Good-Deal Bounds in Asset Pricing

Mingxuan Ge

Abstract

Market incompleteness arises from frictions or transaction costs. When investors trade in such markets, due to non-uniqueness of stochastic discount factor, the standard risk-neutral pricing formula under the set of equivalent martingale measures would be practically too wide to be useful for implementation. In this report, we show the motivations of good-deal bounds in the framework of asset pricing from Cocharane and Saa-Requejo (2000). Then we study mechanisms of stochastic control approach with rigorous math formulations for a factor market model with Hansen-Jagannathan bounds. With this technique of good-deal bounds for pricing assets, we consider a specific application on a partially observed information model with an unobserved Markov chain over a finite time horizon.

1 Introduction

1.1 Background

In Lucas Asset Pricing, an economy populated by infinitely many identical individual consumers, in which assets are all identical, the representative agent is supposed to maximize her utility function along with the abilities to trade stocks in equity markets as an intertemporal problem. In this section, we introduce a similar yet much simpler version of Lucas' model in discrete-time state space for economic illustrations of stochastic discount factor.

Given $(\Omega, \mathscr{F}, \mathbb{P})$, where $\mathbb{P}$ is the underlying objective probability measure, i.e the measure with respect to the real world, then we state the following problem

$$\max_{\xi} \; u(c_t) + E_t^{\mathbb{P}}[\beta u(c_{t+1})]$$
$$\text{such that} \qquad c_t = e_t - p_t \xi$$
$$c_{t+1} = e_{t+1} + x_{t+1} \xi$$

where shortened symbols represent

$u()$: a strictly increasing and concave utility function of investors
β: the measure of agent's impatience
$c()$: the consumption function over the time horizon [0,T]
$e()$: the original consumption level
x: future payoff of assets
p: corresponding prices of the assets
ξ: the amount of assets to trade

Then First-Order-Condition (FOC) is derived by Lagrange multipliers methods as follows:

$$P_t = E_t^{\mathbb{P}}[\frac{\beta u'(c_{t+1})}{u'(c_t)} x_{t+1}]$$

If we denote the stochastic factor factor (pricing kernel) as $m = \frac{\beta u'(c_{t+1})}{u'(c_t)}$, then it leads to the below definition of stochastic discount factor.

Definition 1. *A stochastic discount factor (discrete case) is a stochastic process $\{m_{t,t+s}\}$ such that for any security with payoff x_{t+1} at time t+1 the price of that security at time t is*

$$p_t = E_t^{\mathbb{P}}[m_{t+1} x_{t+1}] \quad \forall t$$

For economic interpretations of consumption models, one may also define it as the marginal utility growth rate: the rate at which the investor is willing to substitute consumption at time $t+1$ for consumption at time t.

1.2 Motivations of Incomplete Markets

However, if the market completeness fails, investors will not be able to have perfect risk sharing for hedging risks away. Mathematically, there does not exist an equivalent martingale measure $\mathbb{Q}$ to $\mathbb{P}$ anymore, where we use Radon-Nikodym theorem to connect equivalence for probability measures.

Definition 2. *Two probability measures $\mathbb{P}$ and $\mathbb{Q}$ are **equivalent** if and only if $\exists$ a random variable X such that $EX = 1, X > 0$ and $\mathbb{Q}$ satisfies that $\mathbb{Q}(A) = EX1_A = \int_A X d\mathbb{P}$. Then, given the payoff function of a contingent claim Z, the corresponding Radon-Nikodym derivative is given by $E_{\mathbb{P}}(Z) = E_{\mathbb{Q}}(Z\frac{d\mathbb{P}}{d\mathbb{Q}})$,*

Therefore $\{m : p(x) = E[mx]\}$ such that $\{m = x' + \varepsilon | E(x'\varepsilon) = 0\}$ are all potential stochastic discount factors.

