Using Infinite Games with Perfect Information to Characterize the Product Perfect Set Property

Terry Guan
Mentor: Iian Smythe, University of Michigan

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Abstract

A set A is said to have the product perfect set property if there exist perfect sets P, Q such that $P \times Q \subseteq A$. Our goal is to create a game that characterizes this property. While we were able to produce a game that characterizes the product perfect set property under certain conditions, we weren't able to produce such a game for any given condition of the payoff set. However, we were able to narrow down the properties of such a game, if it were to exist. We also explore the possibility and methodology of showing that no such game exists as a possible avenue for future research.

1 Introduction

The study of games as a mathematical object began in 1913 with Zermelo's analysis of real-life, finite games like chess and Go. The study of games were then extended to infinite games by the likes of Banach and Mazur in the 1930's and Gale and Stewart in the 1950's. Infinite games became a rich field of study during the 1960's and 1970's when the Axiom of Determinacy, which postulates that every game is determined, was introduced. The power of infinite games lie in their ability to provide a paradigm that can emphasize the dichotomies of certain properties. An example of such a property is the perfect set property, which a set A is said to have if A is either countable or contains a perfect set. This property provides a dichotomy that is emphasized by the perfect set game. Our research is focused on the product perfect set property, where a set A is said to have if there exist perfect sets P, Q such that $P \times Q \subseteq A$. This property has been previously researched by the likes of Galvin who showed a set $P \subseteq X$ where X is a nonempty perfect Polish space and where P has the Baire property and be non-meager, then P contains a product of perfect sets [4]. However, it is not known if there is a game that characterizes the product perfect set property. Our goal is to create such a game. We wish, in doing so, to nicely characterize the property's negation so to provide a dichotomy of the property as in the case of the perfect set game.

2 Background

This section begins by first outlining some notation that will be used throughout the paper. ω denotes the set of natural numbers $\{0, 1, 2, \ldots\}$. A natural number n denotes the set of all natural numbers less than it, such as $2 = \{0, 1\}$. The set X^Y denotes the set $\{f : Y \to X : \ f \text{ a function}\}$. An example of such a set is 2^ω, which denotes the set of functions $f : \omega \to 2$. An equivalent definition of X^ω is the set of sequences of X, so 2^ω is the set of binary sequences. We will denote $X^{<\omega} = \{f : n \to X : \ f \text{ a function} \wedge n \in \omega\}$. Equivalently, $X^{<\omega}$ is the set of finite sequences of elements of X, so $2^{<\omega}$ is the set of finite binary sequences. Let π_0 denote the projection function in the first coordinate and π_1 denote the projection function in the second coordinate. π_0 takes as input some two dimensional object and outputs the first coordinate of that object. Define π_1 similarly. Examples would be $\pi_0((0,1)) = 0, \pi_1((0,1)) = 1$. In addition, for some set set of pairs A, we say $\pi_0[A] = \{x : \exists y((x,y) \in A)\}, \pi_1[A] = \{y : \exists x((x,y) \in A)\}$ as the set of first and second coordinate projections of A, respectively. We will denote finite sequences with $\langle\rangle$ such as $\langle 0,0,0,1\rangle$. For any two sequences s, t, we denote the concatenation of the sequences $s \frown t$ e.g $\langle 0,0,0\rangle \frown \langle 1,1,1\rangle = \langle 0,0,0,1,1,1\rangle$.

A *Gale-Stewart game* is a two player game, in the natural sense, where players alternate playing an element from a given set X for a countably infinite number of turns with player I starting first. We denote this game as G_X. These games are *perfect information* games; that is, at any given move of a player, the player has access to all the information on prior moves made by both players. For convenience, we will refer to Gale-Stewart games as simply games. For the scope of our research, we will be considering only games where players play elements from 2, 2×2, $2^{<\omega}$, and $(2^{<\omega})^2$. The outcome of a game is the infinite sequence produced by concatenating the moves of both players in the order they were played. Associated with each game is a *payoff set* $A \subseteq X^\omega$. Player I wins if the outcome is in A, otherwise Player II wins. We denote a game G_X with its payoff set A as $G_X(A)$.

player I	s_0		s_2		$\ldots$
player II		k_1		k_3	$\ldots$

Figure 1: Note that player I's moves are even indexed and player II's moves are odd indexed. Player I wins if $\langle s_0, k_1, s_2, k_3, \ldots\rangle \in A$. Otherwise, player II wins.

