Right tail of two-point distribution of the KPZ fixed point

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Abstract

A formula for the KPZ fixed point of the multi-point TASEP distribution was very recently obtained in terms of a Fredholm determinant [Liu, 2021]. I will analyze certain right-tail asymptotics of this distribution using the steepest descent method. Specifically I will investigate the asymptotic regime where the height of the second point is a fixed multiple of the height of the first point, and we let both heights go to infinity.

1 Introduction

The TASEP (Totally Asymmetric Simple Exclusion Process) model can be formulated by considering particles which lie a subset of the points on the infinite lattice $\mathbb{Z}$. Each integer can contain at most one particle. Furthermore each particle is assigned an independent clock which rings after a wait time governed by an exponential random variable with parameter 1. Once a particle's clock rings, it moves one unit to the right if the next integer is not already occupied, otherwise it stays at the same integer. Then the particle's clock is immediately reset.

An equivalent way to model TASEP called the corner growth model, is to consider the evolution of some random height function $H(x,t)$. Let $\mathcal{H}$ denote the space of all function $h : \mathbb{Z} \to \mathbb{Z}$ satisfying

1. $h(x+1) - h(x) \in \{-1, 1\}$ for all $x \in \mathbb{Z}$
2. $h(0) \in 2\mathbb{Z}$

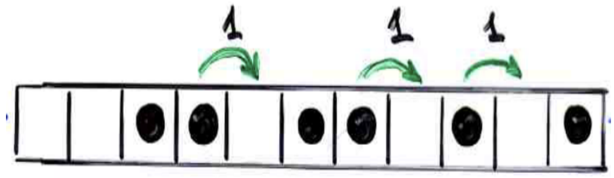


Figure 1: TASEP modeled by evolution of particles in $\mathbb{Z}$

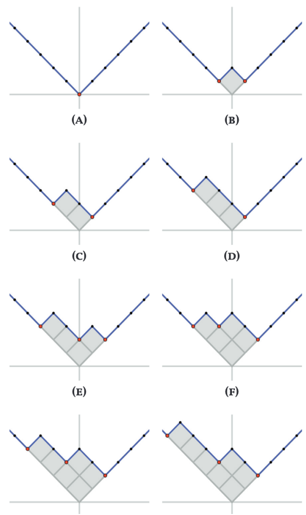


Figure 2: TASEP modeled by evolution of height function $H(x,t)$

We start with an initial height function $H(x,0)$ and evolve $H(x,t)$ by assigning each integer an independent clock with wait time given by an exponential random variable with parameter 1. When the clock associated with i rings, we increase $H(i,t)$ by 2 and keep $H(x,t)$ the same for all other x if the resulting function $H(x,t)$ belongs to $\mathcal{H}$, otherwise we leave $H(x,t)$ unchanged. Then we reset the clock.

We will study this height function with the initial condition $H(x,0) = |x|$ referred to as the "step" initial condition. Figure 2 shows an example of how $H(x,t)$ could evolve.

It is of interest to study the limiting behaviour of the height function $H(x,t)$, and a particularly fruitful scaling to study is the $1:2:3$ scaling for which time is of order T, the space parameter is of order $T^{2/3}$ and the fluctuation of the height is of order $T^{1/3}$.

With this scale it was proven that

Theorem 1.1 ([3])**.** *For every* $(x,\tau,h) \in \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}$*,*

$$\lim_{T\to\infty} \boldsymbol{P}\left(\frac{H(2xT^{2/3}, 2\tau T) - \tau T}{-T^{1/3}} \leq h\right) = F_{step}(h,(x,\tau)) \tag{1.1}$$

Where $F_{step}(h,(x,\tau))$ is defined as a Fredholm determinant $\det(1-K)$. In order to define K, let C_L be an unbounded contour in the left half of the complex plane going from $\infty e^{-\frac{2}{3}\pi i}$ to $\infty e^{\frac{2}{3}\pi i}$, and C_R an unbounded contour in the right half of the complex plane going from $\infty e^{-\frac{\pi}{3}i}$ to $\infty e^{\frac{\pi}{3}i}$. Additionally define the measure $d\mu(z) := \frac{dz}{2\pi i}$. Then $K : L^2(C_L, d\mu) \to L^2(C_L, d\mu)$ is the kernel operator with kernel

$$K(z,z') = \int_{C_R} \exp(-\frac{1}{3}\tau(z^3-w^3)+x(z^2-w^2)+h(z-w)) \cdot \frac{1}{(z-w)(z'-w)}\,d\mu(w) \tag{1.2}$$

Using the identity $\det(1-AB)=\det(1-BA)$ which holds for Fredholm determinants under mild assumptions, it can be shown that $F_{step}(h,(0,1))$ is the Tracy-Widom distribution appearing in the asymptotic formula for the largest eigenvalue of GUE random matrices.
The right tail of $F_{step}(h,(x,\tau))$ is well known. For example, see [3]. It can be obtained from the series formula of the Fredholm determinant

$$\det(1-K) = 1+\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\int_{C_L}\cdots\int_{C_L}\det((K(z_i,z_j))_{i,j=1}^n)\,d\mu(z_1)\ldots d\mu(z_n) \tag{1.3}$$

Note that the trace is given by

$$tr(K) = \int_{C_L} K(z,z)d\mu(z) = \int_{C_L}\int_{C_R}\exp(-\frac{1}{3}\tau(z^3-w^3)+x(z^2-w^2)+h(z-w))\frac{1}{(z-w)^2}\frac{dw}{2\pi i}\frac{dz}{2\pi i} \tag{1.4}$$

Hence, if we set $\hat{h}=\tau^{1/3}(x^2\tau^{-1}+h)$, then,

$$\begin{aligned} tr(K) &= \\ &\int_{C_L}\int_{C_R}\exp(-\frac{1}{3}\tau(z^3-w^3)+x(z^2-w^2)+h(z-w))\frac{1}{(z-w)^2}\frac{dw}{2\pi i}\frac{dz}{2\pi i} \\ &= \int_{C_L}\int_{C_R}\exp(-\frac{1}{3}(\zeta^3-\omega^3)+\hat{h}(\zeta-\omega))\frac{1}{(\zeta-\omega)^2}\frac{d\omega}{2\pi i}\frac{d\zeta}{2\pi i} \\ &= \frac{1}{16\pi}\hat{h}^{-3/2}e^{-\frac{4}{3}\hat{h}^{3/2}}(1+\frac{35}{24}\hat{h}^{-3/2}+O(\hat{h}^{-3})) \end{aligned} \tag{1.5}$$

as $\hat{h}\to+\infty$, where the method of steepest-descent can be used to obtain the last equality. Furthermore it can be show that all the higher order terms in the determinant, have a smaller order contribution $O(e^{-\frac{8}{3}\hat{h}^{3/2}})$ so that

Theorem 1.2. *With* $\hat{h} := \tau^{1/3}(x^2\tau^{-1}+h)$,

$$F_{step}(h,(x,\tau)) = 1-\frac{1}{16\pi}\hat{h}^{-3/2}e^{-\frac{4}{3}\hat{h}^{3/2}}(1+\frac{35}{24}\hat{h}^{-3/2}+O(\hat{h}^{-3})) \tag{1.6}$$

