REALIZABILITY UNDER LINEAR ISOMETRIES OPERADS OF SATURATED TRANSFER SYSTEMS FOR CYCLIC GROUPS

GAVIN BELL, COLIN FEENEY, ETHAN HARR

FACULTY MENTOR: BAR ROYTMAN

ABSTRACT. In this paper, we take steps towards tackling one of the oldest problems in one of mathematics's youngest fields: Rubin's Saturation Conjecture [Rub21]. Namely, we provide a new proof for the realizability of saturated transfer systems of odd order cyclic groups (originally proven by [Mac] and [CGMR]), attacking from a number theoretic perspective, which aids in the discovery of new results in the case of even order cyclic groups.

1. PRELIMINARIES

Let G be a finite group, $\leq$ denote the subgroup relation, and let $\mathrm{Sub}(G)$ denote the subgroup lattice of G. We begin with the definition of a G-transfer system, initially given by [Rub21]:

Definition 1.1 (Transfer System)**.** A *G-transfer system* $\mathcal{R}$ is a binary relation $\to_{\mathcal{R}}$ on $\mathrm{Sub}(G)$ satisfying the following properties

(1) Reflexivity: $\forall K \leq K, K \to_{\mathcal{R}} K$
(2) Transitivity: $L \to_{\mathcal{R}} K$ and $K \to_{\mathcal{R}} H \implies L \to_{\mathcal{R}} H$
(3) Refinement: $K \to_{\mathcal{R}} H \implies K \leq H$
(4) Conjugation: $K \to_{\mathcal{R}} H \implies \forall g \in G$, $gKg^{-1} \to_{\mathcal{R}} gHg^{-1}$
(5) Restriction: $K \to_{\mathcal{R}} H \implies \forall L \leq H, (L \cap K) \to_{\mathcal{R}} L$

We say that $\mathcal{R}$ is *saturated* when it additionally satisfies:

(6) Saturation: $K \to_{\mathcal{R}} H \implies \forall K \leq L \leq H, L \to_{\mathcal{R}} H$

As our notation suggests, it is often helpful to depict a G-transfer system as a set of arrows on $\mathrm{Sub}(G)$, omitting reflexive arrows for brevity.

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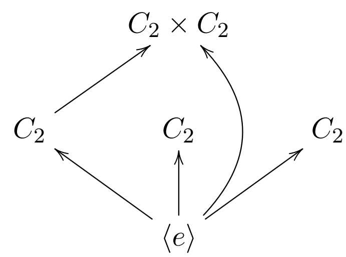
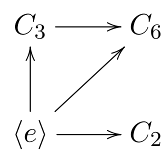
Example 1.2. *Some examples of transfer systems include*





Note that neither of the above transfer systems are saturated. Were we to close them under the saturation condition, both would become there respective complete transfer systems. The transfer system in which $K \to_{\mathcal{R}} H$ whenever $K \leq H$. In what follows, we shall largely restrict our attention to saturated transfer systems.

Observe that any saturated G-transfer system $\mathcal{R}$ is uniquely determined by its arrows $K \to_{\mathcal{R}} H$ where there is no intermediary subgroup $K \lneq L \lneq H$. Given these "covering" arrows, and utilizing the restriction and saturation axioms, any other arrow can be decomposed as the transitive closure of these smaller arrows.

When dealing with G-transfer systems, it is useful to identify subgroups which have no nontrivial (i.e. non-reflexive) arrows to them:

Definition 1.3 (Cofibrant Subgroup)**.** Let $\mathcal{R}$ be a G-transfer system, and $H \leq G$. We say H is *$\mathcal{R}$-cofibrant* provided that for any $K \leq G$, $K \to_{\mathcal{R}} H \implies K = H$.

Given a G-transfer system $\mathcal{R}$, let $\mathcal{C}_{\mathcal{R}}$ denote the set of $\mathcal{R}$-cofibrant subgroups. Note that we always have $\langle e \rangle \in \mathcal{C}_{\mathcal{R}}$.

Lemma 1.4 ([CGMR])**.** *For any G-transfer system $\mathcal{R}$, $\mathcal{C}_{\mathcal{R}} \subseteq \mathrm{Sub}(G)$ is closed under joins and conjugation.*

Proof. Let $H_1, H_2 \in \mathcal{C}_{\mathcal{R}}$, and suppose $\langle H_1, H_2 \rangle \notin \mathcal{C}_{\mathcal{R}}$, where $\langle H_1, H_2 \rangle$ denotes the subgroup generated by H_1 and H_2. Necessarily, there is some $K \lneq \langle H_1, H_2 \rangle$ with $K \to_{\mathcal{R}} \langle H_1, H_2 \rangle$. By the restriction axiom $K \cap H_1 \to_{\mathcal{R}} H_1$, so we must have $H_1 \leq K$ since $H_1 \in \mathcal{C}_{\mathcal{R}}$. But because $K \lneq \langle H_1, H_2 \rangle$ this means that $H_2 \nleq K$. However, the restriction axiom forces $K \cap H_2 \to_{\mathcal{R}} H_2$, contradicting $H_2 \in \mathcal{C}_{\mathcal{R}}$. Closure under conjugation follows immediately from the conjugation axiom. □

In the case of saturated transfer systems, the significance of these subgroups becomes apparent:

Theorem 1.5 ([Mac], Lemma 2.9)**.** *If $\mathcal{R}$ and $\mathcal{R}'$ are transfer systems with $\mathcal{R}'$ saturated, then $\mathcal{R} \subseteq \mathcal{R}'$ if and only if every $\mathcal{R}'$-cofibrant subgroup is $\mathcal{R}$-cofibrant.*

Corollary 1.6. *A saturated transfer system is uniquely determined by its cofibrant subgroups.*

Theorem 1.7 ([CGMR])**.** *Let $\mathcal{C} \subseteq \mathrm{Sub}(G)$ such that $\mathcal{C}$ is closed under joins and conjugation, and $\langle e \rangle \in \mathcal{C}$. Then there is some saturated G-transfer system $\mathcal{R}$ such that $\mathcal{C} = \mathcal{C}_{\mathcal{R}}$.*

Let $\hat{G}$ denote the set of isomorphism classes of irreducible complex G subrepresentations.

Definition 1.8 (G-Universe)**.** A (complex) *G-universe* $\mathcal{U}$ is a complex representation, with $\mathcal{U} \cong \mathcal{U} \oplus \mathcal{U}$, which contains the trivial representation as a subrepresentation, and is a union of its finite dimensional G-representations.

Since, up to isomorphism, a G-universe $\mathcal{U}$ depends only on its irreducible complex G-subrepresentations, we can think of $\mathcal{U}$ as simply some subset of $\hat{G}$ containing the isomorphism class of the trivial representation.

In order to describe the linear isometries transfer system for some G-universe, we define the following functors:

Definition 1.9 (Restricted Universe)**.** Let $\mathcal{U}$ be a complex G-universe with $H \leq G$. We define the complex H-universe $\mathrm{Res}^G_H(\mathcal{U})$ to be the set of isomorphism classes of irreducible subrepresentations of $\mathrm{Res}^G_H(V)$ where $V \in \mathcal{U}$.

Definition 1.10 (Induced Universe)**.** Let $\mathcal{U}$ be complex H-universe with $H \leq G$. We define the complex G-universe $\mathrm{Ind}^G_H(\mathcal{U})$ to be set of isomorphism classes of irreducible subrepresentations of $\mathrm{Ind}^G_H(V)$ with $V \in \mathcal{U}$.

It is an easy exercise to verify that these operations on universes satisfy properties analogous to the ordinary Ind and Res operations.

Definition 1.11 (Linear Isometries Transfer System.)**.** For a G universe $\mathcal{U}$ we define a transfer system $\mathcal{L}(\mathcal{U})$ by:

$$K \to_{\mathcal{L}(\mathcal{U})} H \iff \mathrm{Ind}^H_K(\mathrm{Res}^G_K(\mathcal{U})) \subseteq \mathrm{Res}^G_H(\mathcal{U}).$$

The fact that this gives a valid transfer system was initially shown by [BH15]. Although the definition of a transfer system had not been formalized until [Rub21], Blumberg and Hill showed that the above satisfied all of Rubin's axioms.

In what follows, we shall restrict our attention to C_n-transfer systems, where C_n is the cyclic group of order n. In particular, we investigate which C_n-transfer systems are $\mathcal{L}(\mathcal{U})$ for some C_n.

