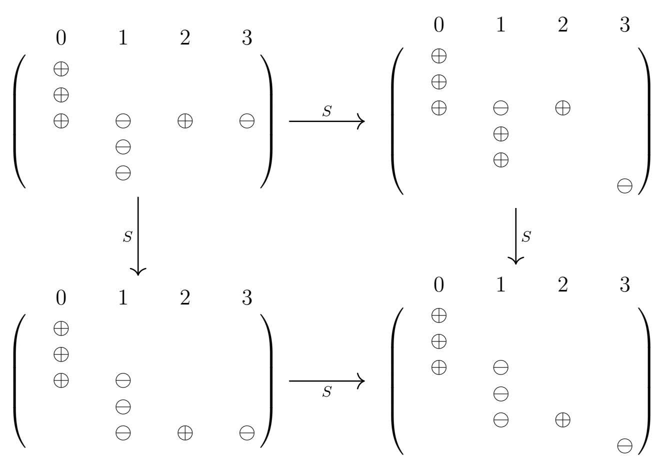
12.2. **Commutativity of Row Operations.** All of the operators of type 3'; $S; M; U; D^1$; and D^2, are, in a sense, 'local.' Each of them affects $1-2$ rows in structurally predictable ways and and leaves the others completely intact, modulo a possible sign change. This means that many of these operators can be performed simultaneously, independently of each other. In a sense, they commute with each other. This idea is the primary mechanism behind Theorem 12.1. We make it precise in the following theorem. We also provide an example to illustrate each specific commutativity result in the theorem.

Theorem 12.3 (Commutativity of Operators)**.** *Let $\mathcal{E}$ be of type $Y_{\mathcal{M}}$, with $\mathcal{M}$ starting at zero.*

1. *Any two operators of type 3' involving distinct rows commute.*
2. *Given a chain r, we have:*

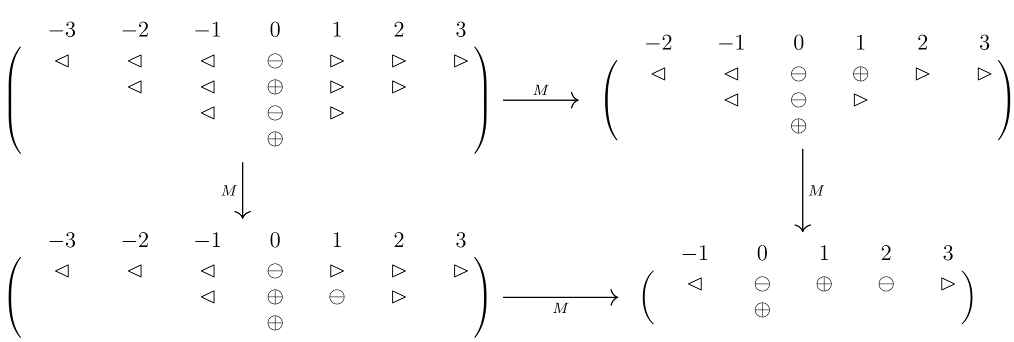
$$S_{r',k_2} \circ S_{r,k_1} = S_{r'',k_1} \circ S_{r,k_1+k_2}.$$

Here, r' is the row remaining after k_1 circles have been taken out of r, while r'' is the row of $k_1 + k_2$ circles removed from r.



3. *Given consecutive hats $h_1 < h_2 < h_3$, we have*

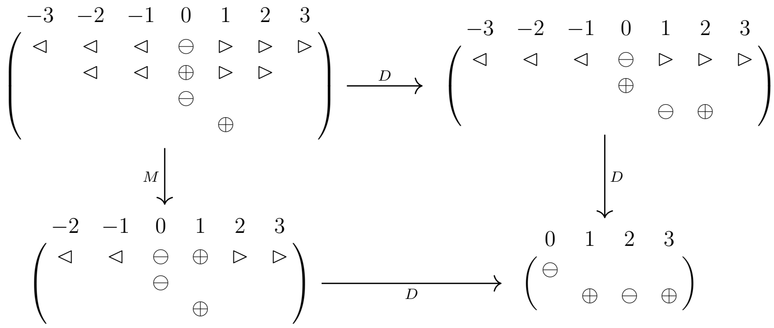
$$(h_1 * h_2) * h_3 = h_1 * (h_2 * h_3).$$



4. *Given mergable hats h_1, h_2 and a row of circles r, we have*

$$D_{h_1,r'} \circ D_{h_2,r} = D_{h_1 * h_2,r} \circ M_{h_1,h_2},$$

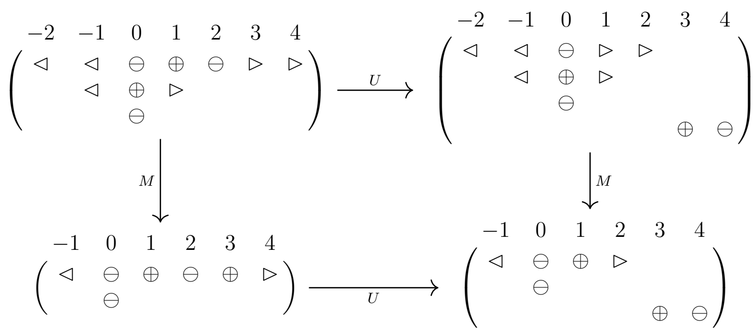
where r' is the image of r under D.



