Modeling Vibrating Strings

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Abstract

This project is aimed to construct a simplified, but justifiable model, for vibrating strings so that we are able to explore questions related to musical instruments. We begin with solutions to the 1-dimensional wave equation with fixed, and later, mixed boundary conditions given by Fourier coefficients. For the mixed boundary condition the eigenvalues are unsolvable, instead we find a numerical approximation for the solution. Following, we investigate the energy and distribution of energies of each solution.

Finally, we construct a mathematical model for the advanced musical performance technique, vibrato. The exact solution of this mathematical model is discussed by Gaffour in [3], [7]. In our paper, we adopted an adiabatic approximation [15] in order to obtain an approximate solution with a simpler form. Additionally, justification is provided that our adiabatic approximation is within good reason of the exact solution given by Gaffour [3], [7].

Contents

1 Introduction

String instruments are played via a plucking or bowing method to produce a sound, which we recognize as string music. The process of playing a string instrument produces an oscillation in the string. These oscillations of the string are described by the famous second-order linear partial differential equation for waves, also known as the 1-dimensional wave equation. The 1-dimensional wave equation is as follows:

$$\frac{\partial^2 u}{\partial t^2}(x,t) = c^2 \frac{\partial^2 u}{\partial x^2}(x,t) \tag{1}$$

where $u(x,t)$ is a valid solution to the 1-dimensional wave equation (1). The c^2 constant value represents physical constants of the string being modeled:

$$c^2 = \frac{T}{\rho}$$

where T is the tension on the string and ρ is the density of the string.

Notice that the 1-dimensional wave equation is a second order differential equation, and thus a unique solution requires two initial conditions:

$$u(x, t = 0) = \Phi(x) \tag{2}$$

$$\frac{\partial u}{\partial t}(x, t = 0) = \Psi(x) \tag{3}$$

where $\Phi(x, t = 0)$ is the initial position and $\Psi(x, t = 0)$ is the initial velocity.

The energy of the string is given by the sum of potential energy and the kinetic energy, and can be defined as the integral:

$$E(u) = \frac{1}{2}\int_0^L \rho \left(\frac{\partial u}{\partial t}(x,t)\right)^2 + T \left(\frac{\partial u}{\partial x}(x,t)\right)^2 dx \tag{4}$$

Beginning with the simplest model for various musical instruments, such as guitar, violin, cello, is to assume both ends are fixed, and the act of plucking or striking the string yields the initial condition of the string. In the following section, we provide a mathematical model with both ends of string being fixed; solutions to this model are found by solving the 1-dimensional wave equation with the Dirichlet Boundary Conditions. In Section 3, we computed the energy of each mode of the solution using equation (4), as well as the distribution of energy. This computation shows the mathematical association between the timbre and the given initial conditions.

However, string instruments are not commonly built with two fixed ends, rather the string rests on the bridge of the instrument. The bridge on violins is made of a softer wood than the rest of the instrument, acting as a spring, yielding our second model: a model with one spring end and one fixed. In Section 4 we provide the formerly defined model, in which, the eigenvalues are unsolvable, so we find a numerical approximation for the solution. Similar to the Dirichlet

Boundary Conditions, we also investigated the energy of each mode and distribution for these boundary conditions.

Finally, music performance is not just about plucking, striking, or bowing strings, as noted by Daniel Leech-Wilkinson, "performing musically, or stylishly, involves modifying those aspects of the sound that our instrument allows us to modify, and doing it in a way that beings to the performance a sense that the score is more than just a sequence of pitches and duration." [14]. Professional musicians adopted various expressive devices, such as tempo variation, dynamic shaping, pitch variation, and timbre modulation to create an emotional performance. One of these techniques is vibrato, which is an expressive device involving continuous pitch modulation. Vibrato is widely employed in string, voice, wind instrumental performance [14]. It is employed by string musicians by moving their finger a small distance back and fourth on the string. In Section 6, we construct a mathematical model for this technique. The exact solution of the 1-dimensional wave equation with moving boundary conditions is discussed by Gaffour in [3], [7] by transforming the original moving boundary domain into a fixed boundary domain using the conformal mapping method. In our paper, we adopted an adiabatic approximation [15] in order to obtain an approximate solution with simpler form. Additionally, justification is provided that the adiabatic approximation is within good reason of the exact solution.

2 Dirichlet Boundary Conditions

We begin with a review of the solution of 1-dimensional wave equation with Dirichlet boundary conditions. First of all, we need to determine the boundary conditions: take some L, the length of the string on the instrument at a specific note, then the Dirichlet Boundary Condition is mathematically defined as: $\forall t \geq 0$

$$u(x=0,t) = 0 = u(x=L,t), \tag{5}$$

Specifying a string with fixed endpoints at the equilibrium, or zero, position of the string.

Now, Equation 5 can be used, in conjunction with Equations 2 and 3 to obtain a solution for the 1-dimensional wave equation, with initial conditions $\Phi(x)$ and $\Psi(x)$, with Dirichlet boundary conditions. To begin, rewrite the partial differential equation as an operator on $u(x,t)$:

$$\left[\frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial x^2}\right]u(x,t) = 0$$

Then, using separation of variables method, we have:

$$u(x,t) = F(x)G(t) \tag{6}$$

Where $F(x)$ and $G(t)$ are functions of x and t respectively and when the operator is applied, we see two Ordinary Differential Equations (below) that can be solved using the eigen-values and eigen-functions of some λ.

$$\begin{aligned} G''(t) &= \lambda c^2 G(t) \\ F''(x) &= \lambda F(x) \end{aligned}$$

With fixed boundary conditions.

In order to satisfy the Dirichlet Boundary Conditions, $\lambda < 0$, and we obtain the eigen-values (top) and eigen-functions (bottom) below.

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2 \tag{7}$$

$$F_n(x) = \sin\left(\frac{n\pi x}{L}\right), \forall n \in \mathbb{N} \tag{8}$$

In order to continue, we must define the inner product and norm of F and G. Let the inner product of two function be defined as:

$$\langle f, g \rangle = \int_0^L f(x)g(x)dx \tag{9}$$

The norm of the eigen-function $F_n(x)$ is given by:

$$\|F_n(x)\| = \left(\int_0^L |F_n(x)|^2 dx\right)^{\frac{1}{2}} = \sqrt{\frac{L}{2}} \tag{10}$$

After normalizing the eigen-function, we obtain the following orthonormal system of functions:

$$\left\{\sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right)\right\} \forall n \in \mathbb{N}$$

Theorem 2.1. *The orthonormal system, is a 'basis' of $C_D^\infty[0, L]$ and of every function, $f \in C_D^\infty$ is a linear combination of the normalized eigen-functions* ($\left\{\sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right)\right\}$).

Proof. A proof can be found in Chapter 2 of Folland's 1992 book: Fourier Analysis and its Applications [2] $\square$

An important note to make, in regards to Theorem 2.1, is that the term 'basis' is not in the usual sense of algebra, because it requires the use of infinite series, rather than finite sums.

Continuing, we apply the same process from $F_n(x)$ to $G_n(t)$:

$$G_n(t) = A_n \cos\left(\frac{cn\pi t}{L}\right) + B_n \sin\left(\frac{cn\pi t}{L}\right) \tag{11}$$

Using principle of superposition to combine equation 7 and equation 11, the general separated solution is obtained by multiplying together the separated solutions, $F_n(x)$ and $G_n(t)$:

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} F_n(x)G_n(t) && (12) \\ &= \sum_{n=1}^{\infty}(A_n \cos\frac{cn\pi t}{L} + B_n \sin\frac{cn\pi t}{L}) \sin\frac{n\pi x}{L} && (13) \end{aligned}$$

To obtain the exact solution, we use the initial conditions to determine the A_n and B_n. In order to determine A_n, recall that the initial position is given by

$$\Phi(x) = u(x,0) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L}, \tag{14}$$

so that the Fourier coefficient, A_n is given by:

$$A_n = \langle\Phi(x), \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)\rangle = \frac{2}{L}\int_0^L \Phi(x)\sin\frac{n\pi x}{L}dx \tag{15}$$

Then, in order to obtain B_n, apply term by term differentiation to get

$$\Psi(x) = \frac{\partial u}{\partial t}(x,0) = \sum_{n=1}^{\infty} B_n\frac{cn\pi}{L}\sin\frac{n\pi x}{L} \tag{16}$$

so that the Fourier coefficient B_n is given by:

$$B_n = \frac{2}{cn\pi}\int_0^L \Psi(x)\sin\frac{n\pi x}{L}dx \tag{17}$$

Finally, evaluating equations 12, 15, and 17, an exact solution to the 1-dimensional wave equation can be found. In order to further understand our solution, we will investigate the periodicity of the general solution.

2.1 Overtones

In order to further understand the periodicity of the general solution to the 1-dimensional wave equation. A function $u(x,t)$ is periodic in time, of period τ_u if and only if $u(x,t+\tau_u) = u(x,t)$. This is saying that, if the period of $u(x,t)$ is added to the time, then the result should be the same, showing that the function is indeed periodic on said period.

Theorem 2.2. *All solutions to the 1-dimensional wave equation (Equation 1) with Dirichlet Boundary Conditions are periodic in time, with period* $\tau_u = \frac{2L}{c} = \frac{1}{f_1}$. *Where* f_1 *is the fundamental frequency and is related to the period by the inverse period-frequency physical relationship.*

Proof.

$$\begin{aligned}
\cos\left(\frac{cn\pi(t+\tau_u)}{L}\right) &= \cos\left(\frac{cn\pi}{L}(t+\frac{2L}{c})\right) \\
&= \cos\left(\frac{cn\pi t}{L}+\frac{cn\pi(2L)}{c}\right) \\
&= \cos\left(\frac{cn\pi t}{L}+2\pi n\right) \\
&= \cos\left(\frac{cn\pi t}{L}\right)
\end{aligned}$$

□

It's also important to note that n, now plays an important role in the general and specific solutions of the 1-dimensional wave equation. For each $n \in \mathbb{N}$, there exists is a wave form, where as n increases, the respective wave form is the $n^{th}-1$ overtone of the fundamental wave form. Harmonically, the $n = 1$ waveform is the fundamental wave form and frequency. With the periodicity, we can now investigate a more specific model.

2.2 Plucked String Model with Dirichlet Boundary Conditions

In order to form a specific model, recall that A_n and B_n are defined as:

$$A_n = \frac{2}{L}\int_0^L \Phi(x)\sin\frac{n\pi x}{L}dx$$

$$B_n = \frac{2}{cn\pi}\int_0^L \Psi(x)\sin\frac{n\pi x}{L}dx$$

A plucked string has non-trivial initial displacement and zero initial velocity, hence $\Phi(x) = f(x)$ and that $\Psi(x) = 0$, for some function $f(x)$.

In contrast, a struck (or hammered) string, such as in a piano, will have zero initial displacement and non-trivial initial velocity, hence $\Phi(x) = 0$, and $\Psi(x) = g(x)$, for some function $g(x)$. This pair of initial conditions exists because the string has an initial velocity from the strike (due to the transfer of mechanical energy from the hammer to the string), but no change initial position from the equilibrium. Models can be obtained for both the plucked and hammered string with relative ease using equations from the previous section, but since the objective of this project is to model orchestral string instruments, the focus will remain on modeling a plucked string.