Recently in [4], an analogous formula for the limiting joint distribution of m points was discovered. Going forward I will only consider the case when $m=2$ and so the result can be stated as

Theorem 1.3 ([4])**.** *The two point limiting distribution satisfies*

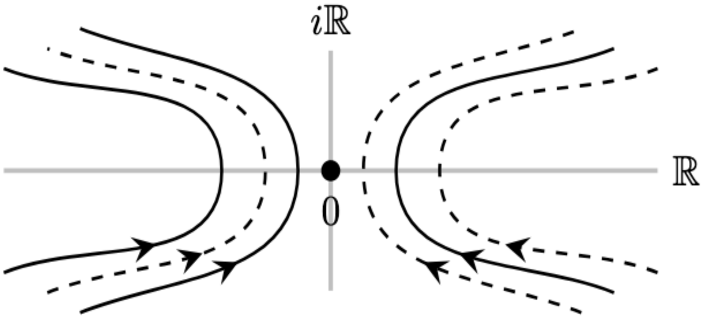
$$\lim_{T\to\infty}\boldsymbol{P}\left(\left\{\frac{H(2x_1T^{2/3},2\tau_1T)-\tau_1T}{-T^{1/3}}\le h_1\right\}\bigcap\left\{\frac{H(2x_2T^{2/3},2\tau_2T)-\tau_2T}{-T^{1/3}}\le h_2\right\}\right) = F_{step}(h_1,h_2,(x_1,\tau_1),(x_2,\tau_2)) \tag{1.7}$$

where $F_{step}(h_1,h_2,(x_1,\tau_1),(x_2,\tau_2))$ *will be defined in the next section.*

The primary objective will be to obtain an analogous result to Theorem 1.2 by calculating certain asymptotics for $F_{step}(h_1, h_2, (x_1, \tau_1), (x_2, \tau_2))$ as $h_1, h_2 \to +\infty$. I will only carry out these calculations with several restrictions on the parameters described in section 2.3.

2 Asymptotic Analysis of F_{step} for $m = 2$

2.1 Definition of F_{step}



Let $C_{2,L}^{in}$, $C_{1,L}$, $C_{2,L}^{out}$ denote three unbounded contours in the left half of the complex plane going from $\infty e^{-2\pi i/3}$ to $\infty e^{2\pi i/3}$ which are nested from innermost to outermost (or equivalently from left to right). Similarly let $C_{2,R}^{in}$, $C_{1,R}$, $C_{2,R}^{out}$ denote three unbounded contours in the left half of the complex plane going from $\infty e^{-\pi i/3}$ to $\infty e^{\pi i/3}$ which are nested from innermost to outermost (or equivalently from left to right).

For convenience we denote

$$C_{2,L} := C_{2,L}^{in} \cup C_{2,L}^{out} \text{ and } C_{2,R} := C_{2,R}^{in} \cup C_{2,R}^{out},$$

$$S_1 := C_{1,L} \cup C_{2,R},$$

and

$$S_2 := C_{1,R} \cup C_{2,L}$$

We also introduce the measure

$$d\mu(w) := \begin{cases} \frac{-z_1}{1-z_1}\, dz, & w \in C_{2,L}^{out} \cup C_{2,R}^{out} \\ \frac{1}{1-z_1}\, dz, & w \in C_{2,L}^{in} \cup C_{2,R}^{in} \\ \frac{dz}{2\pi i} & w \in C_{1,L} \cup C_{1,R} \end{cases}$$

With this setup we define $D_{step}(z_1)$ as the following Fredholm determinant

Definition 2.1.

$$D_{step}(z_1) := \det\left(1 - K_1 K_{step}\right)$$

where the operators

$$K_1 : L^2(S_2, d\mu) \to L^2(S_1, d\mu), \qquad K_{step} : L^2(S_1, d\mu) \to L^2(S_2, d\mu)$$

are integral operators with kernels

$$K_1(w,w') = \begin{cases} \exp(-\frac{1}{3}\tau_1 w^3 + x_1 w^2 + h_1 w)\frac{1}{w-w'}(1-z_1), & w \in C_{1,L}, w' \in C_{1,R} \\ \exp(-\frac{1}{3}\tau_1 w^3 + x_1 w^2 + h_1 w)\frac{1}{w-w'}(1-z_1^{-1}), & w \in C_{1,L}, w' \in C_{2,L} \\ \exp(\frac{1}{3}(\tau_2-\tau_1) w^3 - (x_2-x_1) w^2 - (h_2-h_1) w)\frac{1}{w-w'}(1-z_1), & w \in C_{2,R}, w' \in C_{1,R} \\ \exp(\frac{1}{3}(\tau_2-\tau_1) w^3 - (x_2-x_1) w^2 - (h_2-h_1) w)\frac{1}{w-w'}(1-z_1^{-1}), & w \in C_{2,R}, w' \in C_{2,L} \end{cases}$$

and

$$K_{step}(w',w) = \begin{cases} \exp(\frac{1}{3}\tau_1 (w')^3 + x_1 (w')^2 + h_1 w')\frac{1}{w-w'}, & w \in C_{1,L}, w' \in C_{1,R} \\ \exp(-\frac{1}{3}(\tau_2-\tau_1)(w')^3 + (x_2-x_1)(w')^2 + (h_2-h_1)w')\frac{1}{w-w'}, & w \in C_{2,R}, w' \in C_{2,L} \\ 0 & otherwise \end{cases}$$

Finally, the formula of F_{step} obtained in [Liu, 21] is the following:

Definition 2.2. *We have*

$$F_{step}(h_1,h_2,(x_1,\tau_1),(x_2,\tau_2)) := \oint_{|z_1|=1/2} \frac{1}{1-z_1} D_{step}(z_1)\,\frac{dz_1}{2\pi i z_1} \tag{2.1}$$

where the contour integral is oriented counter-clockwise.

Going forward I will let $K := K_1 K_{step}$

2.2 Main Conjecture

We are interested in the right tail of the two-point function

$$F_{step}((h_1,(x_1,\tau_1)),(h_1+h_2,(x_1+x_2,\tau_1+\tau_2)))$$

We will focus only on the case when $x_1 = x_2 = 0$. Furthermore, we will consider the case that

$$h_2 = \alpha h_1, \quad \tau_1 = \alpha \tau_2$$

for a fixed constant $\alpha \in (0,1)$

Going forward it will also be convenient to define the quantities $\hat{h}_1 := \tau_1^{-1/3} h_1$ and $\hat{h}_2 := \tau_2^{-1/3} h_2$. It is worth noting that $\hat{h}_2 = \alpha^{4/3}\hat{h}_1 < \hat{h}_1$.

With these constraints I will be interested in the asymptotics of $F_{step}((h_1,(x_1,\tau_1)),(h_1+h_2,(x_1+x_2,\tau_1+\tau_2))$ as $h_1 \to +\infty$. I will provide calculations which will lead to the following asymptotic formula.