We refer to the outcome $\langle s_0, k_1, s_2, k_3, \ldots\rangle$ of a game as a *play* of the game. An initial segment of the outcome is referred to as a *partial play*. A player's *strategy* dictates what element to play given the previous moves. Formally, a strategy is a function $\sigma : \bigcup_n X^{2n+k} \to X$ where $k = 0$ if σ is a strategy for player I, and $k = 1$ if σ is a strategy for player II. Let $y \in X^\omega$ be a sequence, or enumeration, of moves played at each turn by player II. For player I's strategy

I	$\sigma(\emptyset)$		$\sigma(\langle\sigma(\emptyset), y_1\rangle)$		$\sigma(\langle\sigma(\emptyset), y_1, \sigma(\sigma(\emptyset), y_1), y_3\rangle)$		$\dots$
II		y_1		y_3		y_5	$\cdots$

Figure 2: A play of the game where player I has strategy σ and player II is playing $y = \langle y_1, y_3, y_5, \ldots\rangle$

σ, we denote $\sigma * y$ as the outcome of the game where player I plays according to the strategy σ against player II's moves y. Similarly, if τ is a strategy for player II and $z \in X^\omega$ is a sequence of moves played at each turn by player I, then $\tau * z$ is the outcome of the game where player II plays according to τ against player I's moves z. We denote the set of outcomes of a strategy σ as $[\sigma] = \{\sigma * y : y \in X^\omega\}$.

Definition 2.1. A strategy σ for player I is said to be *winning* if $[\sigma] \subseteq A$. Similarly, a strategy τ for player II is said to be winning if $[\tau] \subseteq A^c$.

In essence, a winning strategy guarantees that a player will win, regardless of what the opposing player plays.

Definition 2.2. A game $G_X(A)$ is said to be *determined* iff one of the players has a winning strategy.

We will often for convenience say that A is determined, dropping the G_X. An intuitive observation is player I and player II can't both have winning strategies. Otherwise, if these strategies played each other, one must lose, thereby contradicting its winningness. There are also more interesting and significant observations made about games: we let $O(s) = \{x \in X^\omega : s \sqsubseteq x\}$, where $s \sqsubseteq x$ denotes s as an initial segment or initial subsequence of x. $O(s)$ is the set of sequences x where s is an initial segment of it. We define a set $A \subseteq X^\omega$ *open* iff $A = \bigcup_s O(s)$. We say that A is *closed* iff $X^\omega - A$ is open. There are two significant results in regards to closed and open payoff sets.

Theorem 2.1 (Gale-Stewart, [2])**.** *For $A \subseteq X^\omega$, If A is either open or closed, then $G_X(A)$ is determined.*

Theorem 2.2 (Gale-Stewart, [2])**.** *For any set X with at least two elements, there is an $A \subseteq X^\omega$ such that $G_X(A)$ is not determined.*

Associated with a strategy σ is a rooted tree, called the *strategy tree* and denoted T_σ, which illustrates σ response to a given opposing player's moves. T_σ is partially ordered by initial segment with a unique least element, the root, and where every set of predecessors is well-ordered. Refer to Figure 3 for an example of a strategy tree of a player II strategy in a game with $X = 2 \times 2$. We denote the set of infinite paths of a strategy tree T_σ starting from the root, often denoted $\emptyset$, as $[T_\sigma]$. Note that $[\sigma] = [T_\sigma]$ since an infinite path of T_σ starting at the root is exactly an infinite sequence of player I and player II moves, the outcome of a game.

Definition 2.3. A tree is *perfect* if for all nodes t, there exist s_0, s_1 such that $t \sqsubseteq s_0, t \sqsubseteq s_1$ and s_0, s_1 are not comparable, or not initial segments of each other. Equivalently, a tree is *perfect* if every node has a descendant with at least two children.

We will also denote the level-wise product of two trees S, T as $S \otimes T$. The level-wise product of trees S, T is defined as $S \otimes T = \bigcup_{n \in \omega}(S(n) \times T(n))$ where $S(n) = \{s \in S : s \text{ on level } n\}, T(n) = \{t \in T : t \text{ on level } n\}$ where a node is said to be on the nth level if there is a path from the root to the node that is n nodes long. Note that $[S \otimes T] = [S] \times [T]$.

