DIFFERENTIAL CLOSURE OF IDEALS

LILLIAN MCPHERSON, MONROE STEPHENSON, FUXIANG YANG

Abstract. In the following paper the differential closure, inspired by similar closure operations, is explored in its relation to the radical and $\mathcal{D}_R$-modules. Additionally, the differential power is surveyed in respects to its eventual principality and containment in ordinary powers, particularly in the case of monomial ideals.

Contents

Date: August 2021.

1. Introduction

Commutative algebraists are interested in studying closure operations on ideals, primarily as a method to classify the singularities of a ring. For example, closure operations were used to prove the Briançon-Skoda Theorem, which has direct corollaries describing singularities on hypersurfaces [Hoc07]. Most famously Hilbert's Nullstellensatz uses a closure operation to describe the one-to-one correspondence between ideals and varieties. We begin by defining the powers of ideals, which leads us to general closure operations.

Two well-studied notions of powers of an ideal I are the n^{th} *ordinary power* I^n, which is the ideal generated by the products of n elements of I, and the e^{th} *Frobenius power*, $I^{[p^e]}$, which is the ideal generated by p^{eth} powers of elements in I [HH89]. Both of these lead to natural closure operations, the *integral closure* [HS06]:

$$\overline{I} \coloneqq \{r \in R \mid \text{ there exists } c \in R^{\circ} \colon cr^n \in I^n \text{ for all } n \gg 0\}$$

and the *tight closure* [HH89]:

$$I^* \coloneqq \{r \in R \mid \text{ there exists } c \in R^{\circ} \colon cr^{p^e} \in I^{[p^e]} \text{ for all } e \gg 0\}$$

Inspired by the definitions above, we take another notion of power of an ideal, the *differential power* and define a new closure operation, the *differential closure.*

Definition 1.1 (Definition 2.2, [DDSG+18])**.** Let R be a ring and $I \subseteq R$ be an ideal. Define the n-th differential power of I to be

$$I^{\langle n \rangle} = \{r \in R \mid D(r) \in I \ \forall D \text{ differential operators of order less than } n\}$$

Definition 1.2. Let R be a ring and $I \subseteq R$ be an ideal. Define the differential closure of I to be

$$\overline{I}^{\text{diff}} = \{r \in R \mid \text{ there exists } c \in R^{\circ} : cr^n \in I^{\langle n \rangle} \text{ for all } n \gg 0\}$$

The differential power of an ideal coincides with the symbolic power of an ideal for prime ideals in polynomial rings of characteristic zero [DDSG+18].

Commutative algebraists have previously studied properties of symbolic powers, such as in the containment results which inspired sections 4.3 and 4.4 of our paper (see [ELS01]). One major containment result is stated in the following theorem.

Theorem 4.18. *Let $I = (x_1^{\beta_1}, \ldots, x_m^{\beta_m})$ be an ideal in the ring $R = \mathsf{k}[x_1, \ldots, x_z]$ for some characteristic 0 field k. $I^{\langle cn \rangle} \subseteq I^n$ where $c = \max\{\beta_1, \ldots, \beta_m, m+1\}$.*

We also studied the behavior of the differential power $I^{\langle n \rangle}$ as n gets large.

Theorem 4.26, 4.31. *For certain ideals I in a characteristic 0 polynomial ring, there exists a positive integer N such that $I^{\langle N \rangle}$ is principal.*

Finally, we studied properties of the differential closure, proving that it is a closure operation. In fact, we have that the differential closure equals the radical for ideals of simple $\mathcal{D}_R$-modules.

Theorem 5.13. *Let R be a a regular characteristic 0 ring or positive characteristic p ring, and I is an ideal of R, then $\sqrt{I} = \overline{I}^{diff}$ if and only if R is a simple $\mathcal{D}_R-$module.*

2. Background

In order to study the differential power and differential closure, we must familiarize ourselves with definitions of differential operators on a commutative k-algebra. An operator on the ring R is an element of $\operatorname{Hom}_k(R, R)$. Recall the following definitions regarding differential operators. In these definitions, let R be a commutative k-algebra.

Definition 2.1 ([Cou95])**.** A derivation of R is a linear operator D of R which satisfies Leibniz's Rule. (i.e. $D(ab) = aD(b) + bD(a)$ for all $a, b \in R$)

Definition 2.2 ([Cou95])**.** The commutator of two differential operators P, Q is defined as

$$[P, Q] = P \cdot Q - Q \cdot P$$

Consider the following examples of a derivation and commutator in a polynomial ring.

Example 2.3. Let $R = \mathbb{C}[x, y]$. The operator $\frac{\partial}{\partial y}$ is a derivation of R. We know from calculus partial derivatives follow Leibniz's Rule.

Example 2.4. Let $R = \mathbb{C}[x, y]$, $P = \frac{\partial}{\partial x}$, and $Q = x\frac{\partial^2}{\partial y^2}$. We can compute the commutator of P and Q:

$$\begin{aligned} [P, Q] &= \frac{\partial}{\partial x} \cdot x\frac{\partial^2}{\partial y^2} - x\frac{\partial^2}{\partial y^2} \cdot \frac{\partial}{\partial x} \\ &= x\frac{\partial^3}{\partial x \partial y^2} + \frac{\partial^2}{\partial y^2} - x\frac{\partial^3}{\partial x \partial y^2} \end{aligned}$$

With these definitions we have the information we need in order to define a differential operator in a commutative k-algebra.

Definition 2.5 ([Cou95])**.** The order of a differential operator is defined inductively. An operator P has order zero if $[a, P] = 0$ for every $a \in R$. If we have defined all differential operators of order less than n, define the set of operators of order n

$$D_n \coloneqq \{P \in \operatorname{Hom}_k(R, R) \mid P \text{ does not have order} < n \text{ and } [a, P] \text{ has order} < n \text{ for all } a \in R\}$$

Applying this inductive definition, we can explicitly write out the operators of order less than or equal to one in terms of derivations and multiplication by a ring element.

Lemma 2.6 (Chapter 3, Lemma 1.1 [Cou95])**.** *The operators of order ≤ 1 correspond to the elements $Der_{\mathsf{k}}(R) + R$. The elements of order zero are the elements of R. $Der_{\mathsf{k}}(R)$ denotes the derivations of R, and elements of R are considered differential operators through multiplication by that element.*

In more general rings, this inductive notion of differential operators can appear abstract. For many of our results in this paper, we focus on polynomial rings. In characteristic zero, derivations are of the form $\sum\left(a \cdot \frac{\partial}{\partial x_i}\right)$ for x_i a variable in the ring, and a a ring element. A general differential operator is a linear combination of partial derivatives and multiplication by a ring element. In characteristic $p > 0$, derivations are of the same form as characteristic 0. When discussing more general differential operators in positive characteristic, however, we must consider linear combinations which include generators of the form $\frac{1}{p^e!}\frac{\partial^{p^e}}{\partial x_i^{p^e}}$ for e a nonnegative integer. These generators are called divided powers.

Example 2.7. Let $R = \mathbb{C}[x, y]$. The differential operator $\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y}$ acts on $x + y^2$ as follows:

$$\begin{aligned} \left(\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y}\right)(x + y^2) &= \frac{\partial}{\partial x}(x + y^2) + \left(y^2\frac{\partial}{\partial y}\right)(x + y^2) \\ &= 1 + y^2(2y) \\ &= 1 + 2y^3 \end{aligned}$$

3. Properties of Differential Closure and Differential Powers

For this section, let $R = \mathsf{k}[x_1, \ldots, x_z]$ be a polynomial ring where k is a field, and let $I \subseteq R$ be an ideal.

3.1. Differential Closure is a Closure Operation.

Before we discuss properties of the differential closure, we need to justify calling it a closure operation. To do this, we need some facts about differential powers and differential operators.