Conjecture 2.3. *Taking* $h_2 = \alpha h_1$ *and* $\tau_1 = \alpha\tau_2$ *for a fixed* $0 < \alpha < 1$ *and letting* $h_1 \to +\infty$ *we have*

$$\begin{aligned} &F_{step}(h_1, h_1+h_2, (0,\tau_1),(0,\tau_1+\tau_2)) = \\ &F_{step}(h_1,(0,\tau_1)) - \frac{3}{64\pi}\alpha^{-1}\hat{h}_1^{-3/2} e^{-\frac{4}{3}(1+\alpha^2)\hat{h}_1^{3/2}}(1+O(\hat{h}_1^{-3/4})) \end{aligned} \tag{2.2}$$

where $\hat{h}_1 := \tau_1^{-1/3} h_1$

The statement above is labeled as a conjecture, because although the calculations are detailed, there are error terms appearing in the analysis which have been neglected.

As an added consequence if this conjecture holds then we can compute the limiting conditional probability

Conjecture 2.4. *Taking $h_2 = \alpha h_1$ and $\tau_1 = \alpha\tau_2$ for a fixed $0 < \alpha < 1$ and letting $h_1 \to +\infty$ we have*

$$\begin{aligned}
&\lim_{T\to\infty} \boldsymbol{P}\left(\frac{H(0,2\tau_2 T) - \tau_2 T}{-T^{1/3}} \leq h_2 \,\middle|\, \frac{H(0,2\tau_1 T) - \tau_1 T}{-T^{1/3}} \leq h_1\right) \\
&= 1 - \frac{3}{64\pi}\alpha^{-1}\hat{h}_1^{-3/2} e^{-\frac{4}{3}(1+\alpha^2)\hat{h}_1^{3/2}}(1 + O(\hat{h}_1^{-3/4}))
\end{aligned} \tag{2.3}$$

where $\hat{h}_1 := \tau_1^{-1/3} h_1$

Proof of Conjecture 2.4 assuming Conjecture 2.3 holds.

Simply apply Baye's Rule.

$$\begin{aligned}
&\lim_{T\to\infty} \mathbf{P}\left(\tfrac{H(2x_2T^{2/3},2\tau_2 T)-\tau_2 T}{-T^{1/3}} \leq h_2 \,\middle|\, \tfrac{H(2x_1T^{2/3},2\tau_1 T)-\tau_1 T}{-T^{1/3}} \leq h_1\right) \\
&= \tfrac{F_{step}(h_1,h_2,(x_1,\tau_1),(x_2,\tau_2))}{F_{step}(h_1,(x_1,\tau_1))} \\
&= 1 + \tfrac{-\frac{3}{64\pi}\alpha^{-1}\hat{h}_1^{-3/2}e^{-\frac{4}{3}(1+\alpha^2)\hat{h}_1^{3/2}}(1+O(\hat{h}_1^{-3/4}))}{1-O(\hat{h}_1^{-3/2}e^{-\frac{4}{3}\hat{h}_1^{3/2}})} \\
&= 1 - \tfrac{3}{64\pi}\alpha^{-1}\hat{h}_1^{-3/2}e^{-\frac{4}{3}(1+\alpha^2)\hat{h}_1^{3/2}}(1 + O(\hat{h}_1^{-3/4}))
\end{aligned}$$

□

2.3 Trace Computation

The main result of this section is

Proposition 2.5.

$$tr(K) = (1 - z_1)J_1 + (1 - z_1^{-1})J_2 \tag{2.4}$$

where

$$J_1 := \int_{C_{1,L}}\int_{C_{1,R}} \exp(-\frac{1}{3}\tau_1(w^3 - z^3) + h_1(w - z))\frac{1}{(z-w)^2}\frac{dz}{2\pi i}\frac{dw}{2\pi i} \tag{2.5}$$

and

$$J_2 := \int_{C_{1,R}}\int_{C_{1,L}} \exp(-\frac{1}{3}\tau_2(z^3 - w^3) + h_2(z - w))\frac{1}{(w-z)^2}\frac{dz}{2\pi i}\frac{dw}{2\pi i} \tag{2.6}$$

Proof. Simply unraveling the definitions and recalling that for $K_{step}(z, w) \neq 0$ we must have $w \in C_{1,L}$ and $z \in C_{1,R}$ or $w \in C_{2,R}$ and $z \in C_{2,L}$,

$$\begin{aligned}
tr(K) &= \int_{S_1}\int_{S_2} K_1(w, z)K_{step}(z, w)\, d\mu(z)\, d\mu(w) \\
&= I_1 + I_2 + I_3 + I_4 + I_5
\end{aligned}$$

where

$$
\begin{aligned}
I_1 &:= \int_{C_{1,L}} \int_{C_{1,R}} K_1(w,z) K_{step}(z,w) \, d\mu(z) \, d\mu(w) \\
&= \int_{C_{1,L}} \int_{1,R} \exp(-\frac{1}{3}\tau_1(w^3 - z^3) + h_1(w-z)) \frac{1}{(z-w)^2}(1-z_1) \frac{dz}{2\pi i} \frac{dw}{2\pi i} \\
&= (1-z_1) J_1
\end{aligned}
$$

$$
\begin{aligned}
I_2 &:= \int_{C_{2,R}^{in}} \int_{C_{2,L}^{in}} K_1(w,z) K_{step}(z,w) \, d\mu(z) \, d\mu(w) \\
&= \int_{C_{2,R}^{in}} \int_{C_{2,L}^{in}} \exp(-\frac{1}{3}\tau_2(z^3 - w^3) + h_2(z-w)) \\
&\cdot \frac{1}{(w-z)^2}(1-z_1^{-1}) \frac{dz}{2\pi i} \frac{dw}{2\pi i} \frac{1}{(1-z_1)^2} \\
&= (1-z_1^{-1}) \frac{1}{(1-z_1)^2} J_2
\end{aligned}
$$

where the last line follows from using Cauchy's theorem to deform the contours $C_{2,R}^{in}$ and $C_{2,L}^{in}$ to $C_{1,R}$ and $C_{1,L}$ respectively.
Similarly

$$
\begin{aligned}
I_3 &:= \int_{C_{2,R}^{in}} \int_{C_{2,L}^{out}} K_1(w,z) K_{step}(z,w) \, d\mu(z) \, d\mu(w) \\
&= (1-z_1^{-1}) \frac{-z_1}{(1-z_1)^2} J_2, \\
I_4 &:= \int_{C_{2,R}^{out}} \int_{C_{2,L}^{in}} K_1(w,z) K_{step}(z,w) \, d\mu(z) \, d\mu(w) \\
&= (1-z_1^{-1}) \frac{-z_1}{(1-z_1)^2} J_2,
\end{aligned}
$$

and

$$
\begin{aligned}
I_5 &:= \int_{C_{2,R}^{out}} \int_{C_{2,L}^{out}} K_1(w,z) K_{step}(z,w) \, d\mu(z) \, d\mu(w) \\
&= (1-z_1^{-1}) \left(\frac{-z_1}{1-z_1} \right)^2 J_2
\end{aligned}
$$