2.1 Previously Studied Games

Two previously analyzed games are the *binary game*, by Mazur [5], and the *perfect set game* by Davis [1]. In the binary game, denoted $G_2(A)$, each player takes turns playing from 2. $G_2(A)$ has interesting properties in regards to its player strategies and determinacy. Notably, if σ is a strategy for either player, then $[T_\sigma]$ is nonempty and perfect. Since T_σ contains all possible moves for the opposing player and the possible moves are from 2, every other level of the tree contains a node with two children. We observe that if $G_2(A)$ is determined, one of A or A^c contains a nonempty, perfect set. By σ being winning, $[T_\sigma] = [\sigma] \subseteq A$, so A contains a nonempty, perfect set. If player II has a winning strategy τ, by the same argument, A^c contains a nonempty, perfect set.

We denote the perfect set game G_2^*. In G_2^*, player I plays elements from $2^{<\omega}$, the set of finite binary sequences, and player II plays from 2. It has been shown that

Theorem 2.3 (Davis, [1]). *For $A \subseteq 2^\omega$,*

a) Player I has a winning strategy in $G_2^*(A)$ iff A contains a perfect subset

b) Player II has a winning strategy in $G_2^*(A)$ iff A is countable

These games are of importance in our research as they establish relationships between specific games and perfect sets. In particular, the perfect set game establishes a strong characterization of the perfect set property.

3 The Binary Squared Game

Our first attempt to constructing a game was extending the binary game to two dimensions. Each players play elements from 2×2, or pairs of binary digits. Player I plays elements on even turns so their moves form the sequence

$((s_{2n})_{n\in\omega}, (t_{2n})_{n\in\omega})$. Player II plays elements on odd turns so their moves form the sequence $((s_{2n+1})_{n\in\omega}, (t_{2n+1})_{n\in\omega})$. The outcome of the game is the pair of concatenated sequences $((s_n)_{n\in\omega}, (t_n)_{n\in\omega})$, with player I and player II moves alternating. We denote this game with payoff set $A \subseteq (2^\omega)^2$ as $G_{2\times 2}(A)$. Our initial investigation into $G_{2\times 2}$ starts with considering when payoff sets are products of sets.

Theorem 3.1. *Player I has a winning strategy in $G_2(A)$ and $G_2(B)$ iff Player I has a winning strategy in $G_{2\times 2}(A \times B)$*

Proof. $\Rightarrow$: Let σ_A, σ_B be the winning strategies for $G_2(A), G_2(B)$ respectively. Consider σ, where for any partial play s ending with player II's move $\sigma(s) = (\sigma_A(\pi_0(s)), \sigma_B(\pi_1(s)))$. Then, for any sequence of player II's move y, we have $\sigma * y = (\sigma_A * \pi_0(y), \sigma_B * \pi_1(y))$. Since σ_A, σ_B are winning strategies in their respective games, $\sigma_A * \pi_0(y) \in A$ and $\sigma_B * \pi_1(y) \in B$, so $\sigma * y \in A \times B$.
$\Leftarrow$: Suppose that σ is a winning strategy for $G_{2\times 2}(A \times B)$. Fix some $q \in 2^\omega$. Define τ, a strategy for player I in $G_2(A)$, as

$$\tau(\langle s_0, t_1, \ldots, s_{2n}, t_{2n+1}\rangle) = \pi_0 \circ \sigma(\langle s_0, t_1, \ldots, s_{2n}, t_{2n+1}\rangle, \langle q_0, \ldots, q_{2n+1}\rangle)$$

where $\langle s_0, t_1, \ldots, s_{2n}, t_{2n+1}\rangle$ is a partial play of $G_2(A)$ ending with player II's move. τ plays according to σ in the first coordinate when σ plays against q in the second coordinate. τ is a winning strategy since for any $p \in 2^\omega$, $\tau * p = \pi_0 \circ (\sigma * (p, q))$ where $\sigma * (p, q) \in A \times B$ because σ is winning, so $\tau * p \in A$. Thus, $\{\tau * p : p \in 2^\omega\} \subseteq A$. $G_2(B)$ follows similarly with the following alteration: define $\tau = \pi_1 \circ \sigma(\langle q_0, \ldots, q_{2n+1}\rangle, \langle s_0, t_1, \ldots, s_{2n}, t_{2n+1}\rangle)$. □

Theorem 3.2. *If player II has a winning strategy in one of $G_2(A), G_2(B)$, then II has a winning strategy in $G_{2\times 2}(A \times B)$.*