Proposition 3.1 (Proposition 3.8 [BJNB19])**.** *$I^{\langle n \rangle}$ is an ideal.*

Also, we note that $I^{\langle n\rangle} \subseteq I^{\langle m\rangle}$ for all $n > m$. Instead of checking all differential operators of a given order, we write the following two lemmas to simplify which differential operators are sufficient to check when determining if an element is in a given differential power.

Lemma 3.2. *Define D_α^n to be a differential operator of order less than n which is a composition of partial derivatives in characteristic k, and a composition of partial derivatives and divided powers in characteristic k.*

If $y \in R$ satisfies $D_\alpha^n \cdot y \in I$ for all possible compositions of order less than n, then $y \in I^{\langle n\rangle}$.

Recall that divided powers are operators of the form

$$\frac{1}{p^l!}\frac{\partial^{p^l}}{\partial x_i^{p^l}}$$

for $x_i \in R$ and $l \in \mathbb{N}$.

Proof. Let $D = \sum_\alpha r_\alpha D_\alpha^n$, where r_α denotes multiplication by an element of R. Since R is a polynomial ring, D is an arbitrary differential operator of order less than n.

Let $y_\alpha = D_\alpha^n(y)$ and we know that $y_\alpha \in I$. Consider

$$D(y) = \left(\sum_\alpha r_\alpha D_\alpha^n\right)(y) = \sum_\alpha r_\alpha(D_\alpha^n(y)) = \sum_\alpha r_\alpha y_\alpha$$

where each r_α are elements of R. Since $y_\alpha \in I$ and $r_\alpha \in R$ for every possible α, then $D(y) \in I$ which implies that $y \in I^{\langle n\rangle}$. □

In the characteristic zero case, we need the following result on multiplying differential powers.

Lemma 3.3. *If k characteristic zero, $I^{\langle n\rangle} \cdot I^{\langle m\rangle} \subseteq I^{\langle m+n\rangle}$.*

Proof. We know the elements of $I^{\langle n\rangle} \cdot I^{\langle m\rangle}$ are of the form $\sum_{i,j} s_i t_j$ where $s_i \in I^{\langle n\rangle}$ and $t_i \in I^{\langle m\rangle}$. Without loss of generality we can consider only one term. If $st \in I^{\langle n\rangle} \cdot I^{\langle m\rangle}$, then we want to show $st \in I^{\langle n+m\rangle}$. We first note that $D_{<n}(s) \in I$, where $D_{<n}$ denotes an arbitrary differential operator of order less than n. Similarly $D_{<m}(t) \in I$. We want to show $D_{<m+n}(st) \in I$ because this would imply our desired result. Let $\alpha \in \mathbb{N}^z$ such that $|\alpha| \leq m + n - 1$, where $|\alpha|$ denotes the sum of the entries in α. Define an operator $D_\alpha = \prod_{\alpha_i \in \alpha} \frac{\partial^{\alpha_i}}{\partial x_i^{\alpha_i}}$. Lemma 3.2 says in characteristic zero it is sufficient to consider differential operators as compositions of partials, so it is sufficient to check that $D_\alpha(st) \in I$ for all α satisfying the above condition. We can use the General Leibniz rule to write

$$D_\alpha(st) = \sum_{\omega\,:\,\omega \leq \alpha} \binom{\alpha}{\omega} D_\omega(s) D_{\alpha-\omega}(t)$$

where

$$\binom{\alpha}{\beta} = \binom{\alpha_1}{\beta_1}\binom{\alpha_2}{\beta_2}\cdots\binom{\alpha_z}{\beta_z}$$

and we say $\omega \leq \alpha$ for $\omega \in \mathbb{N}^z$ if $\omega_i \leq \alpha_i$ for all i.

Note, we are only considering the order of the differential operators. In this case, we notice that if either $D_\omega(s)$ is in I or $D_{\alpha-\omega}(t)$ is in I for all $\omega \leq \alpha$ then $D_\alpha(st) \in I$. Equivalently, we need to show $|\alpha - \omega| \leq m - 1$ or $|\omega| \leq n - 1$. We know $|\alpha - \omega| = |\alpha| - |\omega| \leq m + n - 1 - |\omega|$. If $|\alpha - \omega| > m - 1$, then $|\omega| \leq n - 1$. If $|\omega| > n - 1$, then $|\alpha - \omega| \leq m - 1$. So, either $|\alpha - \omega| \leq m - 1$ or $|\omega| \leq n - 1$. Thus we have shown $D_\omega(t) \in I$ or $D_{\alpha-\omega}(s) \in I$ for all $\omega : \omega \leq \alpha$. From this, we find that $D_\alpha(st) \in I$, so $st \in I^{\langle n+m\rangle}$. Thus, $I^{\langle n\rangle} \cdot I^{\langle m\rangle} \subseteq I^{\langle n+m\rangle}$. □

The following example illustrates Lemma 3.3.

Example 3.4. Let $R = \mathbb{Q}[x, y, z]$, $I = (x^2y^3)$, $n = 4$, and $m = 3$. Lemma 3.3 states that $I^{\langle 4\rangle} \cdot I^{\langle 3\rangle} \subseteq I^{\langle 7\rangle}$. Indeed, we see

$$\begin{aligned} I^{\langle 4\rangle} \cdot I^{\langle 3\rangle} &= (x^5y^6) \cdot (x^4y^5) = (x^9y^{11}) \\ I^{\langle 7\rangle} &= (x^8y^9) \end{aligned}$$

We can see $I^{\langle 4\rangle} \cdot I^{\langle 3\rangle} \subseteq I^{\langle 7\rangle}$ since $xy^2 \cdot x^8y^9 = x^9y^{11}$.

The following proposition is intuitive in the case of polynomial rings of characteristic zero. Generalizing this fact to polynomial rings of positive characteristic (and, in fact, even more general rings) allows for our next propositions to hold for all polynomial rings.

Proposition 3.5. *Let $r \in R$. For all $m > l$, $r|D_{\leq l}(r^m)$ where $0 \leq l < m$ and $D_{\leq l}$ is any operator of order less than or equal to l.*

Proof. Proof by induction on l.

We begin by considering the base case where $l = 0$. Let $D_{\leq 0}$ be an operator of order zero. We want to show for all $m > 0$ we have r divides $D_{\leq 0}(r^m)$. The only differential operators of order zero are multiplication by a ring element so we may assume $D_{\leq 0}$ is multiplication by a. Then, $D_{\leq 0}(r^m) = ar^m$ and since $m > 0$, we know r divides $D_{\leq 0}(r^m)$.

Next consider the inductive hypothesis. Assume for all $m > l$, r divides $D_{\leq l}(r^m)$ for every operator $D_{\leq l}$ of order less than or equal to l. We want to show for all $m' > l + 1$, r divides $D_{\leq l+1}(r^{m'})$ for any operator $D_{\leq l+1}$ of order order less than or equal to $l + 1$. The inequality $m > l$ implies $m + 1 > l + 1$. So we can rewrite m' as $m + 1$ to use the same m as the inductive hypothesis. After this substitution we want to show for all $m > l$, r divides $D_{\leq l+1}(r^{m+1})$.

Pick a differential operator D and without loss of generality let the order of D be $l + 1$. Then, $[D, r]$ is a differential operator of order less than or equal to l. Use the inductive hypothesis to write $[D, r](r^m) = r(A)$ for some A in R. Then,

$$\begin{aligned} r(A) &= [D, r](r^m) \\ &= (D \cdot r)(r^m) - (r \cdot D)(r^m) \\ &= D(r^{m+1}) - r \cdot (D(r^m)) \end{aligned}$$

Since r divides the left side of the equation and r divides $rD(r^m)$, we have r divides $D(r^{m+1})$. Since D was an arbitrary differential operator, we have proven r divides $D(r^{m+1})$ for all differential operators of order less than or equal to $l + 1$. $\square$

We can see this proposition more clearly through the use of the following example.