(5) *Given mergable hats h_1, h_2, we have*

$$M_{h_1', h_2} \circ U_{h_1,k} = U_{h_1 * h_2, k} \circ M_{h_1,h_2},$$

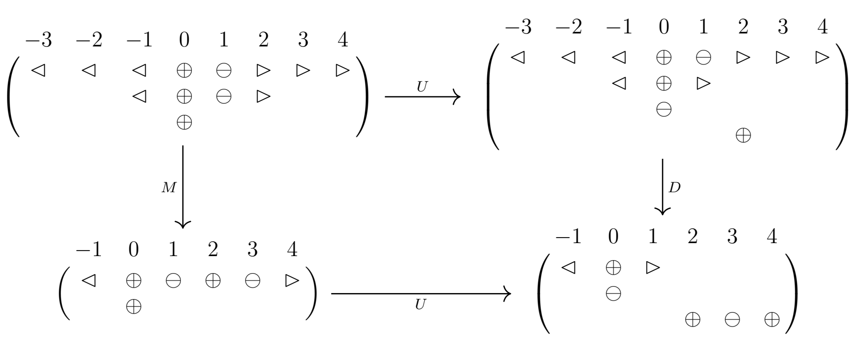
where h_1' is the image of h_1 after U.



(6) *Given mergable hats h_1, h_2, we have*

$$D_{h_1,r} \circ U_{h_2,k} = U_{h_1 * h_2, C(h_1)+k} \circ M_{h_1,h_2},$$

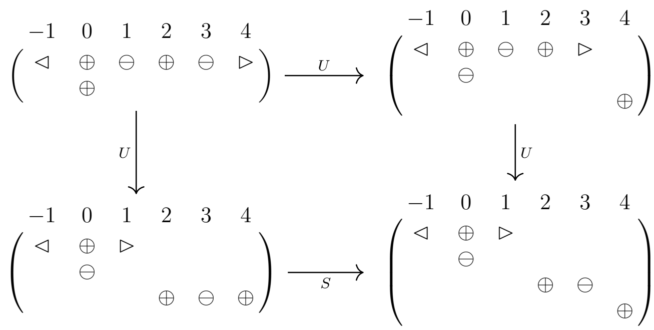
where r is the row of circles in the image of h_2 under U.



(7) *Given a hat h, we have*

$$U_{h',k_2} \circ U_{h,k_1} = S_{r,k_1} \circ U_{h,k_1+k_2},$$

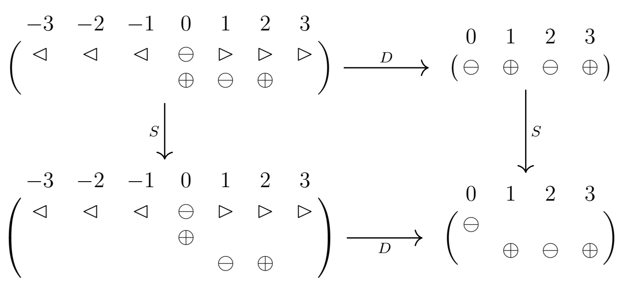
where h' is the remaining part of of h when k_1 circles are pulled out through U, and r is the row of $k_1 + k_2$ circles pulled out.



(8) *Given a hat h and row of circles r, we have*

$$S_{r',C(h)+k} \circ D_{h,r} = D_{h,r_2} \circ S_{r,k},$$

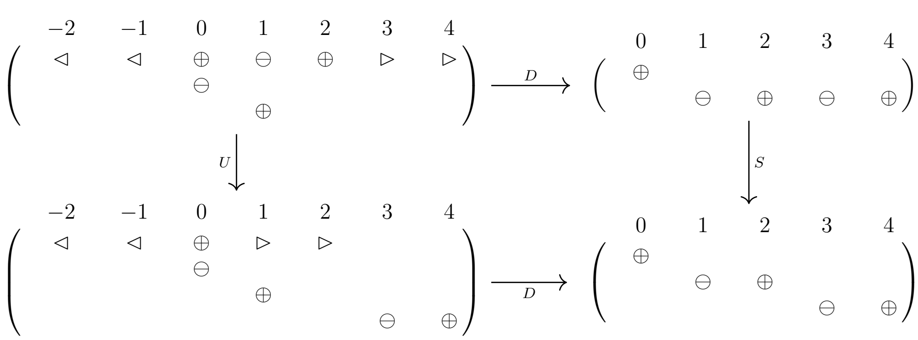
where r' is the image of r after D, and r_2 is the second row in the image of r under S.



(9) *Given a hat h and row of circles r, we have*

$$D_{h',r} \circ U_{h,k} = S_{r',k} \circ D_{h,r},$$

where h' is the remaining hat after U, and r' is the image of r under D.



Proof. We note that for the duration of this proof, it suffices to show that two combinations of operations yield the same virtual extended multi-segments up to the signs. That is, we need only check the effect on the $\mathcal{S}$-data, and since all these operations preserve η and the sign conditions for Definition 7.2, all the signs must match. To do this, we will utilize the descriptions of the effects of S, M, U, and D given by Remarks 7.31, 7.35, 7.41, and 7.43.

Part (1) follows from the fact that all four operations are local. That is, they affect $1 - 2$ segments in a prescribed manner and leave the other segments unaffected up to sign.

For Part (2), let $r = ([A,B],0,\eta)$. We have two cases, the first of which is that $B - A = k_1 + k_2$ and r is a z-chain. In this case, applying $S_{r',k_2} \circ S_{r,k_1}$ impacts the $\mathcal{S}$-data in the following way.

$$\begin{aligned}&\{B,\dots,A,\dots\}\\ \longrightarrow&\{B,\dots,A-k_1\},\{A-k+1,\dots,A,\dots\}\\ \longrightarrow&\{B+1,\dots,A-k_1\},\{A-k+1,\dots,A,\dots\},\end{aligned}$$

while applying $S_{r'',k_1} \circ S_{r,k_1+k_2}$ gives the same result

$$\begin{aligned}\{B,\dots,A,\dots\}\longrightarrow&\{B+1,\dots,A,\dots\}\\ \longrightarrow&\{B+1,\dots,A-k_1\},\{A-k+1,\dots,A,\dots\}.\end{aligned}$$