To continue to model the plucked string, the initial condition function must be defined. Often times a plucked string can be modeled by a tent function as shown below:

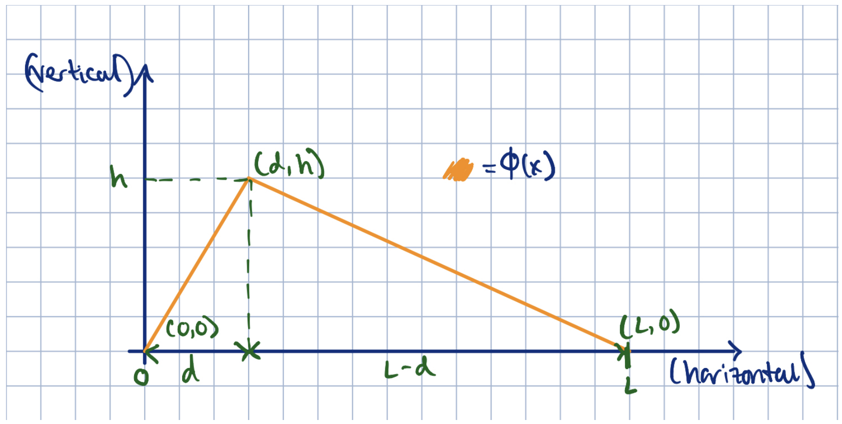

Figure 1: Tent Function for a Plucked String

In the tent function, the height, h, of the pluck is the vertical distance from the equilibrium position of the string, and the horizontal distance, d, from the 0 point on the string (the bridge) is the pluck distance. This results in a plucked string peak at the location (d, h). The orange function in Figure 1 is the tent function for the initial condition, $\Phi(x)$, which can be defined by the following system of equations:

$$\Phi(x) = \begin{cases} \frac{h}{d}x, x \in [0, d] \\ h + \frac{dh-hx}{L-d}, x \in (d, L] \end{cases}$$

The next section investigates the energy of the systems of plucked strings with Dirichlet boundary conditions, which can be further utilized to understand the energy of the overtones and the significance of fourth and fifth overtones.

3 Energy of the Plucked String Model with Dirichlet Boundary Conditions

As mentioned, the energy of all solutions to the 1-dimensional wave equation with Dirichlet Boundary Conditions is given by the following function:

$$E(u) = \frac{1}{2}\int_0^L \rho \left(\frac{\partial u}{\partial t}(x,t)\right)^2 + T\left(\frac{\partial u}{\partial x}(x,t)\right)^2 dx \tag{18}$$

The first portion of the integral represents the potential energy of the wave form, and the second portion represents the kinetic energy.
To begin, we prove that the energy is preserved in the string.

Theorem 3.1. *$E(u)$ is time independent.*

Proof. In order to show $E(u)$ is time independent, we want to show: $\frac{\partial E(u)}{\partial t} = 0$

$$\begin{aligned}
\frac{\partial E(u)}{\partial t} &= \frac{1}{2}\frac{\partial}{\partial t}\int_0^L \rho\left(\frac{\partial u}{\partial t}(x,t)\right)^2 + T\left(\frac{\partial u}{\partial x}(x,t)\right)^2 dx \\
&= \int_0^L \rho\frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial t^2} + T\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial t \partial x} dx
\end{aligned}$$

Then, via integration by parts for the second portion of the integral:

$$\begin{aligned}
\frac{\partial E(u)}{\partial t} &= T\frac{\partial u}{\partial x}\frac{\partial u}{\partial t}|_0^L + \int_0^L \rho\frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial t^2} - T\frac{\partial^2 u}{\partial x^2}\frac{\partial u}{\partial t} dx \\
&= T\frac{\partial u}{\partial x}\frac{\partial u}{\partial t}|_0^L + \frac{1}{\rho}\int_0^L \frac{\partial u}{\partial t}(\frac{\partial^2 u}{\partial t^2} - \frac{T}{\rho}\frac{\partial^2 u}{\partial x^2})dx \\
&= 0
\end{aligned}$$

By the fixed boundary condition, the first portion of the integral is 0, and the latter is 0 by the 1-dimensional wave equation operator.

□

Next, we prove how the total energy is related to the energies of the modes:

Theorem 3.2. *If $u(x,t) = \sum_{n=1}^{\infty} u_n(x,t)$, then $\forall$ n $E(u_n)$ is constant in time and $E(u) = \sum_{n=1}^{\infty} E(u_n)$.*

Proof. The proof for $E(u_n)$ is constant in time is exactly the same as previous proof since u_n also satisfies the 1-D wave equation and the Dirichlet Boundary Condition.
Continuing, we want to show:

$$E(u) = E(\sum_{n=1}^{\infty} u_n(x,t)) = \sum_{n=1}^{\infty} E(u_n(x,t))$$

We prove by induction:

$$E(u_1+u_2) = \frac{1}{2}\int_0^L \rho(\frac{\partial}{\partial t}(u_1+u_2))^2 + T(\frac{\partial}{\partial x}(u_1+u_2))^2 dx$$

Since the partial derivative is linear, then

$$\begin{aligned}
E(u_1+u_2) &= \frac{1}{2}\int_0^L \rho(\frac{\partial}{\partial t}(u_1) + \frac{\partial}{\partial t}(u_2))^2 + T(\frac{\partial}{\partial x}(u_1) + \frac{\partial}{\partial x}(u_2))^2 dx \\
&= \frac{1}{2}\int_0^L \rho(\frac{\partial u_1}{\partial t})^2 + T(\frac{\partial u_1}{\partial x})^2 dx + \frac{1}{2}\int_0^L \rho(\frac{\partial u_2}{\partial t})^2 + T(\frac{\partial u_2}{\partial x})^2 dx + \int_0^L \rho\frac{\partial u_1}{\partial t}\frac{\partial u_2}{\partial t} + T\frac{\partial u_1}{\partial x}\frac{\partial u_2}{\partial x} dx \\
&= E(u_1) + E(u_2) + \int_0^L \rho\frac{\partial u_1}{\partial t}\frac{\partial u_2}{\partial t} + T\frac{\partial u_1}{\partial x}\frac{\partial u_2}{\partial x} dx
\end{aligned}$$

$$\frac{\partial u_1}{\partial t}\frac{\partial u_2}{\partial t} = \frac{dG_1(t)}{dt}F_1(x)\frac{dG_2(t)}{dt}F_2(x) = 0$$

Then, since $F_1(x)$ and $F_2(x)$ are orthogonal:

$$\frac{\partial u_1}{\partial x}\frac{\partial u_2}{\partial x} = \frac{dF_1(x)}{dx}G_1(t)\frac{dF_2(x)}{dx}G_2(t) = 0$$

Since $\frac{dF_n(x)}{dx}$ is of the form $\frac{n\pi}{L}\cos\frac{n\pi x}{L}$, which we know are orthogonal, we can end by induction.

□

Recall that the general solution for the 1-dimensional wave equation for a plucked string is given by:

$$u_n = A_n \cos\frac{cn\pi t}{L}\sin\frac{n\pi x}{L}$$

Then, solving for A_n from equation 15:

$$A_n = \frac{2L}{n^2\pi^2}\left(\frac{h}{d} + \frac{h}{L-d}\right)\sin\left(n\pi\frac{d}{L}\right). \tag{19}$$

Since $E(u_n)$ is independent of time, we can evaluate the integral at $t = 0$, thus obtaining the following general equation for the energy of the n^{th} mode waveform:

$$E(u_n) = \frac{Tn^2A_n^2\pi^2}{4L} \tag{20}$$

Theorem 3.3. *The total energy of the plucked string model is:*

$$E(u) = \frac{Th^2}{2}\frac{L}{d(L-d)} \tag{21}$$

Proof.

$$E(u) = \sum_1^\infty E(u_n) = \sum_1^\infty \frac{Tn^2\pi^2}{4L}A_n^2$$

Since $\Phi'(x) = \sum_{n=1}^\infty A_n \frac{n\pi}{L}\cos\left(\frac{n\pi x}{L}\right)$, when $t = 0$

$$\int_0^L \Phi'^2\, dx = \frac{1}{2}\sum_{n=1}^\infty \frac{n^2\pi^2}{L}A_n^2$$

$$\sum_1^\infty E(u_n) = \frac{T}{2}\int_0^L \Phi'^2 dx$$

Using the initial position for plucked string:

$$\begin{aligned}
\sum_1^\infty E(u_n) &= \frac{T}{2}\left(\int_0^d \frac{h^2}{d^2}dx + \int_d^L \frac{h^2}{(L-d)^2}dx\right)\\
&= \frac{T}{2}\left(\frac{h^2}{d} + \frac{h^2}{L-d}\right)\\
&= \frac{Th^2}{2}\frac{L}{d(L-d)}
\end{aligned}$$

□

Since we have both an equation for the total energy and the energy of the n^{th} mode, it's logical to next investigate the distribution of energies across the modes.

3.1 Distribution of Energies

In order to explore the distribution of the energies, we must first find an expression that defines the distribution of the energies. We use the following definition:

Definition 3.1. *The distribution of energies, $\mathcal{P}_n$, is the ratio of the n^{th} mode energy to the total energy.*

$$\mathcal{P}_n = \frac{E(u_n)}{E(u)} \tag{22}$$

Proposition 3.1. *The distribution of energies for the plucked string model is given by:*

$$\begin{aligned} \mathcal{P}_n &= \frac{E(u_n)}{E(u)} & (23) \\ &= \frac{2L^2}{n^2\pi^2 d(L-d)} \sin^2 \frac{n\pi d}{L} & (24) \end{aligned}$$

The proof of the following is a straightforward evaluation of Equation 20 and 21, as shown below:

Proof.

$$\mathcal{P}_n = \frac{E(u_n)}{E(u)} = \frac{Tn^2\pi^2}{4L} \frac{2d(L-d)}{Th^2 L} \frac{4L^2}{n^4\pi^4} \frac{h^2 L^2}{d^2(L-d)^2} \sin^2 \frac{n\pi d}{L}$$

After simplification:

$$\mathcal{P}_n = \frac{2L^2}{n^2\pi^2 d(L-d)} \sin^2 \frac{n\pi d}{L}$$

□

Let $L = \pi$, Figure 2 shows $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, \mathcal{P}_4$ with varying initial position.