So summing everything,

$$
\begin{aligned}
I_1 + I_2 + I_3 + I_4 + I_5 &= (1-z_1) J_1 + (1-z_1^{-1}) \left(\frac{1}{1-z_1} + \frac{-z_1}{1-z_1} \right)^2 J_2 \\
&= (1-z_1) J_1 + (1-z_1^{-1}) J_2
\end{aligned}
$$

□

Corollary 2.6. *$tr(K)$ is completely independent of τ_2, h_2. Moreover keeping x_1, τ_1 fixed and letting $h_1 \to \infty$,*

$$
\begin{aligned}
&\oint_{|z_1|=1/2} \frac{1}{1-z_1}(1 - tr(K)) \frac{dz_1}{2\pi i z_1} = F_{step}(h_1, (0, \tau_1)) + O(e^{-\frac{8}{3}\hat{h}_1^{3/2}}) \\
&= \frac{1}{16\pi} e^{-\frac{4}{3}\hat{h}_1^{3/2}} \hat{h}_1^{-3/2} (1 - \frac{35}{24}\hat{h}_1^{-3/2} + O(\hat{h}_1^{-3}))
\end{aligned}
\tag{2.7}
$$

where $\hat{h}_1 := \tau_1^{-1/3}\hat{h}_1$

Proof. Using the residue theorem,

$$\oint_{|z_1|=1/2} \frac{1}{1-z_1}\frac{dz_1}{2\pi i z_1} = 1,$$

$$\oint_{|z_1|=1/2} \frac{1}{1-z_1}(1-z_1)\frac{dz_1}{2\pi i z_1} = 1,$$

and

$$\oint_{|z_1|=1/2} \frac{1}{1-z_1}(1-z_1^{-1})\frac{dz_1}{2\pi i z_1} = -\oint_{|z_1|=1/2} \frac{dz_1}{2\pi i z_1^2} = 0.$$

So

$$\oint_{|z_1|=1/2} \frac{1}{1-z_1}(1-tr(K))\frac{dz_1}{2\pi i z_1} = 1 - J_1$$

where

$$J_1 = \int_{C_{1,L}}\int_{C_{1,R}} \exp(-\frac{1}{3}\tau_1(w^3-z^3)+x_1(w^2-z^2)+h_1(w-z))\frac{1}{(z-w)^2}\frac{dz}{2\pi i}\frac{dw}{2\pi i}$$

precisely (1.4), the trace term in the $m=1$ case. The asymptotics (1.5) are then obtained using the steepest descent method and the error term is obtained through bounding the kernel appropiately and applying Hadamard's inequality to the higher order terms in the Fredholm determinant expansion (1.3). □

2.4 Second Order Term

The next term in the Fredholm determinant $\det(1-K)$ is

$$\begin{aligned} &\frac{1}{2}\int_{S_1}\int_{S_1} \det((K(w_i,w_j))_{1\le i,j\le 2})d\mu(w_1)d\mu(w_2) \\ &= \frac{1}{2}[tr(K)^2 - tr(K^2)] \end{aligned} \tag{2.8}$$

From the previous section we know that

$$tr(K)^2 = (1-z_1)^2 J_1^2 + 2(1-z_1)(1-z_1^{-1})J_1J_2 + (1-z_1^{-1})^2 J_2^2$$

so that

$$\begin{aligned} &\oint_{|z_1|=1/2} \frac{1}{1-z_1} tr(K)^2 \frac{dz_1}{2\pi i z_1} \\ &= J_1^2 \oint_{|z_1|=1/2} (1-z_1)\frac{dz_1}{2\pi i z_1} + 2J_1J_2\oint_{|z_1|=1/2}(1-z_1^{-1})\frac{dz_1}{2\pi i z_1} \\ &+ J_2^2\oint_{|z_1|=1/2}(1-z_1^{-1})\frac{dz_1}{-2\pi i z_1^2} \\ &= J_1^2 + 2J_1J_2 = O(\hat{h}_1^{-3}e^{-\frac{4}{3}(1+\alpha^2)\hat{h}_1^{3/2}}) \end{aligned} \tag{2.9}$$

Turning to $tr(K^2)$, we have

$$
\begin{aligned}
tr(K^2) &= \int_{S_1}\int_{S_1} K(w_1,w_2)K(w_2,w_1)\,d\mu(w_1)\,d\mu(w_2) \\
&= \int_{S_1}\int_{S_1}\int_{S_2}\int_{S_2} K_1(w_1,\zeta_1)K_{step}(\zeta_1,w_2)K_1(w_2,\zeta_2)K_{step}(\zeta_2,w_1) \\
& d\mu(\zeta_1)\,d\mu(\zeta_2)\,d\mu(w_1)\,d\mu(w_2)
\end{aligned}
$$

But in order for $K_{step}(\zeta_1,w_2)\neq 0$ either $\zeta_1\in C_{1,R}$ and $w_2\in C_{1,L}$ or $\zeta_1\in C_{2,L}$ and $w_2\in C_{2,R}$. Similarly for $K_{step}(\zeta_2,w_1)\neq 0$ either $\zeta_2\in C_{1,R}$ and $w_1\in C_{1,L}$ or $\zeta_2\in C_{2,L}$ and $w_1\in C_{2,R}$. Therefore

$$
tr(K^2) = A_1+A_2+A_3+A_4
$$

where

$$
A_1 := \int_{C_{1,L}}\int_{C_{1,L}}\int_{C_{1,R}}\int_{C_{1,R}} K_1(w_1,\zeta_1)K_{step}(\zeta_1,w_2)K_1(w_2,\zeta_2)K_{step}(\zeta_2,w_1)\,\frac{d\zeta_1}{2\pi i}\,\frac{d\zeta_2}{2\pi i}\,\frac{dw_1}{2\pi i}\,\frac{dw_2}{2\pi i},
$$

$$
\begin{aligned}
A_2 := \sum_{a,b\in\{in,out\}} & \int_{C_{1,L}}\int_{C_{2,R}^a}\int_{C_{2,L}^b}\int_{C_{1,R}} K_1(w_1,\zeta_1)K_{step}(\zeta_1,w_2)K_1(w_2,\zeta_2)K_{step}(\zeta_2,w_1) \\
& d\mu(\zeta_1)\,d\mu(\zeta_2)\,d\mu(w_1)\,d\mu(w_2),
\end{aligned}
$$

$$
\begin{aligned}
A_3 := \sum_{a,b\in\{in,out\}} & \int_{C_{2,R}^a}\int_{C_{1,L}}\int_{C_{1,R}}\int_{C_{2,L}^b} K_1(w_1,\zeta_1)K_{step}(\zeta_1,w_2)K_1(w_2,\zeta_2)K_{step}(\zeta_2,w_1) \\
& d\mu(\zeta_1)\,d\mu(\zeta_2)\,d\mu(w_1)\,d\mu(w_2),
\end{aligned}
$$

and

$$
\begin{aligned}
A_4 := \sum_{a,b,c,d\in\{in,out\}} & \int_{C_{2,R}^a}\int_{C_{2,R}^b}\int_{C_{2,L}^c}\int_{C_{2,L}^d} K_1(w_1,\zeta_1)K_{step}(\zeta_1,w_2)K_1(w_2,\zeta_2)K_{step}(\zeta_2,w_1) \\
& d\mu(\zeta_1)\,d\mu(\zeta_2)\,d\mu(w_1)\,d\mu(w_2)
\end{aligned}
$$