Given the above background, the motivation of good-deal bounds is to add a layer of economic interpretation and the method aims to rule out not only ridiculously good deals by arbitrage bounds, but also unreasonable good deal bounds (Cochrane & Saa-Requejo 2000). They stated the optimization problem as follows and one may use duality to attain the upper good-deal bound easily.

$$\underline{C} = \min_{|m|} E(mx^c)$$
$$\text{such that} \quad \mathbf{p} = E(m\mathbf{x}); m \geq 0; \sigma(m) \leq \frac{h}{R^f}$$

where x^c is the focus payoff to be valued and R^f stands for risk-free rate.

First two constraints give out well-known arbitrage bounds on the value, where one uses the idea of relative pricing and guarantees the marginal utility growth rate to be non-negative. That said, we have reached the arbitrage-free property, i.e when the payoff is non-negative, the price will also be non-negative. Practically, this bound is too wide to be useful. Adding another constraint shrinks the feasible region on the convex hull. The innovation relies on Hansen and Jagannathan (1991)'s statement that

$$E(mR^e) = 0 \text{ if and only if } \frac{|E(R^e)|}{\sigma(R^e)} \leq \frac{\sigma(m)}{E(m)}$$

Given $E(m) = 1/R^f$ and a pre-specified Sharpe-Ratio h when investors start to trade at the margin for arbitrages, we derived the last constraint for one's marginal utility to satisfy.

2 Asset Pricing in Continuous Time

Until this moment, all equations and statements are specified for discrete time cases, but we could easily transform them into continuous scenarios, which we will do in this section.

2.1 Dynamics of T-period contingent claim

Suppose we are given a martingale probability measure $\mathbb{Q}$ on $(\Omega, \mathscr{F})$ which is equivalent to the underlying objective probability measure $\mathbb{P}$. Due to market incompleteness, $\mathbb{Q}$ may not be unique.

Definition 3. *A T-period contingent claim is a derivative whose future payoff depends on the value of another underlying asset on finite time horizon* $[0,T]$.

Then we assume that an arbitrary T-period contingent claim has price and factor dynamics of underlying objective measure $\mathbb{P}$ as follows

$$\begin{aligned}
&\frac{dS_t}{S_t} = \alpha(S_t,Y_t,t)dt + \sigma(S_t,Y_t,t)dZ_t \\
&dY_t = a(S_t,Y_t,t)dt + b^z(S_t,Y_t,t)dZ_t + b^w(S_t,Y_t,t)dW_t \\
&r_t = r(S_t,Y_t) \quad \text{risk-free rate}
\end{aligned}$$

where Z_t, W_t denote the Brownian Motion(s)

The factor dynamics (dY_t) corresponds to the systematic risk factors in asset pricing of economic models. Furthermore, we define the no-arbitrage price process of the T-period contingent claim Z as:

$$\pi(Z,t) = E^{\mathbb{P}}\left[\frac{\Lambda_T}{\Lambda_t} \cdot Z|\mathscr{F}_t\right] = E^{\mathbb{Q}}\left[e^{-\int_t^T r_u\,du} \cdot Z|\mathscr{F}_t\right]$$

where Λ is a stochastic discount factor with respect to risk-free rate r such that

$$\Lambda_t = e^{-\int_0^t r_u\,du} L_t \quad L_t = E\left[\frac{d\mathbb{Q}}{d\mathbb{P}}|\mathscr{F}_t\right] \qquad \forall t \in [0,T]$$

As the Brownian motion W carries out new information and is orthogonal to the Brownian motion Z, by techniques of decomposition of control processes, we observe that L_t has the following dynamics

$$\frac{dL_t}{L_t} = h_t^z dZ_t + h_t^w dW_t$$

Then using Girsanov Theorem, one also has the following relation

$$\begin{aligned}
dZ_t &= h_t^z dt + dZ_t^{\mathbb{Q}} \\
dW_t &= h_t^w dt + dW_t^{\mathbb{Q}}
\end{aligned}$$

where $Z^{\mathbb{Q}}$ and $W^{\mathbb{Q}}$ are Brownian Motions under the measure $\mathbb{Q}$

Since the Girsanov kernel process can be represented as $(h_z, h_w) = h(t, S_t, Y_t)$, a feedback form including dynamics of both S_t & Y_t, this guarantees the usage of the Dynamic Programming Principle (DPP) later.