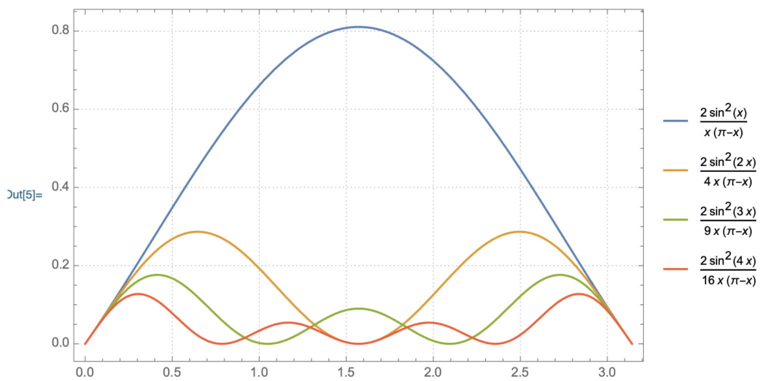

Figure 2: Distribution of energies at $n = 1, 2, 3, 4$ for varying d

We observe that for all $d > 0$, fundamental frequency has the most energy, whereas the each of the rest of the modes has the most energy, until the mode reaches the distance of $d = \frac{L}{n}$, The exception to this is at distances = when d is very very small and at $d = L$. So we observe at the second frequency, or the first overtone, that there is no energy at $d = \frac{L}{n}$. Additionally, we make the observation that $\mathcal{P}_n$ solely depends on d, and further, because $\mathcal{P}_n$ depends on d, we can establish that $E(u)$ and furthermore, $E(u_n)$ also depend on d. The following propositions demonstrate our observation regarding distribution of energy:

Proposition 3.2. *The fundamental frequency, $n = 1$ gets largest distribution of energy, regardless of the pluck position d.*

Proof. W.T.S $\mathcal{P}_1 - \mathcal{P}_n \geq 0, \forall n \in \mathbb{N}, \forall d \in [0, L]$

$$\begin{aligned}\mathcal{P}_1 - \mathcal{P}_n &= \frac{2}{d(\pi - d)} \sin^2 d - \frac{2}{n^2 d(\pi - d)} \sin^2 nd \\ &= \frac{2}{d(\pi - d)} (\sin^2 d - \frac{sin^2 nd}{n^2})\end{aligned}$$

Since $\frac{2}{d(\pi-d)} \geq 0$, we are now left to show that $\sin^2 d - \frac{sin^2 nd}{n^2} \geq 0$:

$$|\sin d| \geq |\frac{\sin nd}{n}|, \forall n \in \mathbb{N}$$

Then a proof by induction:

$$\begin{aligned}|\sin(n+1)x| = |\sin(nx + x)| &= |\sin nx \cos x + \cos nx \sin x| \\ &\leq |\sin nx \cos x| + |\cos nx \sin x| \\ &\leq |\sin nx| + |\sin x|\end{aligned}$$

Hence $\sin^2 d \geq \frac{sin^2 nd}{n^2}$, $\mathcal{P}_1 \geq \mathcal{P}_n, \forall n \in \mathbb{N}$. □

Proposition 3.3. *When $d \to 0$, all overtones will have the same distribution of energy.*

Proof. Choose some arbitrary m,$n \in \mathbb{N}$, then:

$$\mathcal{P}_m - \mathcal{P}_n = \frac{2}{m^2 d(\pi - d)} \sin^2 md - \frac{2}{n^2 d(\pi - d)} \sin^2 nd$$

$$\begin{aligned} \lim_{d\to 0} \mathcal{P}_m - \mathcal{P}_n &= \lim_{d\to 0} \frac{2}{m\pi} \frac{\sin^2 md}{md} - \lim_{d\to 0} \frac{2}{n\pi} \frac{\sin^2 nd}{nd} \\ &= \lim_{dm\to 0} \frac{2}{m\pi} \frac{\sin^2 md}{md} - \lim_{dn\to 0} \frac{2}{n\pi} \frac{\sin^2 nd}{nd} \\ &= 0 \end{aligned}$$

□

Proposition 3.4. *The distribution of the energy, $\mathcal{P}_n$ is independent of height, h, and thus depends on the distance, d. Furthermore, we also observe the energy $E(u)$ only depends on d.*

Proposition 3.5. *If the string is plucked at position d, where $k = \frac{L}{d}$, is an integer, then $E(u_n) = 0$ for all n being multiples of k.*

Proof. We have a missing overtone when $E(u_n) = 0$, and furthermore, when $A_n = 0$

$$E(u_n) = \iff A_n = 0$$

$$A_n = 0 \iff \frac{n\pi d}{L} = k\pi \iff n = \frac{kL}{d}, k = 1, 2, 3...$$

□

This observation also leads to an additional exploration: the missing overtone. From Figure 3 it can be observed that the distribution of energy goes to 0, at each point $d = \frac{L}{n}$. When $\mathcal{P}_n = 0$, then we know that:

$$\begin{aligned} 0 &= \mathcal{P}_n \\ &= \frac{E(u_n)}{E(u)} \\ &= \frac{0}{E(u)} \end{aligned}$$

This observation can also be illustrated as a plot of the discrete energies. We plot the energies of the modes, discretely by $n = 1, 2, ...$, for the plot take $L = \pi$, $d = \frac{\pi}{4}$, $h = 2$, and $T = 60$:

Note that in this figure, the horizontal 0 is not at the bottom of the plot, but rather slightly above, so that 0 energies, or missing overtones, can be easily seen. By a choice of $d = \frac{\pi}{4}$, we observe a missing harmonic at every $n = 4, 8, 12, ...$, which follows with Proposition 3.5. Continuing, the plot of A_n in orange, also

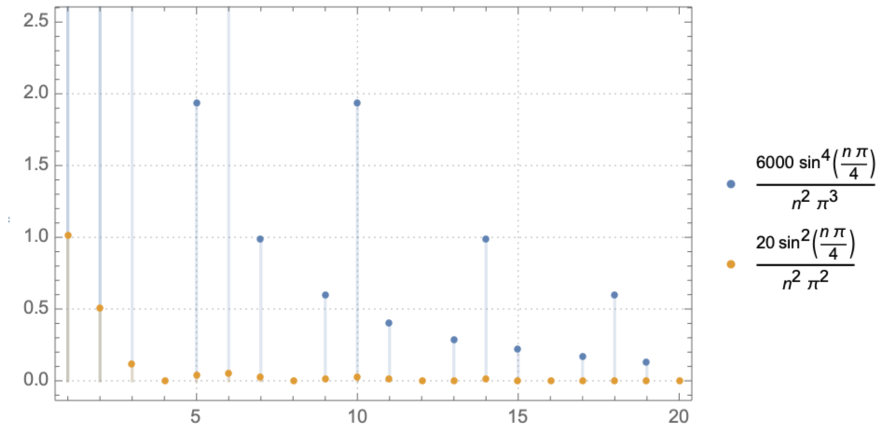

Figure 3: Discrete Plots of $E(u_n)$, in blue, and A_n, in orange.

follows the same patterns of missing harmonics at $d = \frac{L}{n}$. This furthers the conclusion that when $A_n = 0$, then $E(u_n) = 0$, and $\mathcal{P}_n = 0$.

Continuing with the distribution of energies, it is important to understand to what modes most of the energy goes to. Notice, in Figure 4, it seemed as if, in both A_n and $E(u_n)$, most of the energy was in the first few modes, until the missing mode, and then a near negligible amount following that missing mode. After a great deal of experimentation with various initial conditions, string lengths, and physical constants, we were able to make the following conjecture:

Conjecture 3.1. *A minimum of* 80% *of the total energy can be obtained by summing the energy of the modes to the first missing harmonic.*

Rather than verifying this symbolically, we verify numerically using the following cases: $L = \pi$ and $d = \{\frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{10}\}$.

Proof. Case 1: For a large d, we know that, we will need to sum to $n = 2$ to achieve a minimum of 80% of the total energy. We first obtain $\mathcal{P}_n$ for $L = \pi$ and $d = \frac{\pi}{2}$:

$$\mathcal{P}_n = \frac{8\sin^2\frac{n\pi}{2}}{n^2\pi^2}$$

Then, summing from 1 to 2:

$$\begin{aligned} .8 &\leq \sum_{n=1}^{2} \frac{8\sin^2\frac{n\pi}{2}}{n^2\pi^2} \\ &= \frac{8}{\pi^2} \\ .8 &\leq .810 \end{aligned}$$

Case 2: For a smaller d, for this case we will need to sum to $n = 4$ to achieve a minimum of 80% of the total energy. We begin, again by obtaining $\mathcal{P}_n$ for $L = \pi$ and $d = \frac{\pi}{4}$:

$$\mathcal{P}_n = \frac{32 \sin^2 \frac{n\pi}{4}}{3n^2\pi^2}$$

Then summing from 1 to 4:,

$$\begin{aligned} .8 &\leq \sum_{n=1}^{4} \frac{32 \sin^2 \frac{n\pi}{4}}{3n^2\pi^2} \\ &= \frac{232}{27\pi^2} \\ .8 &\leq .870 \end{aligned}$$

Case 3: For a further smaller d, for this case we will need to sum to $n = 10$ to achieve a minimum of 80% of the total energy. We begin, again by obtaining $\mathcal{P}_n$ for $L = \pi$ and $d = \frac{\pi}{4}$:

$$\mathcal{P}_n = \frac{200 \sin^2 \frac{n\pi}{10}}{9n^2\pi^2}$$

Then, summing from 1 to 10:

$$\begin{aligned} .8 &\leq \frac{200 \sin^2 \frac{n\pi}{10}}{9n^2\pi^2} \\ .8 &\leq .892 \end{aligned}$$

We numerically see that this works, and can thus support the conjecture, that by summing to the first missing harmonic, a minimum of 80% of the energy can be obtained. □

3.2 Can We Hear the Initial Conditions?

Moving in a different direction, Mark Kac [6] explores if the different shapes of drums can be heard by an ear that is unknown of the drum shape in his paper "Can we hear the shape of the drum" [6]. This paper prompted our own question, can we hear the initial conditions for the plucked string model with Dirichlet Boundary Conditions? A more formalized version of the problem is presented as follows:

Proposition 3.6. *Suppose a listener has perfect pitch and knows $E(u_n)$ for all n, then they can determine the initial condition for a plucked string with Dirichlet boundary conditions.*

Recall, the period of any solution to the 1-dimensional wave equation is:

$$\tau_u = \frac{2L}{cn}$$

Then, by the inverse period-frequency relationship, the frequency for any solution to the 1-dimensional wave equation is given by:

$$f_n = \frac{cn}{2L}$$

Where the fundamental frequency of the wave is:

$$f_1 = \frac{c}{2L}$$

Furthermore, recall the eigen-values, λ_n, for the wave equation:

$$\lambda_n = -(\frac{n\pi}{L})^2$$

Using algebra, we can define λ_n as a function of the frequency of the wave form:

$$\lambda_n = -(\frac{2\pi f_n}{c})^2 \tag{25}$$

Suppose a listener has perfect pitch and know $\frac{T}{\rho}$, meaning that when they hear a note they can name the note and frequency of the note, and the physical constant c is given, then the listener can determine all λ_n of the solution from the frequency. Furthermore,

$$L = \frac{c}{2f_1} \tag{26}$$

Thus we can determine the length of the string. Suppose we know the $E(u_n)$ for all n, recall from previous section (Equation 20) that A_n is associated to $E(u_n)$ by

$$A_n = \sqrt{\frac{4E(u_n)L}{Tn\pi}} \tag{27}$$

Therefore, by knowing $E(u_n)$ for all n, we can determine all A_n. Note that A_n is the Fourier coefficient of initial position with the orthogonal system $\{\sin\frac{n\pi x}{L}\}$, we can reconstruct the initial position using Fourier series:

$$\Phi(x) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L}$$

Finally, the initial velocity of the plucked string is zero. Thus, by these assumption we can hear the initial condition for a plucked string with the Dirichlet boundary conditions. This conclusion is especially interesting because we've come to a different conclusion than in Kac's [6] paper.