First to estimate A_1, make the usual change of variables $(w_1,w_2,\zeta_1,\zeta_2)\mapsto \hat{h}_1^{1/2}\tau_1^{-1/2}(w_1,w_2,\zeta_1,\zeta_2)$, and recall that $\hat{h}_1=\tau^{-1/3}h_1$. Then deform the updated contours of integration $C_{1,L}$ and $C_{1,R}$ to be the steepest descent contours for the Airy function, so

$$
\begin{aligned}
A_1 =& \frac{1}{16\pi^4}\int_{C_{1,L}}\int_{C_{1,L}}\int_{C_{1,R}}\int_{C_{1,R}} \exp(\hat{h}_1^{3/2}[-\frac{1}{3}(w_1^3+w_2^3-\zeta_1^3-\zeta_2^3)+(w_1+w_2-\zeta_1-\zeta_2)]) \\
&\cdot\frac{dw_1\,dw_2\,d\zeta_1\,d\zeta_2}{(w_1-\zeta_1)(w_2-\zeta_1)(w_1-\zeta_2)(w_2-\zeta_2)}
\end{aligned}
$$

and since $|w_1-\zeta_1|,|w_2-\zeta_1|,|w_1-\zeta_2|,|w_2-\zeta_2|\geq 2$ on the curves of integration, a steepest descent analysis identical to that for the Tracy-Widom distribution shows that

$$
A_1 = O(\hat{h}_1^{-3}e^{-\frac{8}{3}\hat{h}_1^{3/2}})
$$

and therefore

$$
\oint_{|z_1|=1/2}\frac{1}{1-z_1}A_1\,\frac{dz_1}{2\pi i z_1} = A_1 = O(\hat{h}_1^{-3}e^{-\frac{8}{3}\hat{h}_1^{3/2}}) \qquad (2.10)
$$

Next to analyze A_4 we first look at one of the quadruple integrals in the sum,

$$\int_{C_{2,R}^{in}}\int_{C_{2,R}^{in}}\int_{C_{2,L}^{in}}\int_{C_{2,L}^{in}} K_1(w_1,\zeta_1)K_{step}(\zeta_1,w_2)K_1(w_2,\zeta_2)K_{step}(\zeta_2,w_1)$$
$$d\mu(\zeta_1)\,d\mu(\zeta_2)\,d\mu(w_1)\,d\mu(w_2)$$
$$= \frac{1}{16\pi^4}\int_{C_{2,R}^{in}}\int_{C_{2,R}^{in}}\int_{C_{2,L}^{in}}\int_{C_{2,L}^{in}} \exp\left(\frac{\tau_2}{3}(w_1^3+w_2^3-\zeta_1^3-\zeta_2^3)-h_2(w_1+w_2-\zeta_1-\zeta_2)]\right)$$
$$\cdot\frac{dw_1\,dw_2\,d\zeta_1\,d\zeta_2}{(w_1-\zeta_1)(w_2-\zeta_1)(w_1-\zeta_2)(w_2-\zeta_2)}\cdot(1-z_1^{-1})^2\cdot\frac{1}{(1-z_1)^4}$$

The terms in the denominators $|w_1-\zeta_1|, |w_2-\zeta_1|, |w_1-\zeta_2|, |w_2-\zeta_2|$ can remain bounded below by a fixed constant while deforming $C_{2,L}^{in}\to C_{1,L}$ and $C_{2,R}^{in}\to C_{1,R}$.

Arguing similarly for all 16 quadruple integrals in the sum defining A_4, we find that

$$A_4 = (1-z_1^{-1})^2\left(\frac{1}{1-z_1}+\frac{-z_1}{1-z_1}\right)^4 A = (1-z_1^{-1})^2 A$$

where

$$A = \int_{C_{1,R}}\int_{C_{1,R}}\int_{C_{1,L}}\int_{C_{1,L}} \exp\left(\frac{\tau_2}{3}(w_1^3+w_2^3-\zeta_1^3-\zeta_2^3)-h_2(w_1+w_2-\zeta_1-\zeta_2)]\right)\cdot\frac{dw_1\,dw_2\,d\zeta_1\,d\zeta_2}{16\pi^4(w_1-\zeta_1)(w_2-\zeta_1)(w_1-\zeta_2)(}$$

does not depend on z_1. Hence observe that

$$\oint_{|z_1|=1/2}\frac{1}{1-z_1}A_4\,\frac{dz_1}{2\pi i z_1} = A\oint_{|z_1|=1/2}\frac{1-z_1}{z_1^3}\,\frac{dz_1}{2\pi i} = 0 \tag{2.11}$$

Moving on to A_2 and A_3 we first observe that through interchanging w_1 with w_2 and ζ_1 with ζ_2 it is clear that $A_2 = A_3$.

A_2 is the most difficult part to deal with because of the singular integrals involved, so we will need the following lemma

Lemma 2.7. *Suppose* $C_L = \{(-\sqrt{1+\frac{t^2}{3}}, t) : t\in\mathbb{R}\}$ *the steepest descent contour, and let* C_L^L (C_R^R) *be a curve which lies strictly to the left (right) of* C *which also goes from* $e^{-\frac{2}{3}\pi i}\infty$ *to* $e^{\frac{2}{3}\pi i}\infty$*. Then*

$$\int_{C_L^L}\exp(h^{3/2}(-\frac{1}{3}z^3+z)))\frac{1}{z+1}\,dz = -\pi i e^{-\frac{2}{3}h^{3/2}}(1+O(h^{-3/4})) \tag{2.12}$$

and

$$\int_{C_L^R}\exp(h^{3/2}(-\frac{1}{3}z^3+z)))\frac{1}{z+1}\,dz = \pi i e^{-\frac{2}{3}h^{3/2}}(1+O(h^{-3/4})) \tag{2.13}$$

as $h\to+\infty$.

