Wave Kinetic Equation and Kolmogorov-Zakharov Cascade Spectra

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Abstract

In wave turbulence theory, wave systems are governed by nonlinear dispersive equations. The nonlinear Schrödinger (NLS) equation is one example that governs waves in superfluids. In this report, The NLS equation, its corresponding wave kinetic equation (WKE), and the Kolmogorov-Zakharov solutions to the WKE will be discussed.

Contents

1 Introduction

Wave turbulence theory is the theory of statistical mechanics for waves. Those complex partial differential equations (PDE) governing different wave systems are called *nonlinear dispersive equations*. One interesting example that we focused on is the nonlinear Schrödinger (NLS) equation, which describes waves in superfluids:

A function $u : \mathbb{R} \times [0, L]^d \to \mathbb{C}$ satisfies the nonlinear Schrödinger equation if:

$$i\partial_t u + \Delta u = \epsilon |u|^2 u,$$

where $\epsilon > 0$ is a fixed parameter.

Solutions u of NLS have some interesting properties, including phase rotation symmetry, time translation invariance, and space translation invariance. More importantly, mass and energy are conserved in time, where the mass at a given time t of a solution u is defined to be $\|u(t, \cdot)\|_{\mathcal{L}^2(\mathbb{T}_L^d)}$ and the energy is defined as $E(t) := \int_{\mathbb{T}_L^d} \frac{1}{2}|\nabla u(t, x)|^2 + \frac{\epsilon}{4}|u(t, x)|^4 dx$.

2 The WKE Equation

While nonlinear dispersive equations describes the *microscopic* behavior of a wave system, a statistical (or averaged) description, at a very long timescale, is claimed to be given by the wave kinetic equation (WKE) in wave turbulence theory. Every nonlinear dispersive equation has a corresponding wave kinetic equation, and the WKE for the NLS is:

$$\partial_t n(t, \xi_1) = \mathcal{K}(n(t, \cdot)),$$

$$\mathcal{K}(\phi)(\xi_1) := \int_{\substack{(\xi_2,\xi_3,\xi_4) \in \mathbb{R}^{3d}_{>0} \\ \xi_1+\xi_2=\xi_3+\xi_4}} \phi_1\phi_2\phi_3\phi_4 (\frac{1}{\phi_1} + \frac{1}{\phi_2} - \frac{1}{\phi_3} - \frac{1}{\phi_4}) \delta_{\mathbb{R}}(\Omega) \, \mathrm{d}\xi_2 \mathrm{d}\xi_3 \mathrm{d}\xi_4$$

where $\Omega = |\xi_1|^2 + |\xi_2|^2 - |\xi_3|^2 - |\xi_4|^2$ and $\phi_j = \phi(\xi_j)$.

2.1 Conserved Quantities

If $\rho(\xi_1)$ is the density of some physical quantity then

$$\int \rho(\xi_1) n(\xi_1, t) \, \mathrm{d}\xi_1$$

gives the total amount of that physical quantity in the system where $n(\xi_1, t)$ is a solution to the WKE equation. The symmetries of the delta functions in the WKE equation can be used to find quantities that are conserved in time:

$$\frac{d}{dt} \int \rho(\xi_1) n(t, \xi_1) \, d\xi_1 = \int \rho(\xi_1) \partial_t n(t, \xi_1) d\xi_1$$

$$= \int \int_{\xi_1+\xi_2-\xi_3-\xi_4=0} \rho(\xi_1) \phi\phi_1\phi_2\phi_3 (\frac{1}{\phi_1} + \frac{1}{\phi_2} - \frac{1}{\phi_3} - \frac{1}{\phi_4}) \delta_{\mathbb{R}}(\Omega) \, \mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}\xi_3 \mathrm{d}\xi_4$$

We will apply changes of variables in the second, third, and fourth terms. In the second term, we switch ξ_1 and ξ_2. In the third term, we switch ξ_1 and ξ_3

and switch ξ_2 and ξ_4. In the fourth term, we switch ξ_1 and ξ_4 and switch ξ_2 and ξ_3. Assuming that all terms are finite, the resulting integral is

$$= \int\int_{\xi_1+\xi_2-\xi_3-\xi_4=0} (\rho(\xi_1)+\rho(\xi_2)-\rho(\xi_3)-\rho(\xi_4))\phi_1\phi_2\phi_3\phi_4\delta_{\mathbb{R}}(\Omega)\,\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_3\mathrm{d}\xi_4$$

If we choose ρ to be a function such that the quantity in parenthesis vanishes when Ω vanishes or when $\xi_1+\xi_2=\xi_3+\xi_4$ then this integral will be 0 and ρ will be a conserved quantity. Some possible choices are $\rho(\xi)=1$, $\rho(\xi)=\xi$, $\rho(\xi)=|\xi|^2$ or any linear combination of these three. These three choices for ρ correspond to mass, momentum, and energy respectively.