4 Mixed Boundary Conditions

After using the Dirichlet Model, we realize that, while the strings are fixed at the end of the neck of the instrument, they rest on a bridge made of a light wood, which is not nearly as rigid as a fixed end. In this model, the left end of the string rests on a spring system, as shown in the diagram below, and the right remains fixed:

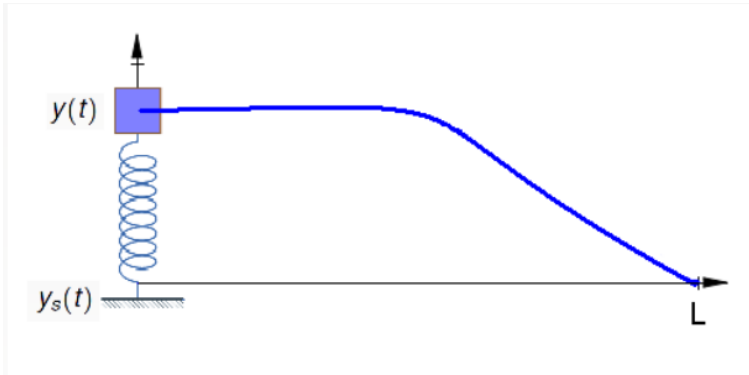

Figure 4: Spring-mass system with an attached string

4.1 Derivation of Boundary Conditions

Suppose we have a string where one end is attached on the system shown in Figure 4, and the other end is fixed such as in the Dirichlet boundary conditions. This yields a set of mixed boundary conditions, one following the Robin (both ends are attached to a spring-mass system) boundary condition, and the other with the Dirichlet boundary conditions (both ends are fixed).

Denote $y(t) = u(0,t)$, where $y(t)$ satisfies the following ODE:

$$m\frac{\partial}{\partial t^2}y(t) = -k[y(t) - y_E(t)] + T(0,t)\sin(\theta(0,t)) + \gamma \tag{28}$$

$y_E(t)$ represents the equilibrium position of the string and k the spring constant (the physical stiffness of the spring). The second term of the ODE in (28) represents the vertical component of the tensile force of the string. Then, since θ is always small:

$$\begin{aligned} T(0,t)\sin(\theta(0,t)) &\approx T(0,t)\frac{\sin(\theta(0,t))}{\cos(\theta(0,t))} \\ &= T(0,t)\tan(\theta(0,t)) \\ &= T(0,t)\frac{\partial}{\partial x}u(0,t) \end{aligned}$$

Next, since the string is resting on the spring and then continues to a fixed end, we can consider the mass in the spring-mass system to be small enough to be negligible. More specifically, the mass that would exist on the spring, is the point mass of the string on the spring, which is so small, that we consider it to be 0. Furthermore, there are no additional external forces in this system, so $\gamma = 0$, as is the $y_E(t)$ term, because the equilibrium position of the system is at $x = 0$. Finally, the vertical component of the tensile force is non-changing because we have previously defined constant tension T in the system and we are only working in 1-dimension, and thus $T(0,t) = T$. With these simplifications, equation (28) becomes the following:

$$T\frac{\partial u}{\partial x}(0,t) = ku(0,t) \tag{29}$$

Furthermore, the boundary conditions for the Mixed Boundary Condition model are now:

$$T\frac{\partial u}{\partial x}(0,t) = ku(0,t) \tag{30}$$

$$0 = u(L,t) \tag{31}$$

Together with the 1-dimensional wave equation:

$$\frac{\partial^2 u}{\partial t^2}(x,t) = c^2 \frac{\partial^2 u}{\partial x^2}(x,t)$$

where $c^2 = \frac{T}{\rho}$ and the initial conditions $\Psi(x)$ and $\Phi(x)$.

$$\begin{aligned} u(x, t=0) &= \Phi(x) \\ \frac{\partial u}{\partial t}(x, t=0) &= \Psi(x) \end{aligned}$$

we can solve for the unique solution to the 1-dimensional wave equation with the mixed boundary conditions.

4.2 Series Solution to the Mixed Boundary Conditions

Using separation of variables method, let $u(x,t)$ be composed of the product of two functions:

$$u(x,t) = F(x)G(t)$$

Then, we apply the same 1-dimensional wave equation operator from the Dirichlet model, resulting:

$$\frac{\partial}{\partial t^2}G(t) - \lambda c^2 G(t) = 0 \tag{32}$$

$$\frac{\partial}{\partial x^2}F(x) - \lambda F(x) = 0 \tag{33}$$

Then, applying the boundary conditions: $F'(0) = \frac{k}{T}F(0)$ and $F(L) = 0$. Now we obtain the general solution for $F(x)$:

$$F(x) = c_1 \cos(v_n x) + c_2 \sin(v_n x) \tag{34}$$

Because $F(L)G(t) = 0$, by the fixed boundary condition at L, and since $G(t) \neq 0$, $F(L) = 0$. Additionally, $\lambda = v_n^2$. Further using the boundary condition explained above, we see that $c_2 = \frac{k}{Tv_n}c_1$, thus the specific solution for $F_n(x)$:

$$F(x) = c_1 \cos v_n x + c_1 \frac{k}{Tv_n} \sin v_n x$$

Multiplying $F(x)$ by $\frac{1}{c_1}$, it will remains to be a solution of 1-dimensional wave equation, hence

$$F(x) = \cos v_n x + \frac{k}{T v_n} \sin v_n x \tag{35}$$

Substituting for $F(L) = 0$, we have v_n is the solution of:

$$\tan v_n L = -\frac{T}{k} v_n \tag{36}$$

Then for $G(t)$, we apply the same process as in the Dirichlet model, resulting in:

$$G(t) = A_n \cos(c v_n t) + B_n \sin(c v_n t)$$

where A_n and B_n are the n-th Fourier coefficients. Now, the solutions to the 1-dimensional wave equation with mixed boundary conditions is:

$$u_n(x,t) = (A_n \cos(c v_n t) + B_n \sin(c v_n t))(\cos(v_n x) + \frac{k}{T v_n} \sin(v_n x)) \tag{37}$$

The following theorem shows the orthogonality of eigenfunction F_n in Mixed boundary condition

Theorem 4.1. *The eigenfunction* $\{\cos v_n x + \frac{k}{T v_n} \sin v_n x, \forall n = 1, 2, ...\}$ *are mutually orthogonal to each other.*

Proof. W.T.S: $\int F_n(x) F_m(x) dx = 0$, if $m \neq n$.
Note that

$$\begin{aligned}
(-F_n' F_m + F_n F_m')' &= -(F_n'' F_m + F_n' F_m') + (F_n' F_m' + F_n F_m'') && (38) \\
&= -F_n'' F_m - F_n' F_m' + F_n' F_m' + F_n F_m'' && (39) \\
&= -F_n'' F_m + F_n F_m'' && (40)
\end{aligned}$$

Using $F_n'' = \lambda_n F_n$, and $F_m'' = \lambda_m F_m$

$$(-F_n' F_m + F_n F_m')' = -\lambda_n F_n F_m + \lambda_m F_n F_m = (\lambda_m - \lambda_n) F_n F_m$$

$$\begin{aligned}
\int F_m F_n dx &= \int_0^L \frac{1}{\lambda_m - \lambda_n} (F_n F_m' - F_n' F_m)' dx && (41) \\
&= \frac{1}{\lambda_m - \lambda_n} (F_n F_m' - F_n' F_m)|_{x=0}^{x=L} && (42) \\
&= \frac{1}{\lambda_m - \lambda_n} (F_n'(0) F_m(0) - F_n(0) F_m'(0)) && (43)
\end{aligned}$$

Applying the boundary conditions:

$$\int F_m F_n dx = \frac{1}{\lambda_m - \lambda_n} (\frac{k}{T} F_n(0) F_m(0) - F_n(0) \frac{k}{T} F_m(0)) = 0 \tag{44}$$

Hence $\{F_n(x)\} = \{\cos v_n x + \frac{k}{T v_n} \sin v_n x\} n = 1, 2, ...$ are orthogonal sets. □

4.3 Numerical Approximations for v_n

Recall v_n's are the solution of the following equation:

$$\tan v_n L \quad = \quad -\frac{T}{k} v_n$$

Notice, that v_n appears on both sides of the last equation and thus cannot be symbolically solved, but rather numerical approximation must be used. In order to obtain a numerical approximation, we began by plotting $\tan(v_n L)$ against $\frac{-T}{k} v_n$. In the following plot, figure 5, v_n has just been called v for ease of plotting, additionally, just for this plot $T = 300$ and $k = 200$.

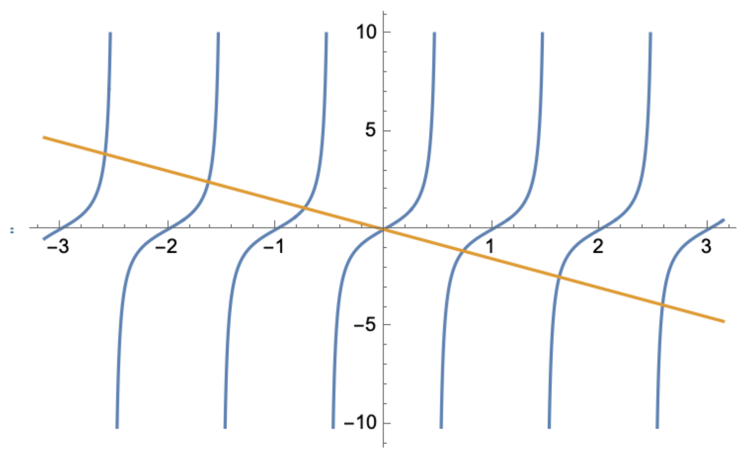

Figure 5: $\tan(v_n L)$ in blue, plotted against $\frac{-T}{k} v_n$, in orange in order to determine a numerical approximation for v_n.