Similarly for $C_R = \{(\sqrt{1+\frac{t^2}{3}}, t) : t \in \mathbb{R}\}$ the steepest descent contour on the right, and C_R^L (C_R^R) be a curve which lies strictly to the left (right) of C which also goes from $e^{-\frac{\pi}{3}i}\infty$ to $e^{\frac{\pi}{3}i}\infty$

$$\int_{C_R^L} \exp(h^{3/2}(\frac{1}{3}z^3 - z)))\frac{1}{z-1}\,dz = -\pi i e^{-\frac{2}{3}h^{3/2}}(1+O(h^{-3/4})) \qquad (2.14)$$

and

$$\int_{C_R^R} \exp(h^{3/2}(\frac{1}{3}z^3 - z)))\frac{1}{z-1}\,dz = \pi i e^{-\frac{2}{3}h^{3/2}}(1+O(h^{-3/4})) \qquad (2.15)$$

as $h \to +\infty$

Proof. Addressing (1), Using the opening and ending angle condition, it is easy to deform the contour C_L^L to be near enough C_L, so that outside of $B_1(-1)$, for $\Phi(z) = -\frac{1}{3}z^3 + z$, $\Re\Phi(z) < -(\frac{2}{3}+\epsilon)$ for some fixed $\epsilon > 0$. Furthermore this contour can enter $B_1(-1)$ at $-1-i$ and exit at $1+i$ since $\Re\Phi(1+i) = \Re\Phi(1-i) = -5/3$. Then for $h \geq 1$, on this newly deformed C_L^L,

$$\begin{aligned}
&\int_{C_L^L \cap B_1(-1)^c} \exp(h^{3/2}(-\frac{1}{3}z^3 + z)))\frac{1}{z+1}\,dz \\
&= e^{-(\frac{2}{3}+\epsilon)h^{3/2}} \int_{C_L^L \cap B_1(-1)^c} \exp(h^{3/2}(\Phi(z) + (\frac{2}{3}+\epsilon)))\frac{1}{z+1}\,dz \\
&\leq e^{-(\frac{2}{3}+\epsilon)h^{3/2}} \int_{C_L^L \cap B_1(-1)^c} \exp(\Phi(z) + (\frac{2}{3}+\epsilon))\frac{1}{z+1}\,dz = O(e^{-(\frac{2}{3}+\epsilon)h^{3/2}})
\end{aligned}$$

So focusing on the behaviour near $z_0 = -1$, we Taylor expand $\Phi(z) = -\frac{1}{3}z^3 + z = -\frac{2}{3} + (z-1)^2 - \frac{1}{3}(z-1)^3$, and making the substitution $w = h^{3/4}(z-1)$,

$$\begin{aligned}
&\int_{C_L^L \cap B_1(-1)} \exp(h^{3/2}(-\frac{1}{3}z^3 + z)))\frac{1}{z+1}\,dz \\
&= e^{-\frac{2}{3}h^{3/2}} \int_\Gamma \exp(w^2 - h^{-3/4}w^3)\,\frac{dw}{w} \\
&= e^{-\frac{2}{3}h^{3/2}} \int_\Gamma \exp(w^2)(1 + h^{-3/4}O(w^3))\,\frac{dw}{w}
\end{aligned}$$

where $\Gamma = h^{3/4}((C_L^L - 1) \cap B_1(0))$. Notice that Γ starts at $-ih^{3/4}$ and ends at $ih^{3/4}$ so we can use Cauchy's theorem to deform Γ to be a contour composed of a line segment from $-ih^{3/4}$ to $-i$ followed by a radius 1 clockwise semicircle from $-i$ to i, finally followed by another line segment from i to $ih^{3/4}$.
It's clear that $h^{-3/4}\int_\Gamma \exp(w^2)O(w^2)dw = O(h^{-3/4})$ so it suffices to consider the integral $\int_\Gamma \exp(w^2)\,\frac{dw}{w}$.
We immediately see that the integral on the two parts of Γ on the imaginary line cancel. Then using this symmetry again we can use Cauchy's theorem to shrink the remaining semicircle, so that

$$\int_\Gamma \exp(w^2)\,\frac{dw}{w} = \lim_{\delta \downarrow 0} \int_{3\pi/2}^{\pi/2} e^{2\delta e^{i\theta}}\, i\, d\theta = -\pi i$$

This completes the proof for (1). The other asymptotics are obtained in a very similar manner. □

Returning to A_2, we analyze one of the terms in the sum,

$$I_{in,in} := \int_{C_{1,L}} \int_{C_{2,R}^{in}} \int_{C_{2,L}^{in}} \int_{C_{1,R}} K_1(w_1,\zeta_1) K_{step}(\zeta_1,w_2) K_1(w_2,\zeta_2) K_{step}(\zeta_2,w_1)$$
$$d\mu(\zeta_1)\, d\mu(\zeta_2)\, d\mu(w_1)\, d\mu(w_2)$$

First making our usual change of variables,

$$I_{in,in} = \int_{C_{1,L}} \int_{C_{2,R}^{in}} \int_{C_{2,L}^{in}} \int_{C_{1,R}} \exp(-\frac{\tau_1}{3}(\zeta_2^3 - w_1^3) + h_1(\zeta_2 - w_1))$$
$$\cdot \exp(-\frac{\tau_2}{3}(w_2^3 - \zeta_1^3) + h_2(w_2 - \zeta_1)) \cdot \frac{dw_1\, dw_2\, d\zeta_1\, d\zeta_2}{(w_1-\zeta_1)(w_1-\zeta_2)(w_2-\zeta_1)(w_2-\zeta_2)}$$
$$= \int_{\tau_1^{1/2} h_1^{-1/2} C_{1,L}} \int_{\tau_2^{1/2} h_2^{-1/2} C_{2,R}^{in}} \int_{\tau_2^{1/2} h_2^{-1/2} C_{2,L}^{in}} \int_{\tau_1^{1/2} h_1^{-1/2} C_{1,R}} \exp(\hat{h}_1^{3/2}(-\frac{1}{3}(\zeta_2^3 - w_1^3) + (\zeta_2 - w_1)))$$
$$\cdot \exp(\hat{h}_2(-\frac{1}{3}(w_2^3 - \zeta_1^3) + (w_2 - \zeta_1)))$$
$$\cdot \frac{\tau_1^{-1/2} h_1^{1/2} dw_1\, \tau_2^{-1/2} h_2^{1/2} dw_2}{(\tau_1^{-1/2} h_1^{1/2} w_1 - \tau_2^{-1/2} h_2^{1/2} \zeta_1)(\tau_1^{-1/2} h_1^{1/2} w_1 - \tau_1^{-1/2} h_1^{1/2} \zeta_2)}$$
$$\cdot \frac{\tau_2^{-1/2} h_2^{1/2} d\zeta_1\, \tau_1^{-1/2} h_1^{1/2} d\zeta_2}{(\tau_2^{-1/2} h_2^{1/2} w_2 - \tau_2^{-1/2} h_2^{1/2} \zeta_1)(\tau_2^{-1/2} h_2^{1/2} w_2 - \tau_1^{-1/2} h_1^{1/2} \zeta_2)}$$

Now from the key assumption that $h_2/h_1 = \tau_2/\tau_1$ it follows that $h_1^{1/2}\tau_1^{-1/2} = h_2^{1/2}]tau_2^{-1/2}$ so that the order of the contours remains unchanged and the integrand greatly simplifies. Therefore after deforming back to the original contours.