Example 3.6. Let $R = \mathbb{Z}/3\mathbb{Z}[x, y, z]$, $r = x^2z$. Consider the differential operators of order less than or equal to five. We will look at $m = 7$ and the differential operator $\left(\frac{1}{3!}\frac{\partial^3}{\partial x^3}\right)\frac{\partial}{\partial x}$ which has order four.

$$\begin{aligned} \left(\frac{1}{3!}\frac{\partial^3}{\partial x^3}\right)\frac{\partial}{\partial x}((x^2z)^7) &= \left(\frac{1}{3!}\frac{\partial^3}{\partial x^3}\right)\frac{\partial}{\partial x}(x^{14}z^7) \\ &= \left(\frac{1}{3!}\frac{\partial^3}{\partial x^3}\right)(2x^{13}z^7) \\ &= 2x^{10}z^7 \end{aligned}$$

We can see that x^2z divides $2x^{10}z^7$.

Finally, we have the information needed to prove the differential closure is a closure operation. In fact, we will later show in Proposition 5.12 that $\overline{I}^{\text{diff}} = \sqrt{I}$ for any ideal I in a polynomial ring of characteristic zero and all of the properties below follow from this fact. However, we include a direct proof here for some properties which hold more generally.

Proposition 3.7. *The differential closure is a closure operation on ideals in a polynomial ring of characteristic zero.*

Equivalently, $\overline{I}^{diff}$ *satisfies the following properties where* I, J *are ideals in a polynomial ring of characteristic zero.*

(1) $\overline{I}^{diff}$ *is an ideal*
(2) $I \subseteq \overline{I}^{diff}$
(3) *If* $I \subseteq J$*, then* $\overline{I}^{diff} \subseteq \overline{J}^{diff}$
(4) $\overline{\overline{I}^{diff}}^{diff} = \overline{I}^{diff}$

Proof.

Proof of (1):

First, we want to show that the differential closure is closed under multiplication by a ring element. If $a \in \overline{I}^{\text{diff}}$, want to show $ra \in \overline{I}^{\text{diff}}$. This means we want to find a nonzero $c \in R$ such that $c(ra)^n \in I^{\langle n\rangle}$ for all n sufficiently large. We have $(ra)^n = r^n a^n$. We know there exists a nonzero c such that $ca^n \in I^{\langle n\rangle}$. Since r^n is a ring element and $I^{\langle n\rangle}$ is an ideal, $r^n(ca^n) \in I^{\langle n\rangle}$.

Second, we want to show that the differential closure is closed under addition. For $a, b \in \overline{I}^{\text{diff}}$, we want to show that $a+b \in \overline{I}^{\text{diff}}$. We know from our definitions that there exist nonzero $c_a, c_b \in R$ such that $c_a a^n \in I^{\langle n\rangle}$ and $c_b b^n \in I^{\langle n\rangle}$ for n sufficiently large. We want to show $c_{(a+b)}(a+b)^m \in I^{\langle m\rangle}$ for some nonzero $c_{(a+b)} \in R$ for all m sufficiently large. We know by the binomial theorem that

$$c_{(a+b)}(a+b)^m = c_{(a+b)} \sum_{k=0}^{m} \binom{m}{k} a^k b^{m-k} = \sum_{k=0}^{m} \binom{m}{k} c_{(a+b)} a^k b^{m-k}$$

When the characteristic of k is zero, we can invoke Lemma 3.3, so it suffices to show that there exists some $c_{(a+b)}$ such that for each k, $c_{(a+b)} a^k b^{m-k} \in I^{\langle k\rangle} \cdot I^{\langle m-k\rangle} \subseteq I^{\langle m\rangle}$. By the definition of differential power, we know there exists an N such that $c_a a^n \in I^{\langle n\rangle}$ and $c_b b^n \in I^{\langle n\rangle}$ for all $n \geq N$, so we know that $(c_a a^N) \cdot a^i \in I^{\langle i+N\rangle} \subseteq I^{\langle i\rangle}$ and $(c_b b^N) \cdot b^i \in I^{\langle i+N\rangle} \subseteq I^{\langle i\rangle}$ for all i greater than zero. In this case, let $c_{(a+b)} = c_a c_b a^N b^N$ then we have $c_{(a+b)} a^k b^{m-k} = c_a a^{k+N} c_b b^{m-k+N}$. We know $c_a a^{k+N} \in I^{\langle k\rangle}$ from our choice of N. Similarly, $c_b b^{m-k+N} \in I^{\langle m-k\rangle}$. So, $c_{(a+b)} a^k b^{m-k} \in I^{\langle k\rangle} \cdot I^{\langle m-k\rangle} \subseteq I^{\langle m\rangle}$ and we are done.

Proof of (2):

Suppose $r \in I$. We want to show there exists a nonzero c such that $cr^n \in I^{\langle n\rangle}$ for all n sufficiently large. This means $D(cr^n) \in I$ for all differential operators D of order less than n. Choose $c = 1$. Using Proposition 3.5, $D(r^n) = c_1 r$ where c_1 might be zero. If c_1 is zero, then $c_1 r \in I$ since $0 \in I$. If c_1 is nonzero, then $c_1 r \in I$ since $r \in I$.

Proof of (3):

Let $a \in \overline{I}^{\text{diff}}$. By definition there exists a nonzero $c \in R$ such that $ca^n \in I^{\langle n\rangle}$ for all positive integers n sufficiently large. This implies that $D \cdot ca^n \in I$ for all differential operators D with order less than n. Since $I \subseteq J$, then $D \cdot ca^n \in I \subseteq J$. This means that $ca^n \in J^{\langle n\rangle}$ for all positive integers n sufficiently large. By definition $a \in \overline{J}^{\text{diff}}$. Thus, $\overline{I}^{\text{diff}} \subseteq \overline{J}^{\text{diff}}$.

Proof of (4):

Later we will prove that $\overline{I}^{\text{diff}} = \sqrt{I}$ if and only if R is a simple $\mathcal{D}$-module. Since polynomial rings are simple $\mathcal{D}$-modules, this equality holds. By properties of the radical, we know $\sqrt{\sqrt{I}} = \sqrt{I}$ (Exercise 1.13, [AM69]), so we have the idempotence property for the differential closure.

$\square$

3.2. **Other Properties.** Now we can discuss other properties of the differential closure and differential power. As it is often useful to consider decompositions of ideals, we want to prove that the differential power and differential closure respect intersections.

Lemma 3.8 (Exercise 2.13 [DDSG$^+$18]). *Let $\{I_\alpha\}_{\alpha\in A}$ be an indexed family of ideals. Then,*

$$\bigcap_{\alpha\in A} {I_\alpha}^{\langle n\rangle} = \left(\bigcap_{\alpha\in A} I_\alpha\right)^{\langle n\rangle}$$

for every positive integer n.

Proof. First, we want to show that $\bigcap_{\alpha\in A} {I_\alpha}^{\langle n\rangle} \subseteq \left(\bigcap_{\alpha\in A} I_\alpha\right)^{\langle n\rangle}$. Let s be an element of $\bigcap_{\alpha\in A} {I_\alpha}^{\langle n\rangle}$. By definition, we have s in ${I_\alpha}^{\langle n\rangle}$ for all $\alpha \in A$. By the definition of differential power, we have $D \cdot s \in I_\alpha$ for all $\alpha \in A$ and for all differential operators D with order less than n. This implies that $D \cdot s$ is an element of $\bigcap_{\alpha\in A} I_\alpha$. Thus, s is an element of $\left(\bigcap_{\alpha\in A} I_\alpha\right)^{\langle n\rangle}$.
Now let t be an element of $\left(\bigcap_{\alpha\in A} I_\alpha\right)^{\langle n\rangle}$. By the definition of differential power, we have $D \cdot t \in \bigcap_{\alpha\in A} I_\alpha$ for all differential operators D with order less than n. By the definition of intersection, we have $D \cdot t \in I_\alpha$ for all $\alpha \in A$. This implies that t is an element of ${I_\alpha}^{\langle n\rangle}$ for all $\alpha \in A$. Thus, t is an element of $\bigcap_{\alpha\in A} {I_\alpha}^{\langle n\rangle}$. We have shown both direction of containment, so this proves

$$\bigcap_{\alpha\in A} {I_\alpha}^{\langle n\rangle} = \left(\bigcap_{\alpha\in A} I_\alpha\right)^{\langle n\rangle}$$

for every positive integer n. $\square$

Lemma 3.9. *Let $\{I_\alpha \mid \alpha \in A\}$ be a family of ideals with an index set A. Then*

$$\bigcap_{\alpha\in A} \overline{I_\alpha}^{\textit{diff}} \supseteq \overline{\bigcap_{\alpha\in A} I_\alpha}^{\textit{diff}},$$

and when A is finite, we have

$$\bigcap_{\alpha\in A} \overline{I_\alpha}^{\textit{diff}} = \overline{\bigcap_{\alpha\in A} I_\alpha}^{\textit{diff}}.$$