4.3.1 Approximation of v_n when n is Large

By further observation, we see that when n is sufficiently large, or when k is very small, v_n lies approximately on the vertical asymptotes of the tangent line function. Note that v_n can not be 0, since if v_n is 0, then solution with 0 as an eigenvalue does not satisfy the boundary condition. From the plot in Figure 5, the vertical asymptotes of $\tan{(v_n L)}$ occur at $\frac{1}{2}+n$. Thus. our first method of numerical approximation for $v_n L$ (and v_n) is as follows:

$$\begin{aligned} v_n L &\approx \frac{\pi}{2}+(n-1)\pi \\ v_n &\approx \frac{\pi}{2L}+\frac{(n-1)\pi}{L} \end{aligned}$$

Although, at higher degrees, a correction is required. Let $v_n = \frac{\pi}{2L}+\frac{(n-1)\pi}{L} - \epsilon_n$, where ϵ_n is the correction factor, and $n \in \mathbb{Z}$:

$$\tan v_n L = \tan{(\frac{\pi}{2}+(n-1)\pi - \epsilon_n L)} = \tan{(\frac{\pi}{2} - \epsilon_n L)}$$

Notice that $\tan{(\frac{\pi}{2}-x)}$ is not smooth, meaning not continuously differentiable, at $x=0$ and thus we can rewrite $\tan\frac{\pi}{2}-x$ as:

$$\tan{(\frac{\pi}{2}-x)} = \frac{1}{x}\frac{x\cos x}{\sin x}$$

Now, taking the Taylor Expansion of $\frac{x\cos x}{\sin x}$, in order to find a better approximation:

$$\frac{x\cos x}{\sin x} = 1-\frac{x^2}{3}-\frac{x^4}{45}-\frac{2x^6}{925}-...$$

We find through the Taylor Expansion, that $\tan{(\frac{\pi}{2}-x)} \approx \frac{1}{x} = -\frac{Tv_n}{k}$. Then further substituting into the equation for the correction, we have:

$$\begin{aligned} \frac{1}{\epsilon_n L} &= -\frac{T}{k}(\frac{\pi}{2}+(n-1)\pi-\epsilon_n) \\ \iff \frac{TL}{k}\epsilon_n^2 &- \frac{\pi+(2n-2)\pi}{2}\frac{TL}{k}\epsilon_n - 1 = 0 \end{aligned}$$

hence, let $a=\frac{TL}{k}$, the solution of the quadratic formula is given by:

$$\epsilon_n = \frac{(\pi+2(n-1)\pi)a+\sqrt{a^2(\pi+2(n-1)\pi)^2+a}}{4a}$$

$$\epsilon_n = \frac{(\pi+2(n-1)\pi)a-\sqrt{a^2(\pi+2(n-1)\pi)^2+a}}{4a}$$

With a positive and a negative answer, we chose the sign of ϵ_n based on the sign of v_n that we are trying to approximate. If we are trying to approximate positive v_n, we will chose positive ϵ_n, while if we are trying to approximate negative v_n, we will chose negative ϵ_n, since we want to use the correction to

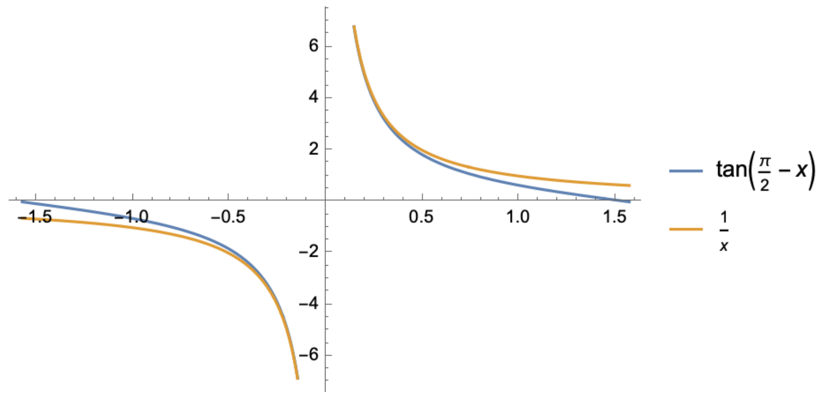

Figure 6: Taylor expansion of $\tan x$ around $\epsilon_n \approx 0$

make v_n as close as possible. Then we plot $\tan \frac{\pi}{2} - x$ and $\frac{1}{x}$ in Figure 6 in order to see how the approximation performs.

Exmaple 1: Suppose k = 1, T = 1, L = 3, the numerical solution of v_{10} is given by mathematica as

$$v_{10} \approx 9.98166025783878,$$

where our numerical approximation methods gives

$$v'_{10} \approx 9.949074768037809.$$

Exmaple 2: Suppose k = 1, T = 1, L = 3, the numerical solution of v_{20} is given by mathematica as

$$v_{20} \approx 20.436649815855155,$$

where our numerical approximation methods gives

$$v'_{20} \approx 20.420692321110568$$

In Figure 6, we can see that the approximation performs very well for small correction, i.e $\epsilon_n \approx 0$. Thus we can consider this our numerical approximation for v_n when n is large. We will use the other approximation for when n is small, and this is more important to our project since fundamental frequency plays the largest role in sound spectrum.

4.3.2 Approximation of Fundamental Frequency

Noted that in real life, k is substantially large and the slope of the linear line, $-\frac{T}{k}$, is close to zero, hence $\frac{v_1}{L}$ is approximately π and v_1 is approximately $\frac{\pi}{L}$. The numerical approximation method is follows:

Let $y = v_1 L - \pi$, hence y is a function that depends on v_1.
Consider the following equation:

$$y + \frac{1}{3}y^3 + \frac{2}{15}y^5 + \cdots = -\kappa\frac{y+\pi}{L} \tag{45}$$

Where κ is $\frac{T}{k}$. Note that the right hand side is equivalent to $-\frac{T}{k}v_1$ and the left hand side of the equation is the Taylor expansion of $\tan x$ around $x = v_1 L - \pi$. Regard (45) as a function of y that depends on κ:

$$y = y(\kappa) \text{ with } y(0) = 0 \tag{46}$$

By definition of y, v_1 can be written as a function of y

$$v_1 = \frac{y+\pi}{L}; \text{ hence } v_1(0) = \frac{\pi}{L} \tag{47}$$

Next, we are trying to expand $y(\kappa)$ around $\kappa \approx 0$ with only the first order correction.
Using implicit differentiation, and apply $\frac{d}{d\kappa}$ to both sides of the defining equation (45), we obtain

$$y' + y'y^2 + \frac{2}{3}y'y^4 + \cdots = -\frac{y+\pi}{L} \tag{48}$$

Evaluating at $\kappa = 0$, and note that y(0)=0, we obtain $y'(0) = -\frac{\pi}{L}$. Hence the first order Taylor expansion of $y(\kappa)$ is as follows

$$y(\kappa) = y(0) + \kappa\frac{y'(0)}{1} = -\kappa\frac{\pi}{L}$$

by equation (47), when κ is small (i.e when k is large)

$$v_1(\kappa) \approx \frac{1}{L}(\pi - \kappa\frac{\pi}{L}) \tag{49}$$

Equation (45) is interpreted as follows: when κ is zero, then $v_1 = \frac{\pi}{L}$, when κ is close to zero, then the correction term is $-\frac{\kappa\pi}{L^2}$.

Exmaple 3: Suppose k = 500, T = 1, L = 3, the numerical solution of v_1 is given by mathematica as

$$v_1 \approx 1.0464998856249224,$$

where our numerical approximation methods gives

$$v_1' \approx 1.0464994194958.$$

Exmaple 4: Suppose k = 50, T = 1, L = 3, the numerical solution of v_1 is given by mathematica as

$$v_1 \approx 1.040263461830458,$$

where our numerical approximation methods gives

$$v_1' \approx 1.0402162341886205.$$

Therefore, the numerical solution supports that our approximation method works well when k is large. Moreover, the larger k is, the better aprroximation this method performs.

Observation 4.1. *u_n has period $\frac{2\pi}{cv_n}$.*

Proof. Modes of solution has the form:

$$u_n(x,t) = (A_n \cos cv_n t + B_n \sin cv_n t)(\cos v_n x + \frac{k}{Tv_n} \sin v_n x)$$

$$\begin{aligned} u_n(x, t + \frac{2\pi}{cv_n}) &= \left(A_n \cos cv_n(t + \frac{2\pi}{cv_n}) + B_n \sin cv_n(t + \frac{2\pi}{cv_n})\right)(\cos v_n x + \frac{k}{Tv_n} \sin v_n x) \\ &= (A_n \cos(cv_n t + 2\pi) + B_n \sin(cv_n t + 2\pi))(\cos v_n x + \frac{k}{Tv_n} \sin v_n x) \\ &= (A_n \cos cv_n t + B_n \sin cv_n t)\frac{k}{Tv_n} \sin v_n x) = u_n(x,t) \end{aligned}$$

□

Hence the frequency of each solution is given by:

$$f_n = \frac{cv_n}{2\pi}$$

Observation 4.2. *Since v_n's are not multiple of each other, we don't have overtone anymore.*

Furthermore, by our approximation method, the fundamental frequency of the vibrating string with Mixed Boundary Conditions is given by:

$$f_{1_M} = \frac{c(\pi - \kappa\frac{\pi}{L})}{2\pi L} = \frac{c}{2L}(1 - \frac{\kappa}{L})$$

Recall from a previous section that the fundamental frequency in Dirichlet boundary condition is

$$f_{1_D} = \frac{c}{2L}$$

Theorem 4.2. *With same physical constants ρ, T and L, the fundamental frequency of the mixed boundary condition is approximately a factor of that of the Fixed boundary condition, i.e*

$$f_{1_M} \approx f_{1_D}(1 - \frac{\kappa}{L}) \tag{50}$$

Remark: since $\frac{T}{kL}$ is positive, the fundamental frequency of the mixed boundary condition is always less than the fundamental frequency in Dirichlet boundary condition, with same physical constants ρ, T and L.

4.4 Plucked String Model with Mixed Boundary Conditions

Now that the wave equation for the mixed boundary conditions is complete and we have a numerical approximation for v_n, we can build a model for plucked strings. Recall that the initial position of the plucked string is the tent function and the initial velocity of the plucked string is 0, thus $B_n = 0$.
The solution of 1D wave equation with mixed boundary condition for plucked string is given by:

$$u(x,t) = \sum_{n=1}^{\infty} \left(A_n \cos\left(cv_n t\right)\right)\left(\cos\left(v_n x\right) + \frac{k}{Tv_n}\sin\left(v_n x\right)\right) \tag{51}$$

where A_n is related to the initial position as:

$$\Phi(x) = u(x,0) = \sum_{n=1}^{\infty} u_n(0,t) = \sum_{n=1}^{\infty} A_n(\cos\left(v_n x\right) + \frac{k}{Tv_n}\sin\left(v_n x\right)) \tag{52}$$

Given $(\cos(v_n x) + \frac{k}{Tv_n}\sin(v_n x))$ as the eigen-function of $F_n''(x) = \lambda F_n(x)$, therefore A_n is the Fourier coefficient of initial position given by inner product of initial position and the eigenfunction:

$$A_n = \langle \Phi(x), \frac{F_n(x)}{\|F_n(x)\|} \rangle \tag{53}$$

$$\begin{aligned} A_n &= \frac{1}{\|F_n(x)\|}\left(\int_0^d \frac{h}{d} x F_n(x)dx + \int_d^L \left(\frac{hL}{L-d} - \frac{h}{L-d}x\right)F_n(x)dx\right) \\ &= \frac{1}{\|F_n(x)\|}\Big(\frac{h}{d}\frac{(T-dk)v_n\cos(dv_n) + (k+Tv_n^2)\sin(dv_n) - Tv_n}{Tv_n^3} \\ &\qquad + \frac{h}{d-L}\frac{k\sin v_n L - v_n(T+kL-kd)\cos v_n d + Tv_n\cos v_n L - (k+(d-L)Tv_n^2)\sin v_n d}{Tv_n^3}\Big) \end{aligned}$$

Where the norm of the eigen-function is given by:

$$\|F_n(x)\| = \frac{1}{2}\sqrt{\frac{-k^2\sin 2v_n L + v_n(2k(kL + T - T\cos 2v_n L) + T^2 v_n(\sin 2v_n L + 2v_n L))}{T^2 v_n^3}} \tag{54}$$

Note that when k is large, $v_n L$ is approximately $\frac{\pi}{2} + n\pi$, hence

$$\sin 2v_n L \approx \sin(\pi + 2n\pi) = 0,$$

additionally

$$\cos 2v_n L \approx \cos(\pi + 2n\pi) = -1$$

Thus, $\|F_n(x)\|$ is approximately

$$\|F_n(x)\| \approx \frac{1}{2}\sqrt{\frac{2k(kL+2T)}{T^2 v_n^2} + 2L}$$

As n tends to infinity,

$$\lim_{n\to\infty} \|F_n(x)\| = \sqrt{\frac{L}{2}}$$

Thus, when k is large, the norm of eigen-function in the mixed boundary condition model will converge to the norm of eigen-function in fixed boundary condition model (Dirichlet Model).