Proof. Suppose without loss of generality II has a winning strategy τ' in $G_2(B)$. Define τ to be the strategy for II in $G_{2\times 2}(A \times B)$ that always plays 0 in the first coordinate and according to τ' in the second coordinate, so for a partial play m ending with player I's move, $\tau(m) = (0, \tau'(\pi_1(m))$. Then, for any sequence of I's move $z \in (2^\omega)^2$, $\tau * z = ((x_i), (y_i)) \notin A \times B$ since $(y_i) \notin B$. □

Theorem 3.3. *If at least one of $G_2(A), G_2(B)$ is determined, and II has a winning strategy for $G_{2\times 2}(A \times B)$, then II has a winning strategy for one of $G_2(A)$ or $G_2(B)$.*

Proof. If $G_2(A), G_2(B)$ are both determined: Suppose that II does not have a winning strategy in either $G_2(A)$ or $G_2(B)$. Since they are determined, I has a winning strategy in both, so I has a winning strategy in $G_{2\times 2}(A \times B)$, contradicting the assumption that II has a winning strategy.
If only one of $G_2(A)$ or $G_2(B)$ is determined: without loss of generality, suppose that $G_2(A)$ is determined and $G_2(B)$ is undetermined. Then, one of player I or II has a winning strategy in $G_2(A)$. Suppose that player I has winning strategy σ in $G_2(A)$. Let τ be II's winning strategy in $G_{2\times 2}(A \times B)$. Since $G_2(B)$ is

undetermined, every strategy for II has at least one outcome in B. Consequently, τ cannot be a strategy such that for all $b \in 2^\omega$, $\tau * (a, b) = (x, y)$ for $y \in B$. Otherwise, by similar methods as in Theorem 3.1, τ can be used to construct a strategy in $G_2(B)$ that is winning. So, we have that there exists $b \in 2^\omega$ where when τ plays against b in the second coordinate its outcome y is in B. Suppose that there exists $a \in 2^\omega$ such that $\tau * (a, b) = (x, y) \in A \times B$. Then τ is not winning, so it must be the case that $\forall a \in 2^\omega, \tau * (a, b) \in A^c \times B$. Let x_σ be the sequence of moves played by σ when it plays against the moves played by τ in the first coordinate. Then, $\tau * (x_\sigma, b) \in A^c \times B$, so $x_\sigma \in A^c$, contradicting σ being winning. By determinacy in $G_2(A)$, player II has a winning strategy. □

It is a noteworthy observation that player I having a winning strategy in $G_{2\times2}(A \times B)$ implies determinacy of $G_2(A), G_2(B)$ while determinacy is being used to say something meaningful for when II has a winning strategy in $G_{2\times2}(A \times B)$.

$G_{2\times2}(A)$ is a a game that describes a method for obtaining products of perfect sets for particular payoff sets. But, the statement does not generally hold true for all strategies. In fact, if σ is any strategy for one of the players, there won't necessarily exist a product of perfect trees in the payoff set. Consider the following: Let σ be a strategy for player II that plays as such:

1. player I plays $(0, 0)$, σ plays $(0, 0)$
2. player I plays $(0, 1)$, σ plays $(1, 0)$
3. player I plays $(1, 0)$, σ plays $(0, 1)$
4. player I plays $(1, 1)$, σ plays $(1, 1)$

Theorem 3.4. *There does not exist any nonempty, perfect $P, Q \subseteq 2^{<\omega}$ such that $P \otimes Q \subseteq T_\sigma$.*

Proof. Suppose that there exists such S, T. We claim that if $S \otimes T$ is contained in T_σ and S, T are pruned, that is a tree with no leaves, then they both must be a single branch. In particular, they cannot be perfect.