Proof. First, we want to show that $\overline{\bigcap_{\alpha\in A} I_\alpha}^{\text{diff}} \subseteq \bigcap_{\alpha\in A} \overline{I_\alpha}^{\text{diff}}$. We have $\bigcap_{\alpha\in A} I_\alpha \subseteq I_\alpha$ for every $\alpha \in A$. By the property of closure operations, this implies that $\overline{\bigcap_{\alpha\in A} I_\alpha}^{\text{diff}} \subseteq \overline{I_\alpha}^{\text{diff}}$ for every $\alpha \in A$. Thus, by the definition of intersection, we have

$$\overline{\bigcap_{\alpha\in A} I_\alpha}^{\text{diff}} \subseteq \bigcap_{\alpha\in A} \overline{I_\alpha}^{\text{diff}}.$$

Now we want to show that $\overline{\bigcap_{\alpha\in A} I_\alpha}^{\text{diff}} \supseteq \bigcap_{\alpha\in A} \overline{I_\alpha}^{\text{diff}}$. Let $b \in \bigcap_{\alpha\in A} \overline{I_\alpha}^{\text{diff}}$ be an arbitrary element. This means that $b \in \overline{I_\alpha}^{\text{diff}}$ for all $\alpha \in A$. By definition, for all $\alpha \in A$ there exists an element $c_\alpha \in R$ such that $c_\alpha b^n \in {I_\alpha}^{\langle n\rangle}$ for all n sufficiently large. This implies that $(\prod_{\alpha\in A} c_\alpha) b^n \in {I_\alpha}^{\langle n\rangle}$ for all $\alpha \in A$ and n sufficiently large. By definition of intersection and Lemma 3.8, we have

$$\left(\prod_{\alpha\in A} c_\alpha\right) b^n \in \bigcap_{\alpha\in A} {I_\alpha}^{\langle n\rangle} = \left(\bigcap_{\alpha\in A} I_\alpha\right)^{\langle n\rangle}.$$

By the definition of differential closure, $b \in \overline{\bigcap_{\alpha\in A} I_\alpha}^{\text{diff}}$. This implies that $\overline{\bigcap_{\alpha\in A} I_\alpha}^{\text{diff}} \supseteq \bigcap_{\alpha\in A} \overline{I_\alpha}^{\text{diff}}$. Thus, $\bigcap_{\alpha\in A} \overline{I_\alpha}^{\text{diff}} = \overline{\bigcap_{\alpha\in A} I_\alpha}^{\text{diff}}$. □

Next, we look at an interesting application of Lemma 3.2.

Proposition 3.10. *If k is of characteristic zero, ${I^{\langle n\rangle}}^{\langle m\rangle} = I^{\langle n+m-1\rangle}$ for all n, m.*

If k is of characteristic $p > 0$, then ${I^{\langle n\rangle}}^{\langle m\rangle} \neq I^{\langle n+m-1\rangle}$ if, for some $e > 0$, $n \leq p^e$ and $m \leq p^e$ and $n + m - 1 > p^e$.

Proof. Let r be an arbitrary element in $I^{\langle n+m-1\rangle}$. This means that $D(r) \in I$ for all differential operators D with order less than or equal to $n+m-2$. Let D_1 be a differential operator of order less than or equal to $n-1$ and let D_2 be a differential operator of order less than or equal to $m-1$. Then $D = D_1 \circ D_2$, has order less than $n+m-2$, so $(D_1 \circ D_2)(r) = D(r) \in I$. Since D_1 and D_2 were arbitrary, we have $r \in {I^{\langle n\rangle}}^{\langle m\rangle}$. This implies $I^{\langle n+m-1\rangle} \subseteq {I^{\langle n\rangle}}^{\langle m\rangle}$.

To prove ${I^{\langle n\rangle}}^{\langle m\rangle} \subseteq I^{\langle n+m-1\rangle}$, we want to show for all D of order less than or equal to $n+m-2$, there exists a D_1 and D_2 of orders less than n and m respectively which satisfy the equality $D(r) = (D_1 \circ D_2)(r)$ for all $r \in R$.

First consider the case where k is of characteristic zero. Recall for polynomial rings of characteristic zero, Lemma 3.2 states that when considering differential operators of order N it is sufficient to consider compositions of N first-order partial derivatives. Thus, write $D = \prod_{i\in[z]} \frac{\partial^{c_i}}{\partial x_i^{c_i}}$ where each c_i is a nonnegative integer, $0 \leq \sum_{i\in[z]} c_i \leq n+m-2$. If we restrict D_1 and D_2 to the forms $D_1 = \prod_{i\in[z]} \frac{\partial^{a_i}}{\partial x_i^{a_i}}$ and $D_2 = \prod_{i\in[z]} \frac{\partial^{b_i}}{\partial x_i^{b_i}}$ where each a_i, b_i is a nonnegative integer, $0 \leq \sum_{i\in[z]} a_i \leq n-1$, and $0 \leq \sum_{i\in[z]} b_i \leq m-1$, we can find a D_1 and D_2 such that $D = D_1 \circ D_2$, since partial derivatives commute.

Second consider the case where k is of characteristic $p > 0$. Recall for polynomial rings of characteristic p, Lemma 3.2 states that when considering differential operators of order N, it is sufficient to consider compositions of partial derivatives and operators of the form $\frac{1}{p^e!}\frac{\partial^{p^e}}{\partial x_i^{p^e}}$ where the sum of orders of operators in the composition equals N. Partial derivatives all have order 1, and the other operators have order p^e.

So, we consider all the values of e such that $n+m-1 > p^e$. Let $D = \frac{1}{p^e!}\frac{\partial^{p^e}}{\partial x_i^{p^e}}$. We want to write $\frac{1}{p^e!}\frac{\partial^{p^e}}{\partial x_i^{p^e}}$ as a composition of an operator of order less than n and an operator of order less than m. But since $\frac{1}{p^e!}\frac{\partial^{p^e}}{\partial x_i^{p^e}}$ is a generator in the ring of differential operators, the only way we can write it as a composition is $1 \circ \frac{1}{p^e!}\frac{\partial^{p^e}}{\partial x_i^{p^e}}$ or $\frac{1}{p^e!}\frac{\partial^{p^e}}{\partial x_i^{p^e}} \circ 1$. So, either $p^e < n$ or $p^e < m$, or both. So, for the two ideals not to be equal, we have $p^e \geq n$ and $p^e \geq m$. □

4. Differential Powers and Closures of Monomial Ideals

In this section, let $R = \mathsf{k}[x_1, \ldots, x_z]$ where k is a field of characteristic 0. The goal of this section is to understand the properties of differential power and closure when the ideal is a monomial ideal. We want to know whether or not I being a monomial ideal implies $\overline{I}^{\text{diff}}$ and $I^{\langle n\rangle}$ are monomial ideals. We are also interested in studying the conditions of containment problems between differential power $I^{\langle n\rangle}$ and ordinary power I^n.