If we are not in the previous case, then applying $S_{r',k_2} \circ S_{r,k_1}$ gives

$$\begin{aligned}&\{B,\dots,A,\dots\}\\ \longrightarrow&\{B,\dots,A-k_1\},\{A-k+1,\dots,A,\dots\}\\ \longrightarrow&\{B,\dots,A-k_1-k_2\},\{A-k_1-k_2+1,\dots,A-k_1\},\{A-k+1,\dots,A,\dots\},\end{aligned}$$

and applying $S_{r'',k_1} \circ S_{r,k_1+k_2}$ gives

$$\begin{aligned}&\{B,\dots,A,\dots\}\\ \longrightarrow&\{B,\dots,A-k_1-k_2\},\{A-k_1-k_2+1,\dots A,\dots\}\\ \longrightarrow&\{B,\dots,A-k_1-k_2\},\{A-k_1-k_2+1,\dots,A-k_1\},\{A-k+1,\dots,A,\dots\},\end{aligned}$$

which is the same result.

Part (3) is straightforward.

For Part (4), let $h_1 = ([A,-B],B,\eta_1)$ and $h_2 = ([B-1,-C],C,\eta_2)$. We have two possible cases: either r is a chain, in which case all applications of D are D^1, or r is a multiple, in which case all the D's except $D_{h_1,r'}$ are D^2. For the first case, let $r = ([C-1,D],0,\eta')$. Then applying $D_{h_1,r'} \circ D_{h_2,r}$ influences the $\mathcal{S}$-data in the following way

$$\begin{aligned}&\{D,\dots,C-1,\overline{C,\dots,B-1},\overline{B,\dots,A},\dots\}\\ \longrightarrow&\{D,\dots,C-1,C,\dots,B-1,\overline{B,\dots,A},\dots\}\\ \longrightarrow&\{D,\dots,C-1,C,\dots,B-1,B,\dots,A,\dots\}.\end{aligned}$$

On the other hand, performing $D_{h_1*h_2,r} \circ M_{h_1,h_2}$ gives

$$\begin{aligned}&\{D,\dots,C-1,\overline{C,\dots,B-1},\overline{B,\dots,A},\dots\}\\ \longrightarrow&\{D,\dots,C-1,\overline{C,\dots,B-1,B,\dots,A},\dots\}\\ \longrightarrow&\{D,\dots,C-1,C,\dots,B-1,B,\dots,A,\dots\},\end{aligned}$$

which is indeed the same.

In the case where r is a multiple, let $r = ([C-1,C-1],0,\eta')$. Then implementing $D_{h_1,r'} \circ D_{h_2,r}$ gives

$$\begin{aligned}&\{D,\dots,C-1,\overline{C,\dots,B-1},\overline{B,\dots,A},\dots\}\\ \longrightarrow&\{D,\dots,C-1\},\{C-1,C,\dots,B-1,\overline{B,\dots,A},\dots\}\\ \longrightarrow&\{D,\dots,C-1\},\{C-1,C,\dots,B-1,B,\dots,A,\dots\}.\end{aligned}$$

Similarly, implementing $D_{h_1 * h_2, r} \circ M_{h_1,h_2}$ gives

$$\begin{aligned}
&\{D, \dots, C-1, \overline{C, \dots, B-1}, \overline{B, \dots, A}, \dots\} \\
\longrightarrow &\{D, \dots, C-1, \overline{C, \dots, B-1, B, \dots, A}, \dots\} \\
\longrightarrow &\{D, \dots, C-1\}, \{C-1, C, \dots, B-1, B, \dots, A, \dots\},
\end{aligned}$$

which proves Part (4).

For Part (5), let $h_1 = ([A, -B], B, \eta_1)$ and $h_2 = ([B-1, -C], C, \eta_2)$. Implementing $M_{h_1', h_2} \circ U_{h_1,k}$ changes the $\mathcal{S}$-data as follows

$$\begin{aligned}
&\{\dots, \overline{C, \dots, B-1}, \overline{B, \dots, A}, \dots\} \\
\longrightarrow &\{\dots, \overline{C, \dots, B-1}, \overline{B, \dots, A-k}\}, \{A-k+1, \dots, A, \dots\} \\
\longrightarrow &\{\dots, \overline{C, \dots, B-1, B, \dots, A-k}\}, \{A-k+1, \dots, A, \dots\}.
\end{aligned}$$

Meanwhile, applying $U_{h_1 * h_2, k} \circ M_{h_1,h_2}$ gives

$$\begin{aligned}
&\{\dots, \overline{C, \dots, B-1}, \overline{B, \dots, A}, \dots\} \\
\longrightarrow &\{\dots, \overline{C, \dots, B-1, B, \dots, A}, \dots\} \\
\longrightarrow &\{\dots, \overline{C, \dots, B-1, B, \dots, A-k}\}, \{A-k+1, \dots, A, \dots\},
\end{aligned}$$

which is evidently the same.

For Part (6), let $h_1 = ([A, -B], B, \eta_1)$ and $h_2 = ([B-1, -C], C, \eta_2)$. We note that implementing U always creates a new chain, so $D_{h_1,r}$ must be a D^1. Implementing $D_{h_1,r} \circ U_{h_2,k}$ changes the $\mathcal{S}$-data as follows

$$\begin{aligned}
&\{\dots, \overline{C, \dots, B-1}, \overline{B, \dots, A}, \dots\} \\
\longrightarrow &\{\dots, \overline{C, \dots, B-1-k}\}, \{B-k, \dots, B-1, \overline{B, \dots, A}, \dots\} \\
\longrightarrow &\{\dots, \overline{C, \dots, B-1-k}\}, \{B-k, \dots, B-1, B, \dots, A, \dots\},
\end{aligned}$$

while implementing $U_{h_1 * h_2, C(h_1)+k} \circ M_{h_1,h_2}$ gives

$$\begin{aligned}
&\{\dots, \overline{C, \dots, B-1}, \overline{B, \dots, A}, \dots\} \\
\longrightarrow &\{\dots, \overline{C, \dots, B-1, B, \dots, A}, \dots\} \\
\longrightarrow &\{\dots, \overline{C, \dots, B-1-k}\}, \{B-k, \dots, B-1, B, \dots, A, \dots\},
\end{aligned}$$

which is the same result.