2.2 Derivation Of The Isotropic Equation

In this section we will simplify the the WKE equation in the case where the input function ϕ is isotropic, i.e., $\phi(\xi_i)=f(|\xi_i|^2)$ for some f. In order to make all integrals converge nicely we will also assume that ϕ is in the Schwartz class. Recall the right hand side of the WKE equation,

$$\int_{\xi_1+\xi_2-\xi_3-\xi_4=0} \phi_1\phi_2\phi_3\phi_4(\frac{1}{\phi_1}+\frac{1}{\phi_2}-\frac{1}{\phi_3}-\frac{1}{\phi_4})\delta_{\mathbb{R}}(\Omega)\,\mathrm{d}\xi_2\mathrm{d}\xi_3\mathrm{d}\xi_4$$

By multiplying by a delta function we can remove the need to integrate over the surface $\{\xi_1+\xi_2-\xi_3-\xi_4=0\}$. We obtain the expression

$$\int_{\mathbb{R}^{3d}} \phi_1\phi_2\phi_3\phi_4(\frac{1}{\phi_1}+\frac{1}{\phi_2}-\frac{1}{\phi_3}-\frac{1}{\phi_4})\delta_{\mathbb{R}}(\Omega)\delta(\xi_1+\xi_2-\xi_3-\xi_4)d\xi_2d\xi_3d\xi_4 \quad (2.1)$$

The theory of Fourier transformation yields that

$$\delta(\xi_1+\xi_2-\xi_3-\xi_4)=\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{i\lambda\cdot(\xi_1+\xi_2-\xi_3-\xi_4)}d\lambda$$

in the sense of distributions. Making this substitution into 2.1 we obtain the expression

$$\frac{1}{(2\pi)^d}\int_{\mathbb{R}^{4d}} \tilde{\phi}e^{i\lambda\cdot(\xi_1+\xi_2-\xi_3-\xi_4)}d\xi_1d\xi_2d\xi_3d\lambda \quad (2.2)$$

where $\tilde{\phi}=\phi_1\phi_2\phi_3\phi_4(\frac{1}{\phi_1}+\frac{1}{\phi_2}-\frac{1}{\phi_3}-\frac{1}{\phi_4})\delta_{\mathbb{R}}(\Omega)$ is isotropic. Note that the norm of the gradient of the function $|\cdot|$ is 1 almost everywhere. By applying the coarea formula, 2.2 is equal to

$$\frac{1}{(2\pi)^d}\int_0^\infty\int_0^\infty\int_0^\infty \tilde{\phi}\int_0^\infty \left(\int_{S^{d-1}_{|\lambda|}} e^{i\lambda\cdot\xi_1}\prod_{j=2}^{4}\int_{S^{d-1}_{|\xi_j|}} e^{i\lambda\cdot\xi_j}\right)d|\xi_2|d|\xi_3|d|\xi_4|d|\lambda| \quad (2.3)$$

Where S^{d-1}_R denotes the $d-1$-dimensional sphere of radius R. Now consider the function on $S^{d-1}_{|\lambda|}$ that gives the angle θ between a point λ and ξ_1. The level sets of this function are spheres of radius $|\lambda|\sin(\theta)$ and the function θ has gradient $|\lambda|^{-1}$. The first fact can be seen by noting that in a coordinate system where $\xi_1 = (0, \dots, 0, |\xi_1|)^T$ the θ-level sets are the set of all $(x_1, \dots, x_{d-1}, |\lambda|\cos(\theta))$ such that

$$x_1^2 + \cdots + x_{d-1}^2 + |\lambda|^2 \cos^2(\theta) = |\lambda|^2$$

or equivalently

$$x_1^2 + \cdots + x_{d-1}^2 = |\lambda|^2 - |\lambda|^2 \cos^2(\theta) = |\lambda|^2 \sin^2(\theta)$$