$$I_{in,in} =$$
$$= \int_{C_{1,L}} \int_{C_{2,R}^{in}} \exp(\hat{h}_1^{3/2}(-\frac{1}{3}\zeta_2^3 + \zeta_2)) \exp(\hat{h}_2^{3/2}(\frac{1}{3}\zeta_1^3 - \zeta_1)) g_1(\zeta_1,\zeta_2) g_2(\zeta_1,\zeta_2)\, d\zeta_1\, d\zeta_2$$
$$\times (1-z_1)(1-z_1^{-1}) \frac{1}{2\pi i} \left(\frac{1}{1-z_1}\right) \frac{1}{2\pi i} \frac{1}{2\pi i} \left(\frac{1}{1-z_1}\right) \frac{1}{2\pi i}$$
$$=: -\frac{1}{16\pi^4 z_1^2} I'_{in,in}$$

where

$$g_1(\zeta_1,\zeta_2) := \int_{C_{1,R}} \exp(\hat{h}_1^{3/2}(\frac{1}{3}w_1^3 - w_1)) \frac{dw_1}{(w_1-\zeta_1)(w_1-\zeta_2)}$$
$$g_2(\zeta_1,\zeta_2) := \int_{C_{2,L}^{in}} \exp(\hat{h}_2^{3/2}(-\frac{1}{3}w_2^3 + w_2)) \frac{dw_2}{(w_2-\zeta_1)(w_2-\zeta_2)}$$

Taylor expanding $\frac{1}{(w_1-\zeta_1)(w_1-\zeta_2)}$ up to first order in w_1, and $\frac{1}{(w_2-\zeta_1)(w_2-\zeta_2)}$ up to first order in w_2, and further assuming $C_{2,R}^{in}$ and $C_{1,L}$ avoid the steepest descent contours so that we can deform $C_{1,R}$ and $C_{2,L}^{in}$ to the steepest descent contours,

$$g_1(\zeta_1,\zeta_2) \sim \frac{1}{(1-\zeta_1)(1-\zeta_2)} \int_{C_{1,R}} \exp(\hat{h}_1^{3/2}(\frac{1}{3}w_1^3 - w_1))\, dw_1 \sim i\sqrt{\pi}\hat{h}_1^{-3/4} e^{-\frac{2}{3}\hat{h}_1^{3/2}} \frac{1}{(\zeta_1-1)(\zeta_2-1)}$$

$$g_2(\zeta_1,\zeta_2) \sim \frac{1}{(-1-\zeta_1)(-1-\zeta_2)} \int_{C_{2,L}^{in}} \exp(\hat{h}_2^{3/2}(-\frac{1}{3}w_2^3+w_2))\, dw_2 \sim i\sqrt{\pi}\hat{h}_2^{-3/4} e^{-\frac{2}{3}\hat{h}_2^{3/2}} \frac{1}{(\zeta_1+1)(\zeta_2+1)}$$

where I have used the asymptotic analysis of the Airy function to evaluate each of the asymptotics above.

So

$$I'_{in,in} = -\pi \hat{h}_1^{-3/4}\hat{h}_2^{-3/4} e^{-\frac{2}{3}(\hat{h}_1^{3/2}+\hat{h}_2^{3/2})} I^{(1)}_{in,in} I^{(2)}_{in,in}$$

where

$$I^{(1)}_{in,in} := \int_{C_{2,R}^{in}} \exp(\hat{h}_1^{3/2}(\frac{1}{3}\zeta_1^3 - \zeta_1))\, \frac{d\zeta_1}{(\zeta_1-1)(\zeta_1+1)}$$

and

$$I^{(2)}_{in,in} := \int_{C_{1,L}} \exp(\hat{h}_1^{3/2}(-\frac{1}{3}\zeta_2^3 + \zeta_2))\, \frac{d\zeta_2}{(\zeta_2-1)(\zeta_2+1)}$$

Now Taylor expanding $1/(\zeta_1+1)$ and $1/(\zeta_2-1)$ up to first order, and applying Lemma 2.5,

$$I^{(1)}_{in,in} \sim \frac{1}{2}\int_{C_{2,R}^{in}} \exp(\hat{h}_1^{3/2}(\frac{1}{3}\zeta_1^3 - \zeta_1))\, \frac{d\zeta_1}{\zeta_1-1} \sim \frac{\pi}{2} i e^{-\frac{2}{3}\hat{h}_1^{3/2}}$$

and

$$I^{(2)}_{in,in} \sim -\frac{1}{2}\int_{C_{1,L}} \exp(\hat{h}_1^{3/2}(-\frac{1}{3}\zeta_2^3 + \zeta_2))\, \frac{d\zeta_2}{(\zeta_2+1)} - \frac{\pi}{2} i e^{-\frac{2}{3}\hat{h}_1^{3/2}}$$

so that

$$I'_{in,in} \sim -\frac{\pi^3}{4}\hat{h}_1^{-3/4}\hat{h}_2^{-3/4} e^{-\frac{4}{3}(\hat{h}_1^{3/2}+\hat{h}_2^{3/2})}$$
$$I_{in,in} \sim \frac{1}{64\pi z_1^2}\hat{h}_1^{-3/4}\hat{h}_2^{-3/4} e^{-\frac{4}{3}(\hat{h}_1^{3/2}+\hat{h}_2^{3/2})}$$

Next we can similarly define $I_{in,out}, I_{out,in}, I_{out,out}$.

Going through the same analysis for each we case we find that, $I'_{out,out} = I'_{in,in}$ and $I'_{in,out} = I'_{out,in} = -I'_{in,in}$. Now accounting for the measures and the $Q_1(j)$ factors, $I_{in,out} = I_{out,in} = (1-z_1)(1-z_1^{-1})\frac{-z_1}{(1-z_1)^2}\frac{1}{16\pi^4}I'_{in,out}$, and $I_{out,out} = (1-z_1)(1-z_1^{-1})\left(\frac{-z_1}{1-z_1}\right)^2 \frac{1}{16\pi^4}I'_{out,out}$. So adding all the contributions

$$\begin{aligned} A_2 + A_3 = 2A_2 &= 2(I_{in,in} + I_{in,out} + I_{out,in} + I_{out,out}) \\ &= \frac{1}{8\pi^4}(1-z_1)(1-z_1^{-1})\left(\frac{1}{(1-z_1)^2} - 2\frac{-z_1}{(1-z_1)^2} + \frac{z_1^2}{(1-z_1)^2}\right) I'_{in,in} \\ &= -\frac{(1+z_1)^2}{8\pi^4 z_1} I_{in,in} \end{aligned}$$

Hence

$$\begin{aligned} \oint_{|z_1|=1/2} \frac{1}{1-z_1}(A_2+A_3)\, \frac{dz_1}{2\pi i z_1} &= -\frac{1}{8\pi^4} I'_{in,in} \oint_{|z_1|=1/2} \frac{(1+z_1)^2}{1-z_1}\, \frac{dz_1}{2\pi i z_1^2} \\ &= -\frac{3}{8\pi^4} I'_{in,in} \sim \frac{3}{32\pi}\hat{h}_1^{-3/4}\hat{h}_2^{-3/4} e^{-\frac{4}{3}(\hat{h}_1^{3/2}+\hat{h}_2^{3/2})} \end{aligned}$$