For Part (7), let $h = ([A, -B], B, \eta_1)$ and let $r = ([B-1, C], 0, \eta_2)$. Then applying $U_{h', k_2} \circ U_{h,k_1}$ impacts the $\mathcal{S}$-data in the following way

$$\begin{aligned}
&\{C, \dots, B-1, \overline{B, \dots, A}, \dots\} \\
\longrightarrow &\{C, \dots, B-1, \overline{B, \dots, A-k}\}, \{A-k_1+1, \dots, A, \dots\} \\
\longrightarrow &\{C, \dots, B-1, \overline{B, \dots, A-k_1-k_2}\}, \{A-k_1-k_2+1, \dots, A-k_1\}, \\
&\qquad \{A-k_1+1, \dots, A, \dots\}.
\end{aligned}$$

Meanwhile, we note that the row r is produced by the U operation and therefore cannot be a z-chain. Therefore, the S operation is an S^1, so applying $S_{r,k_1} \circ$

U_{h,k_1+k_2} gives

$$\begin{aligned}
&\{C,\dots,B-1,\overline{B,\dots,A},\dots\}\\
\longrightarrow&\{C,\dots,B-1,\overline{B,\dots,A-k_1-k_2}\},\{A-k_1-k_2+1,\dots,A,\dots\}\\
\longrightarrow&\{C,\dots,B-1,\overline{B,\dots,A-k_1-k_2}\},\{A-k_1-k_2+1,\dots,A-k_1\},\\
&\qquad\qquad\qquad\qquad\qquad\{A-k_1+1,\dots,A,\dots\},
\end{aligned}$$

which produces the same result.

For Part (8), we prove the statement assuming that the S operation is an S^1. The proof is exactly the same in the S^2 case, except that one of the sets $\mathcal{S}_i$ must be removed, so we will exclude it. We observe that r must be a chain since it is possible to apply S, so all D's must be D^1. We let let $h=([A,-B],B,\eta_1)$ and let $r=([B-1,C],0,\eta_2)$. Applying $S_{r',C(h)+k}\circ D_{h,r}$ gives

$$\begin{aligned}
&\{C,\dots,B-1,\overline{B,\dots,A},\dots\}\\
\longrightarrow&\{C,\dots,B-1,B,\dots,A,\dots\}\\
\longrightarrow&\{C,\dots,B-1-k\},\{B,\dots,A,\dots\},
\end{aligned}$$

while applying $D_{h,r_2}\circ S_{r,k}$ gives

$$\begin{aligned}
&\{C,\dots,B-1,\overline{B,\dots,A},\dots\}\\
\longrightarrow&\{C,\dots,B-1-k\},\{B-k,\dots,B-1,\overline{B,\dots,A},\dots\}\\
\longrightarrow&\{C,\dots,B-1-k\},\{B,\dots,A,\dots\},
\end{aligned}$$

which is the same result.

Finally, for Part (9), we again only provide the proof in the case where the S is an S^1. Let $h=([A,-B],B,\eta_1)$. We must separately consider the cases where r is a chain or a multiple. If r is a chain, let $r=([B-1,C],0,\eta_2)$. All D's will be D_1, so applying $D_{h',r}\circ U_{h,k}$ gives

$$\begin{aligned}
&\{C,\dots,B-1,\overline{B,\dots,A},\dots\}\\
\longrightarrow&\{C,\dots,B-1,\overline{B,\dots,A-k}\},\{A-k+1,\dots,A,\dots\}\\
\longrightarrow&\{C,\dots,B-1,B,\dots,A-k\},\{A-k+1,\dots,A,\dots\},
\end{aligned}$$

while applying $S_{r',k}\circ D_{h,r}$ gives the same result

$$\begin{aligned}
&\{C,\dots,B-1,\overline{B,\dots,A},\dots\}\\
\longrightarrow&\{C,\dots,B-1,B,\dots,A,\dots\}\\
\longrightarrow&\{C,\dots,B-1,B,\dots,A-k\},\{A-k+1,\dots,A,\dots\}.
\end{aligned}$$

On the other hand, if r is a multiple $([B-1,B-1],0,\eta_2)$, then all the D's will be D^2's. Here, applying $D_{h',r}\circ U_{h,k}$ gives

$$\begin{aligned}
&\{C,\dots,B-1,\overline{B,\dots,A},\dots\}\\
\longrightarrow&\{C,\dots,B-1,\overline{B,\dots,A-k}\},\{A-k+1,\dots,A,\dots\}\\
\longrightarrow&\{C,\dots,B-1\},\{B-1,B,\dots,A-k\},\{A-k+1,\dots,A,\dots\},
\end{aligned}$$

while applying $S_{r',k} \circ D_{h,r}$ gives the same result

$$\begin{aligned} &\{C, \dots, B-1, \overline{B, \dots, A}, \dots\} \\ \longrightarrow &\{C, \dots, B-1\}, \{B-1, B, \dots, A, \dots\} \\ \longrightarrow &\{C, \dots, B-1\}, \{B-1, B, \dots, A-k\}, \{A-k+1, \dots, A, \dots\}. \end{aligned}$$ □

12.3. **Commutativity Results for Theta Correspondence.** We now prove Theorem 12.1. We do this by proving the theorem for the case where T is a raising operator of type $3'$. From this, we immediately obtain the result for the case where T is a lowering operator of type $3'$. All other operators are compositions of such operators of type 3'. The proof for raising operators of type 3' follows from the following series of lemmas.