The second fact can be seen by noting that when an angle of $d\theta$ is traversed away from $\frac{|\lambda|}{|\xi_1|}\xi_1$ an arc length of $2\pi|\lambda| \cdot \frac{d\theta}{2\pi} = |\lambda| d\theta$ is traversed. The gradient of the theta function will be $\lim_{\ell \to 0} \frac{d\theta}{\ell} = \frac{1}{|\lambda|}$ where ℓ is the arc length of a displacement of λ in the direction away from ξ_1. Note that this construction doesn't work at the antipodal point $\frac{|\lambda|}{|\xi_1|}\xi_1$. But this single point has measure 0 so we can ignore it.

(2.3) becomes the following:

$$\frac{1}{(2\pi)^d} \int_0^\infty \int_0^\infty \int_0^\infty \tilde{\phi} \int_0^\infty \Bigg(\int_0^\pi \mathrm{VOL}(S^{d-2})$$
$$|\lambda|^{d-1} \sin^{d-2}(\theta_1) e^{i|\lambda||\xi_1|\cos(\theta_1)} d\theta_1 \prod_{j=2}^{4} \int_{S^{d-1}_{|\xi_j|}} e^{i\lambda\cdot\xi_j} \Bigg) d|\xi_2| d|\xi_3| d|\xi_4| d|\lambda| \tag{2.4}$$

The same process can be done for each of the integrals over $S^{d-1}_{|\xi_j|}$ except this time we may consider the angle that ξ_j makes with λ. (2.4) becomes

$$\frac{1}{(2\pi)^d} \int_0^\infty \int_0^\infty \int_0^\infty \tilde{\phi} \int_0^\infty |\lambda|^{d-1} |\xi_2|^{d-1} |\xi_3|^{d-1} |\xi_4|^{d-1}$$
$$\Bigg(\prod_{j=1}^{4} \int_0^\pi \mathrm{VOL}(S^{d-2}) \sin^{d-2}(\theta_j) e^{i|\lambda||\xi_j|\cos(\theta_j)} d\theta_j \Bigg) d|\xi_2| d|\xi_3| d|\xi_4| d|\lambda| \tag{2.5}$$

To simplify the delta function that is a part of $\tilde{\phi}$, we make the changes of coordinates $|\lambda| = r$ and $\omega_j = |\xi_j|^2$. We have that $\frac{1}{2} d\omega_j = |\xi_j| d|\xi_j|$ so (2.5) becomes

$$\frac{\mathrm{VOL}(S^{d-2})^4}{2^3 (2\pi)^d} \int_0^\infty \int_0^\infty \int_0^\infty \tilde{\phi} \int_0^\infty$$
$$r^{d-1} \sqrt{\omega_2 \omega_3 \omega_4}^{d-2} \Bigg(\prod_{j=1}^{4} \int_0^\pi \sin^{d-2}(\theta_j) e^{ir\sqrt{\omega_j}\cos(\theta_j)} d\theta_j \Bigg) d\omega_2 d\omega_3 d\omega_4 dr \tag{2.6}$$

We will study the case where $d = 3$ because this integral can be simplified tremendously in this case. Set

$$D := \int_0^\infty r^2\sqrt{\omega_2\omega_3\omega_4}\Big(\prod_{j=1}^4\int_0^\pi \sin(\theta_j)e^{ir\sqrt{\omega_j}\cos(\theta_j)}d\theta_j\Big)$$

$$= \frac{1}{\sqrt{\omega_1}}\int_0^\infty \frac{1}{r^2}\Big(\prod_{j=1}^4 e^{ir\sqrt{\omega_j}\cos(\pi)} - e^{ir\sqrt{\omega_j}\cos(0)}\Big)$$

$$= \frac{1}{\sqrt{\omega_1}}\int_0^\infty \frac{1}{r^2}\Big(\prod_{j=1}^4 -2i\sin\big(\sqrt{\omega_j}r\big)\Big)\ \mathrm{dr}$$

$$= \frac{16}{\sqrt{\omega_1}}\int_0^\infty \frac{1}{r^2}\prod_{j=1}^4 \sin\big(\sqrt{\omega_j}r\big)\ \mathrm{dr}$$

WLOG suppose that ω_1 is the largest of ω_1, ω_2, ω_3, and ω_4. It follows that ω_2 is the smallest. WLOG assume that $\omega_3 \geq \omega_4$. Thus, $\omega_1 \geq \omega_3 \geq \omega_4 \geq \omega_2$