Then the price and factor dynamics of equivalent measure $\mathbb{Q}$ become

$$dS_t = S_t\left[\alpha(S_t,Y_t,t)+\sigma h_t^z\right]dt + S_t\sigma(S_t,Y_t,t)dZ_t^{\mathbb{Q}}$$
$$dY_t = \left[a(S_t,Y_t,t)+b^z h_t^z + b^w h_t^w\right]dt + b^z(S_t,Y_t,t)dZ_t^{\mathbb{Q}} + b^w(S_t,Y_t,t)dW_t^{\mathbb{Q}}$$

2.2 Stochastic Control Problem of Upper Good-Deal Bounds

$$\begin{aligned} V(S_t,Y_t,t) := \sup_{\mathbf{h}} \quad & E^{\mathbb{Q}}\left[e^{-\int_t^T r_u du}\cdot\Phi(S_T,Y_T)|\mathscr{F}_t\right] \\ & s.t\ \alpha+\sigma h - r = 0 \quad \text{(arbitrage-free bounds)} \\ & \|\, h \,\|^2_{R^d} \le A^2 \quad \text{(volatility bounds)} \end{aligned}$$

where $\Phi(S_T,Y_T)$ is a $\mathscr{F}_t$-measurable random variable which represents the payoff of a derivative security, e.g $(S_T-K)^+$ is a standard format for European call option with strike price K.

Note that the measure $\mathbb{Q}$ is a martingale measure if and only if the local rate of return of the asset under the measure $\mathbb{Q}$ equals the short rate r, which is captured by the arbitrage-free bounds.

Also, to avoid tedious computations, a trick used here is that under continuous time, imposing the bound on h rather than on Sharpe ratio directly is mathematically correct (Bjork & Irina 2006):

$$|\, SR_t \,|^2 \le \|\, h_t \,\|^2_{R^d} \le A^2 \quad \forall t\in[0,T]$$

where $-h_t$ = market price vector of $(W+Z)$-risk, i.e $\frac{dL_t}{L_t} = h_t^w dW_t + h_t^z dZ_t$

Then by DPP, the problem is reduced to

$$\begin{aligned} V(s,y,t) &= \sup_{\mathbf{h}} E^{\mathbb{Q}}\left[e^{-\int_t^{t+\delta} r_u du} e^{-\int_{t+\delta}^T r_u du}\cdot\Phi(S_T,Y_T)|\mathscr{F}_t\right] \\ &= \sup_{\mathbf{h}} E^{Q}\left[E^{\mathbb{Q}}[e^{-\int_t^{t+\delta} r_u du} e^{-\int_{t+\delta}^T r_u du}\cdot\Phi(S_T,Y_T)|\mathscr{F}_{t+\delta}]|\mathscr{F}_t\right] \\ &= \sup_{\mathbf{h}} E^{\mathbb{Q}}\left[e^{-\int_t^{t+\delta} r_u du}\cdot V(s_{t+\delta},y_{t+\delta},t+\delta)|\mathscr{F}_t\right], \quad \text{for any} \delta>0 \qquad \text{(a)} \end{aligned}$$

Apply Itô's lemma to $V(s_{t+\delta},y_{t+\delta},t+\delta)$, then we have

$$\begin{aligned} V(s_{t+\delta},y_{t+\delta},t+\delta) = V(s,y,t) &+ \int_t^{t+\delta}\left[\frac{\partial V}{\partial t} + S_t(\alpha+\sigma h_t^z)\frac{\partial V}{\partial s} + (a+b^z h_t^z + b^w h_t^w)\frac{\partial V}{\partial y}\right]dt \\ &+ \int_t^{t+\delta}\left[\frac{S_t^2\sigma^2}{2}\frac{\partial^2 V}{\partial s^2} + \frac{b_z^2+b_w^2}{2}\frac{\partial^2 V}{\partial y^2} + \frac{S_t\sigma b^z}{2}\frac{\partial^2 V}{\partial s\partial y}\right]dt \\ &+ \int_t^{t+\delta}\left[S_t\sigma\frac{\partial V}{\partial s} + b^z\frac{\partial V}{\partial y}\right]dZ_t^{\mathbb{Q}} + \int_t^{t+\delta} b^w\frac{\partial V}{\partial y}dW_t^{\mathbb{Q}} \quad \text{(b)} \end{aligned}$$