Lemma 12.4. *Let $\mathcal{E}$ be of type $Y_{\mathcal{M}}$ with $\mathcal{M}$ starting at zero. Suppose there exists a raising operator $T \in \{dual \circ ui \circ dual, ui^{-1}\}$ of type 3' from $\Theta_1(\mathcal{E})$ to $\mathcal{E}'$. If T is not a $dual \circ ui \circ dual$ involving the first row of $\Theta_1(\mathcal{E})$, then there exists $\mathcal{E}'' \sim \mathcal{E}$ such that the following diagram commutes.*

$$\begin{array}{ccc} \Theta_1(\mathcal{E}) & \xrightarrow{T} & \mathcal{E}' = \Theta_1(\mathcal{E}'') \\ \theta\uparrow & & \theta\uparrow \\ \mathcal{E} & \xrightarrow{\quad T \quad} & \mathcal{E}'' \end{array}$$

Proof. Let $h = ([c_{\max}+1, -c_{\max}-1], c_{\max}+1, -\eta(\mathcal{E}))$ be the first row of $\Theta_1(\mathcal{E})$. Note that there is only one possible row operation, a $dual \circ ui \circ dual$ involving h. Therefore, T must involve rows in $\mathcal{E}$. It suffices to prove that such an operation can be implemented regardless of whether h is present. We need to check the four cases:

- If T is of the form U, then Lemma 7.42 implies that such an operation is possible regardless of the presence of h.
- If T is of the form D, then Lemma 7.36 implies that such an operation is possible regardless of the presence of h.
- If T is of form M, then the possibility of M relies purely on the structure of the hats in $\mathcal{E}$ and not on the presence of h.
- If T is of the form S, then Lemma 7.29 implies that this S is possible regardless of the presence of h. □

The above lemma essentially states that almost all raising operators on $\Theta_1(\mathcal{E})$ produce other multi-segments of the form Θ_1 and descend to be operators on $\mathcal{E}$. It is also true that case 3' operators on $\mathcal{E}$ can ascend to $\Theta_1(\mathcal{E})$, as stated in the following lemma.

Lemma 12.5. *Let $\mathcal{E}$ be of type $Y_{\mathcal{M}}$ with $\mathcal{M}$ starting at zero. Suppose there exists a raising operator $T \in \{dual \circ ui \circ dual, ui^{-1}\}$ of type 3' from $\mathcal{E}$ to $\mathcal{E}'$. Then the following diagram commutes.*

$$\begin{array}{ccc} \Theta_1(\mathcal{E}) & \xrightarrow{T} & \Theta_1(\mathcal{E}') \\ \theta\uparrow & & \theta\uparrow \\ \mathcal{E} & \xrightarrow{\quad T \quad} & \mathcal{E}' \end{array}$$

Proof. We have already shown in the proof to Lemma 12.4 that any raising operator T can be implemented regardless of the presence of $h = [c_{\max}+1, -c_{\max}-1], c_{\max}+1, -\eta(\mathcal{E}))$. This suffices to prove the lemma. $\square$

In the next two lemmas, we show that operators can similarly be lifted between Θ_1 and Θ_2.

Lemma 12.6. *Let $\mathcal{E}$ be of type $Y_{\mathcal{M}}$ with $\mathcal{M}$ starting at zero and let $h = ([c_{\max}+1, -c_{\max}-1], c_{\max}+1, -\eta(\mathcal{E}))$ be the first row of $\Theta_1(\mathcal{E})$. Let $T \in \{dual \circ ui \circ dual, ui^{-1}\}$ be a raising operator of type 3' on $\Theta_1(\mathcal{E})$ which is not equal to the $dual \circ ui \circ dual$ that gives $\Theta_2(\mathcal{E})$ (or $\Theta_4(\mathcal{E})$, should it exist). Then the following diagram commutes.*

$$\begin{CD}
\Theta_2(\mathcal{E}) @>T>> \Theta_2(\mathcal{E}') \\
@A{dual\circ ui\circ dual}AA @AA{dual\circ ui\circ dual}A \\
\Theta_1(\mathcal{E}) @>T>> \Theta_1(\mathcal{E}')
\end{CD}$$

Proof. Firstly, we know that there exists $\mathcal{E}'$ such that T goes from $\Theta_1(\mathcal{E})$ to $\Theta_1(\mathcal{E}')$ by Lemma 12.4. Let r be the row that h is combined with in the $dual \circ ui \circ dual$ going from $\Theta_1(\mathcal{E})$ to $\Theta_2(\mathcal{E})$. If T does not involve h or r then the lemma follows from Part (1) of Theorem 12.3. Since there are no other raising operators that involve h, we reduce to the case where T involves r.

If the unique $dual \circ ui \circ dual$ involving h is of the form M and r is a hat, then there are three possibilities for T as follows.

- If T is of the form U, then Part (6) of Theorem 12.3 guarantees the existence of the the following commutative diagram.

$$\begin{CD}
\Theta_2(\mathcal{E}) @>U>> \Theta_2(\mathcal{E}') \\
@A{M}AA @AA{D}A \\
\Theta_1(\mathcal{E}) @>U>> \Theta_1(\mathcal{E}')
\end{CD}$$

- If T is of the form D, then Part (4) of Theorem 12.3 gives the following commutative diagram.

$$\begin{CD}
\Theta_2(\mathcal{E}) @>D>> \Theta_2(\mathcal{E}') \\
@A{M}AA @AA{D}A \\
\Theta_1(\mathcal{E}) @>D>> \Theta_1(\mathcal{E}')
\end{CD}$$

- If T is of the form M, then Part (3) of Theorem 12.3 gives a commutative diagram.