Let $A_j := \sqrt{\omega_j}r$. Apply $\sin X \sin Y = \frac{1}{2}[\cos(X-Y) - \cos(X+Y)]$ and $\cos X \cos Y = \frac{1}{2}[\cos(X-Y) + \cos(X+Y)]$, we get

$$g(r) := \big(\sin(A_1)\sin(A_2)\big)(\sin(A_3)\sin(A_4))$$

$$= \frac{1}{2^3}\sum_{\epsilon\in\{-1,1\}^3} s(\epsilon)\cos(b(\epsilon)r),$$

where $s(\epsilon) := \prod_{j=1}^3 \epsilon_j$, and $b(\epsilon) := \sqrt{\omega_1} + \sum_{j=1}^3 \epsilon_j\sqrt{\omega_{j+1}}$. Now we compute

$$\int_0^\infty \frac{g(r)}{r^2}dr = \int_0^\infty \frac{1}{2^3r^2}\sum_{\epsilon\in\{-1,1\}^3} s(\epsilon)\cos(b(\epsilon)r)dr$$

$$= \lim_{a\to 0^+}\int_a^\infty \frac{1}{2^3r^2}\sum_{\epsilon\in\{-1,1\}^3} s(\epsilon)\cos(b(\epsilon)r)dr$$

$$= \frac{1}{2^3}\lim_{a\to 0^+}\sum_{\epsilon\in\{-1,1\}^3} s(\epsilon)\int_a^\infty \frac{1}{r^2}cos(b(\epsilon)r)\ \mathrm{dr}$$

$$= \frac{1}{2^3}\lim_{a\to 0^+}\sum_{\epsilon\in\{-1,1\}^3} s(\epsilon)\Big(\frac{-1}{r}\cos(b(\epsilon)r)\Big]_a^\infty - \int_a^\infty \frac{b(\epsilon)}{r}\sin(b(\epsilon)r)\ \mathrm{d}r\Big)$$

The boundary terms are equal to $-g(a)/a$. The definition of $g(r)$ in terms of sin-functions makes it clear that these boundary terms will vanish as $a \to 0$. We are left with

$$-\frac{1}{8}\sum_{\epsilon\in\{-1,1\}^3} s(\epsilon)\lim_{a\to 0^+}\int_a^\infty \frac{b(\epsilon)}{r}\sin(b(\epsilon)r)\ \mathrm{dr}$$

$$= -\frac{1}{8}\sum_{\epsilon\in\{-1,1\}^3} s(\epsilon)\lim_{a\to 0^+}\int_a^\infty \frac{|b(\epsilon)|}{r}\sin(|b(\epsilon)|r)\,\mathrm{d}r$$
$$= -\frac{1}{8}\sum_{\epsilon\in\{-1,1\}^3} s(\epsilon)|b(\epsilon)|\lim_{a\to 0^+}\int_a^\infty \frac{\sin(u)}{u}\,\mathrm{d}u$$
$$= \frac{-\pi}{16}\sum_{\epsilon\in\{-1,1\}^3} s(\epsilon)|b(\epsilon)|$$

In order to awake the reader, we suggest to check that this quantity is equal to $\frac{\pi}{8}\sqrt{\omega_2}$. Recall that we were assuming WLOG that ω_2 was the smallest of ω_1, ω_2, ω_3, and ω_4. It follows that D is equal to a constant times

$$\sqrt{\frac{\min(\omega_1,\omega_2,\omega_3,\omega_4)}{\omega_1}}$$

Hence, $D = \frac{(4\pi)^4}{\sqrt{\omega_1}}\frac{\pi}{2}\frac{1}{2^3}(-4\sqrt{\omega_2}) = C\frac{\sqrt{\omega_2}}{\sqrt{\omega_1}} = C\frac{\min\{\sqrt{\omega_1},\sqrt{\omega_2},\sqrt{\omega_3},\sqrt{\omega_4}\}}{\sqrt{\omega_1}}$ for some constant C. Making this substitution into (2.6) we obtain the Isotropic equation.