2.3 The Purely-Wiener Driven PDE

Theorem 1 (Hamilton-Bellman-Jacobi Equation)**.**

$$\frac{\partial V(s,y,t)}{\partial t} + \sup_h \{\mathbf{A}^h V(s,y,t)\} - rV(s,y,t) = 0$$
$$V(s,y,T) = \Phi(s,y)$$

The infinitesimal operator $\mathbf{A}^h$ is given by

$$\begin{aligned}
\mathbf{A}^h V(s,y,t) &= \frac{\partial V(s,y,t)}{\partial s} S(\alpha + \sigma h_z) \\
&+ \frac{\partial V(s,y,t)}{\partial y}\{a(s,y) + b^z(s,y)h_z(t,s,y) + b^w(s,y)h_w(t,s,y)\} \\
&+ \frac{1}{2}\frac{\partial V^2(s,y,t)}{\partial^2 s} S^2\sigma^2(s,y) \\
&+ \frac{1}{2}\frac{\partial V^2(s,y,t)}{\partial^2 y} b_z^2(s,y) + b_w^2(s,y) \\
&+ \frac{1}{2}\frac{\partial V^2(s,y,t)}{\partial s \partial y} S\sigma(s,y)b^z(s,y)
\end{aligned}$$

Proof. Plugging equation (b) into equation (a) while taking the limit of $\delta \to 0$, only integrands remain. Then, under expectation $E^{\mathbb{Q}}[\cdot]$, all martingale terms $dZ_t^{\mathbb{Q}}$ and $dW_t^{\mathbb{Q}}$ are zero. Lastly, after rearranging equations, we have the desired theorem. □

2.4 Optimal Girsanov Kernel Process

As the kernel process is decomposed as $h = (h_z, h_w)$ and Z is only correlated with σ, we have $h_z = \sigma^{-1}(r-a)$.

Given the HJB, the problem is reduced to a linear-quadratic optimization one with optimal kernel process as

$$\max_h \frac{\partial V(s,y,t)}{\partial y}\{b^w(s,y)h_w(s,y,t)\}$$
$$h_w^2 \leq A^2 - (\alpha - r)^2$$

$$h_w = \frac{\sqrt{A^2-(\alpha-r)^2}}{\sqrt{b^2V_y^2}} \cdot bV_y; \quad h_z = \sigma^{-1}(r-a)$$

3 A Partially Observed Model

3.1 General Setup

In this section, we consider a specific application on a partially observed information model with an unknown Markov chain X in short rate dynamics over a finite time horizon. Given the complete probability space $(\Omega, \mathscr{F}, \mathbb{P})$, the Brownian motion Z is orthogonal to the Brownian motion W and the Markov chain is an independent process with respect to Brownian motions, which it takes values in a finite state space

$I = \{1, \cdots, D\}$ for some integer $D \geq 2$.

As usual, we let $\mathscr{F}_t$ denote the natural (full) filtration generated by the Brownian motion Z, Brownian motion W and the Markov chain X. Furthermore, as the information available to investors in the market at time t does not include the Markov chain, we also need the observation filtration, which is represented by

$$\mathscr{F}_t^o := \sigma\{Z(s), W(s), s \in [0,T]\} \vee \mathscr{N}(\mathbb{P}), \quad \forall t \in [0,T]$$

where $\mathscr{N}(\mathbb{P})$ denotes the collection of all $\mathbb{P}$-null events in the probability space ($\Omega, \mathscr{F}, \mathbb{P}$).

Economically, Donnelly (2011) interprets the Markov chain as sources of long-term macro-economic changes and models the framework under regime-switching diffusion markets, but in their work, an investor knows what regime the market is in at each time point. However, this is unlikely to happen especially if we do not restrict on investors who specialize trades in certain fields. We then discard this assumption, yet following a similar setup of Markov chain as Donnelly (2011). Moreover, our Markov chain would be put into the short rate dynamics only to indicate the unknown policy changes by governments or institutions.