$$\begin{CD}
\Theta_2(\mathcal{E}) @>M>> \Theta_2(\mathcal{E}') \\
@A{M}AA @AA{M}A \\
\Theta_1(\mathcal{E}) @>M>> \Theta_1(\mathcal{E}')
\end{CD}$$

Alternatively, if the unique $dual \circ ui \circ dual$ is of the form D and r is a chain, then the only possible raising operator involving r is one of the form S. Here, Part (8) of Theorem 12.3 gives a commutative diagram.

$$\begin{array}{ccc} \Theta_2(\mathcal{E}) & \xrightarrow{S} & \Theta_2(\mathcal{E}') \\ {\scriptstyle D}\uparrow & & \uparrow{\scriptstyle D} \\ \Theta_1(\mathcal{E}) & \xrightarrow{S} & \Theta_1(\mathcal{E}') \end{array}$$

Having considered all possible operators that interact with the same rows as the $dual \circ ui \circ dual$, we have proved the result. □

Next, we demonstrate the reverse direction.

Lemma 12.7. *Let $\mathcal{E}$ be of type $Y_{\mathcal{M}}$ with $\mathcal{M}$ starting at zero. Let T be a raising operator of type 3' from $\Theta_2(\mathcal{E})$ to $\mathcal{E}'$ which is not equal to the ui^{-1} on $\Theta_2(\mathcal{E})$ which gives $\Theta_3(\mathcal{E})$. Then there exists $\mathcal{E}''$ of type X_n such that the following diagram commutes.*

$$\begin{array}{ccc} \Theta_2(\mathcal{E}) & \xrightarrow{T} & \mathcal{E}' = \Theta_2(\mathcal{E}'') \\ {\scriptstyle dual\circ ui\circ dual}\uparrow & & \uparrow{\scriptstyle dual\circ ui\circ dual} \\ \Theta_1(\mathcal{E}) & \xrightarrow{T} & \Theta_1(\mathcal{E}'') \end{array}$$

Proof. We claim that the following diagram commutes.

$$\begin{array}{ccc} \Theta_2(\mathcal{E}) & \xrightarrow{T} & \mathcal{E}' \\ {\scriptstyle dual\circ ui\circ dual}\uparrow & & \uparrow{\scriptstyle dual\circ ui\circ dual} \\ \Theta_1(\mathcal{E}) & \xrightarrow{T} & T(\Theta_1(\mathcal{E})) \end{array}$$

To see this, let $h = ([c_{\max}+1, -c_{\max}-1], c_{\max}+1, -\eta(\mathcal{E}))$ be the first row of $\Theta_1(\mathcal{E})$, let $r \in \Theta_1(\mathcal{E})$ be the row combined with h through the unique $dual \circ ui \circ dual$, and let $r' \in \Theta_2(\mathcal{E})$ be the image of r. If T does not involve r', then the existence of the commutative diagram follows from Part (1) of Theorem 12.3.If T involves r', then the $dual \circ ui \circ dual$ is either M or D. If M, then r' is a merged hat, and T is either M, D, or U. If D, then r' is a row of circles and T is of the form S. In either case, the existence of the four commutative diagrams used in the proof of Lemma 12.6 verify the existence of the desired diagram.

The operation T is not the $dual \circ ui \circ dual$ operation from $\Theta_1(\mathcal{E})$ to $\Theta_2(\mathcal{E})$. Therefore by Lemma 12.5, it is clear that $T(\Theta_1(\mathcal{E})) = \Theta_1(\mathcal{E}'')$ for some $\mathcal{E}'' \sim \mathcal{E}$. The fact that $(dual \circ ui \circ dual)(\Theta_1(\mathcal{E}'')) = \Theta_2(\mathcal{E}'')$ is evident. □

Next, we verify a similar result for $\Theta_1(\mathcal{E})$ and $\Theta_4(\mathcal{E})$.

Lemma 12.8. *Let $\mathcal{E}$ be of type $Y_{\mathcal{M}}$ with $\mathcal{M}$ beginning at zero, and let $h = ([n+1, -n-1], n+1, -\eta(\mathcal{E}))$ be the first row of $\Theta_1(\mathcal{E})$. Let $T \in \{dual \circ ui \circ dual, ui^{-1}\}$ be a raising operator of type 3' on $\Theta_1(\mathcal{E})$ not equal to the $dual \circ ui \circ dual$(s) going* *to* $\Theta_2(\mathcal{E})$ *or* $\Theta_4(\mathcal{E})$*. Then the following diagram commutes.*

$$\begin{array}{ccc} \Theta_4(\mathcal{E}) & \xrightarrow{T} & \Theta_4(\mathcal{E}') \\ {\scriptstyle D}\uparrow & & \uparrow{\scriptstyle D} \\ \Theta_1(\mathcal{E}) & \xrightarrow{T} & \Theta_1(\mathcal{E}') \end{array}$$

Proof. Again, we know that there exists $\mathcal{E}'$ such that T goes from $\Theta_1(\mathcal{E})$ to $\Theta_1(\mathcal{E}')$ by Lemma 12.4. Let r be the last row of $\Theta_1(\mathcal{E})$. The operator T cannot involve h, so we can reduce to the case where T involves r. Indeed, otherwise, the lemma follows from Part (1) of Theorem 12.3. But r is a multiple, and the definitions of S, M, U, and D; the only type 3' raising operators, imply that they do not involve multiples. □

Conversely, raising operators in the Θ_4 layer can descend to the Θ_1 layer.