We recall some facts about monomial ideals. First, we know that every monomial ideal has a decomposition in terms of ideals which are pure powers of variables. In particular,

Theorem 4.1 (Theorem 1.3.1 [HH11]). *Let $I \subset S = K[x_1, \dots, x_n]$ be a monomial ideal. Then $I = \bigcap_{i=1}^m Q_i$ where each Q_i is generated by pure powers of the variables. In other words, each Q_i is of the form $(x_{i_1}^{a_1}, \dots, x_{i_k}^{a_k})$. Moreover, an irredundant presentation of this form is unique.*

In the particular case of squarefree monomial ideals, we see that this decomposition consists of ideals generated by variables.

Corollary 4.1.1. *If I is a squarefree monomial ideal, then $I = \bigcap_{i=1}^m Q_i$ where each Q_i is of the form $(x_{i_1}, \dots, x_{i_k})$.*

Furthermore, the any finite intersection of monomial ideals is still a monomial ideal, and we can explicitly describe its generators.

Definition 4.2. Let I be an ideal of R. $G(I)$ is the set of minimal generators of I.

Proposition 4.3 (Proposition 1.2.1 [HH11]). *Let I and J be monomial ideals. Then $I \cap J$ is a monomial ideal, and $\{\operatorname{lcm}(u, v) \colon u \in G(I), v \in G(J)\}$ is a set of generators of $I \cap J$.*

Corollary 4.3.1. $\bigcap_{i=1}^n I_i$ *is monomial ideal if I_i is a monomial ideal and n is a finite positive integer.*

From these results, it is clear that understanding differential powers of ideals generated by pure powers of variables is the first step to understanding differential powers of monomial ideals.

4.1. **Ideals Generated by Pure Powers of Variables.** In this section, we are studying properties of ideals generated by pure powers of variables, which is equivalent to studying irreducible monomial ideals. Consider an ideal $I = (x_{\alpha_1}^{\beta_1}, x_{\alpha_2}^{\beta_2}, \dots, x_{\alpha_m}^{\beta_m}) \subset R = k[x_1, \dots, x_z]$ where each β_i is a positive integer. We can explicitly write the generators of $I^{\langle n \rangle}$ for all positive integers n and $\overline{I}^{\mathrm{diff}}$.

Proposition 4.4. *$I^{\langle n \rangle}$ is generated by the elements of the forms $x_{\alpha_1}^{\gamma_1} x_{\alpha_2}^{\gamma_2} \dots x_{\alpha_m}^{\gamma_m}$ where γ_i is either 0 or greater than or equal to β_i that satisfies $\sum_{i \in [m], \gamma_i \geq \beta_i} \gamma_i = \sum_{i \in [m], \gamma_i \geq \beta_i} (\beta_i - 1) + n$.*

Proof. Consider an arbitrary element X of the form $x_{\alpha_1}^{\gamma_1} x_{\alpha_2}^{\gamma_2} \cdots x_{\alpha_m}^{\gamma_m}$ where γ_i which satisfies the given condition. This means that there is an induced set S where $S = \{i \in [m] \mid \gamma_i \geq \beta_i\}$. Let D be an arbitrary differential operator of order less than n. By Lemma 3.2, we only need to consider the case where D is a composition of at most $n-1$ differential operators of the form $\frac{\partial}{\partial x_i}$. Now we can let

$$D = \prod_{i \in S} \frac{\partial^{\omega_i}}{\partial x_i^{\omega_i}} \prod_{j \in [m], j \notin S} \frac{\partial^{\theta_j}}{\partial x_j^{\theta_j}}$$

where ω_i, θ_j are positive integers such that $\sum_{i \in S} \omega_i + \sum_{j \in [m], j \notin S} \theta_j \leq n - 1$. If there exists an index $j \in [m]$ such that $\omega_j > \gamma_j$ or $\theta_j > 0$, then $D(X) = 0$. Now we can assume that $\omega_i \leq \gamma_i$ and $\theta_j = 0$ for all $i \in S$ and $j \in [m] \setminus S$. Since θ_js are 0, then we can assume that $D = \prod_{i \in S} \frac{\partial^{\omega_i}}{\partial x_i^{\omega_i}}$. Now we have the inequality $\sum_{i \in S} \omega_i \leq n - 1$. Consider

$$D(X) = c \prod_{i \in S} x_{\alpha_i}^{\gamma_i - \omega_i}$$

where c is some constant in k. Since

$$\sum_{i \in S} (\gamma_i - \omega_i) = \sum_{i \in S} \gamma_i - \sum_{i \in S} \omega_i = \sum_{i \in S} (\beta_i - 1) + n - \sum_{i \in S} \omega_i \geq \sum_{i \in S} (\beta_i - 1) + 1,$$

then by the Pigeonhole Principle, there exists an index $l \in S$ such that $\gamma_l - \omega_l \geq \beta_l$. This implies that $D(X) \in I$. By Lemma 3.2, $X \in I^{\langle n \rangle}$.

Suppose there exists an element $Y = x_{\alpha_1}^{\gamma_1} x_{\alpha_2}^{\gamma_2} \cdots x_{\alpha_m}^{\gamma_m}$ where γ_i is either 0 or greater than or equal to β_i such that $\sum_{i \in S} \gamma_i \leq \sum_{i \in S} (\beta_i - 1) + n - 1$ where $S = \{i \in [m] \mid \gamma_i \geq \beta_i\}$. We can find a differential operator of the form

$$D = \prod_{i \in S} \frac{\partial^{\omega_i}}{\partial x_{\alpha_i}^{\omega_i}}$$

where $\omega_i = \gamma_i - \beta_i + 1$ for all i. We need to check that this operator is of order less than n. We can sum the powers of each partial in the product. Doing this gives us

$$\begin{aligned}\sum_{i\in S}(\gamma_i - \beta_i + 1) &= \sum_{i\in S}\gamma_i - \sum_{i\in S}\beta_i + |S| \\ &\leq \sum_{i\in S}(\beta_i - 1) + n - 1 - \sum_{i\in S}\beta_i + |S| \\ &= n-1\end{aligned}$$

so D is a differential operator of order less than or equal to $n-1$, which is less than n.

Calculating $D(Y)$ gives us

$$\begin{aligned}D(Y) &= c\prod_{i\in S} x_{\alpha_i}^{\gamma_i-(\gamma_i-\beta_i+1)} \\ &= c\prod_{i\in S} x_{\alpha_i}^{\beta_i-1}\end{aligned}$$

where $c \in \mathsf{k}$.

We can see that the power of each x_{α_j} appearing in the sum is less than β_j for all $j \in [m]$. This implies that $D(Y)$ is not in I and $Y \notin I^{\langle n\rangle}$. Therefore, $I^{\langle n\rangle}$ is generated by the elements of the forms $x_{\alpha_1}^{\gamma_1}x_{\alpha_2}^{\gamma_2}\dots x_{\alpha_m}^{\gamma_m}$ where γ_i is either 0 or greater than or equal to β_i that satisfies $\sum_{i\in[m],\gamma_i\geq\beta_i}\gamma_i = \sum_{i\in[m],\gamma_i\geq\beta_i}(\beta_i-1)+n$. □

Proposition 4.5. $\overline{I}^{\text{diff}} = (x_{\alpha_1}, \dots, x_{\alpha_m})$.