Suppose it were not the case that S, T has only one branch. Then, there exists node $p \in S$ with two child nodes. Suppose that p is in an odd level k. Let c_1, c_2 be p children which are in even level $k + 1$. Suppose that $n \in T$ is a node in level $k + 1$. We know such node exists because S, T are both infinite. Then, we have pairs $(c_1, n), (c_2, n) \in S \otimes T$ in level $k + 1$. The even level of T_σ contain pairs of sequences of the form:

- $\langle (\ldots 00), (\ldots 00) \rangle$
- $\langle (\ldots 01), (\ldots 10) \rangle$
- $\langle (\ldots 10), (\ldots 01) \rangle$
- $\langle (\ldots 11), (\ldots 11) \rangle$

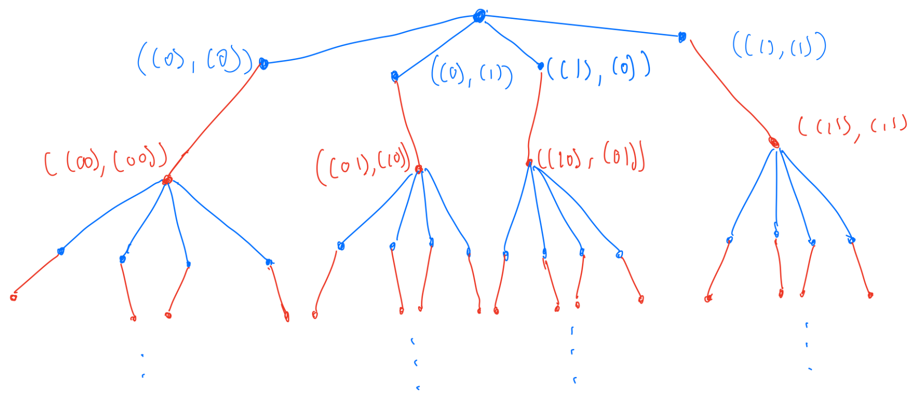

Figure 3: T_σ. The 0th level is the root, often denoted $\emptyset$. Blue is player I's moves. Red is player II's moves. Note that all possibilities of player I's moves at any given turn are considered in the tree.

Since $S \otimes T \subseteq T_\sigma$ and $k+1$ is even, n corresponds to one of the sequences in the right coordinate. Notice that for any sequence in the right coordinate, it is matched to a unique sequence in the left coordinate. So, if $(c_1, n), (c_2, n) \in S \otimes T$, then $c_1 = c_2$, but $c_1 \neq c_2$, a contradiction. Consequently, p cannot be in an odd level.

Suppose then that p is in an even level k. Let c_1, c_2 be the children of p which are in odd level $k+1$. Let g_1, g_2 be children of c_1, c_2 respectively, which are in even level $k+2$. The argument proceeds just as before: since T infinite, there exists node n in level $k+2$. So, we have $(g_1, n), (g_2, n) \in S \otimes T$. Since $(g_1, n), (g_2, n)$ are on an even level in T_σ, n is a sequence in the right coordinate of one of the above listed pairs, so it is uniquely matched. Then, $g_1 = g_2$, but $g_1 \neq g_2$, a contradiction. Thus, p cannot belong in either odd or even levels, so there cannot exist a node in S with two children. The same argument applies for T.

Thus, S, T contain only a single branch, but this contradicts S, T being perfect, so there cannot be any nonempty, perfect S, T such that $S \otimes T \subseteq T_\sigma$ for σ a strategy for player II.

The argument can be adapted for player I. Define σ as above, interchanging the roles of player I and player II and having player I make an arbitrary first move. The argument follows similarly, with the even and odd levels interchanged. $\square$

Thus, $G_{2\times 2}(A)$ is not a game that sufficiently characterizes the product perfect set property. These results naturally lead to a class of strategies with interesting properties.

4 Independent strategies

Definition 4.1. A strategy σ is *independent* if there exist unique strategies σ_x, σ_y such that for all $n \in \omega$.

$$\sigma(\langle(x_0, y_0), \ldots (x_n, y_n)\rangle) = (x_{n+1}, y_{n+1}) = (\sigma_x(\langle x_0, \ldots, x_n\rangle, \sigma_y(\langle y_0, \ldots, y_n\rangle)$$

If σ is a strategy for player I, then $n = 2k + 1$ for all $k \in \omega$. If σ is a strategy for player II, then $n = 2k$ for all $k \in \omega$. We say that $\sigma = (\sigma_x, \sigma_y)$.