Consider C_n with some fixed generator g. Since C_n is abelian, we can identify the isomorphism classes of irreducible complex representations of C_n with the characters of g, which are the nth roots of unity ζ_n^i, $i \in \{0, ..., n-1\}$. As such, we can take a C_n universe to be some subset $\mathcal{U} \subseteq \{1, \zeta_n, \zeta_n^2, ..., \zeta_n^{n-1}\}$ by identifying ζ_n^i with the irreducible complex representation sending g to ζ_n^i. To streamline this notation across subgroups of C_n, we will implicitly take the chosen generator of any $C_m \leq C_n$ to be $g^{\frac{n}{m}}$. For a C_n-universe $\mathcal{U}$ we can then write

$$\mathrm{Res}_{C_m}^{C_n}(\mathcal{U}) = \{\zeta_m^i \mid \zeta_n^i \in \mathcal{U}\}.$$

Using Frobenius Reciprocity, a similar expression holds for Ind: if $\mathcal{U}'$ is some C_m-universe, then

$$\mathrm{Ind}_{C_m}^{C_n}(\mathcal{U}') = \{\zeta_n^i \mid \zeta_m^i \in \mathcal{U}'\}.$$

Lemma 1.12. *Given some C_n-universe $\mathcal{U}$ and $C_k \leq C_m \leq C_n$, we see that $C_k \to_{\mathcal{L}(\mathcal{U})} C_m$ if and only if*

$$\{\zeta_m^i \mid \text{there exists } \zeta_n^d \in \mathcal{U} \text{ with } d \equiv i \bmod k\} \subseteq \{\zeta_m^j \mid \zeta_n^j \in \mathcal{U}\}.$$

Since the reverse inclusion always holds, we have that a subgroup $C_m \leq C_n$ is cofibrant if and only if for all k properly dividing m, we have that

$$\{\zeta_m^i \mid \text{there exists } \zeta_n^d \in \mathcal{U} \text{ with } d \equiv i \bmod k\} \supsetneq \{\zeta_m^j \mid \zeta_n^j \in \mathcal{U}\}.$$

In terms of modular arithmetic, we can recast this condition as follows: $C_k \to_{\mathcal{L}(\mathcal{U})} C_m$ if and only if for all $0 \leq r < m$, one of the following two conditions hold:

(1) There exists some $j \in \mathcal{U}$ such that $j \equiv r \mod m$.
(2) For all $i \in \mathcal{U}$, $i \not\equiv r \mod k$.

We now give some known results regarding the realizability of C_n-transfer systems

Theorem 1.13 ([Mac], Theorem 3.5)**.** *If G is a cyclic group with order coprime to 6, then all saturated G-transfer systems are $\mathcal{L}(\mathcal{U})$ for some real G-universe $\mathcal{U}$.*

This result was improved in [CGMR], by expanding to consider complex representations we have

Theorem 1.14 ([CGMR])**.** *If G is a cyclic group with odd order, then all saturated G-transfer systems are $\mathcal{L}(\mathcal{U})$ for some (complex) G-universe $\mathcal{U}$.*

Furthermore [CGMR] conjectured the following, and proved that should the conjecture hold, it would extend to a complete characterization of the realizability problem for cyclic groups.

Conjecture 1.15 ([CGMR])**.** *Let $\mathcal{R}$ be a saturated G-transfer system for cyclic group G of even order. Then if $C_2 \leq G$ is cofibrant, $\mathcal{R} = \mathcal{L}(\mathcal{U})$ for some G-universe $\mathcal{U}$*

Theorem 1.16 ([CGMR])**.** *Let $\mathcal{R}$ be a saturated $C_{2^k m}$-transfer system, where m is odd. Suppose $\mathcal{R} = \mathcal{L}(\mathcal{U})$ for some $C_{2^k m}$-universe $\mathcal{U}$. If ℓ is minimal such that C_{2^ℓ} is not cofibrant. Then the ℓ first C_m-transfer systems layers are identical. If no such ℓ exists, then all of the C_m transfer system layers are identical.*

Using the above theorem, [CGMR] proved the equivalency of 1.15 to the following

Conjecture 1.17. *Let $\mathcal{R}$ be a saturated $C_{2^k m}$-transfer system where m is odd. $\mathcal{R} = \mathcal{L}(\mathcal{U})$ for some $C_{2^k m}$-universe $\mathcal{U}$ **if and only if** the first ℓ C_m-transfer system layers are identical, where ℓ is minimal such that C_{2^ℓ} is not cofibrant, and if no such minimal ℓ exists, we instead require all layers be identical.*

2. Combinatorics of Transfer Systems

Before we begin to realize transfer systems, it is sensible to ask if there are enough C_n-universes to realize every saturated C_n-transfer system. There are, of course, 2^{n-1} C_n-universes. In this section, we provide some results regarding the enumerative Combinatorics of transfer systems.

Theorem 2.1 ([SBR21], Theorem 20)**.** *The number of C_{p^n} transfer systems (with p prime) is counted by the $n+1$th Catalan number:* Cat_{n+1}.

Proof. We first will show that the number of C_{p^n} transfer systems T_n satisfies the Catalan-like recurrence.

$$T_n = \sum_{i=1}^{n} T_{i-1} \cdot T_{n-i}$$

For each transfer system $\mathcal{R}$ let k be minimal such that $C_{p^{k+1}}$ is cofibrant, if there exists no such k we say $k = n$. We mustn't have any arrow $K \to_{\mathcal{R}} H$ where $K \leq C_{p^{k+1}} \leq H$, as after restriction we would receive $K \to_{\mathcal{R}} C_{p^{k+1}}$ contradicting its cofibrancy. $\mathcal{R}$ can now be decomposed into two transfer systems. A $C_{p^n}/C_{p^{k+1}}$-transfer system rooted at $C_{p^{k+1}}$ and a C_{p^k} transfer system rooted at the trivial subgroup. In fact, these can be chosen completely independently!

Before we continue, we need two lemmas:

Lemma 2.2. *In a C_p^n transfer system with k defined as above, $\langle e \rangle \to_{\mathcal{R}} C_{p^k}$.*

Proof of lemma. Suppose for contradiction $\langle e \rangle \not\to_{\mathcal{R}} C_{p^k}$. Let $m < k$ be maximal such that $\langle e \rangle \to_{\mathcal{R}} C_{p^m}$. By minimality of k we know that $C_{p^{m+1}}$ is not cofibrant. We therefore have some $0 < s \leq m$ such that $C_{p^s} \to_{\mathcal{R}} C_{p^{m+1}}$. However, our arrow connecting $\langle e \rangle$ to C_{p^m} restricts to an arrow $\langle e \rangle \to_{\mathcal{R}} C_{p^s}$. By transitivity we have $\langle e \rangle \to_{\mathcal{R}} C_{p^s} \to_{\mathcal{R}} C_{p^{m+1}}$, and therefore $\langle e \rangle \to_{\mathcal{R}} C_{p^{m+1}}$ contradicting m's maximality. We must have $\langle e \rangle \to_{\mathcal{R}} C_{p^k}$. □

Lemma 2.3. *There are exactly T_{k-1} C_{p^k} transfer systems $\mathcal{R}$, which have $\langle e \rangle \to_{\mathcal{R}} C_{p^k}$.*

Proof of lemma. We construct a bijection f between C_{p^k}-Transfer systems of the above kind, and $C_{p^{k-1}}$ Transfer systems.
Let $\mathcal{R}$ be a C_{p^k}-transfer system. We define $f(\mathcal{R})$ as the $C_{p^{k-1}}$ Transfer system whose arrows are given by

$$C_{p^i} \to_{f(\mathcal{R})} C_{p^j} \iff C_{p^{i+1}} \to_{\mathcal{R}} C_{p^{j+1}}$$

where we simply forget all arrows of $\mathcal{R}$ which originate at the trivial subgroup.

This map is injective. If $f(\mathcal{R}_1) = f(\mathcal{R}_2)$ then $\mathcal{R}_1$ and $\mathcal{R}_2$ can differ only in arrows originating at the trivial subgroup, however because $\langle e\rangle \to_{R_1} C_{p^k}$ and $\langle e\rangle \to_{R_2} C_{p^k}$ both transfer systems have *all* arrows which originate at the trivial subgroup and therefore $\mathcal{R}_1 = \mathcal{R}_2$.

f is also surjective. Fix a $C_{p^{k-1}}$ transfer system $\mathcal{R}$. The transfer system generated under restriction by $\mathcal{R}$ rooted at C_p and all arrows originating at $\langle e\rangle$ is an inverse of $\mathcal{R}$. This is because the arrows added all restrict to arrows only from $\langle e\rangle$. Our piece rooted at C_p remains unchanged and therefore applying f to this new transfer system will output $\mathcal{R}$ □

We now return to the proof of the theorem. Thanks to the two preceding lemmas, we know that for a fixed k, the number of transfer systems with $k+1$ minimally cofibrant is $T_{k-1} \cdot T_{n-k}$ summing over all disjoint possibilities of k grants

$$T_n = \sum_{i=0}^{n} T_{n-1} \cdot T_{n-k}$$

Which after a change of indices is identical to the formula

$$\mathrm{Cat}_{n+1} = \sum_{i=1}^{n+1} \mathrm{Cat}_{i-1} \cdot \mathrm{Cat}_{n-i}$$

Finally it is clear that $T_0 = 1 = \mathrm{Cat}_1$. □

Unfortunately, counting the transfer systems even involving only two primes (i.e. $C_{p^n q^m}$) is much more challenging. Perhaps imposing the saturation condition will ease this pain.