Proof. Let i be an arbitrary index in $[m]$, by Proposition 4.4, we know that $x_{\alpha_i}^{\beta_i-1+n}$ is an element of $I^{\langle n\rangle}$. Notice that if we let $c = x_{\alpha_i}^{\beta_i-1} \neq 0$ which is a constant, we have

$$c(x_{\alpha_i})^n = x_{\alpha_i}^{\beta_i-1+n} \in I^{\langle n\rangle}$$

for all positive integer n sufficiently large. This implies that $x_{\alpha_i} \in \overline{I}^{\text{diff}}$ for all $i \in [m]$. Now we must show that every element not in $(x_{\alpha_1}, \dots, x_{\alpha_m})$ is also not in $\overline{I}^{\text{diff}}$. First, we show a field element is not in $\overline{I}^{\text{diff}}$. We just check that 1 is not in $\overline{I}^{\text{diff}}$. Suppose $1 \in \overline{I}^{\text{diff}}$. Then, there exists a nonzero $c \in R$ such that $c(1)^n \in I^{\langle n\rangle}$ for n sufficiently large. This implies that $c \in I^{\langle n\rangle}$ for all n sufficiently large, we need to show that there exists at least one monomial term b of c such that $b \notin I^{\langle n\rangle}$ for all n sufficiently large. we can write $b = r\prod_{i\in[m]} x_{\alpha_i}^{\gamma_i}$ where each γ_i is a constant positive integer, and r is a monomial with variables not in $\{x_{\alpha_1}, \dots, x_{\alpha_m}\}$. Let $S = \{i \in [m] \mid \gamma_i \geq \beta_i\}$, and suppose that $b \in I^{\langle n\rangle}$. By Proposition 4.4, this implies that $\sum_{i\in S}\gamma_i \geq \sum_{i\in S}(\beta_i - 1) + n$. Here we can see that $b \notin I^{\langle n\rangle}$ for all $n > \sum_{i\in S}(\gamma_i - \beta_i + 1)$. This shows that $b \notin I^{\langle n\rangle}$ for all n sufficiently large, which implies that $c \notin I^{\langle n\rangle}$ for all n sufficiently large. This is a contradiction, so $1 \notin \overline{I}^{\text{diff}}$.

Next, we show that a linear combination of variables which do not appear in I are not in $\overline{I}^{\text{diff}}$. It suffices to check monomial terms. Let $X = \prod_{j\in[z],j\notin\alpha} x_j^{\omega_j}$ where ω_j are positive integers. Suppose that there exists $c \in R$ such that $cX^n \in I^{\langle n\rangle}$ for all n sufficiently large. Consider a monomial term b of c and we can show that $X^n b \notin I^{\langle n\rangle}$ for all n sufficiently large for the same reason. Thus, a linear combination of variables not in $\{x_{\alpha_1}, \dots, x_{\alpha_m}\}$ is not in $\overline{I}^{\text{diff}}$. We showed that $\overline{I}^{\text{diff}} = (x_{\alpha_1}, \dots, x_{\alpha_m})$. □

Remark 4.6. Notice that if we have any irreducible monomial ideal I, then $\overline{I}^{\text{diff}}$ and $I^{\langle n\rangle}$ will be monomial ideals for all n.

Example 4.7. Let $R = \mathbb{Q}[x, y]$, $I = (x^3, y^3)$. We find $I^{\langle 3\rangle}$ by considering differential operators. According to Lemma 3.2, we only need to consider operators which are compositions of partial derivatives. In this case, that means we need to check $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial^2}{\partial x\partial y}, \frac{\partial^2}{\partial x^2}, \frac{\partial^2}{\partial y^2}$. Since constants are not affected by partial derivatives, we check the behavior of these partials on an element x^iy^j.

$$\frac{\partial}{\partial x}(x^iy^j) = ix^{i-1}y^j \qquad \frac{\partial}{\partial y}(x^iy^j) = jx^iy^{j-1}$$
$$\frac{\partial^2}{\partial x^2} = i(i-1)x^{i-2}y^j \qquad \frac{\partial^2}{\partial y^2} = j(j-1)x^iy^{j-2}$$
$$\frac{\partial^2}{\partial x \partial y} = ijx^{i-1}y^{j-1}$$

From these equations, we can see that x^iy^j is in $I^{\langle 3\rangle}$ if $i \geq 5$ or $j \geq 5$ or ($i = 4$ and $j = 3$) or ($i = 3$ and $j = 4$). Thus, $I^{\langle 3\rangle} = (x^5, y^5, x^3y^4, x^4y^3)$, which agrees with Proposition 4.4.

Example 4.8. Let $R = \mathbb{C}[x, y, z]$, $I = (x^2, y^3)$. Applying Proposition 4.4 and Proposition 4.5, we have $I^{\langle 2\rangle} = (x^3, y^4, x^2y^3)$, $I^{\langle 3\rangle} = (x^4, y^5, x^3y^3, x^2y^4)$, and $\overline{I}^{\text{diff}} = (x, y)$.

Example 4.9. Let $R = \mathbb{C}[x, y, z]$, $I = (x^3, y^3, z^5)$. Applying Proposition 4.4 and Proposition 4.5, we have $I^{\langle 2\rangle} = (x^4, y^4, z^6, x^3y^3, x^3z^5, y^3z^5)$, $I^{\langle 3\rangle} = (x^5, y^5, z^7, x^4y^3, x^3y^4, x^4z^5, x^3z^6, y^4z^5, y^3z^6, x^3y^3z^5)$, and $\overline{I}^{\text{diff}} = (x, y, z)$

4.2. **Monomial Ideals.** Since monomial ideals can be written as an intersection of finitely many ideals generated by pure powers of variables by Theorem 4.1, we can use our results in the previous section to study more general monomial ideals.

Theorem 4.10. *Given a monomial ideal I in R, $I^{\langle n\rangle}$ is also a monomial ideal for every positive integer n.*

Proof. By Theorem 4.1, we have
$$I = \bigcap_{i=1}^{m} Q_i.$$
Consider the n-th differential power of I and apply Lemma 3.8,
$$I^{\langle n\rangle} = \left(\bigcap_{i=1}^{m} Q_i\right)^{\langle n\rangle} = \bigcap_{i=1}^{m} {Q_i}^{\langle n\rangle}.$$
Since Q_i is an irreducible monomial ideal for all i, then by Proposition 4.4, ${Q_i}^{\langle n\rangle}$ is a monomial ideal for all i. By Proposition 4.3, we know that $\bigcap_{i=1}^{m} {Q_i}^{\langle n\rangle}$ is a monomial ideal. Thus, $I^{\langle n\rangle}$ is a monomial ideal. □

Since Lemma 3.8 tells us the differential power respects intersections, we are able to compute the n-th differential power of any monomial ideal by breaking it up into irreducible monomial ideals. When I is generated by a single monomial, we can explicitly write the generators of $I^{\langle n\rangle}$ for all n.

Proposition 4.11. *Consider an ideal $I = (\prod_{i\in[m]} x_{\alpha_i}^{\beta_i}) \subset \mathsf{k}[x_1, \ldots, x_z]$ where each β_is are positive integers. Then $I^{\langle n\rangle} = (\prod_{i\in[m]} x_{\alpha_i}^{\beta_i+n-1})$.*

Proof. First notice that $I = \bigcap_{i\in[m]}(x_{\alpha_i}^{\beta_i})$. From Lemma 3.8, we know that differential powers respect finite intersections. Thus,
$$I^{\langle n\rangle} = \bigcap_{i\in[m]} (x_{\alpha_i}^{\beta_i})^{\langle n\rangle}$$
We know how to compute each of these differential powers by Proposition 4.4. This gives us
$$I^{\langle n\rangle} = \bigcap_{i\in[m]} (x_{\alpha_i}^{\beta_i-1+n})$$
Finally, we rewrite this intersection as a single ideal
$$I^{\langle n\rangle} = \left(\prod_{i\in[m]} x_{\alpha_i}^{\beta_i+n-1}\right)$$

□

Now we know that the n-th differential power of a monomial ideal is a monomial ideal, the next question will be whether or not the differential closure of a monomial ideal is a monomial ideal. Since we showed that differential closure commutes with finite intersections in Lemma 3.9, then we can show this property using a very similar proof.