Theorem 4.1. *If σ is an independent strategy for game $G_X(A)$ on any set X, then $[T_\sigma] = [T_{\sigma_x}] \times [T_{\sigma_y}]$.*

Proof. $[T_\sigma] \subseteq [T_{\sigma_x}] \times [T_{\sigma_y}]$ follows from the definition of σ. By independence, $\sigma = (\sigma_x, \sigma_y)$ so for any outcome $(p, q) \in [T_\sigma]$, we have that $(p, q) = (\sigma_x * a, \sigma_x * b)$ for some $a, b \in X^\omega$, so $(p, q) \in [T_{\sigma_x}] \times [T_{\sigma_y}]$.
It remains to show to the other containment. Suppose σ is a strategy for player I and that there exists $(p, q) \in [T_{\sigma_x}] \times [T_{\sigma_y}]$ such that $(p, q) \notin [T_\sigma]$. Then, there exists a pair of finite sequences $(s, t) \in X^{<\omega}$ with length n where (s, t) is an initial segment of some element in $[T_\sigma]$ but $(s \frown s_{n+1}, t \frown t_{n+1}) \not\sqsubseteq (p, q)$ for $(s_{n+1}, t_{n+1}) \in T_\sigma$. (s, t) is the shortest initial segment where the next element in its sequence according to T_σ deviates from (p, q). $(s, t) \neq (\emptyset, \emptyset)$ since $\sigma(\emptyset) = (\sigma_x(\emptyset), \sigma_y(\emptyset)) \sqsubseteq (s, t)$. s_{n+1}, t_{n+1} are also not moves by player II since all of player II moves are included in T_σ by definition, so s_{n+1}, t_{n+1} must be within a move made by player I. Let $s' \sqsubseteq s, t' \sqsubseteq t$ denote the sequences where the last element in the sequences is the last move played by player II in s, t respectively. Then, s_{n+1}, t_{n+1} are elements played according to $\sigma_x(s'), \sigma_y(t')$ respectively, so $s \frown s_{n+1} \sqsubseteq s' \frown \sigma_x(s')$ and $t \frown t_{n+1} \sqsubseteq t' \frown \sigma_y(t')$. We have that $s' \frown \sigma_x(s') \sqsubseteq p$ since $s' \sqsubseteq p$ and by definition p contains player I's response to s', namely $\sigma_x(s')$. Similarly, $t' \frown \sigma_x(t') \sqsubseteq q$. So, $s \frown s_{n+1} \sqsubseteq p, t \frown t_{n+1} \sqsubseteq q$, contradicting (s, t) being a sequence deviating from (p, q). In the case where σ is a strategy for player II, the argument is the same with the change that $(s, t) \neq (\emptyset, \emptyset)$ because T_σ contains all of player I's possible first moves and flipping player I and player II in the analysis. Thus, $[T_{\sigma_x}] \times [T_{\sigma_y}] \subseteq [T_\sigma]$. □

Corollary 4.1.1. *For σ an independent strategy, $T_\sigma = T_{\sigma_x} \otimes T_{\sigma_y}$.*

Proof. σ independent so $[T_\sigma] = [T_{\sigma_x}] \times [T_{\sigma_y}] = [T_{\sigma_x} \otimes T_{\sigma_y}]$. Thus, $T_\sigma = T_{\sigma_x} \otimes T_{\sigma_y}$. □

By considering independent strategies in $G_{2\times 2}$, we can better understand our previous results of products of sets to general sets:

Corollary 4.1.2. *If σ is winning independent strategy for player I in $G_{2\times 2}(A)$, then there exist nonempty, perfect sets $P, Q \subseteq 2^\omega$ such that $P \times Q \subseteq A$. If σ is a winning independent strategy for player II, then there exist nonempty, perfect sets $P, Q \subseteq 2^\omega$ such that $P \times Q \subseteq A^c$. Particularly, $P = T_{\sigma_x}, Q = T_{\sigma_y}$.*

Proof. Since σ independent, we have $\sigma = (\sigma_x, \sigma_y)$ for some σ_x, σ_y. Then, $[T_\sigma] = [T_{\sigma_x} \otimes T_{\sigma_y}] \subseteq A$ by σ being winning. If σ is a strategy for player II, then adapt the same argument but for A^c. □

We now see that independent strategies are the exact strategies needed to produce a product of perfect sets in $G_{2\times 2}$. In fact, the strategy provided for figure 3 is necessarily non-independent. We move on to our next game, which attempts to take the perfect set game to two dimensions.

5 Perfect Set Squared Game

The perfect set squared game refers to the game where player I plays elements of $(2^{<\omega})^2$ and player II plays elements of 2×2. We denote this game $G^*_{2\times 2}(A)$.