Proposition 2.4 ([SBR21])**.** *The number of saturated C_{p^n}-transfer systems is 2^n*

Proof. This is simply a corollary of 1.6 and 1.7. We create a bijection between saturated C_{p^n} transfer systems, and subsets of the nontrivial subgroups of C_{p^n}, simply mapping each transfer system to its set of nontrivial cofibrant subgroups. Because all we require is that our set of cofibrant subgroups be closed under meets and contain $\langle e\rangle$, and the fact that $\mathrm{Sub}(C_{p^n})$ is a chain poset, all such subsets are valid sets of cofibrant subgroups.

□

In the saturated case, we are able to count the $C_{p^m q^n}$ case. With the aid of a computer program written to count "by hand," we evaluated the number of saturated $C_{p^n q^m}$ transfer systems (a number which we call $\Gamma(m+1, n+1)$) for some small values of m and n. The results of which are documented in Table 1.

$\Gamma(m,n)$	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
1	1	2	4	8	16	32	64	128	256
2	2	7	23	73	227	697	2123	6433	19427
3	4	23	115	533	2359	10133	42655	177053	727639
4	8	73	533	3451	20753	118843	657833	3553651	18856433
5	16	227	2359	20753	164731	1220657	8617219	58718873	389652571
6	32	697	10133	118843	1220657	11467387	101114633	850870003	6910640657
7	64	2123	42655	657833	8617219	101114633	1096832395	11223603953	109840403299
8	128	6433	177053	3553651	58718873	850870003	11223603953	138027417451	1608470722553
9	256	19427	727639	18856433	389652571	6910640657	109840403299	1608470722553	22111390122811
10	512	58537	2969093	98725963	2533853537	54607433227	1038159911993	17967115841923	289166888374337

TABLE 1. $\Gamma(m,n)$ computed by brute force.

Recalling that we do not care about arrows longer than between subgroups with no intermediates, the algorithm first works by fixing stacks of horizontal arrows as below

$$\begin{array}{ccccccccc}
\cdot & \longrightarrow & \cdot & & \cdot & & \cdot & & C_{p^4q^2} \\
\\
\cdot & \longrightarrow & \cdot & & \cdot & & \cdot & \longrightarrow & \cdot \\
\\
\langle e\rangle & \longrightarrow & \cdot & \longrightarrow & \cdot & & \cdot & \longrightarrow & \cdot
\end{array}$$

These stacks can be chosen independently from each other. For each of these choices of stacks, the algorithm iterates through the possible vertical arrows. Noting that in the above example an arrow $C_q \to_{\mathcal{R}} C_{q^2}$ forces $C_{pq} \to_{\mathcal{R}} C_{pq^2}$ by saturation, and further, there can be no arrows $C_{p^a} \to_{\mathcal{R}} C_{p^a q}$ for $a \geq 2$ this is because any such arrow would restrict to the arrow $C_{p^2} \to_{\mathcal{R}} C_{p^2 q}$ which by saturation further forces $C_{pq} \to_{\mathcal{R}} C_{p^2 q}$ which violates the fact that our horizontal stack were pre-fixed. The program counts all combinations of these vertical arrows for each set of horizontal arrows.

At first, the values in the table look without a pattern (besides the cases where $m, n = 1$). However, upon closer inspection, a pattern can be found in the difference sequences along a row or column. In fact it is possible to fit a degree $n+1$ linear recurrence to the difference sequence along row n. If we now consider the closed forms of these recurrences we see

$$\begin{aligned}
300 \cdot 6^n - 480 \cdot 5^n + 225 \cdot 4^n - 30 \cdot 3^n + \frac{1}{2} \cdot 2^n &= \Gamma(5, n+1) - \Gamma(5, n) \\
48 \cdot 5^n - 54 \cdot 4^n + 14 \cdot 3^n - \frac{1}{2} \cdot 2^n &= \Gamma(4, n+1) - \Gamma(4, n) \\
9 \cdot 4^n - 6 \cdot 3^n + \frac{1}{2} \cdot 2^n &= \Gamma(3, n+1) - \Gamma(3, n) \\
2 \cdot 3^n - \frac{1}{2} \cdot 2^n &= \Gamma(2, n+1) - \Gamma(2, n) \\
\frac{1}{2} \cdot 2^n &= \Gamma(1, n+1) - \Gamma(1, n)
\end{aligned}$$

Now we are cooking with gas! Besides the obvious pattern in the exponential bases, some patterns emerge in the coefficients of these recurrences. The coefficients in each equation are simply a constant multiple of a corresponding row of Stirling numbers of the second kind. In fact, such a pattern continues for the remaining values which we were able to calculate. Assuming this pattern holds, we can use this to compute $\Gamma(n, m)$ as a running total of the above differences. Such an assumption implies the formula.

$$\Gamma(m,n) = 2^{m-1} + \sum_{j=1}^{n-1}\left(\frac{1}{2}(-1)^{m+1}2^j + \sum_{i=2}^{m}\left\{{m \atop i}\right\}\frac{i\cdot i!}{2}(i+1)^j(-1)^{i+m}\right)$$

where $\left\{{a \atop b}\right\}$ denotes Stirling numbers of the second kind.

Proposition 2.5. *The above sum simplifies to*

$$\Gamma(m,n) = \frac{1}{2}\sum_{i=1}^{m}\left\{{m \atop i}\right\} i!(-1)^{i+m}(i+1)^n$$

Proof.

$$\Gamma(m,n) = 2^{m-1} + \sum_{j=1}^{n-1}\left(\frac{1}{2}(-1)^{m+1}2^j + \sum_{i=2}^{m}\left\{{m \atop i}\right\}\frac{i\cdot i!}{2}(i+1)^j(-1)^{i+m}\right)$$

To start, we can recognize the first expression in the outer sum as the $i = 1$ term of the inner sum:

$$= 2^{m-1} + \sum_{j=1}^{n-1}\sum_{i=1}^{m}\left\{{m \atop i}\right\}\frac{i!\cdot i}{2}(i+1)^j(-1)^{i+m}$$

Swapping the order of the sums, we see that j only appears in $(i+1)^j$, allowing us to simplify the inner sum.

$$= 2^{m-1} + \sum_{i=1}^{m}\sum_{j=1}^{n-1}\left\{{m \atop i}\right\}\frac{i!\cdot i}{2}(i+1)^j(-1)^{i+m}$$

$$= 2^{m-1} + \sum_{i=1}^{m}\left(\left\{{m \atop i}\right\}\frac{i!\cdot i}{2}(-1)^{i+m}\sum_{j=1}^{n-1}(i+1)^j\right)$$

Denote the inner sum by $S = \sum_{j=1}^{n-1}(i+1)^j$. After observing that

$$(i+1)S - S = \sum_{j=1}^{n-1}(i+1)^{j+1} - (i+1)^j$$

is telescoping, we see that $S((i+1)-1) = (i+1)^n - (i+1)$. Hence $S = \frac{(i+1)^n-(i+1)}{i}$ Substituting back into the original expression

$$\Gamma(m,n) = 2^{m-1} + \sum_{i=1}^{m} \left\{ {m \atop i} \right\} \frac{i! \cdot i}{2} (-1)^{i+m} \frac{(i+1)^n - (i+1)}{i}$$

Note that n appears only once in the sum, and we can isolate it into a separate sum.

$$= 2^{m-1} + \frac{1}{2} \sum_{i=1}^{m} \left\{ {m \atop i} \right\} i!(-1)^{i+m}(i+1)^n - \frac{1}{2} \sum_{i=1}^{m} \left\{ {m \atop i} \right\} i!(-1)^{i+m}(i+1)$$