Let the Markov chain start at a fixed state $i_0 \in I$ with a generator G, a $D \times D$ matrix $G = (g_{ij})_{i,j=1}^D$ where $g_{ij} \geq 0, \forall i \neq j$ and $g_{ii} = -\sum_{j \neq i} g_{ij}$. Then the martingale condition compensated by the intensity process becomes (Donnelly 2011)

$$M_{ij}(t) = N_{ij}(t) - \int_0^t \lambda_{ij}(s) ds$$

where

$$\begin{aligned} N_{ij}(t) &= \sum_{0<s\leq t} \mathbf{1}_{\{(X_{s-})=i\}} \mathbf{1}_{\{X(s)=j\}} \quad \forall t \in [0,T] \quad \text{(counting process)} \\ \lambda_{ij}(t) &= g_{ij} \mathbf{1}_{\{X(t_-)=i\}} \quad \text{(intensity process)} \end{aligned}$$

Finally, we define all definitions of Brownian motions and the general setup of the price and factor dynamics similar as Section 2 (Asset Pricing in Continuous Time) when the model is fully observable.

3.2 New Dynamics of T-period contingent claim

By Girsanov theorem, the likelihood process L corresponding to the measure $\mathbb{Q}$ becomes follows. Note that for simplicity, we still denote the martingale measure used in this partially observable case as $\mathbb{Q}$, though it should not coincide with the fully observable case in general.

$$\begin{aligned} \frac{dL_t}{L_{t-}} &= h_t^z dZ_t + h_t^w dW_t + \sum_{i=1}^{D} \sum_{j=1, j\neq i}^{D} \eta_{ij}(t) dM_{ij}(t) \\ dZ_t &= h_t^z dt + dZ_t^{\mathbb{Q}} \\ dW_t &= h_t^w dt + dW_t^{\mathbb{Q}} \\ dM_{ij}(t) &= \eta_{ij}(t)\lambda_{ij}(t) dt + dM_{ij}^{\mathbb{Q}}(t) \\ \text{where} \quad M_{ij}^{\mathbb{Q}}(t) &= N_{ij}(t) - \int_0^t (1 + \eta_{ij}(s))\lambda_{ij}(s) ds \end{aligned}$$

We claim that price, factor and short rate dynamics with respect to the underlying objective measure $\mathbb{P}$ when the risk-free rate is affected by the Markov chain as follows

$$
\begin{aligned}
&\frac{dS_t}{S_t} = \alpha(S_t, Y_t, t)dt + \sigma(S_t, Y_t, t)dZ_t \\
&dY_t = a(S_t, Y_t, t)dt + b^z(S_t, Y_t, t)dZ_t + b^w(S_t, Y_t, t)dW_t \\
&r_t = r(S_t, Y_t, X(t))
\end{aligned}
$$

Again, all the processes $\alpha, a, b, \sigma, \gamma$ are suitably integral and measurable with the condition. Then using Girsanov theorem, we have dynamics as follows

$$
\begin{aligned}
&\frac{dS_t}{S_t} = [\alpha(S_t, Y_t, t) + \sigma h_t^z]\,dt + \sigma(S_t, Y_t, t)dZ_t^{\mathbb{Q}} \\
&dY_t = [a(S_t, Y_t, t) + b^z h_t^z + b^w h_t^w]\,dt + b^z(S_t, Y_t, t)dZ_t^{\mathbb{Q}} + b^w(S_t, Y_t, t)dW_t^{\mathbb{Q}}
\end{aligned}
$$

where $Z^{\mathbb{Q}}$ and $W^{\mathbb{Q}}$ are Brownian Motions under the measure $\mathbb{Q}$