Theorem 5.1. *If player I has a winning strategy σ, then there exist nonempty $P \subseteq A$ perfect.*

Proof. We have that $[\sigma] \subseteq A$ by it being winning. $[\sigma]$ is perfect since in T_σ, after every one of player I's moves, player II creates four incomparable nodes, or four children. □

Note that the converse need not hold true. Consider the following complete binary tree in $(2^\omega)^2$ as A:

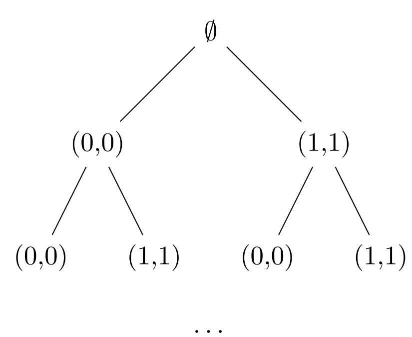

Figure 4: the left child at any junction is $(0,0)$ and the right child is $(1,1)$. The tree extends infinitly.

A itself is a perfect tree, but player II has a winning strategy: play one of $(0,1), (1,0)$. In fact, we must enforce independence on player I's a winning strategy:

Theorem 5.2. *If there exist nonempty, perfect P, Q such that $P, Q \subseteq A$, then there exist a winning strategy for player I. In particular, the strategy is independent.*

Proof. We construct Player I winning strategy σ to play according to P and Q. For player I's $2n$th move, where the concatenation of the previous $2n-1$ moves is (s,t), player I plays $(p,q) \in P \times Q$ where $p,q \in 2^{<\omega}$, $s \frown p \in P, t \frown q \in Q$ and both p,q are sequences such that $p \frown 0, p \frown 1 \in P$ and $q \frown 0, q \frown 1 \in Q$. σ is winning:
At the 0th move of the game, $\sigma(\emptyset) = (a,b)$ for some $(a,b) \in (2^{<\omega})^2$ where both $a \frown 0, a \frown 1 \in P$ and $b \frown 0, b \frown 1 \in Q$. Player II plays either $(i,j) \in 2 \times 2$. The partial play of the game is $(a \frown i, b \frown j)$. By the condition on a,b, we have that $(a \frown i, b \frown j) \in P \times Q$.
Suppose that the partial play (s,t) up to the $2n$th move is in $P \times Q$ and player I up till now has played so that for any of its moves p,q, $p \frown 0, p \frown 1 \in P$ and $q \frown 0, q \frown 1 \in Q$. At the $2n+1$ move, player II plays from $(i,j) \in 2 \times 2$. By the condition on player I's moves, $(s \frown i, t \frown j) \in P \times Q$. At the $2n+2$th play of the game, σ plays a pair of sequences $(p,q) \in P \times Q$ such that $p \frown 0, p \frown 1 \in P$ and $q \frown 0, q \frown 1 \in Q$. We have up till moves $2(n+1)$ that $(s \frown i \frown p, t \frown j \frown q) \in P \times Q$. Thus, at every nth move of the game played with σ, the pair of sequences is in $P \times Q$, so σ winning. In particular, $T_\sigma = P \times Q$, so $[\sigma_x] = P, [\sigma_y] = Q$ and $\sigma = (\sigma_x, \sigma_y)$, thus σ is independent. □

Theorem 5.3. *Player I has a winning independent strategy σ in $G^*_{2\times 2}(A)$ iff there exist nonempty perfect $P, Q \subseteq 2^\omega$ such that $P \times Q \subseteq A$. In particular, $[T_{\sigma_x}] \times [T_{\sigma_y}] \subseteq A$.*

Proof. Suppose that player I has a winning independent strategy σ. By σ being independent, there exist strategies σ_x, σ_y such that $\sigma = (\sigma_x, \sigma_y)$. More importantly, $[T_\sigma] = [T_{\sigma_x}] \times [T_{\sigma_y}]$. Both σ_x, σ_y are winning strategies for player I in the perfect set game, namely $G^*_2(\pi_0[A])$ and $G^*_2(\pi_1[A])$, so $[T_{\sigma_x}], [T_{\sigma_y}]$ are perfect sets themselves. We have that $[T_\sigma]$ is perfect, and by σ being winning, $[T_\sigma] = [\sigma] \subseteq A$. □

Theorem 5.4. *If A is countable, then player II has a winning strategy.*

Proof. We proceed as in Kanamori [3]: enumerate and diagonalize. Since A is countable, we enumerate $A = \langle (a_i, b_i) : i \in \omega \rangle$. Player II plays the ith to ensure that it differs with (a_i, b_i). □