We first look to simplify $\sum_{i=1}^{m} \left\{ {m \atop i} \right\} i!(-1)^{i+m}(i+1)$:

$$\sum_{i=1}^{m} \left\{ {m \atop i} \right\} i!(-1)^{i+m}(i+1) = (-1)^m \sum_{i=0}^{m} \left\{ {m \atop i} \right\} i!(-1)^{i}(i+1).$$

Recall that the Stirling numbers of the second kind have an exponential generating function

$$\sum_{m=j}^{\infty} \left\{ {m \atop j} \right\} \frac{x^m}{m!} = \frac{(e^x-1)^j}{j!},$$

and so

$$\sum_{m=j}^{\infty} \left\{ {m \atop j} \right\} j! \frac{x^m}{m!} = (e^x-1)^j.$$

This expression lets us find an exponential generating function for $\sum_{i=0}^{m} \left\{ {m \atop i} \right\} i!(-1)^i(i+1)$:

$$f(x) := \sum_{m=0}^{\infty} \left(\sum_{i=0}^{m} \left\{ {m \atop i} \right\} i!(-1)^i(i+1) \right) \frac{x^m}{m!} = \sum_{i=0}^{\infty} \sum_{m=i}^{\infty} \left\{ {m \atop i} \right\} i!(-1)^i(i+1) \frac{x^m}{m!}$$

$$= \sum_{i=0}^{\infty} (-1)^i(i+1) \left(\sum_{m=i}^{\infty} \left\{ {m \atop i} \right\} i! \frac{x^m}{m!} \right) = \sum_{i=0}^{\infty} (-1)^i(i+1)(e^x-1)^i$$

$$= \sum_{i=0}^{\infty} (i+1)(1-e^x)^i$$

Let $u = 1 - e^x$, so

$$f(x) = \sum_{i=0}^{\infty} (i+1)u^i = \sum_{i=0}^{\infty} \frac{d}{du} \left[u^{i+1} \right] = \frac{d}{du} \left[\sum_{i=0}^{\infty} u^{i+1} \right] = \frac{d}{du} \left[\frac{u}{1-u} \right]$$

$$= \frac{1}{(1-u)^2} = \frac{1}{e^{2x}}$$

by the quotient rule. Using the Taylor series expansion for $\frac{1}{e^{2x}}$, we see that

$$f(x) = \sum_{m=0}^{\infty} \frac{(-2x)^m}{m!} = \sum_{m=0}^{\infty} (-2)^m \frac{x^m}{m!}$$

But then

$$\sum_{m=0}^{\infty} \left(\sum_{i=0}^{m} \left\{ {m \atop i} \right\} i!(-1)^i(i+1) \right) \frac{x^m}{m!} = \sum_{m=0}^{\infty} (-2)^m \frac{x^m}{m!}$$

and so we can conclude $\sum_{i=0}^{m} \left\{ {m \atop i} \right\} i!(-1)^i(i+1) = (-2)^m$

Returning to our original expression, we get the desired formula

$$\Gamma(m,n) = 2^{m-1} + \frac{1}{2} \sum_{i=1}^{m} \left\{ {m \atop i} \right\} i!(-1)^{i+m}(i+1)^n - \frac{1}{2}(-1)^m(-2)^m$$

$$= \frac{1}{2} \sum_{i=1}^{m} \left\{ {m \atop i} \right\} i!(-1)^{i+m}(i+1)^n$$

□

An equivalent sum was originally found and proven to hold in general by [HMOO22].[1]

While the algorithm we created for brute force counting, *could* be expanded to counting saturated transfer systems of the type $C_{p^n q^m r^\ell}$ which could then be examined using similar methods to try and uncover a closed form solution, it would take significantly more computational power to compute enough data to analyze in this fashion.

3. Realizability of Saturated Transfer Systems of Odd Order Cyclic Groups

Recall Theorem 1.14. In this section, we develop the tools required to present an alternative proof of this theorem. These tools also begin to reveal new information about even order cyclic groups. In line with lemma 1.12 after identifying the n-th roots of unity we retrieve the further equivalent definition for cofibrancy in terms of modular arithmetic presented below.

Lemma 3.1. *$C_m \leq C_n$ is cofibrant if and only if for all k properly dividing m, there exists some $0 \leq r < m$ such that*

- *For all $j \in \mathcal{U}$, $j \not\equiv r \mod m$.*
- *There exists some $i \in \mathcal{U}$ with $i \equiv r \mod k$.*

[1] It should be noted that this grows significantly slower than the number of $C_{p^n q^m}$-universes, $2^{(p^n q^m - 1)}$, and gives us no immediate counting argument as for why certain transfer systems should not be realizable as $\mathcal{L}(\mathcal{U})$

Note that here, r can depend on the proper divisor k. An initial question one might ask is when there is some r satisfying the above condition for all proper divisors k of m. The answer is rather straightforward:

Theorem 3.2. *Suppose that $\mathcal{C}$ is the set of cofibrant subgroups for some saturated C_n transfer system $\mathcal{L}(\mathcal{U})$, where $\mathcal{U}$ is a C_n-universe. Then for all $C_m \in \mathcal{C}$ with $m > 1$, we have that either*

(1) *$m = \operatorname{lcm}(a, b)$ for $a, b < m$ with $C_a, C_b \in \mathcal{C}$*
(2) *There exists some $0 \leq d < m$ satisfying both:*
  (a) *For all $u \in \mathcal{U}$, $u \not\equiv d \mod m$*
  (b) *For all k properly dividing m, there exists $v \in \mathcal{U}$ with $v \equiv d \mod k$*

Proof. Assume that $m \neq \operatorname{lcm}(a, b)$ for any $C_a, C_b \in \mathcal{C}$ with $a, b < m$. We will show that (2) must hold. Let

$$a = \max(\{k \text{ properly dividing } m \mid C_k \in \mathcal{C}\}).$$

We know such an a exists since $m > 1$ and $C_1 \in \mathcal{C}$. By 3.1, we have that there exists some $0 \leq d < m$ such that both for all $u \in \mathcal{U}$, $u \not\equiv d \mod m$, and there exists some $v_a \in \mathcal{U}$ with $v_a \equiv d \mod a$. We will show that this particular d will work for all proper divisors k of m. Suppose for contradiction that there exists some k properly dividing m such that for all $u \in \mathcal{U}$, $u \not\equiv d \mod k$. Let

$$b = \min(\{k \text{ properly dividing } m \mid \forall u \in \mathcal{U}, u \not\equiv d \mod k\}).$$

Since b is minimal, we have by 3.1 that d satisfies the requisite conditions for the cofibrancy of C_b, and so $C_b \in \mathcal{C}$. We cannot have $b \mid a$, since then $v_a \equiv d \mod a$ would imply $v_a \equiv d \mod b$; hence $a < \operatorname{lcm}(a, b)$. By 1.4 both $C_a, C_b \in \mathcal{C}$, $C_{\operatorname{lcm}(a,b)} \in \mathcal{C}$. Note that since both a and b properly divide m, we see that $a, b < m$ and $\operatorname{lcm}(a, b) \mid m$. But by the maximality of a, we necessarily have that $\operatorname{lcm}(a, b) = m$, a contradiction with our original assumption. So for all k properly dividing m, there is some $v \in \mathcal{U}$ with $v \equiv d \mod k$. □

These results motivated a variety of approaches and algorithms involving taking the universe $\mathcal{U}_0 := \{0, 1, ..., n-1\}$ and removing cosets of cofibrant subgroups. Unfortunately, in these primitive algorithms, the union of these cosets would occasionally cover some coset of another subgroup that was not cofibrant. After computational verification revealed this, leading to such a subgroup becoming cofibrant, many such algorithms were discarded. The following algorithm, presented in Theorem 3.4 was finally found.

Let $\mathcal{R}$ be a saturated C_n-transfer system for $n > 0$ and let $\mathcal{C}$ denote the set of $\mathcal{R}$-cofibrant subgroups. In light of Corollary 1.6, in order to prove the realizability of $\mathcal{R}$ by the linear isometries operad, it suffices to find a universe $\mathcal{U}$ such that the set of $\mathcal{L}(\mathcal{U})$-cofibrant subgroups is exactly the same as $\mathcal{C}$. As such, in our proof of realizability in the odd case, we explicitly construct a universe where the above condition holds for C_m if and only if $C_m \in \mathcal{C}$.

How do we ensure that these cosets of $C_m \in \mathcal{C}$ which we schedule for removal do not accidentally form a covering set for some coset of C_k which is not in $\mathcal{C}$? Since removing an entire coset of C_k could cause C_k to be cofibrant, we must enforce some additional structure on these cosets C_m.