Furthermore, the $|A_n|$ is bounded:

$$\frac{1}{\|F_n(x)\|}\left(\frac{h(k+Tv_n+(dk+T)v_n+dTv_n^2)}{dTv_n^3}+\frac{h(2k+Tv_n+(dk+kL+T)v_n+(L-d)Tv_n^2)}{(L-d)Tv_n^3}\right)$$

Since when k is large, $\|F_n(x)\|$ has lower bound $\sqrt{\frac{L}{2}}$, hence $\frac{1}{\|F_n(x)\|}$ has an upper bound $\sqrt{\frac{2}{L}}$

The absolute value of A_n is bounded by:

$$\sqrt{\frac{2}{L}}\left(\frac{h(k+Tv_n+(dk+T)v_n+dTv_n^2)}{dTv_n^3}+\frac{h(2k+Tv_n+(dk+kL+T)v_n+(L-d)Tv_n^2)}{(L-d)Tv_n^3}\right)$$

For example, suppose $L=\pi$, $d=\frac{\pi}{3}$, $h=0.1$, $T=60$, $\rho=1.15$, $k=500$, then we observe in Figure 7 Since the upper bound of absolute value of A_n converges

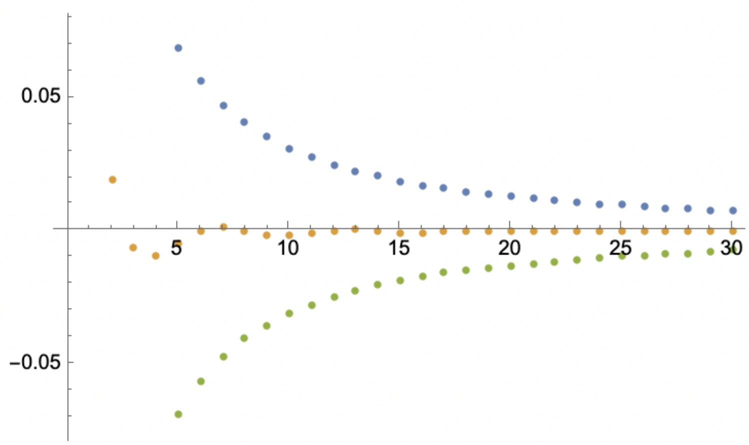

Figure 7: A_n plotted in orange, against the upper and lower bounds, in blue and green respectively. In this plot, n is plotted on the horizontal axis.

to zero when n tends to infinity, we can conclude that A_n also converges to 0 when n tends to infinity. From these observations, we can go on to investigate the energy of this model.

5 Energy of the String with the Mixed Boundary Conditions

Similar to the Dirichlet model, we investigate the energy of the model with the mixed boundary conditions. In the Dirichlet model, the energy is sum of the potential and kinetic energies of the string. However, because of the added spring, the energy is not preserved in the string. Therefore, we add the potential energy of the spring into the system, giving us with a modified energy equation:

$$E(u) = \frac{k}{2}u(0,t)^2 + \frac{1}{2}\left(\int_0^L \rho\left(\frac{\partial u}{\partial t}\right)^2 + T\left(\frac{\partial u}{\partial x}\right)^2 dx\right) \tag{55}$$

$$E(u_{string}) = \frac{1}{2}\left(\int_0^L \rho\left(\frac{\partial u}{\partial t}\right)^2 + T\left(\frac{\partial u}{\partial x}\right)^2 dx\right) = E(u) - \frac{k}{2}u(0,t)^2 \tag{56}$$

In equation 55, the first term is the potential energy of the spring, and the latter two terms are the sum of the potential and kinetic energies of the string. Making the second two terms in equation 55, the energy of the string. Furthermore, we call 56, as the energy of the string. In order to further understand the energy of the model with the mixed boundary conditions, we begin with a similar theorem and proof to the Dirichlet, for the time independence of the energy.

Theorem 5.1. *$E(u)$ is time independent.*

Proof.

$$\begin{aligned}\frac{\partial E(u)}{\partial t} &= ku(0,t)\frac{\partial u(0,t)}{\partial t} + T\frac{\partial u}{\partial x}\frac{\partial u}{\partial t}|_0^L + \frac{1}{\rho}\int_0^L \frac{\partial u}{\partial t}(\frac{\partial^2 u}{\partial t^2} - \frac{T}{\rho}\frac{\partial^2 u}{\partial x^2})dx \\ &= ku(0,t)\frac{\partial u(0,t)}{\partial t} - T\frac{\partial u}{\partial x}\frac{\partial u}{\partial t}|_{x=0}\end{aligned}$$

Using the boundary conditions from above, we have:

$$\frac{\partial E(u)}{\partial t} = T\frac{\partial u}{\partial x}(0,t)\frac{\partial u(0,t)}{\partial t} - T\frac{\partial u}{\partial x}\frac{\partial u}{\partial t}|_{x=0} = 0$$

Thus $E(u)$ is constant in time. □

Remark: the sum of the energies of the spring and the string is constant in time, but each of those energies are not as it is specifies in 56.

Continuing, we will prove that the total energy is equal to the sum of the energies of the modes.

Theorem 5.2. *If u(x,t) =$\sum_{n=1\infty} u_n(x,t)$as above, then $\forall n$, $E(u_n)$ is constant in time and the total energy is equal to the sum of the energies of the modes.*

Proof. Prove by induction:

$$
\begin{aligned}
E(u_1+u_2) &= \frac{k}{2}(u_1(0,t)^2 + 2u_1(0,t)u_2(0,t) + u_2(0,t)^2) \\
&+ \frac{1}{2}\int_0^L \rho((\frac{\partial u_1}{\partial t})^2 + 2\frac{\partial u_1}{\partial t}\frac{\partial u_2}{\partial t} + (\frac{\partial u_2}{\partial t})^2) + T((\frac{\partial u_1}{\partial x})^2 + 2\frac{\partial u_1}{\partial x}\frac{\partial u_2}{\partial x} + (\frac{\partial u_2}{\partial x})^2)dx \\
&= E(u_1) + E(u_2) + ku_1(0,t)u_2(0,t) + \rho\int_0^L \frac{\partial u_1}{\partial t}\frac{\partial u_2}{\partial t}dx + T\int_0^L \frac{\partial u_1}{\partial x}\frac{\partial u_2}{\partial x}dx
\end{aligned}
$$

$\int_0^L \frac{\partial u_1}{\partial t}\frac{\partial u_2}{\partial t}dx = 0$, by Theorem (4.1).

$$\int_0^L F_n' F_m' dx = Fn'Fm|_{x=0}^{x=L} - \lambda_n \int_0^L F_n F_m dx = -\frac{k}{T}F_n(0)F_m(0)$$

$$T\int_0^L \frac{\partial u_1}{\partial x}\frac{\partial u_2}{\partial x}dx = TG_1(t)G_2(t)[-\frac{k}{T}F_1(0)F_2(0)] = -ku_1(0,t)u_2(0,t)$$

$$E(u_1+u_2) = E(u_1) + E(u_2)$$

□

5.1 Energies of the n^{th} mode of the Plucked String

Recall from previous section that:

$$u(x,t) = \sum_{n=1}^{\infty} (A_n \cos(cv_n t))(\cos(v_n x) + \frac{k}{Tv_n}\sin(v_n x)) \tag{57}$$

Where A_n is:

$$
\begin{aligned}
A_n = \frac{1}{\|F_n(x)\|}(&\frac{h}{d}\frac{(T-dk)v_n\cos(dv_n) + (k+Tv_n^2)\sin(dv_n) - Tv_n}{Tv_n^3} \\
&+ \frac{h}{d-L}\frac{k\sin v_n L - v_n(T+kL-kd)\cos v_n d + Tv_n\cos v_n L - (k+(d-L)Tv_n^2)\sin v_n d}{Tv_n^3})
\end{aligned}
$$

and the norm of the eigen-function is given by:

$$\|F_n(x)\| = \frac{1}{2}\sqrt{\frac{-k^2\sin 2v_n L + v_n(2k(kL+T-T\cos 2v_n L) + T^2 v_n(\sin 2v_n L + 2v_n L))}{T^2 v_n^3}} \tag{58}$$

With these equations, we can evaluate $E(u_n)$ in order to understand the energy of each mode. Recall, that $E(u_n)$ is time independent, and we can further evaluate at $t=0$, giving the energy of each mode:

$$E(u_n) = A_n^2\left(\frac{k}{2} + \rho\frac{-2kT + 2kT\cos 2v_n L + \sin 2v_n L\frac{k^2 - T^2 v_n^2}{v_n} + 2L(k^2 + T^2 v_n^2)}{4T^2}\right) \tag{59}$$

Recall, when k is large, $v_n L \approx \frac{\pi}{2} + n\pi$, thus:

$$\sin 2v_n L \approx \sin(\pi + 2n\pi) = 0$$

additionally,

$$\cos 2v_n L \approx \cos(\pi + 2n\pi) = -1$$

Then, with these evaluations, we can simplify $E(u_n)$ into the following:

$$E(u_n) = A_n^2\left(\frac{k}{2} + \rho\frac{-4kT + 2L(k^2 + T^2 v_n^2)}{4T^2}\right) \tag{60}$$

Next, we plot $E(u_n)$ to visually see the energy of the string at each mode:

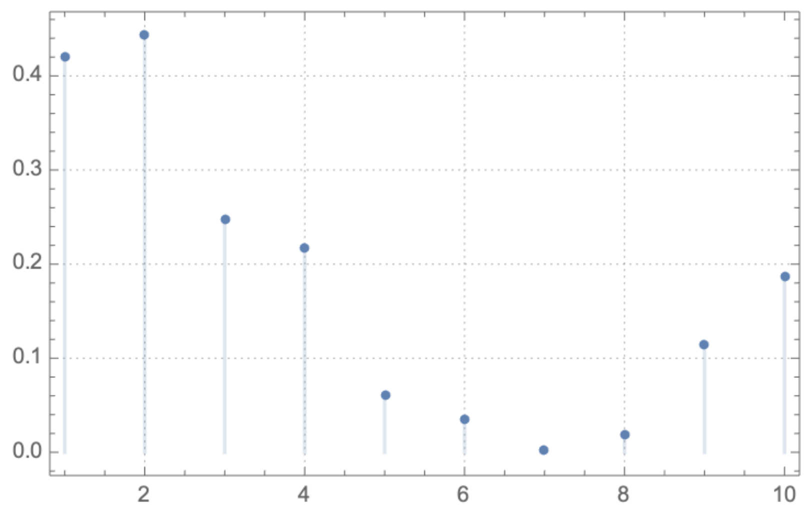

Figure 8: Plot of $E(u_n)$ for $n = (1, 10)$

Additionally, we can plot $E(u_n)$ against d, in order to see how $E(u_n)$ changes with d. In the following plot, we plotted $E(u_1), E(u_2), E(u_3),$ and $E(u_4)$

From Figure 9, it's difficult to understand the relationship between d and $E(u_n)$, we can change the bounds of the graph, to get a closer look of what happens as d approaches 0.