Lemma 12.9. *Let* $\mathcal{E}$ *be of type* $Y_{\mathcal{M}}$ *with* $\mathcal{M}$ *beginning at zero and* $m_{c_{\max}} \geq 3$*. Let* T *be a raising operator of type 3' from* $\Theta_4(\mathcal{E})$ *to* $\mathcal{E}'$ *not equal to the* ui^{-1} *from* $\Theta_4(\mathcal{E})$ *to* $\Theta_3(\mathcal{E})$*. Then there exists* $\mathcal{E}''$ *of type* $Y_{\mathcal{M}}$ *such that the following diagram commutes.*

$$\begin{array}{ccc} \Theta_4(\mathcal{E}) & \xrightarrow{T} & \mathcal{E}' = \Theta_4(\mathcal{E}'') \\ {\scriptstyle D}\uparrow & & \uparrow{\scriptstyle D} \\ \Theta_1(\mathcal{E}) & \xrightarrow{T} & \Theta_1(\mathcal{E}'') \end{array}$$

Proof. Let $h = ([c_{\max}+1, -c_{\max}-1], n+1, -\eta(\mathcal{E}))$ be the first row of $\Theta_1(\mathcal{E})$ and let the multiple $r \in \Theta_1(\mathcal{E})$ be the last row. Let r' be the image of r under the dualization: a row of two circles. If T does not involve r', then the existence of the commutative diagram follows from part (1) of Theorem 12.3. The only row operation involving the row of two circles r' is $S_{r,1}$, which we have assumed T is not. □

Finally, we show that most operations T can ascend and descend between Θ_2 and Θ_3.

Lemma 12.10. *Let* $\mathcal{E}$ *be of type* $Y_{\mathcal{M}}$ *with* $\mathcal{M}$ *beginning at zero. Let* $T \in \{dual \circ ui \circ dual, ui^{-1}\}$ *be a raising operator of type 3' on* $\Theta_2(\mathcal{E})$ *not equal to the* ui^{-1} *from* $\Theta_2(\mathcal{E})$ *to* $\Theta_3(\mathcal{E})$*. Then the following commutative diagram commutes.*

$$\begin{array}{ccc} \Theta_3(\mathcal{E}) & \xrightarrow{T} & \Theta_3(\mathcal{E}') \\ {\scriptstyle ui^{-1}}\uparrow & & \uparrow{\scriptstyle ui^{-1}} \\ \Theta_2(\mathcal{E}) & \xrightarrow{T} & \Theta_2(\mathcal{E}') \end{array}$$

Proof. Let r be the row that is split by the ui^{-1} from $\Theta_2(\mathcal{E})$ to $\Theta_3(\mathcal{E})$. If T does not directly involve r, then the result follows from Part (1) of Theorem 12.3. Otherwise, we consider a number of cases.

If the ui^{-1} from $\Theta_2(\mathcal{E})$ to $\Theta_3(\mathcal{E})$ is of the form U and r is a hat, then there are three possibilities for T as follows.

- If T is of the form U, then Part (7) of Theorem 12.3 guarantees the existence of the following commutative diagram.

$$\begin{array}{ccc} \Theta_3(\mathcal{E}) & \xrightarrow{U} & \Theta_3(\mathcal{E}') \\ {\scriptstyle U}\uparrow & & \uparrow{\scriptstyle S} \\ \Theta_2(\mathcal{E}) & \xrightarrow{U} & \Theta_2(\mathcal{E}_2) \end{array}$$

- If T is of the form D, then Part (9) of Theorem 12.3 gives the following commutative diagram.

$$\begin{array}{ccc} \Theta_3(\mathcal{E}) & \xrightarrow{D} & \Theta_3(\mathcal{E}') \\ {\scriptstyle U}\uparrow & & \uparrow{\scriptstyle S} \\ \Theta_2(\mathcal{E}) & \xrightarrow{D} & \Theta_2(\mathcal{E}') \end{array}$$

- If T is of the form M, then Part (5) of Theorem 12.3 gives a commutative diagram.

$$\begin{array}{ccc} \Theta_3(\mathcal{E}) & \xrightarrow{M} & \Theta_3(\mathcal{E}') \\ {\scriptstyle U}\uparrow & & \uparrow{\scriptstyle U} \\ \Theta_2(\mathcal{E}) & \xrightarrow{M} & \Theta_2(\mathcal{E}') \end{array}$$

Alternatively, if the ui^{-1} is of the form S and r is a chain, then the only possible raising operator involving r is one of the form S. Here, Part (2) of Theorem 12.3 gives a commutative diagram.

$$\begin{array}{ccc} \Theta_3(\mathcal{E}) & \xrightarrow{S} & \Theta_3(\mathcal{E}') \\ {\scriptstyle S}\uparrow & & \uparrow{\scriptstyle S} \\ \Theta_2(\mathcal{E}) & \xrightarrow{S} & \Theta_2(\mathcal{E}') \end{array}$$

Having considered all possible operators that interact with the same rows as the ui^{-1} we have proved the result. □

Conversely, all operations on the Θ_3 layer descend to the Θ_2 layer.

Lemma 12.11. *Let $\mathcal{E}$ be of type $Y_{\mathcal{M}}$, with $\mathcal{M}$ starting at zero. Let T be a raising operator of type 3' from $\Theta_3(\mathcal{E})$ to $\mathcal{E}$.' Then there exists $\mathcal{E}''$ of type X_n such that the following diagram commutes.*

$$\begin{array}{ccc} \Theta_3(\mathcal{E}) & \xrightarrow{T} & \mathcal{E}' = \Theta_3(\mathcal{E}'') \\ {\scriptstyle ui^{-1}}\uparrow & & \uparrow{\scriptstyle ui^{-1}} \\ \Theta_2(\mathcal{E}) & \xrightarrow{T} & \Theta_2(\mathcal{E}'') \end{array}$$

Proof. Let $r \in \Theta_2(\mathcal{E}_1)$ affected by the ui^{-1} from $\Theta_2(\mathcal{E})$ to $\Theta_3(\mathcal{E})$, and let r' be its image after the last circle is separated by the ui^{-1}. If T does not involve r', then the existence of the commutative diagram follows from Part (1) of Theorem 12.3.