Player II winning strategy is not necessarily independent. Consider $A = \{(\langle 000\ldots\rangle, \langle 000\ldots\rangle), (\langle 111\ldots\rangle, \langle 111\ldots\rangle)\}$. A winning non-independent strategy for player II can be described as such:

1. I plays (0,0), II plays (1,1)
2. I plays (0, 1), II plays (0, 0)
3. I plays (1, 0), II plays (0, 1)
4. I plays (1, 1), II plays (1, 0)

The strategy is necessarily not independent since for each $i \in 2$ in the left coordinate, i flips between 0 and 1 depending on the value of the second coordinate.

Theorem 5.5. *If player II has a winning independent strategy τ, then A is countable.*

Proof. Since τ is independent, $\tau = (\tau_x, \tau_y)$. τ_x, τ_y are winning strategies for $G_2^*(\pi_0[A]), G_2^*(\pi_1[A])$ respectively, $\pi_0[A], \pi_1[A]$ are both countable. Then, $|A| \leq |\pi_0[A] \times \pi_1[A]|$, so we have that A is countable. $\square$

It is not true that in general player II having a strategy implies A being countable. Consider the complete binary tree as in figure 4. Player II has a winning strategy: after player I's move, play $(0,1)$ or $(1,0)$. But, $|A| = |2^\omega|$, so it is not countable.

$G_{2\times2}^*(A)$ characterizes the product perfect set property under the condition that the strategy be independent. However, this game does not nicely characterize the property since it only holds for particular strategies and not general strategies. Furthermore, it does not provide a nice characterization of the property's negation unlike the perfect set game does for the perfect set property.

6 Conclusion

While we were able to produce a game that characterizes the product perfect set property under certain conditions, we weren't able to produce such a game for any given condition of the payoff set. We narrowed down the properties of such a game, if such a game exists. The game must have player I playing elements from $2^{<\omega}$ and player II playing at least from 2×2, possibly $2^{<\omega}$. Player I must play from $2^{<\omega}$ because if player I could only play from 2×2, then payoff set A could be constructed as a perfect tree where the tree splits only at even level, forcing player I to choose one of the nodes present at the split, ensuring that the outcome was not a product of nonempty perfect sets. Player II controls the splitting for any of player I's strategy trees, so player II must play at least from 2×2. However, the more important property such a game would have to enforce is the structure of the trees the strategies produced. This is problematic because strategies only have access to information on a given partial play, not every possible partial play, so defining a game with this property would be against the definition of a strategy. A possible future avenue of our research is to show that there does not exist such a game by utilizing mechanics of descriptive set theory. In the perfect set game, we know that a set A having the perfect set property is Σ_2^1. In fact, the statement is Δ_2^1. As in the case of the perfect set game, we know that a set A having the product perfect set property is Σ_2^1, but it is not obvious that the statement is in fact Δ_2^1. This is related to the negation of the product perfect set property having multiple possibilities. If we can show that the statement of A having the product perfect set property necessarily not Δ_2^1 by showing that the statement is Σ_2^1-Hard, we can show that in fact there does not exist such a game.

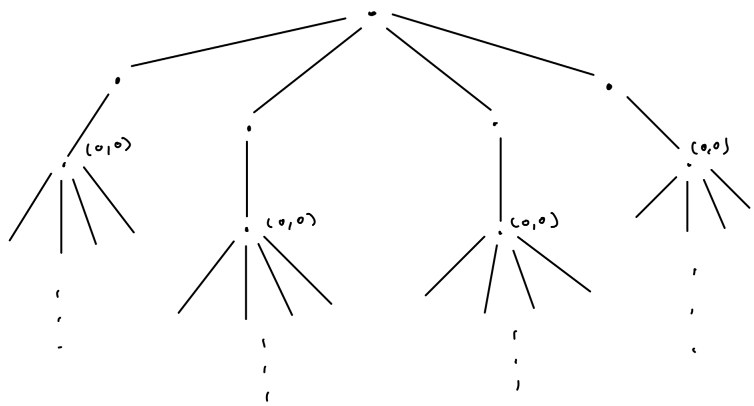

Figure 5: If player I played from 2×2, then a winning strategy would be play $(0, 0)$ for all moves. But, the strategy would not produce a perfect sets.

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