Theorem 4.12. *I is a monomial ideal in R implies that $\overline{I}^{diff}$ is a monomial ideal.*

Proof. By Theorem 4.1, we have

$$I = \bigcap_{i=1}^{m} Q_i$$

where each Q_i is an irreducible monomial ideal. Consider the differential closure of I and apply Lemma 3.9,

$$\overline{I}^{\text{diff}} = \overline{\left(\bigcap_{i=1}^{m} Q_i\right)}^{\text{diff}} = \bigcap_{i=1}^{m} \overline{Q_i}^{\text{diff}}.$$

Since Q_i is an irreducible monomial ideal for all i, then by Proposition 4.5, $\overline{Q_i}^{\text{diff}}$ is a monomial ideal for all i. By Theorem 4.3, we know that $\bigcap_{i=1}^{m} \overline{Q_i}^{\text{diff}}$ is a monomial ideal. Thus, $\overline{I}^{\text{diff}}$ is a monomial ideal. □

Example 4.13. Let $R = \mathbb{C}[x, y, z]$ and $I = (x^2y, xz) = (x) \cap (x^2, z) \cap (y, z)$.

$$\begin{aligned}
I^{\langle 2\rangle} &= (x)^{\langle 2\rangle} \cap (x^2, z)^{\langle 2\rangle} \cap (y, z)^{\langle 2\rangle} \\
&= (x^2) \cap (x^3, z^2, x^2z) \cap (y^2, z^2, yz) \\
&= (x^3y^2, x^2z^2, x^2yz) \\
\overline{I}^{\text{diff}} &= \overline{(x)}^{\text{diff}} \cap \overline{(x^2, z)}^{\text{diff}} \cap \overline{(y, z)}^{\text{diff}} \\
&= (x) \cap (x, z) \cap (y, z) \\
&= (xy, xz).
\end{aligned}$$

4.3. **Containment Problem on Irreducible Monomial Ideals.** This section is inspired by previous work on containment problems of symbolic powers, namely Theorem 2.2 from [ELS01] by Ein, Lazarsfeld, and Smith which states that in the ring $R = \mathsf{k}[x_1, \ldots, x_z]$ where k is a field with characteristic 0, we have $I^{(zn)} \subseteq I^n$ for all positive integer n and radical ideals I. This leads us to ask the same question between differential powers and ordinary power. When is the differential power contained in the ordinary power ($I^{\langle a\rangle} \subseteq I^b$)? When is the differential power containing the ordinary power($I^{\langle a\rangle} \supseteq I^b$)? Since we have an explicit way (Proposition 4.4) to write out the generators of the n-th differential power of an irreducible monomial ideal, then we want to approach these two questions by restricting to I being an irreducible monomial ideal first before generalizing to other ideals. Recall that in the irreducible monomial ideal section, we defined $I = (x_{\alpha_1}^{\beta_1}, x_{\alpha_2}^{\beta_2}, \ldots, x_{\alpha_m}^{\beta_m})$ where each β_is are positive integers. First, we want to study the conditions in which $I^a \subseteq I^{\langle b\rangle}$.

Proposition 4.14. *$I^n \subseteq I^{\langle n\rangle}$.*

Proof. We can write the set of generators of I^n as $\{x_{\alpha_1}^{\beta_1\omega_1} \cdots x_{\alpha_m}^{\beta_m\omega_m} \mid \sum_{i=1}^{m} \omega_i = n, \omega_i \in \mathbb{Z}_{\geq 0} \forall i \in [m]\}$. Let $x_{\alpha_1}^{\beta_1\omega_1} \cdots x_{\alpha_m}^{\beta_m\omega_m}$ be an arbitrary element from I^n, $S \subseteq [m]$ be the set of indexes such that $\omega_j \neq 0$ for all $j \in S$. We can see that this S construction is same as Proposition 4.4 because when $\omega_j = 0$, $\omega_j\beta_j < \beta_j$, and when $\omega_j > 0$, $\omega_j\beta_j \geq \beta_j$. Consider

$$\begin{aligned}\sum_{i\in S}\beta_i\omega_i &= \sum_{i\in S}(\beta_i-1)+|S|+\sum_{i\in S}\beta_i(\omega_i-1)\\ &\geq \sum_{i\in S}(\beta_i-1)+|S|+\sum_{i\in S}(\omega_i-1)\\ &= \sum_{i\in S}(\beta_i-1)+\sum_{i\in S}1+\sum_{i\in S}(\omega_i-1)\\ &= \sum_{i\in S}(\beta_i-1)+\sum_{i\in S}\omega_i\\ &= \sum_{i\in S}(\beta_i-1)+n.\end{aligned}$$

By Proposition 4.4, we know that every element in I^n is in $I^{\langle n\rangle}$. Thus, $I^n \subseteq I^{\langle n\rangle}$. □

Next, we want to find a tighter bound on I^n using the differential power. We have found that $I^n \subseteq I^{\langle n+c\rangle}$ where c depends on n. This c is defined explicitly in the following proposition.

Proposition 4.15. *Fix an arbitrary positive integer n. Let T be the subset of $\mathbb{Z}^m_{\geq 0}$ such that $\boldsymbol{\omega}$ is an element of T if and only if $\sum_{i\in[m]}\boldsymbol{\omega}_i = n$. Then $I^n \subseteq I^{\langle n+c\rangle}$ for all $c \leq \min\{\sum_{i\in[m],\boldsymbol{\omega}_i>0}(\beta_i-1)(\omega_i-1) \mid \boldsymbol{\omega}\in T\}$.*

Proof. With the same construction as the proof of Lemma 4.14, consider

$$\begin{aligned}\sum_{i\in S}\beta_i\omega_i &= \sum_{i\in S}(\beta_i-1)+|S|+\sum_{i\in S}\beta_i(\omega_i-1)\\ &= \sum_{i\in S}(\beta_i-1)+\sum_{i\in S}1+\sum_{i\in S}(\omega_i-1)+\sum_{i\in S}(\beta_i-1)(\omega_i-1)\\ &= \sum_{i\in S}(\beta_i-1)+n+\sum_{i\in S}(\beta_i-1)(\omega_i-1)\\ &\geq \sum_{i\in S}(\beta_i-1)+n+c.\end{aligned}$$

This implies that the generators of I^n are elements of $I^{\langle n+c\rangle}$. Thus, $I^n \subseteq I^{\langle n+c\rangle}$. □

From [ELS01], we have the containment is $I^{(zn)} \subseteq I^n$ for all positive integers n and radical ideals I. Our goal is to find a constant c that depends only on the ring R, such that $I^{\langle cn\rangle} \subseteq I^n$ for some certain type of ideal I. First, we want to simplify the condition by letting I be any irreducible monomial ideal, and we want to find a constant c that depends on n and see if the sequence of c on n converges.

Proposition 4.16. *Let $I = (x_{\alpha_1}^{\beta_1}, x_{\alpha_2}^{\beta_2}, \dots, x_{\alpha_m}^{\beta_m})$. Then $I^{\langle cn^b\rangle} \subseteq I^n$ where $c = \lceil \frac{(n-1)\beta_{\max}+m+1}{n^b}\rceil$ for all positive integers b and $\beta_{\max} = \max\{\beta_i \mid i\in[m]\}$ for all n.*

Proof. By Proposition 4.4, $I^{\langle cn^b\rangle}$ is generated by $\prod_{i\in[m]}x_i^{\gamma_i}$ where γ_i is either 0 or greater than or equal to β_i which induces a subset $S\in\mathcal{P}([m])$ for each generator where $S=\{s\in[m]\mid \gamma_s\geq\beta_s\}$, and the generators satisfy $\sum_{i\in S}\gamma_i = \sum_{i\in S}(\beta_i-1)+cn^b$. We also know that I^n is generated by $\prod_{i\in[m]}x_i^{\beta_i\omega_i}$ where each ω_i is a nonnegative integer and $\sum_{i\in[m]}\omega_i = n$.