3.3 New Upper Good-Deal Bounds Price Processes

Fix the payoff of contingent claim to be $\Phi(S_T, Y_T)$, a $\mathscr{F}_t$-measurable random variable. Since the underlying Markov chain X is unobservable, by the Tower Property, we get

$$
\begin{aligned}
\pi^*(S_t, Y_t, t) &= E^{\mathbb{Q}}\left[e^{-\int_t^T r_u\,du} \cdot \Phi(S_T, Y_T) | \mathscr{F}_t^o\right] \\
&= E^{\mathbb{Q}}\left[E^{\mathbb{Q}}[e^{-\int_t^T r_u\,du} \cdot \Phi(S_T, Y_T) | \mathscr{F}_t] | \mathscr{F}_t^o\right] \\
&= E^{\mathbb{Q}}\left[F(S_t, Y_t, X_t, t) | \mathscr{F}_t^o\right]
\end{aligned}
$$

where we define $F(S_t, Y_t, X_t, t) = E^{\mathbb{Q}}\left[e^{-\int_t^T r_u\,du} \cdot \Phi(S_T, Y_T) | \mathscr{F}_t\right]$

Then we have the following stochastic control problem where the second constraint guarantees the martingale measure $\mathbb{Q}$ to be non-negative. Note that due to the newly added Markov chain, our control processes include both $(\mathbf{h}, \eta)$, yet there is no traded asset in the market based on the Markov chain. Therefore only the market price of the diffusion risk $-h(t)$ appears in the first constraint without the market price of regime change risk $-\eta_{ij}$.

$$
\begin{aligned}
V(S_t, Y_t, t) := \sup_{\mathbf{h}, \eta} \quad & E^{\mathbb{Q}}\left[F(S_t, Y_t, X_t, t) | \mathscr{F}_t^o\right] \\
\text{such that} \quad & \alpha + \sigma h - r = 0 \\
& \eta_{ij}(t) \geq -1, \quad \forall i, j = 1, \cdots, D, j \neq i \\
& \|\, \mathbf{h}(t) \,\|_{R^d}^2 + \sum_{i=1}^{D} \sum_{j=1, j\neq i}^{D} |\eta_{ij}(t)|^2 \lambda_{ij}(t) \ \leq A^2
\end{aligned}
$$

3.4 The Partially Observed Model PDE

Under the partially observed model, we are supposed to apply Itô's formula to the observed state process $F(S_t,Y_t,X_t,t)$ first and then acquire the filtered estimate.

Clearly, **Theorem 1 (Hamilton-Bellman-Jacobi Equation)** with respect to F should still hold true.

Corollary 1.1 (Modified Hamilton-Bellman-Jacobi Equation)**.**

$$\begin{aligned}&\frac{\partial F(s,y,x,t)}{\partial t}+\mathbf{A}^{\boldsymbol{h},\eta}F(s,y,x,t)-rF(s,y,x,t)=0\\&F(s,y,x,T)=\Phi(s,y)\end{aligned}$$

The infinitesimal operator $\mathbf{A}^{\mathbf{h},\eta}$ is given by

$$\begin{aligned}&\mathbf{A}^{\mathbf{h},\eta}F(s,y,x,t)=\mathbf{A}_t^H+\mathbf{A}_t^\eta\\&\text{where}\\&\mathbf{A}_t^H=\frac{\partial F(s,y,x,t)}{\partial s}S(\alpha+\sigma h_z)\\&+\frac{\partial F(s,y,x,t)}{\partial y}\{a(s,y)+b^z(s,y)h_z(t,s,y)+b^w(s,y)h_w(t,s,y)\}\\&+\frac{1}{2}\frac{\partial F^2(s,y,x,t)}{\partial^2 s}S^2\sigma^2(s,y)\\&+\frac{1}{2}\frac{\partial F^2(s,y,x,t)}{\partial^2 y}b_z^2(s,y)+b_w^2(s,y)\\&+\frac{1}{2}\frac{\partial F^2(s,y,x,t)}{\partial s\partial y}S\sigma(s,y)b^z(s,y);\\&\mathbf{A}_t^\eta=-\sum_{i=1,i\neq j}^{D}g_{ij}(1+\eta_{ij}(t,x)\left[F(s,y,x,t,i)-F(s,y,x,t,j)\right]\end{aligned}$$

Proof. A similar application of Itô's formula to the partially observed model. □

As $V(S_t,Y_t,t)$ is defined over the observation filtration $\mathscr{F}_t^o$, we define the filtered estimate to proceed further