From Figure 10, we can see when d is very small the energy is very high, and oscillates across the modes, and this pattern continues until the energies of the modes are nearly equal as d approaches 1. This observation is also verified by Figure 9.

Following the structure of the Dirichlet Model, it would be logical to investigate the distribution of the energies across the modes for the mixed boundary

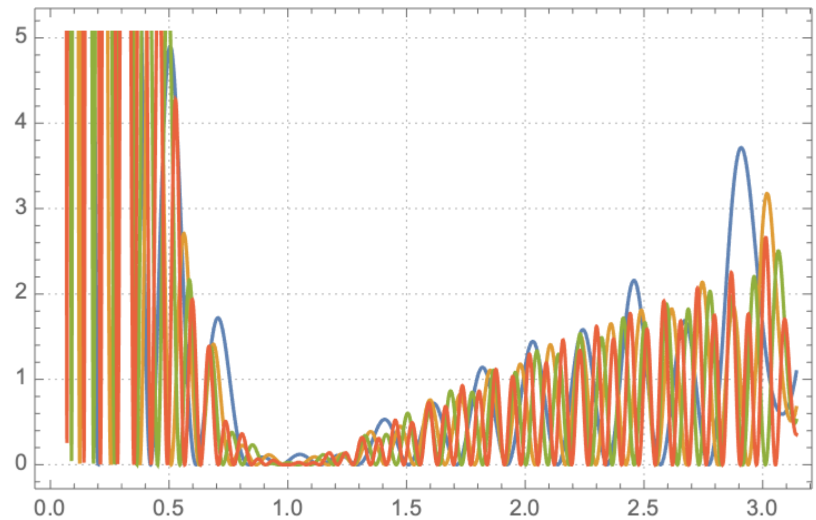

Figure 9: Plot of $E(u_1), E(u_2), E(u_3),$ and $E(u_4)$ against d. $E(u_1)$ is plotted in blue, $E(u_2)$ in orange, $E(u_3)$ in green, and $E(u_4$ in red.

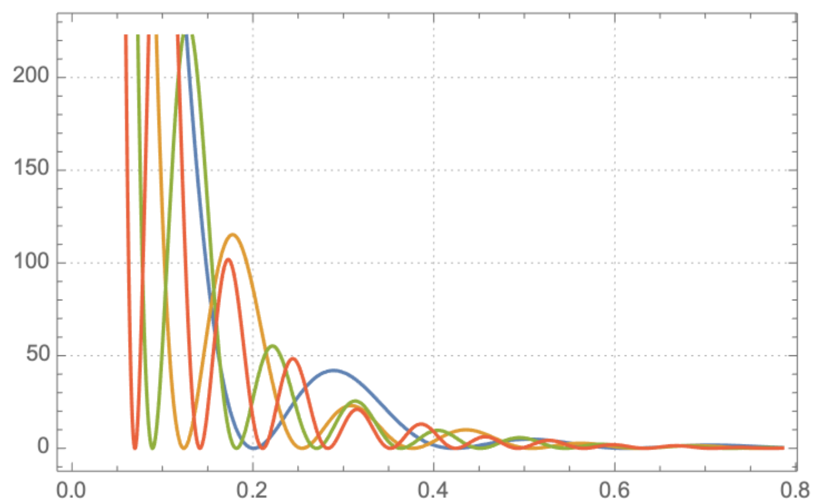

Figure 10: Same plot as in Figure 9
, but with x-axis changed to 0, $\frac{\pi}{4}$

condition model. In order to do this, we would need to again determine $\mathcal{P}_n$ as the distribution of energies. More specifically, we would need to define the average energy of the string across the period. Unfortunately, each solution has a different period, and we are thus unable to take an average across the period.

Furthermore, since we are unable to take this average, we are unable to simply define the distribution of the energies for this model.

6 Vibrato Modeling

Vibrato is employed by string musicians by moving their finger a small distance back and fourth on the string. Suppose the distance/amplitude of vibrato is characterized as α and the frequency of vibrato is characterized as β, then the boundary of the string is instead $L(t) = L + \alpha \sin \beta(2\pi t)$, where L is the original length of the string.

The proposed mathematical model of the string is as following:
$u(x,t)$ satisfies the 1-dimensional wave equation (1)

$$\frac{\partial^2 u}{\partial t^2}(x,t) = c^2 \frac{\partial^2 u}{\partial x^2}(x,t) \tag{61}$$

when $0 \leq x \leq L(t)$, $u(x,t) = 0$ for $x \leq 0$, and $x \geq L(t)$. with boundary condition

$$u(0,t) = 0,\ u(L(t),t) = 0$$

6.1 Conformal Mapping Approach

One method to solve this second order linear partial derivatives is discussed systematically by Gaffour in [3], [7] by transforming the original moving boundary domain into a fixed boundary domain using a conformal map.
Consider the 1-dimensional wave equation (1):

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial \tau^2} = 0$$

with the following initial conditions:

$$u(x,0) = \Psi(x)$$

$$\frac{\partial u}{\partial x}(x,0) = \Phi(x)$$

and the following moving boundary conditions:

$$u(0,\tau) = 0;\ u(L(t),\tau) = 0;\ \tau \geq 0 \text{ and } 0 \leq x \leq L(t)$$

This approach uses the analogy between Laplace's equation and the wave equation. The change $\tilde{x} = ix$ transforms Equation (61) into the following elliptic equations:

$$\frac{\partial^2 u}{\partial \tilde{x}^2} + \frac{\partial^2 u}{\partial \tau^2} = 0 \tag{62}$$

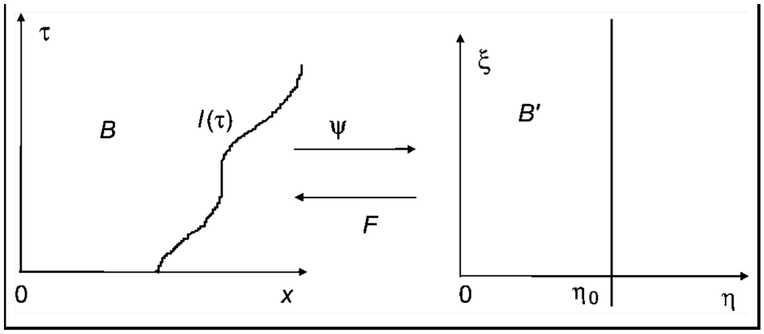

Figure 11: Conformal Map

It is well known that that Laplace's equation is invariant when subjected to a conformal map transformation. More precisely, if:

$$Z = F(W) = \tilde{f}(\xi, \tilde{\eta}) + i\tilde{g}(\xi, \tilde{\eta})$$

with $W = \xi + i\tilde{\eta}$ and $Z = \tau + i\tilde{x}$, then the change of variables:

$$\tau = \tilde{f}(\xi, \tilde{\eta}); \ \tilde{x} = \tilde{g}(\xi, \tilde{\eta}) \tag{63}$$

transform equation (62) into the following equation:

$$\frac{\partial^2 u}{\partial \tilde{\eta}^2} + \frac{\partial^2 u}{\partial \xi^2} = 0 \tag{64}$$

We impose the following condition upon F:

$$\bar{F}(W) = F(\bar{W}). \tag{65}$$

Then deduce by use of the McLaurin expansion that

$$\tilde{f}(\xi, i\eta) = f(\xi, \eta); \ \tilde{g}(\xi, i\eta) = ig(\xi, \eta)$$

where $f(\xi, \eta)$ and $g(\xi, \eta)$ are real functions of the real variable ξ, η.

Setting $\tilde{\eta} = i\eta$ in Equation (63), we find the original variables τ and x:

$$\tau = f(\xi, \eta); \ x = g(\xi, \eta) \tag{66}$$

Then Equation (64) takes the following form:

$$\frac{\partial^2 u}{\partial \eta^2} + \frac{\partial^2 u}{\partial \xi^2} = 0$$

The right choice of F, produced a conformal map of the time domain variable $0 \le x \le L(t)$ to a band $0 \le \eta \le \eta_0$. In this case, the boundary conditions can be expressed by:

$$u(\xi, 0) = 0; \ u(\xi, \eta_0) = 0$$

The solution for $u(\xi, \eta)$ in the fixed domain is well known and can be expressed in terms of the complex Fourier series.

$$u(\xi, \eta) = \sum_{-\infty}^{\infty} A_n \left[\exp\left(\frac{i\pi n}{\eta_0}(\xi + \eta) \right) - \exp\left(\frac{i\pi n}{\eta_0}(\xi - \eta) \right) \right] \tag{67}$$

To find the solution with the original variables τ and x, we consider the inverse function $\psi(Z) = F^{-1}(W)$, note that this ψ is not the same as the initial position function Ψ.
Finally, the exact solution with the original variables is given in terms of the functional Fourier series:

$$u(\tau, x) = \sum_{-\infty}^{\infty} A_n \left[\exp\left(\frac{i\pi n}{\eta_0}\psi(\tau + x) \right) - \exp\left(\frac{i\pi n}{\eta_0}\psi(\tau - x) \right) \right] \tag{68}$$

where $\psi(\tau + x) = \xi + \eta, \psi(\tau - x) = \xi - \eta$, then we deduce:

$$\eta_0 = \frac{\psi(\tau + L(t)) - \psi(\tau - L(t))}{2}; \; \eta_0 = \eta(x = L(t)). \tag{69}$$

The expression for the coefficients (for the symmetric initial conditions) is:

$$A_n = \frac{1}{4i\pi n} \int_{-L(0)}^{L(0)} \kappa(x) \exp\left(i\pi n \psi(x)\right) dx \tag{70}$$

$$\kappa(x) = \begin{cases} \Phi(x) + \Psi(x), x \geq 0 \\ \Phi(x) - \Psi(x), x \leq 0 \end{cases}$$

6.2 Adiabatic Approach

After some investigations using the conformal mapping method, we realized that the solution yielding by the conformal map is complicated to calculate. Therefore, we adapted the adiabatic approximation approach.

A very "slow" change in the boundary conditions of a problem defines an **"adiabatic" process**. More explanation of the "adiabatic" process is explained in [15].

With this intuition, the frequency of the vibrating string for A_4 is 440 Hz, while the frequency of vibrato is about 6-7 Hz, hence the boundary is changing "slowly" comparing to the behavior of the vibrating string. An intuitive "solution" to the moving boundary condition is given by the solution of the Dirichlet Boundary Condition, but replace L with L(t).

$$u_{adiabatic} = \sum_{n=1}^{\infty} A_n \cos \frac{cn\pi t}{L(t)} \sin \frac{n\pi x}{L(t)} \tag{71}$$

where L(t) $= L + \alpha \sin \beta 2\pi t$.

We still need to justify that the adiabatic approximation is "close" to the original solution.