We seek to develop a reservation function, which for each subgroup C_m (regardless of cofibrancy), reserves one of its cosets $d_m C_m$, such that, if any element of this special coset is present, then the subgroup is not cofibrant, and vice versa. We do this by assigning to each subgroup a representative, which we call d_m, which we specially choose to ensure lies in no other subgroups reserved coset. We can then use these special elements to determine the cofibrancy of all subgroups, and make sure that no evil conspiracies ruin things.

Notation 3.3. *For $a \in \{0, ..., n-1\}$ and b dividing n we denote the equivalence class of a modulo b as $[a]_b$*

Theorem 3.4 (Realizability in the Odd Case)**.** *Let $n > 0$ be odd. Then every saturated C_n-transfer system is realizable as $\mathcal{L}(\mathcal{U})$ for some complex C_n-universe $\mathcal{U}$.*

Proof. Let $n > 0$ be odd and let $\mathcal{R}$ be a saturated C_n-transfer system. Let $\mathcal{C}$ be the set of $\mathcal{R}$-cofibrant subgroups.

Let $n = p_1^{e_1} \cdots p_r^{e_r}$ be the prime factorization[2] of n. For $C_m \in \mathcal{C}$, let $m = p_{j_1}^{a_{j_1}} \cdots p_{j_{\omega(m)}}^{a_{\omega(m)}}$ be the prime factorization of m. Here, the prime factorization of m is indexed by $J = \{j_1, \ldots, j_{\omega(m)}\}$ where j_i is the index of the i-th smallest prime dividing m in the prime factorization of n.

For each $C_m \in \mathcal{C}$ with $m \neq 1$, we define $0 \leq d_m < n$ as follows:

$$d_m \equiv 0 \mod p_i^{e_i} \text{ for all } i \in \{0, \ldots, r\} \setminus J$$

$$d_m \equiv c_{j_\ell} p_{j_\ell}^{a_{j_\ell}-1} \mod p_{j_\ell}^{e_{j_\ell}} \text{ for all } j_\ell \in J,$$

where $c_{j_\ell} = \min(\omega(m), \ell + 1)$. Note that defining these equivalences determines a unique $0 \leq d_m < n$ by the Chinese Remainder Theorem. Let $\mathcal{U}_0 = \{0, \ldots, n-1\}$. Define

$$\mathcal{U} = \mathcal{U}_0 \setminus \bigcup_{C_m \in \mathcal{C}} [d_m]_m.$$

We will prove that $\mathcal{U}$ is a universe for which $\mathcal{R} = \mathcal{L}(\mathcal{U})$. First, we will show that our choice of d_m satisfies various desirable properties.

Lemma 3.5. *For all $j_\ell \in J$, we have that $c_{j_\ell} \not\equiv 0 \mod p_{j_\ell}$. Additionally, if $\omega(m) > 1$, then for all $j_\ell \in J$, $c_{j_\ell} - 1 \not\equiv 0 \mod p_{j_\ell}$.*

Proof. Suppose there were some $j_\ell \in J$ such that $c_{j_\ell} \equiv 0 \mod p_{j_\ell}$. Then $p_{j_\ell} \mid c_{j_\ell}$ and $c_{j_\ell} > 0$ by construction. In particular, then, $p_{j_\ell} \leq c_{j_\ell}$. Letting p' denote the $(\ell+1)$-th prime, then, since n is odd, we have

$$p' \leq p_{j_\ell} \leq c_{j_\ell} \leq \ell + 1.$$

[2]For the purpose of this proof, we will assume that our prime factorizations are indexed in ascending order. That is, if $i < j$, then $p_i < p_j$.

This gives us that $p \leq \ell + 1$, which is a contradiction. We conclude that $c_{j_\ell} \not\equiv 0 \mod p_{j_\ell}$ for all $j_\ell \in J$.

Now let $C_m \in \mathcal{C}$ such that $\omega(m) > 1$ and suppose for contradiction that there exists $j_\ell \in J$ such that $c_{j_\ell} - 1 \equiv 0 \mod p_{j_\ell}$. Since $\omega(m) > 1$ we have that $c_{j_\ell} > 1$, so we can conclude, by similar reasoning to the previous case, that $p_{j_\ell} \leq c_{j_\ell} - 1$. Leting p' denote the $(\ell+1)$-th prime. We have

$$p' \leq p_{j_\ell} \leq c_{j_\ell} - 1 \leq \ell < \ell + 1.$$

This gives us that $p < \ell + 1$, which is a contradiction. We conclude that $c_{j_\ell} - 1 \not\equiv 0 \mod p_{j_\ell}$ for all $j_\ell \in J$. $\square$

The fact that n is odd is used in the above argument when we claim $p \leq p_{j_\ell}$. Since n is odd, the first prime number, 2, does not show up in its prime factorization. Therefore, if $n = p_1^{e_1} \cdots p_r^{e_r}$, then each p_i is at least the $(i+1)$-th prime. Note that this condition does not hold for even n; consider $n = m = 6$. Then $c_1 = c_2 = 2$, but $2 \equiv 0 \mod 2$.

As a consequence of Lemma 3.5, we must have $0 < c_{j_\ell} < p_{j_\ell}$ for all $j_\ell \in J$. Therefore, $0 < c_{j_\ell} p_{j_\ell}^{a_{j_\ell}-1} < p_{j_\ell}^{e_{j_\ell}}$, meaning $d_m \not\equiv 0 \mod p_{j_\ell}^{e_{j_\ell}}$ for all $j_\ell \in J$.

Lemma 3.6. *Let $C_m, C_{m'} \in \mathcal{C}$ such that $m, m' \neq 1$. If $d_{m'} \equiv d_m \mod m$, then $m = m'$.*

Proof. Let $m = p_{j_1}^{a_{j_1}} \cdots p_{j_{\omega(m)}}^{a_{j_{\omega(m)}}}$ and $m' = p_{i_1}^{b_{i_1}} \cdots p_{i_{\omega(m')}}^{b_{i_{\omega(m')}}}$ be indexed by $J = \{j_1, \ldots, j_{\omega(m)}\}$ and $I = \{i_1, \ldots, i_{\omega(m')}\}$ respectively. Suppose $d_{m'} \equiv d_m \mod m$ and fix some index $j_\ell \in J$. By CRT, we get $d_{m'} \equiv d_m \mod p_{j_\ell}^{a_{j_\ell}}$. By our construction, this means

$$d_{m'} \equiv c_{j_\ell} p_{j_\ell}^{a_{j_\ell}-1} \mod p_{j_\ell}^{a_{j_\ell}}.$$

By Lemma 3.5, $d_m \not\equiv 0 \mod p_{j_\ell}^{a_{j_\ell}}$, so $d_{m'} \not\equiv 0 \mod p_{j_\ell}^{e_{j_\ell}}$. Note that by our construction and Lemma 3.5, $d_{m'} \equiv 0 \mod p_{j_\ell}^{e_{j_\ell}}$ if and only if $j_\ell \notin I$. Therefore, $j_\ell \in I$, and, since j_ℓ was arbitrary, $J \subseteq I$.

Let $\ell \mapsto c_{j_\ell}$ for $\ell \in \mathbb{N}$ be the previously defined sequence of coefficients for the construction of d_m and $s \mapsto c'_{i_s}$ be the coefficients used in the construction of $d_{m'}$. We will show that for all $j_\ell \in J$, $a_{j_\ell} = b_{j_\ell}$.

Fix $j_\ell \in J$. Then there exists some $i_s \in I$ with $j_\ell = i_s$. Suppose for contradiction that $a_{j_\ell} < b_{i_s}$. Since $d_{m'} \equiv d_m \mod p_{j_\ell}^{a_{j_\ell}}$, we get

$$c'_{i_s} p_{i_s}^{b_{i_s}-1} \equiv c_{j_\ell} p_{j_\ell}^{a_{j_\ell}-1} \mod p_{j_\ell}^{a_{j_\ell}}.$$

Multiplying through gives us,

$$c'_{i_s} p_{i_s}^{b_{i_s}-a_{j_\ell}} \equiv c_{j_\ell} \mod p_{j_\ell},$$

implying $c_{j_\ell} \equiv 0 \mod p_{j_\ell}$. This is a contradiction with Lemma 3.5.

Now suppose for contradiction that $a_{j_\ell} > b_{i_s}$. Using a similar set-up to the previous step and multiplying through gives us

$$c'_{i_s} \equiv c_{j_\ell} p_{j_\ell}^{a_{j_\ell}-b_{i_s}} \mod p_{i_s}^{a_{j_\ell}-b_{i_s}+1}.$$

This implies that $c'_{i_s} \equiv 0 \mod p_{i_s}$, which is a contradiction with Lemma 3.5.