We may then assume that T involves r'. The ui^{-1} from $\Theta_2(\mathcal{E})$ to $\Theta_3(\mathcal{E})$ is either a U or an S. If U, then r' is a hat and T is either M, D, or D. If S, then r' is a row of circles and T is an S. In either case, the existence of the four commutative diagrams used in the proof of Lemma 12.11 verify the existence of the desired diagram.

Once we know that the diagram commutes, it is clear from Lemma 12.7 that T must go from $\Theta(\mathcal{E})$ to $\Theta_2(\mathcal{E}'')$ for some $\mathcal{E}''$. This then implies that $\mathcal{E}' = \Theta_2(\mathcal{E}'')$, completing the proof. □

Theorem 12.1 now follows directly from the above lemmas.

Remark 12.12. *These commutativity results have practical use apart from providing a conceptual understanding of Theorems 8.4, 8.5, and 9.5. They also provide a mechanism by which one could prove that sets like the ones detailed in this theorem are closed under row operations. In the above proofs of these theorems, we proved this closure property with less difficulty by examining the effect of row operations on $\mathcal{S}$-data. However, cases more general than those considered in this article (e.g., non-tempered and non-anti-tempered) may still involve operations like S, M, U, and D while not admitting a structural description like the $\mathcal{S}$-data. In such cases, results similar to the above eight lemmas and Theorem 12.3 (which may be proved without reference to the $\mathcal{S}$-data) may prove useful.*

13. Appendix: Notation

We use the following conventions. Generally, letters refer to the following.

- A refers to the right endpoint of a row.
- a is $A + B + 1$.
- α refers to $2m - 2n - 1$ in the theta correspondence; with a subscript it can also refer to $\sum_{j<i} a_j$ in the calculation of dual (see Definition 3.11).
- B refers to the left endpoint of a row.
- b is $A - B + 1$
- $\mathcal{B}$ refers to a block (see Definition 4.4).
- $\beta_i = \sum_{j>i} b_j$ in the calculation of dual (see Definition 3.11).
- C denotes the number of circles in a row.
- c is for columns of an extended multi-segment (e.g. c_{max}).
- $D_{h,r}$ denotes a certain $dual \circ ui \circ dual$ operation with rows h and r (see Definition 7.28).
- down refers to the sign for the going-down tower.
- η denotes the sign of a row, and also the sign of an extended multi-segment (Definition 6.2).
- ϵ denotes a sign (e.g. in the calculation of row exchange, union-intersection).
- $\mathcal{E}$ refers to an extended multi-segment.
- $\mathrm{Eseg}(G)$ denotes extended multi-segments of G.
- h refers to hats (see Definition 6.13).
- G without a subscript is $G_n = \mathrm{Sp}_{2n}(F)$ or $H_m^{\pm} = \mathrm{O}_{2m}^{\pm}(F)$.
- i is an index.
- j is an index.
- k is an index.

- l refers to the number of triangles in a row.
- m refers to (half of) the dimension of the space underlying the group $\mathrm{O}_{2m}^{\pm}$.
- $m^{\mathrm{up}}(\pi)$ denotes the first occurrence in the going-up tower. Note that the definition is that the lift is to $H^{\mathrm{up}}_{m^{\mathrm{up}}(\pi)}$, not $H^{\mathrm{up}}_{2m^{\mathrm{up}}(\pi)}$. $m^{\mathrm{down}}(\pi)$ is similar.
- $m^{\alpha,\mathrm{up}}(\pi) = m^{\mathrm{up}}(\pi) - 2n - 1$ is the corresponding odd integer for the theta lift to the going-up tower at the first occurrence.
- m_c denotes the multiplicity of a circle in column c.
- $M_{h,r}$ denotes a certain $dual \circ ui \circ dual$ operation with rows h and r (see Definition 7.28).
- n refers to (half) the dimension of the underlying space of the group Sp_{2n}.
- (P) and (P') are conditions on the order of an extended multi-segment.
- r denotes a row.
- $\mathrm{Rep}(G)$ denotes the subset of $\mathcal{E} \in \mathrm{Eseg}(G)$ such that $\pi(\mathcal{E}) \neq 0$.
- s denotes a row.
- $S_{r,k}$ denotes a certain ui^{-1} operator on row r (see Definition 7.28).
- $\mathcal{S} = (\mathcal{S}_i)$ is the data of an extended multi-segment equivalent to a block.
- T refers to an operation on an extended multi-segment.
- $\theta^{\pm}_{-\alpha}(\pi)$ denotes the theta lift of $\pi \in \Pi(G_n)$ to $H^{\pm}_m$ where $2m = \alpha + 2n + 1$.
- up refers to the sign for the going-up tower.
- $U_{h,k}$ denotes a certain ui^{-1} operation on row h (see Definition 7.28).
- $\mathrm{Vseg}(G)$ denotes virtual extended multi-segments, i.e., we drop the sign condition.
- $\mathrm{Vseg}^{\mathbb{Z}}(G)$ denotes the subset of $\mathrm{Vseg}(G)$ for which every $A_i \in \mathbb{Z}$.
- $\mathrm{VRep}(G)$ denotes the subset of $\mathrm{Vseg}(G)$ which satisfies the nonvanishing conditions for each ρ.
- $\mathrm{VRep}^{\mathbb{Z}}(G)$ denotes $\mathrm{VRep}(G) \cap \mathrm{Vseg}^{\mathbb{Z}}(G)$.
- X_n is an extended multi-segment of a certain form.
- Y_n is an extended multi-segment of a certain form.
- Z_{k_1,k_2} is an extended multi-segment of a certain form.
- z-chains are rows of a certain form.

We also have the following other conventions.

- Extended multi-segments and related objects are in `\mathcal{}` font.
- r' usually denotes the image of r after some row operation is applied.
- $\widehat{r}$ denotes the image of a row r after the dual operation is applied.

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