Theorem 6.1. *If the adiabatic condition is satisfied, then the norm of the difference* $||u - u_{adiabatic}||$ *is small for a bounded interval of time.*

Proof. Recall that the 1D wave equation (1) is

$$c^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t}$$

where u(x,t) is the exact solution of the 1-dimensional wave equation with moving boundary condition.
Introduce v-another functor of (x,t), and consider

$$\begin{cases} \frac{\partial u}{\partial t} = v \\ \frac{\partial v}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \end{cases} \tag{72}$$

u(x,t) satisfies the above system if and only if it satisfies the 1-dimensional wave equation.
Introduce the vector $\overrightarrow{f}(x,t) = \begin{pmatrix} u(x,t) \\ v(x,t) \end{pmatrix}$
(72) is satisfied if and only if

$$\frac{\partial \overrightarrow{f}}{\partial t} = \begin{pmatrix} 0 & In \\ c^2 \frac{\partial^2}{\partial x^2} & 0 \end{pmatrix} \overrightarrow{f} \tag{73}$$

Denote $A = \begin{pmatrix} 0 & In \\ c^2 \frac{\partial^2}{\partial x^2} & 0 \end{pmatrix}$
Consider u(x,t) as the solution of the 1D wave equation with moving boundary condition, then

$$\begin{cases} \frac{d\overrightarrow{f}}{dt} = A\overrightarrow{f}, \forall t \\ \overrightarrow{f}|_{t=0} = \overrightarrow{f_0} \text{ is given} \end{cases}$$

Then the solution is written formally as

$$\overrightarrow{f}(t) = e^{tA}\overrightarrow{f_0}$$

Let $g(x,t)$ denote the solution given by the adiabatic approximation such that it satisfies:

$$\begin{cases} \frac{\partial \overrightarrow{g}}{\partial t} = A\overrightarrow{g} + \overrightarrow{r}(t) \\ \overrightarrow{g}|_{t=0} = \overrightarrow{f}_0 \text{ same as before} \end{cases}$$

According to Duhamel's formula

$$\overrightarrow{g} = \int_0^t e^{(t-s)A}\overrightarrow{r}(s)ds + e^{tA}\overrightarrow{f_0} = \int_0^t e^{(t-s)A}\overrightarrow{r}(s)ds + \overrightarrow{f} \tag{74}$$

Our objective is to estimate $||\overrightarrow{f} - \overrightarrow{g}||$

$$||\overrightarrow{f} - \overrightarrow{g}|| = ||\int_0^t e^{(t-s)A}\overrightarrow{r}ds|| \leq \int_0^t ||e^{(t-s)A}\overrightarrow{r}||ds$$

by triangle inequality.
We choose energy as the norm of the vector functions, the operator $e^{\tau A}$ is bounded within a bounded interval of time, since the energy can not be infinity, hence,

$$||e^{\tau A}\overrightarrow{r}|| \leq c_\tau ||r||$$

where c_τ is a constant that depends on τ. In the end, in order to show $||\overrightarrow{f} - \overrightarrow{g}||$ is small, we are left to show $||\overrightarrow{r}||$ is small, where $\overrightarrow{r} = c^2 \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2}$, or in other words, $\overrightarrow{r}$ is the 1D wave operator acting on the adiabatic approximation.
Let

$$f[x,t] = A \sin \frac{n\pi x}{L} \cos \frac{cn\pi t}{L}$$

$$g[x,t] = A \sin \frac{n\pi x}{\alpha \sin \beta t + L} \cos \frac{cn\pi t}{\alpha \sin \beta t} + L$$

$$r[x,t] = c^2 \frac{\partial^2 g}{\partial x^2} - \frac{\partial^2 g}{\partial t^2}$$

Norm of r is given by energy of r, which is

$$rNorm[x,t] = \int_0^L \rho(\frac{\partial r}{\partial t})^2 + T(\frac{\partial r}{\partial x})^2 dx$$

Consider the integrand:

$$\rho(\frac{\partial r}{\partial t})^2 + T(\frac{\partial r}{\partial x})^2$$

By Mathematica, the integrand is given by Figure 12:
Notice that when the adiabatic condition is satisfied $L >> \alpha$, $1 >> \beta > 0$. Hence

$$\frac{\alpha}{L} \approx 0 \text{ and } \frac{\beta}{L} \approx 0 \tag{75}$$

It is suffice to let L=1, and α be the proportion of the length that vibrato oscillates, hence $\alpha << 1$.
Applying the adiabatic condition, we have Figure 13

Notice that all terms without α, β cancel out nicely, which is Figure 14:

After cancellation, the upper bound of the solution is given by

$$\rho(2An^3\pi^3x\alpha\beta + An^3\pi^3t\alpha\beta + 8An^2\pi^2\alpha\beta)^2 + T(2An^2\pi^2\alpha\beta + 2An^3\pi^3x\alpha\beta)^2$$

Integral the upper bound from 0 to L, we have

$$(\alpha^2\beta^2)\frac{1}{3}A^2n^4(4*(12+n\pi(6+n\pi))T + (192+n\pi(48(1+t)+n\pi(4+3t(2+t))))\rho)$$

Since $\alpha << 1, \beta << 1$ and the above integral has coefficients $\alpha^2\beta^2$, the integral is approximately 0. Hence we proof that the norm of the difference $||u - u_{adiabatic}||$ is approximately 0 for a bounded interval of time $\square$

```
trial2[x_, t_] =
 ρ
   (A c^2 n^3 π^3 x α β Cos[t β] Cos[c n π t] Cos[n π x] - 2 A n π x α β Cos[t β] Cos[c n π t] Cos[n π x] (-c n π t α β Cos[t β] + c n π)^2 -
     2 A n π x α β Cos[t β] Cos[n π x] (2 c n π t α^2 β^2 Cos[t β]^2 - 2 c n π α β Cos[t β] + c n π t α β^2 Sin[t β]) Sin[c n π t] +
     4 A n π x α^2 β^2 Cos[t β]^2 Cos[n π x] (-c n π t α β Cos[t β] + c n π) Sin[c n π t] +
     2 A n π x α β^2 Cos[n π x] Sin[t β] (-c n π t α β Cos[t β] + c n π) Sin[c n π t] +
     A n π x α β Cos[t β] Cos[n π x]
      (-Cos[c n π t] (-c n π t α β Cos[t β] + c n π)^2 - (2 c n π t α^2 β^2 Cos[t β]^2 - 2 c n π α β Cos[t β] + c n π t α β^2 Sin[t β]) Sin[c n π t]) +
     2 A c^2 n^2 π^2 α β Cos[t β] Cos[c n π t] Sin[n π x] - 2 A n^2 π^2 x^2 α^2 β^2 Cos[t β]^2 (-c n π t α β Cos[t β] + c n π) Sin[c n π t] Sin[n π x] +
     A c^2 n^2 π^2 (-c n π t α β Cos[t β] + c n π) Sin[c n π t] Sin[n π x] -
     A (-3 Cos[c n π t] (2 c n π t α^2 β^2 Cos[t β]^2 - 2 c n π α β Cos[t β] + c n π t α β^2 Sin[t β]) (-c n π t α β Cos[t β] + c n π) -
        (-6 c n π t α^3 β^3 Cos[t β]^3 + 6 c n π α^2 β^2 Cos[t β]^2 - 6 c n π t α^2 β^3 Cos[t β] Sin[t β] + c n π t α β^3 Cos[t β] + 3 c n π α β^2 Sin[t β]) Sin[c n π t] +
        (-c n π t α β Cos[t β] + c n π)^3 Sin[c n π t]) Sin[n π x] +
     A (-c n π t α β Cos[t β] + c n π) Sin[c n π t]
      (2 n π x α^2 β^2 Cos[t β]^2 Cos[n π x] + n π x α β^2 Cos[n π x] Sin[t β] - n^2 π^2 x^2 α^2 β^2 Cos[t β]^2 Sin[n π x]) -
     A Cos[c n π t] (n^3 π^3 x^3 α^3 β^3 Cos[t β]^3 Cos[n π x] - 6 n π x α^3 β^3 Cos[t β]^3 Cos[n π x] - 6 n π x α^2 β^3 Cos[t β] Cos[n π x] Sin[t β] +
        n π x α β^3 Cos[t β] Cos[n π x] + 6 n^2 π^2 x^2 α^3 β^3 Cos[t β]^3 Sin[n π x] + 3 n^2 π^2 x^2 α^2 β^3 Cos[t β] Sin[t β] Sin[n π x]))^2 +
  T
   (-A c^2 n^3 π^3 Cos[c n π t] Cos[n π x] - 2 A n π α β Cos[t β] Cos[n π x] (-c n π t α β Cos[t β] + c n π) Sin[c n π t] -
     A n π Cos[n π x] (-Cos[c n π t] (-c n π t α β Cos[t β] + c n π)^2 - (2 c n π t α^2 β^2 Cos[t β]^2 - 2 c n π α β Cos[t β] + c n π t α β^2 Sin[t β])
         Sin[c n π t]) + 2 A n^2 π^2 x α β Cos[t β] (-c n π t α β Cos[t β] + c n π) Sin[c n π t] Sin[n π x] -
     A Cos[c n π t] (-n^3 π^3 x^2 α^2 β^2 Cos[t β]^2 Cos[n π x/(L + α Sin[t β])] + 2 n π α^2 β^2 Cos[t β]^2 Cos[n π x/(L + α Sin[t β])] +
        n π α β^2 Cos[n π x/(L + α Sin[t β])] Sin[t β] - 4 n^2 π^2 x α^2 β^2 Cos[t β]^2 Sin[n π x/(L + α Sin[t β])] - n^2 π^2 x α β^2 Sin[t β] Sin[n π x/(L + α Sin[t β])]))^2
```

Figure 12: integrand given by Mathematica

```
ρ
  (A n^3 π^3 x α β Cos[t β] Cos[c n π t] Cos[n π x] - 2 A n^3 π^3 x α β Cos[t β] Cos[c n π t] Cos[n π x] - A n^3 π^3 x α β Cos[t β] Cos[n π x] Cos[c n π t] +
    2 A n^2 π^2 α β Cos[t β] Cos[c n π t] Sin[n π x] + A n^2 π^2 (-c n π t α β Cos[t β] + c n π) Sin[n π t] Sin[n π x] -
    A (-3 Cos[c n π t] (2 n^2 π^2 α β Cos[t β]) + n^3 π^3 Sin[n π t]) Sin[n π x])^2 +
 T
  (-A n^3 π^3 Cos[c n π t] Cos[n π x] - 2 A n^2 π^2 α β Cos[t β] Cos[n π x] Sin[c n π t] -
    A n π Cos[n π x] (-Cos[c n π t] (-c n π t α β Cos[t β] + c n π)^2 + 2 c n π α β Cos[t β] Sin[c n π t]) +
    2 A n^3 π^3 x α β Cos[t β] Sin[c n π t] Sin[n π x])^2;
```

Figure 13: Applying the adiabatic condition to the solution

```
(*after some cancelation*)
ρ (-2 A n^3 π^3 x α β Cos[t β] Cos[c n π t] Cos[n π x] - A n^2 π^2 n π t α β Cos[t β] Sin[n π t] Sin[n π x] + 8 A n^2 π^2 α β Cos[n π t] Cos[t β] Sin[n π x])^2 +
  T (-4 A n^2 π^2 α β Cos[t β] Cos[n π x] Sin[n π t] + 2 A n^3 π^3 x α β Cos[t β] Sin[c n π t] Sin[n π x])^2;
≤ ρ (2 A n^3 π^3 x α β + A n^2 π^2 n π t α β + 8 A n^2 π^2 α β)^2 + T (4 A n^2 π^2 α β + 2 A n^3 π^3 x α β)^2;
```

Figure 14: After cancellation

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