We now have that every prime that appears in m also appears in m' with the same multiplicity. Therefore, it now suffices to show that $\omega(m) = \omega(m')$.

Suppose $\omega(m) = 1$ and let j be the unique element in J. There exists $i_s \in I$ with $j = i_s$, and since we know that $a_j = b_{i_s}$, the above equalities give us

$$c'_{i_s} \equiv c_j \mod p_{i_s}.$$

Since $\omega(m) = 1$, $c_j = 1$, meaning

$$c'_{i_s} \equiv 1 \mod p_{i_s}.$$

This forces $\omega(m') = 1$ by the contrapositive of Lemma 3.5.

Now suppose $\omega(m') > 1$. Note that for all $1 \le q < \omega(m')$, $c'_{i_q} = q + 1$. Since $J \subseteq I$, there exists $i_s \in I$ such that $i_s = j_{\omega(m)} = \omega(m)$. By our construction on the sequence of coefficients, we must have $\omega(m) \le s$, and therefore that $c'_{i_s} = \omega(m') = \omega(m)$.

This gives us that $\omega(m) = \omega(m')$. In conjunction with the fact that the primes which show up in m also show up in m' with the same multiplicity, we conclude that $m = m'$. □

This condition, along with the lemma below, guarantees that if $C_k \notin \mathcal{C}$, then whenever some class $[x]_k$ is missing from $\mathcal{U}$, we can identify a cofibrant divisor of k which removed that class.

Lemma 3.7. *Let m be a proper divisor of n and let $0 \le d < m$. If for all $\rho \in \mathcal{U}$, $\rho \not\equiv d \mod m$, then $d \equiv d_k \mod k$ for some $C_k \in \mathcal{C}$ properly dividing m.*

Proof. We will prove the contrapositive. Let $0 \le d < m$ and suppose for all $C_k \in \mathcal{C}$ with k properly dividing m, $d \not\equiv d_k \mod k$. We will show that there exists $\rho \in \mathcal{U}$ with $\rho \equiv d \mod m$.

Let $m = p_{j_1}^{a_{j_1}} \cdots p_{j_{\omega(m)}}^{a_{j_{\omega(m)}}}$ be indexed by $J = \{j_1, \ldots, j_{\omega(m)}\}$. Define $0 \le \varphi < n$ to be the unique element satisfying the following equivalences:

$$\varphi \equiv d \,\%\, p_{j_\ell}^{a_{j_\ell}} \mod p_{j_\ell}^{k_{j_\ell}} \text{ for all } j \in J$$

$$\varphi \equiv 0 \mod p_i^{k_i} \text{ for all } i \in \{0, \ldots, r\} \setminus J.$$

Here, $\%$ denotes the modulo operator, where $a \,\%\, b := \min\{x \in \mathbb{Z}_{\ge 0} \mid x \equiv a \mod b\}$. Note that $\varphi \equiv d \mod p_{j_\ell}^{a_{j_\ell}}$ by construction for all $j_\ell \in J$. We claim that this $\varphi \in \mathcal{U}$.

Suppose for contradiction that $\varphi \notin \mathcal{U}$. By our construction of $\mathcal{U}$, there exists some $C_{m'} \in \mathcal{C}$ with m' properly dividing n such that $\varphi \in [d_{m'}]_{m'}$. Let $m' = p_{i_1}^{b_{i_1}} \cdots p_{i_{\omega(m')}}^{b_{i_{\omega(m')}}}$ be indexed by $I = \{i_1, \ldots, i_{\omega(m')}\}$. Fix $i_s \in I$. We know that $\varphi \equiv d_{m'} \mod m'$, so

$\varphi \equiv d_{m'} \mod p_{i_s}^{b_{i_s}}$. In particular, by Lemma 3.5, $\varphi \not\equiv 0 \mod p_{i_s}^{b_{i_s}}$. This gives us that $\varphi \not\equiv 0 \mod p_{i_s}^{e_{i_s}}$, so $i_s \in J$. Therefore, $I \subseteq J$.

We now have that all of the primes which show up in m' also show up in m. To show that $m' \mid m$, it suffices to show that the multiplicity of each of these primes in m' is less or equal to than their multiplicity in m. Suppose there were some $i_s \in I$ such that $a_{i_s} < b_{i_s}$.

We know that $\varphi \equiv d_{m'} \mod m'$, so by our construction,

$$\varphi \equiv c_{i_s} p_{i_s}^{b_{i_s}-1} \mod p_{i_s}^{b_{i_s}}.$$

Since a_{i_s} is at most $b_{i_s} - 1$, this gives us

$$\varphi \equiv 0 \mod p_{i_s}^{a_{i_s}}.$$

By our definition of φ, this means that

$$d \% p_{i_s}^{a_{i_s}} \equiv 0 \mod p_{i_s}^{a_{i_s}}.$$

Therefore, $d \% p_{i_s}^{a_{i_s}} = 0$, so $\varphi \equiv 0 \mod p_{i_s}^{e_{i_s}}$. This implies that $d_{m'} \equiv 0 \mod p_{i_s}^{b_{i_s}}$, which contradicts Lemma 3.5. We now have that $m' \mid m$.

Notice that $\varphi \equiv d \mod m$ implies that $\varphi \equiv d \mod m'$, and therefore that $d \equiv d_{m'} \mod m'$. This contradicts our initial assumption that $d \not\equiv d_k \mod k$ for all $C_k \in \mathcal{C}$ with k properly dividing m. We conclude that $\varphi \in \mathcal{U}$. □

We now have all the tools we need to prove that $\mathcal{L}(\mathcal{U}) = \mathcal{R}$. Let $\mathcal{C}_{\mathcal{L}}$ denote the set of cofibrant subgroups in $\mathcal{L}(\mathcal{U})$. We will prove that $\mathcal{C} = \mathcal{C}_{\mathcal{L}}$.

Let $C_m \in \mathcal{C}$. If $m = 1$, then $C_m \in \mathcal{C}_{\mathcal{L}}$, as desired. Otherwise, fix a proper divisor k of m. Since $C_m \in \mathcal{C}$, by our construction, for all $z \in \mathcal{U}$, $z \not\equiv d_m \mod m$. We must show that there exists $z \in \mathcal{U}$ such that $z \equiv d_m \mod k$. If $k = 1$, note that $0 \in \mathcal{U}$ by Lemma 3.5, so we can choose $z = 0 \in \mathcal{U}$. Otherwise, suppose for contradiction that for all $z \in \mathcal{U}$, $z \not\equiv d_m \mod k$. Then by Lemma 3.7, there exists some $C_{k'} \in \mathcal{C}$ with $k' \mid k$ such that $d_m \equiv d_{k'} \mod k'$. Since $k \mid m$, k' is a proper divisor of m, so this equivalence contradicts Lemma 3.6. We conclude that C_m satisfies the condition for cofibrancy in $\mathcal{L}(\mathcal{U})$. Therefore, $\mathcal{C} \subseteq \mathcal{C}_{\mathcal{L}}$.

Suppose $C_m \in \mathcal{C}_{\mathcal{L}}$. Suppose for contradiction that $C_m \notin \mathcal{C}$. Let k be the maximal divisor of m such that $C_k \in \mathcal{C}$. k must be strictly less than m, so k is a proper divisor of m. By the condition for cofibrancy in $\mathcal{L}(\mathcal{U})$, there exists some $0 \leq d < m$ such that for all $z \in \mathcal{U}$, $z \not\equiv d \mod m$ and there exists some $z_k \in \mathcal{U}$ such that $z_k \equiv d \mod k$. By lemma 3.7, there exists some $C_{k'} \in \mathcal{C}$ with $k' > 1$ and $k' \mid m$ such that $d \equiv d_{k'} \mod k'$. If $k' \mid k$, then $z_k \in \mathcal{U}$ and $z_k \equiv d \equiv d_{k'} \mod k'$, which cannot happen since $C_{k'} \in \mathcal{C}$. Since $C_k, C_{k'} \in \mathcal{C}$, $C_{\mathrm{lcm}(k,k')} \in \mathcal{C}$. Additionally, since $k' \nmid k$, $\mathrm{lcm}(k, k') < k$, which contradicts the maximality of k. We conclude that $C_m \in \mathcal{C}$.