Definition 4. *If $E^{\mathbb{P}}(|F(S_t,Y_t,X_t)|)\leq\infty$, then the filtered estimate $\pi_t(F)$ is given by*

$$\pi_t(F)=E^{\mathbb{P}}(F(S_t,Y_t,X_t)|\mathscr{F}_t^o)=\frac{E_t^{\mathbb{Q}}(F(S_t,Y_t,X_t)\Lambda_t|\mathscr{F}_t^o)}{E_t^{\mathbb{Q}}(\Lambda_t|\mathscr{F}_t^o)}$$

where Λ_t is the Radon-Nikydom derivative

Corollary 1.2.

$$
\begin{aligned}
&V_t(F) = E_t^{\mathbb{Q}}\left[F(S_t,Y_t,X_t)|\mathscr{F}_t^o\right] \\
&= E_t^{\mathbb{Q}}\left[F(S_0,Y_0,X_0)|\mathscr{F}_t^o\right] + \int_0^t E_t^{\mathbb{Q}}\left[\mathbf{A}_u^H F(S_u,Y_u,X_u)|\mathscr{F}_u^o\right] du \\
&\quad + \int_0^t E_t^{\mathbb{Q}}\left[F(S_u,Y_u,X_u)|\mathscr{F}_u^o\right](dZ_u - h_u^z du) \\
&\quad + \int_0^t E_t^{\mathbb{Q}}\left[F(S_u,Y_u,X_u)|\mathscr{F}_u^o\right](dW_u - h_u^w du) \\
&\textit{where now we have} \quad V(S_t,Y_t,t) \equiv \sup_{\mathbf{h}} V_t(F)
\end{aligned}
$$

Proof. Given the payoff $\phi(S_t,Y_t)$ of the fixed T-period contingent claim, the integrability condition for $F(S_t,Y_t,X_t)$ is automatically satisfied. Moreover, we don't need to define Λ_t because the optimal value function V is already defined over the measure $\mathbb{Q}$. Ultimately, since X is unobservable in $\mathscr{F}_t^o$, after re-expressing the infinitesimal operator in the expectation under probability measure $\mathbb{Q}$, it leads to the corollary. □

3.5 Conclusion

The key difference of this optimal control problem is that due to the nature of partially observed model, a general stochastic filtration theory is needed to be taken into account, which now has been manually simplified. After the enrichment of the model, one may conduct some numerical tests to compare with results derived by other techniques in asset pricing. Unfortunately, due to the time limit, we are unable to finish this part at the moment.

Acknowledgement

The author wishes to appreciate Department of Mathematics at the University of Michigan for this research opportunity and meticulous support from my mentor Prakash Chakraborty during this tough COVID-19 period. Though unfortunately I did't make any significant contribution, the author was exposed to this exciting field of optimal control and learnd a lot throughout the process. This report would never be possible without mentor and department's support.

References

1. Cochrane, John H., and Jesus Saa-Requejo. ”Beyond arbitrage: Good-deal asset price bounds in incomplete markets.” Journal of political economy 108.1 (2000): 79-119.

2. Landen, Camilla. ”Bond pricing in a hidden Markov model of the short rate.” Finance and Stochastics 4.4 (2000): 371-389.

3. Björk, Tomas, and Irina Slinko. ”Towards a general theory of good-deal bounds.” Review of Finance 10.2 (2006): 221-260.

4.Van Handel, Ramon. ”Stochastic calculus, filtering, and stochastic control.” Course notes., URL http://www. princeton. edu/rvan/acm217/ACM217. pdf 14 (2007).

5. Cochrane, John H. Asset pricing: Revised edition. Princeton university press, 2009.

6. Donnelly, Catherine. ”Good-deal bounds in a regime-switching diffusion market.” Applied Mathematical Finance 18.6 (2011): 491-515.

7. Murgoci, Agatha. ”Pricing counter-party risk using good deal bounds.” Available at SSRN 1096590 (2014).

8. Chen, Jun-Home, Yu-Lieh Huang, and Jow-Ran Chang. ”Robust Good-Deal Bounds In Incomplete Markets: The Case Of Taiwan.” Hitotsubashi Journal of Economics (2017): 53-67.