3 The Isotropic Equation

The isotropic equation is

$$\partial_t f(\omega_1) = \int_{\omega_1+\omega_2=\omega_3+\omega_4,\,\omega_i>0} W f_1 f_2 f_3 f_4[\frac{1}{f_1}+\frac{1}{f_2}-\frac{1}{f_3}-\frac{1}{f_4}]\,\mathrm{d}\omega_3\mathrm{d}\omega_2$$

where $W = \sqrt{\frac{\min(\omega_1,\omega_2,\omega_3,\omega_4)}{\omega_1}}$ and $f_i = f(\omega_i)$. We can think of this integral as an integral over $\{(\omega_3,\omega_2)\in\mathbb{R}^2 : \omega_2,\omega_3,\omega_4 > 0\}$ where $\omega_4 := \omega_1+\omega_2-\omega_3$. This region can be split up into four subregions which we will call W_1, W_2, W_3, and W_4. The W_i region will be defined to be the set where $0 < \omega_i = \min(\omega_1,\omega_2,\omega_3,\omega_4)$. These regions are important because the Zacharov transformations, which are defined in section 3.1, define bijections between these regions.

3.1 Zakharov Transformations

The Zakharov Transformations (ZT) are a collection of transformations between the four subregions W_1, W_2, W_3, and W_4. They are useful for finding power law stationary solutions to the isotropic equation.

We can express these four regions by $\{(\omega_3,\omega_2) : \omega_3,\omega_2,\omega_4 > 0$, and some additional conditions$\}$, where $\omega_4 := \omega_1+\omega_2-\omega_3$ and "some additional conditions" is listed below:

W_1	$\omega_2 > \omega_3 > \omega_1$
W_2	$\omega_1 > \omega_3 > \omega_2$
W_3	$\omega_2 > \omega_3,\ \omega_1 > \omega_3$
W_4	$\omega_3 > \omega_2,\ \omega_3 > \omega_1,$ or equivalently $\omega_1 > \omega_4, \omega_2 > \omega_4$

The Zakharov Transformations (ZT) T_2, T_3, and T_4 are defined as below, while T_1 is the identity transformation, i.e. keep everything unchanged.

$$T_2 = \begin{cases} \omega_2 = \frac{\omega_1^2}{\tilde{\omega_2}} \\ \omega_3 = \frac{\omega_1 \tilde{\omega_4}}{\tilde{\omega_2}} \\ \omega_4 = \frac{\omega_1 \tilde{\omega_3}}{\tilde{\omega_2}} \end{cases}$$

$$T_3 = \begin{cases} \omega_2 = \frac{\omega_1 \tilde{\omega_4}}{\tilde{\omega_3}} \\ \omega_3 = \frac{\omega_1^2}{\tilde{\omega_3}} \\ \omega_4 = \frac{\omega_1 \tilde{\omega_2}}{\tilde{\omega_3}} \end{cases}$$

$$T_4 = \begin{cases} \omega_2 = \frac{\omega_1 \tilde{\omega_3}}{\tilde{\omega_4}} \\ \omega_3 = \frac{\omega_1 \tilde{\omega_2}}{\tilde{\omega_4}} \\ \omega_4 = \frac{\omega_1^2}{\tilde{\omega_4}} \end{cases}$$

The following table summarizes T_j's actions on W_i by entry at (i, j). Namely, applying T_j on W_i, we get W_k on the spot (i, j). For example, applying T_3 on W_2 results in W_4.

	T_1	T_2	T_3	T_4
W_1	W_1	W_2	W_3	W_4
W_2	W_2	W_1	W_4	W_3
W_3	W_3	W_4	W_1	W_2
W_4	W_4	W_3	W_2	W_1

Moreover, $\{T_1, T_2, T_3, T_4\}$ forms a group that is isomorphic to the Klein 4 group. The multiplication table is given below:

	T_1	T_2	T_3	T_4
T_1	T_1	T_2	T_3	T_4
T_2	T_2	T_1	T_4	T_3
T_3	T_3	T_4	T_1	T_2
T_4	T_4	T_3	T_2	T_1

With the help of ZT, we can transform the integral on the RHS (right hand side) of the WKE to integrate over only W_i for any $i \in \{1, 2, 3, 4\}$.