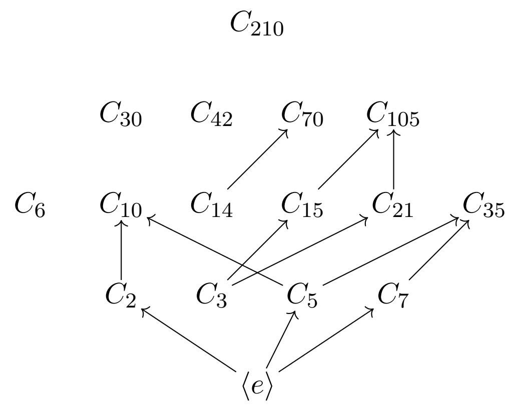
We have $\mathcal{C} = \mathcal{C}_{\mathcal{L}}$, proving $\mathcal{R} = \mathcal{L}(\mathcal{U})$. □

Notably, this method of proof circumvents the use of subinductors, as employed in [Rub21] and [CGMR], opting instead for a concrete, number theoretic construction of a universe.

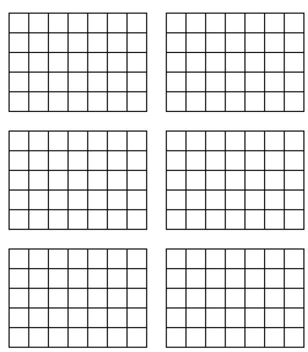
4. Extension of Odd Results to Even Order Cyclic Groups.

The existence of 2 as a prime factor gives this method lots of trouble. Naively extending this method to even order cyclic groups fails almost immediately. This is because we must reserve all of the odd numbers for C_2, and we simply run out of space to fit the remaining d_m.[3] However, there is yet hope. Thanks to the work of [CGMR] we know we need only consider transfer systems for which C_2 is not cofibrant, meaning we don't really need to make a reservation for it. Unfortunately, the method still fails, in particular when C_6 is also a subgroup of G, and is cofibrant. For motivation of our main result, consider the following transfer system:

Example 4.1.



Let us try to construct C_{210} universe $\mathcal{U}$ which realizes the above. We draw the elements of C_{210} schematically with respect to their Chinese Remainder Theorem decomposition modulo $2, 3, 5$, and 7.

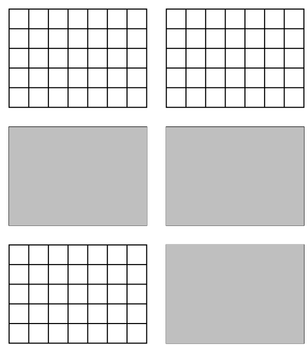


Here, the boxes on the left hand side contain all the even elements, and those on the right contain the odds. Similarly, the boxes from top to bottom stratify the classes mod 3,

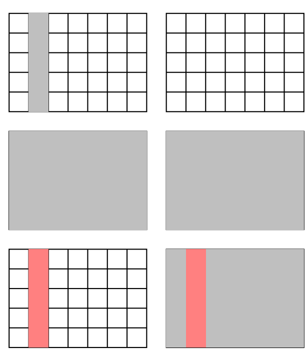
[3]Effectively, in C_{2n}, reserving this giant portion for C_2 leaves us with only C_n's worth of space to work with, yet we still have almost twice as many reservations to make as in the odd case. It will prove impossible to make similar reservations independent of cofibrancy.

with the 0 mod 3 elements on top, and the 2 mod 3 elements on the bottom. Each box is then split into a 5×7 grid to represent their class mod 5 and 7 respectively.

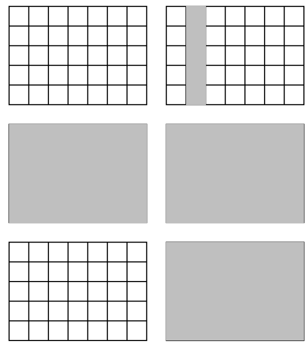
We will say that this picture represents the universe $\mathcal{U}$ with all elements, and we will shade those which we must remove from $\mathcal{U}$. First, $C_3, C_6 \in \mathcal{C}$ so we must remove an entire class modulo 3 and another modulo 6. (It doesn't particularly matter which classes we do, remembering that we must leave $0 \in \mathcal{U}$.)



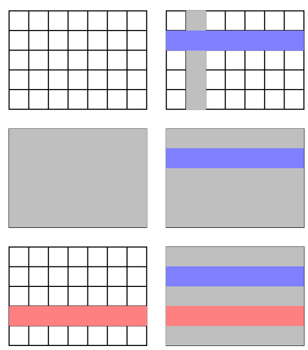
Now, we want to shade a class modulo 14; suppose we try to put it on the left.



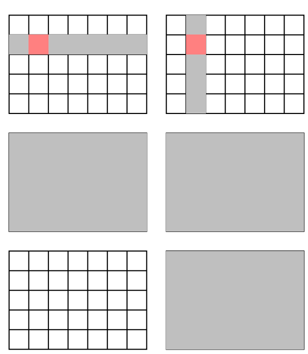
We see that this class mod 14 and the class we cut mod 6 have worked together to cover a class mod 21, but C_{21} is not cofibrant, so we instead will cut this class mod 14 on the right side, where we can avoid this problem.



But now, a similar problem arises when we try to place our class for C_{30} (represented by a single horizontal line). Trying to place it in the top right box (shown, along with its co-conspirators, in blue) covers a class mod 10, and trying to place it in the bottom left covers a class mod 15 (shown similarly, in red).



Since C_{15} and C_{10} are not $\mathcal{R}$-cofibrant, we want to avoid this, and so we place this class in the top left box, but this gives us a new problem as this, and our class mod 14 have come together to cover a class mod 105.



It seems as though 14 and 30 are incompatible in a sense, and when we are low on space they like to force other subgroups to become cofibrant. The following definition and theorem seek to capture this behavior in general.

Definition 4.2 (Conspiring Pair)**.** Let $\mathcal{C}$ be the set of cofibrant subgroups for some saturated C_n-transfer system where $6 \mid n$. We say that $(C_a, C_b) \in \mathcal{C} \times \mathcal{C}$ is a *$\mathcal{R}$-conspiring pair* if the following conditions hold:

(1) $a = 2a'$ for $a' > 1$
(2) $b = 3b'$ for $b > 1$
(3) $\gcd(a, b) \leq 3$
(4) $4 \nmid b$
(5) $9 \nmid a$
(6) $\gcd(a', 2) = 1$
(7) $\gcd(b', 3) = 1$
(8) either $6 \mid a$ or $6 \mid b$
(9) $a, b \neq 6$

For efficiency, the following lemma will be useful:

Lemma 4.3. *For a C_n-universe $\mathcal{U}$, $1 < m \mid n$, if there is some r such that for all $u \in \mathcal{U}$, $u \not\equiv r \mod m$, then there is some divisor k of m with $k > 1$ such that C_k is cofibrant.*

Proof. Let D be the set of proper divisors of m, and fix an r as described. If for all $k \in D$, there is some $u \in \mathcal{U}$ such that $r \equiv u \mod k$, we have by lemma 3.1 that C_m is cofibrant. Otherwise, there is some $k \in D$ so that there is no $u \in \mathcal{U}$ with $r \equiv u \mod k$. Let $l = \min(\{k \in D \mid \forall u \in \mathcal{U}, u \not\equiv r \mod k\})$. Since $0 \in \mathcal{U}$, we know that 1 is not a member of the previous set, and therefore $l > 1$. By the minimality of l, we have that for all k' properly dividing l there is some $u \in \mathcal{U}$ so that $r \equiv u \mod k'$. By 3.1, we can then conclude that C_l is cofibrant. □

Theorem 4.4. *Let $\mathcal{R}$ be a realizable saturated transfer system with C_3 and C_6 cofibrant, but C_2 not cofibrant. Suppose (C_a, C_b) is an $\mathcal{R}$-conspiring pair. In the case $6 \mid b$ then there is some nontrivial cofibrant C_k so that either*

- $2k \mid b'$ *with* $k \neq 2$
- $k \mid \frac{a'b}{2}$ *with* $k \neq 3$ *and* $k \nmid a'$

In the case where $6 \mid a$, then there exists some nontrivial cofibrant C_k such that either

- $k \mid a'$ *with* $k \neq 3$
- $k \mid \frac{ab'}{3}$ *with* $k \neq 2$

Proof. For each $\mathcal{R}$-cofibrant subgroup C_d and t properly dividing d, fix some representative $r_{d,t}$ satisfying lemma 3.1. We first do the case where $6 \mid b$.