Now recalling that $\hat{h}_2 = \alpha^{4/3}\hat{h}_1$ with $0 < \alpha < 1$,

$$\oint_{|z_1|=1/2} \frac{1}{1-z_1}(A_2 + A_3)\,\frac{dz_1}{2\pi i z_1} \sim \frac{3}{32\pi}\alpha^{-1}\hat{h}_1^{-3/2}e^{-\frac{4}{3}(1+\alpha^2)\hat{h}_1^{3/2}} \tag{2.16}$$

Therefore collecting all the terms, thus far, (2.7), (2.9), (2.10), and (2.16) we have

$$\oint_{|z_1|=1/2} \frac{1}{1-z_1}(1 - tr(K) + \frac{1}{2}[tr(K)^2 - tr(K^2)])\,\frac{dz_1}{2\pi i z_1}$$
$$= F_{step}(h_1, (x_1, \tau_1)) - \frac{3}{64\pi}\alpha^{-1}\hat{h}_1^{-3/2}e^{-\frac{4}{3}(1+\alpha^2)\hat{h}_1^{3/2}}(1 + O(\hat{h}_1^{-3/4}))$$

So in order to further justify Conjecture 2.3, the higher order terms in the Fredholm determinant defining F_{step} must be estimated. The beginning of this analysis is carried out in the next section.

2.5 Higher Order Terms

Throughout this subsection we assume $C_{1,L} = \{(-\sqrt{1+\frac{t^2}{3}}, t) : t \in \mathbb{R}\}$ and $C_{1,L} = \{(\sqrt{1+\frac{t^2}{3}}, t) : t \in \mathbb{R}\}$, the steepest descent contours.
First we need a lemma similar to lemma 2.7 to deal with singular integral terms.

Lemma 2.8. *For $w_1 \notin C_{1,R}, w_2 \in C_{1,L}$*

$$\int_{C_{1,R}} \exp(\hat{h}_1^{3/2}(\frac{1}{3}z^3 - z))\frac{1}{(w_1 - z)(w_2 - z)}dz = O(\hat{h}_1^{3/4}e^{-\frac{2}{3}\hat{h}_1^{3/2}}) \tag{2.18}$$

where the implied constant is independent of $d(w_1, C_{1,R})$.
Similarly for $w_1 \notin C_{1,L}, w_2 \in C_{1,R}$

$$\int_{C_{1,L}} \exp(\hat{h}_2^{3/2}(-\frac{1}{3}z^3 + z))\frac{1}{(w_1 - z)(w_2 - z)}dz = O(\hat{h}_2^{3/4}e^{-\frac{2}{3}\hat{h}_2^{3/2}}) \tag{2.19}$$

where the implied constant is independent of $d(w_1, C_{1,L})$.

Proof. I will only prove (1) since the proof of (2) is identical. I will also let $\Phi(z) = \frac{1}{3}z^3 - z$
First suppose $d(w_1, C_{1,R}) \geq \hat{h}_1^{-3/2}$. Then

$$\left|\int_{C_{1,R}} \exp(\hat{h}_1^{3/2}\Phi(z))\frac{1}{(w_1 - z)(w_2 - z)}dz\right|$$
$$\leq \frac{1}{2}\hat{h}_1^{3/2}\int_{C_{1,R}} \exp(\hat{h}_1^{3/2}\Phi(z))dz$$
$$= \pi\hat{h}_1 Ai(\hat{h}_1) = O(\hat{h}_1^{3/4}e^{-\frac{2}{3}\hat{h}_1^{3/2}})$$

I have used the fact that $\Phi(z)$ is real on $C_{1,R}$ and the asymptotics for the Airy function above.

Now in the event that $d(w_1, C_{1,R}) < \hat{h}_1^{-3/2}$, we deform the contour $C_{1,R}$ to a new contour $C'_{1,R}$ which replaces the part where $d(w_1, z) < \hat{h}_1^{-3/2}$ with a circular arc which stays at distance $\hat{h}_1^{-3/2}$ from w_1. On this new contour, $d(w_1, C'_{1,R}) \geq \hat{h}_1^{-3/2}$, Furthermore choosing any $z = (\sqrt{1+\frac{t^2}{3}}, t) \in C_{1,R}$ with $d(z, w_1) < \hat{h}_1^{-3/2}$, then for any z' in the deformed part of $C'_{1,R}$,

$$\begin{aligned}\Re\Phi(z') &\leq \Re\Phi(z) + |\Phi'(z^*)|\hat{h}_1^{-3/2} \\ &\leq -(\frac{2}{3} + C|t|^3) + C'(1+|t|^2)\hat{h}_1^{-3/2}\end{aligned}$$

so

$$\hat{h}_1^{3/2}\Re\Phi(z') \leq -\frac{2}{3}\hat{h}_1^{3/2} + C''$$

for $\hat{h}_1 > 1$, and as a result

$$\begin{aligned}&\left|\int_{C'_{1,R}} \exp(\hat{h}_1^{3/2}\Phi(z))\frac{1}{(w_1 - z)(w_2 - z)}dz\right| \\ &\leq \frac{1}{2}\hat{h}_1^{3/2}\int_{C'_{1,R}} \exp(\hat{h}_1^{3/2}\Re\Phi(z))|dz| \\ &= O(\hat{h}_1^{3/4}e^{-\frac{2}{3}\hat{h}_1^{3/2}})\end{aligned}$$

□

Using this lemma to get bounds on the kernel $K(z, z')$ and applying Hadamard's inequality one should be able to show that with the given restrictions on the parameters,

$$\begin{aligned}&|F_{step}(h_1, h_2, (0, \tau_1), (0, \tau_2)) - \\ &\oint_{|z_1|=1/2} \frac{1}{1-z_1}(1 - tr(K) + \frac{1}{2}[tr(K)^2 - tr(K^2)])\frac{dz_1}{2\pi i z_1}\Bigg| \\ &= O(e^{-\frac{4}{3}(\min(2, 1+2\alpha^2)-\epsilon)})\end{aligned} \tag{2.20}$$

for any $\epsilon > 0$. This would guarantee that the higher order terms would not contribute to the leading asymptotics provided in Conjecture 2.3 or 2.4.

3 Discussion and Further Directions

Initially it was expected that the interesting asymptotic behaviour for $F_{step}(h_1, h_2, (x_1, \tau_1), (x_2, \tau_2))$ would come from the trace term of the Fredholm determinant, but it turned out to come from the second term in the expansion. This also produced an interesting result because it seems that the value of the height function and two different points at different times are quite far from being independent in the 1:2:3 scaling limit.

The relationship $h_2/h_1 = \tau_1/\tau_2$ does not seem necessary, but in other cases, the order of the contours will have to be switched when computing A_2 in order to complete a steepest descent analysis, forcing us to compute residue terms from the beginning.

Going forward I will be filling in details of the proof and working on generalizations. For instance what happens when $h_2/h_1 \neq \tau_1/\tau_2$, $h_2 = \alpha h_1$, $h_1 \to +\infty$, and similar generalizations in the case when $m > 2$. Additionally I am exploring the behaviour as $h_1 \to +\infty$ while holding h_2 fixed.

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