3.2 Applying ZT to Find Power Law Solutions

We make the Ansatz $f_i = \omega_i^\nu$ note that we will assume that every integral that shows up in this derivation will be finite even if this may not be the case. We want to solve for ν such that

$$\int_{\omega_1+\omega_2=\omega_3+\omega_4} W[f_2 f_3 f_4 + f_1 f_3 f_4 - f_1 f_2 f_4 - f_1 f_2 f_3]\, \mathrm{d}\omega_2 \mathrm{d}\omega_3 \mathrm{d}\omega_4 = 0 \quad (3.1)$$

Step 1: Denote (3.1) by I_{34}^{12} where the 1, 2, 3, and 4 refer to the positions that ω_1, ω_2, ω_3, and ω_4 appear in (3.1). Notice that

$$I_{34}^{12} = I_{34}^{21} = -I_{14}^{32} = -I_{31}^{42} \quad (3.2)$$

The first equality comes from the fact that switching ω_1 and ω_2 leaves the expression defining (3.1) unchanged. The third equality can be obtained by switching ω_1 with ω_3 and ω_2 with ω_4 in the expression defining I_{34}^{12} to obtain $-I_{34}^{12}$. Then ω_2 and ω_4 can be switched back via a change of variables. The third equality is similar.

Step 2: Next we apply Zacharov transformations to the second, third, and fourth terms of (3.2). Apply T_2 to the second term, T_3 to the third term, and T_4 to the fourth term. Note that T_2 can be written as $(\omega_1, \omega_2, \omega_3, \omega_4) = c(\tilde{\omega}_2, \omega_1, \tilde{\omega}_3, \tilde{\omega}_4)$ where $c = \frac{\omega_1}{\tilde{\omega}_2}$. T_3 and T_4 can be written the same way except ω_1 is switched with ω_3 in for T_3, ω_4 for T_4 and the constant c is equal to $\frac{\omega_i}{\omega_1}$ for T_i. This property of the Zacharov transformation guarantees that the Zacharov transformation maps the plane defined by the equation $\omega_1 + \omega_2 = \omega_3 + \omega_4$ to itself. In fact, the i'th Zacharov transformation is a diffeomorphism of the region of the plane where each ω_i is positive. The Jacobian associated to this transformation and the euclidean volume form on this plane is $-(\frac{\omega_1}{\omega_i})^3$. We can also pull out factors of $(\frac{\omega_1}{\omega_i})^{1/2}$ coming from the function W and a factor of $(\frac{\omega_1}{\omega_i})^{3\nu}$ coming from the the expression in brackets in (3.1). We obtain that

$$I_{34}^{12} = \pm\frac{1}{4}\int_{\omega_1+\omega_2-\omega_3-\omega_4} W(\frac{\omega_1}{\omega_i})^{3\nu+3.5}[f_2f_3f_4+f_1f_3f_4-f_1f_2f_4-f_1f_2f_3]\, d\omega_2 d\omega_3 d\omega_4$$

where the plus or minus is a plus if $i \in \{1,2\}$ and a minus if $i \in \{3,4\}$.

Step 3: By adding the transformed versions of I_{34}^{21}, I_{14}^{32}, and I_{31}^{42} to I_{34}^{12} we obtain that

$$I_{34}^{12} = \int_{\omega_1+\omega_2-\omega_3-\omega_4=4} W(1 + (\frac{\omega_1}{\omega_2})^{3\nu+3.5} - (\frac{\omega_1}{\omega_3})^{3\nu+3.5} - (\frac{\omega_1}{\omega_4})^{3\nu+3.5})$$

$$[f_2f_3f_4 + f_1f_3f_4 - f_1f_2f_4 - f_1f_2f_3]\, d\omega_2 d\omega_3 d\omega_4$$

By making the expression $(1 + (\frac{\omega_1}{\omega_2})^{3\nu+3.5} - (\frac{\omega_1}{\omega_3})^{3\nu+3.5} - (\frac{\omega_1}{\omega_4})^{3\nu+3.5})$ vanish we can obtain the so called Kolmogorov-Zacharov solutions. To do this we set $-3\nu - 3.5 \in \{0, 1\}$. This corresponds to choosing $\nu \in \{-3/2, -7/6\}$.

4 Future Work

In the future, we want to continue exploring the following questions:

1. Is there fifth power law solutions? Based on our current result, the answer is "No".
2. For which class of f we could make every step of transforming WKE into the isotropic equation mathematically rigorously?
3. Is the solution $f(\omega) := \omega^{-\frac{7}{6}}$ mathematically valid? In other words, if we plug it into WKE, would the integral converge? Based on our current result, the answer is "Yes".

Given our current progress, we are expected to publish those new results someday.

5 References

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