First we show that if $r_{b,6} \equiv r_{6,3} \mod 2$, then there is some $C_k \in \mathcal{C}$, with $k > 1$ and $k \mid b'$. Let $r \equiv r_{b,6} \mod b$. Suppose for contradiction that there is some $u \in \mathcal{U}$ with

$u \equiv r \mod b'$. Note that $r_{b,6} \not\equiv r_{6,3} \mod 6$, so having $r_{b,6} \equiv r_{6,3} \mod 2$ necessitates $r_{b,6} \not\equiv r_{6,3} \mod 3$. Since both $r_{6,3} \not\equiv r_{3,1} \mod 3$ and $r_{b,6} \not\equiv r_{3,1} \mod 3$, it follows that $r_{3,1}$, $r_{6,3}$, and $r_{b,6}$ are representatives of each of the distinct classes modulo 3. Clearly $u \equiv r_{3,1} \mod 3$ cannot happen. Since $6 \mid b$ implies $2 \mid b'$, and so having $u \equiv r_{b,6} \mod b'$ gives $u \equiv r_{b,6} \equiv r_{6,3} \mod 2$. If $u \equiv r_{6,3} \mod 3$, then we have $u \equiv r_{6,3} \mod 6$ which cannot happen. As described above, the only other possibility is $u \equiv r_{b,6} \mod 3$; since $u \equiv r_{b,6} \mod 2$ we have $u \equiv r_{b,6}$ which cannot happen. So for all $u \in \mathcal{U}$, $u \not\equiv r \mod b'$, and 4.3 gives us the desired result.

Next we show that if $r_{a,a'} \not\equiv 2$, then there is some $C_k \in \mathcal{C}$, with $k > 1$ and $k \mid 3a'$. Since $6 \mid b$ and $\gcd(a,b) \leq 3$, necessarily $3 \nmid a'$. We then have unique r modulo $3a'$ satisfying

$$r \equiv r_{a,a'} \mod a'$$

$$r \equiv r_{6,3} \mod 3$$

Suppose for contradiction that there is some $u \in \mathcal{U}$ with $u \equiv r \mod 3a'$. If $u \equiv r_{6,3} \mod 2$ then $u \equiv r_{6,3} \mod 6$ which cannot happen. Otherwise $u \equiv r_{a,a'} \mod 2$, where since $\gcd(a',2) = 1$ we have $u \equiv r_{a,a'} \mod a$. We get the desired result by 4.3.

To finish this case we shall show that if $r_{b,6} \not\equiv r_{6,3} \mod 2$ and $r_{a,a'} \equiv r_{6,3} \mod 2$, then there is some cofibrant $C_k \in \mathcal{C}$, with $k > 1$ and $k \mid \frac{a'b}{2}$. We know that since $6 \mid b$ and $\gcd(a,b) \leq 3$, then $3 \nmid a$. Since $\gcd(a',2) = 1$, it follows that $\gcd(a', \frac{b}{2}) = 1$. We then have a unique r modulo $\frac{a'b}{2}$ satisfying:

$$r \equiv r_{a,a'} \mod a'$$
$$r \equiv r_{b,6} \mod \frac{b}{2}$$

Similar to before, suppose for contradiction that there is some $u \in \mathcal{U}$ so that $u \equiv r \mod \frac{a'b}{2}$. If $u \equiv r_{6,3} \equiv r_{a,a'} \mod 2$, then $u \equiv r_{a,a'} \mod a$ (since $\gcd(a',2) = 1$), which is impossible. So instead $u \equiv r_{b,6} \mod 2$ by our above assumption. Since $4 \nmid b$, $\gcd(b/2, 2) = 1$ and hence $u \equiv r_{b,6} \mod b$. So for all $u \in \mathcal{U}$, $u \not\equiv r \mod \frac{a'b}{2}$, and we can conclude the desired result.

Now we work through the case where $6 \mid a$.

If $b = 3b' = \operatorname{lcm}(k_1, k_2)$ for $C_{k_1}, C_{k_2} \in \mathcal{C}$ we are done, since either k_1 or k_2 must divide b', and $3 \mid a$. Otherwise, from 3.2 we have some class r_b mod b such that for all $u \in \mathcal{U}$, $u \not\equiv r_b \mod b$ but for all k properly dividing b there is some $v \in \mathcal{U}$ so that $v \equiv r_b \mod k$. Since $C_2 \notin \mathcal{C}$, we can also find such a class r_6 satisfying the same condition for all proper divisors of 6.

We begin with the case where $r_{a,6} \equiv r_6 \mod 3$. Pick any $r \equiv r_{a,6} \mod a'$, and suppose for contradiction there is some $u \in \mathcal{U}$ with $u \equiv r \mod a'$. Note that $r_{a,6} \not\equiv r_6 \mod 6$, so $r_{a,6} \equiv r_6 \mod 3$ necessitates $r_{a,6} \not\equiv r_6 \mod 2$. If $u \equiv r_{a,6} \mod 2$, then $\gcd(a',2) = 1$ gives $u \equiv r_{a,6} \mod a$. So $u \equiv r_6 \mod 2$. However, $6 \mid a$ means $3 \mid a'$, and so $u \equiv r_{a,6} \equiv r_6 \mod 3$ results in $u \equiv r_6 \mod 6$—a contradiction. So for all $u \in \mathcal{U}$, $u \not\equiv r \mod a'$ and we have our desired cofibrant subgroup by 4.3.

We now look at when $r_b \not\equiv r_6 \mod 3$. Since $6 \mid a$ and $\gcd(a,b) \leq 3$, we know $6 \nmid b$ and so $2 \nmid b'$. We are thus able to find a unique class r modulo $2b'$ satisfying:

$$r \equiv r_b \mod b'$$
$$r \equiv r_6 \mod 2$$

Note that since $3 \mid b$, our initial assumption gives $r_b \not\equiv r_{3,1} \mod b$, $r_b \not\equiv r_6 \mod 3$ and $r_6 \not\equiv r_{3,1} \mod 3$. Hence r_b, r_6, and $r_{3,1}$ are representatives of distinct classes mod 3. Suppose for contradiction there exists some $u \in \mathcal{U}$ with $u \equiv r \mod 2b'$. Clearly $u \equiv r_{3,1} \mod 3$ is impossible. If $u \equiv r_b \mod 3$, then $\gcd(b', 3) = 1$ yields $u \equiv r_b \mod b$ which cannot happen. So the only other possibility is when $u \equiv r_6 \mod 3$, but since $u \equiv r_6 \mod 2$ we get $u \equiv r_6 \mod 6$. By 4.3 we get the desired cofibrant divisor.

We are left with the case where both $r_{a,6} \not\equiv r_6 \mod 3$ and $r_b \equiv r_6 \mod 3$. Since $6 \mid a$ and $\gcd(a,b) \leq 3$, we know $2 \nmid b$. It then follows from $\gcd(b', 3) = 1$ that $\gcd(b', \frac{a}{3}) = 1$. We can thus find a unique class r modulo $\frac{ab'}{2}$ satisfying

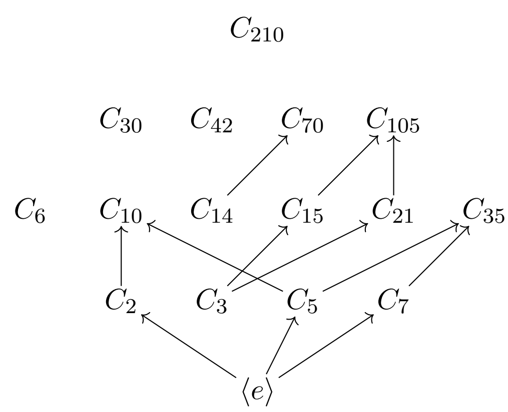
$$r \equiv r_b \mod b'$$
$$r \equiv r_{a,6} \mod \frac{a}{3}$$

Suppose again for contradiction that there is some $u \in \mathcal{U}$ with $u \equiv r \mod \frac{b'a}{3}$. Similar to the previous case, we see that $r_{a,6}$, r_6, and $r_{3,1}$ represent each of the distinct classes modulo 3. Obviously $u \equiv r_{3,1}$ cannot happen, and $u \equiv r_6 \equiv r_b \mod 3$ implies $u \equiv r_b \mod b$ since $\gcd(b', 3) = 1$. So $r \equiv r_{a,6} \mod 3$ and $r \equiv r_{a,6} \mod \frac{a}{3}$, where $9 \nmid a$ gives us $u \equiv r_{a,6} \mod a$. So for all $u \in \mathcal{U}$, $u \not\equiv r \mod \frac{b'a}{3}$, and we have the desired result by 4.3.

□

Corollary 4.5. *Conjecture 1.15 is false.*

Proof. The transfer system from example 4.1 (shown below) is an example of a transfer system which has C_2 non-cofibrant, yet it has conspiring pair (C_{14}, C_{30}) and does not otherwise satisfy Theorem 4.4, and therefore must not be $\mathcal{L}(\mathcal{U})$ for any C_{210}-universe $\mathcal{U}$.



□

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