We want to show that given any generator $\prod_{i\in[m]}x_i^{\gamma_i}$ from $I^{\langle cn^b\rangle}$, this element is also in I^n. In other words, we want to show that there exists $\omega\in\mathbb{Z}^m_{\geq 0}$ such that $\sum_{i\in[m]}\omega_i = n$, and it satisfies that $\gamma_i \geq \beta_i\omega_i$ for all $i\in S$ where S is the induced set from earlier. Let $\gamma_i = \beta_i+\delta_i$ for every $i\in S$, and let $\beta_{\max} = \max\{\beta_i\mid i\in[m]\}$. Since $\gamma_i\geq\beta_i$ for all $i\in S$, then $\delta_i\geq 0$ for all $i\in S$. By Proposition 4.4, we know that $\sum_{i\in S}\gamma_i = \sum_{i\in S}(\beta_i-1)+cn^b$. This implies that $\sum_{i\in S}\delta_i = cn^b-|S|$. Since $\gamma_i\geq\beta_i\omega_i$ and

$\frac{\gamma_i}{\beta_i} = 1 + \frac{\delta_i}{\beta_i} \geq 1 + \frac{\delta_i}{\beta_{\max}}$, then it is sufficient to show that there exist ω_i's that satisfy $1 + \frac{\delta_i}{\beta_{\max}} \geq \omega_i$. Let $\omega_i = \lceil \frac{\delta_i}{\beta_{\max}} \rceil \leq 1 + \frac{\delta_i}{\beta_{\max}}$, we want to show that $\sum_{i \in S} \omega_i \geq n$. Consider

$$\begin{aligned}
\sum_{i \in S} \omega_i &= \sum_{i \in S} \left\lceil \frac{\delta_i}{\beta_{\max}} \right\rceil \\
&\geq \left\lceil \sum_{i \in S} \frac{\delta_i}{\beta_{\max}} \right\rceil \\
&= \left\lceil \frac{cn - |S|}{\beta_{\max}} \right\rceil \\
&\geq \left\lceil \frac{\frac{(n-1)\beta_{\max}+m+1}{n^b} n^b - |S|}{\beta_{\max}} \right\rceil \\
&= \left\lceil \frac{(n-1)\beta_{\max} + m + 1 - |S|}{\beta_{\max}} \right\rceil \\
&= \left\lceil n - 1 + \frac{m + 1 - |S|}{\beta_{\max}} \right\rceil \\
&\geq n.
\end{aligned}$$

Therefore, we showed $I^{\langle cn^b \rangle} \subseteq I^n$ where $c = \lceil \frac{(n-1)\beta_{\max}+m+1}{n^b} \rceil$ for all n. ☐

Remark 4.17. When $b = 1$, $c = \lceil \frac{(n-1)\beta_{\max}+m+1}{n} \rceil$. Notice that this sequence of c depending on n is converging to $\beta_{\max}$, so we can find a constant c that is only depending on the ideal I.

Theorem 4.18. *$I^{\langle cn \rangle} \subseteq I^n$ where $c = \max\{\beta_1, \ldots, \beta_m, m + 1\}$.*

Proof. By Proposition 4.4, $I^{\langle cn \rangle}$ is generated by $\prod_{i \in [m]} x_i^{\gamma_i}$ where γ_i is either 0 or greater than or equal to β_i which induces a subset $S \subseteq [m]$ for each generator where $S = \{s \in [m] \mid \gamma_s \geq \beta_s\}$, and each generator satisfies $\sum_{i \in S} \gamma_i = \sum_{i \in S}(\beta_i - 1) + cn$. We also know that I^n is generated by $\prod_{i \in [m]} x_i^{\beta_i \omega_i}$ where ω_is are nonnegative integers that satisfies $\sum_{i \in [m]} \omega_i = n$.

We want to show that if $\sum_{i \in S} \gamma_i = \sum_{i \in S}(\beta_i - 1) + cn$ with $c = \max\{\beta_1, \ldots, \beta_m, z + 1\}$, then there exist ω_i's such that $\gamma_i \geq \beta_i \omega_i$ for all $i \in S$ where $\sum_{i \in S} \omega_i = n$. It is sufficient to show there exists a set of ω_i's such that $\sum_{i \in S} \omega_i \geq n$ because we can always decrease the power to equal n. This is equivalent in showing $\frac{\gamma_i}{\beta_i} \geq \omega_i$. Since $\gamma_i \geq \beta_i$ for all $i \in S$ by the construction of S, then $\gamma_i = \beta_i + \delta_i$ for some nonnegative integer x_i for all $i \in S$. This means that $\sum_{i \in S} \delta_i = cn - |S|$. Notice

$$\frac{\gamma_i}{\beta_i} = 1 + \frac{\delta_i}{\beta_i} \geq 1 + \frac{\delta_i}{c} \geq \left\lceil \frac{\delta_i}{c} \right\rceil.$$

Choose each $\omega_i = \lceil \frac{\delta_i}{c} \rceil$, we need to check that $\sum_{i \in S} \omega_i \geq n$. Now we have

$$\sum_{i \in S} \omega_i = \sum_{i \in S} \left\lceil \frac{\delta_i}{c} \right\rceil \geq \left\lceil \sum_{i \in S} \frac{\delta_i}{c} \right\rceil = \left\lceil n - \frac{|S|}{c} \right\rceil = n$$

because $|S| < m + 1 \leq c$. We have shown that each generator of $I^{\langle cn \rangle}$ is in I^n. Thus, $I^{\langle cn \rangle} \subseteq I^n$. ☐

4.4. **Containment Problem on Other Monomial Ideals.** Here are some more results on other monomial ideals. First, the goal is to study whether there can exist a constant c which does not depend on the ideal I such that $I^{\langle cn \rangle} \subseteq I^n$. We can construct a counterexample with a monomial ideal to show that there can't exist such constant c.

Proposition 4.19. *There does not exist a constant c where c is only depending on R such that $I^{\langle cn \rangle} \subseteq I^n$ for all ideals I in R and for all $n \in \mathbb{Z}_{>0}$.*

Proof. Suppose that there exists such c, then we can find this counterexample with $I = (x_1^{c+1})$. By Proposition 4.4, $I^{\langle cn\rangle}$ is generated by the element x_1^{c+cn}. We know that I^n is generated by x_1^{cn+n}. Since $I^{\langle cn\rangle} \subseteq I^n$ implies that $c + cn \geq cn + n$, then we have $c \geq n$ which implies that c depends on n. This contradicts with the assumption that c only depend on the ring R. Therefore, such c does not exist. $\square$

On the other hand, we can also use the same counterexample to show the following result.

Proposition 4.20. *The only constant c such that $I^n \subseteq I^{\langle cn\rangle}$ for all ideal I in R and for all $n \in \mathbb{Z}_{>0}$ is $c = 1$.*

Proof. Suppose that there exists such c where $c > 1$, then we have the counterexample $I = (x_1^{c+1})$. By Proposition 4.4, $I^{\langle cn\rangle}$ is generated by the element x_1^{c+cn}. We know that I^n is generated by x_1^{cn+n}. Since $I^n \subseteq I^{\langle cn\rangle}$ implies that $cn + n \geq c + cn$, then we have $n \geq c$ for all $n \in \mathbb{Z}_{>0}$ which implies that $n > 1$. Therefore, only $c = 1$ satisfies that condition. $\square$

Since we can't find a constant c which does not depend on I or on n such that $I^{\langle cn\rangle} \subseteq I^n$, then the next idea would be considering if the statement is true for $I^{\langle cn^2\rangle} \subseteq I^n$, more generally whether does the statement still hold for $I^{\langle p(n)\rangle} \subseteq I^n$ where $p(n)$ is a polynomial with constant coefficient which does not depend on I and with variable n. Now we have this new statement.

Proposition 4.21. *There does not exist a polynomial $p(n) = c_m n^m + c_{m-1}n^{m-1} + \ldots + c_1 n + c_0$ with a set of constant positive integers c_i and a nonnegative integer m such that $I^{\langle p(n)\rangle} \subseteq I^n$ for all $n \in \mathbb{Z}_{\geq 0}$.*

Proof. Suppose that there exists a polynomial $p(n)$ such that $I^{\langle p(n)\rangle} \subseteq I^n$, then we have the counterexample $I = (x_1^{\beta})$ where $\beta = p(2)$. By Proposition 4.4, we have this inequality,

$$\begin{aligned} \beta - 1 + p(n) &\geq \beta n \\ p(n) &\geq \beta(n-1) + 1. \end{aligned}$$

Notice that when $n = 2$ the inequality is $p(2) \geq p(2) + 1$ which leads to a contradiction. Therefore, there does not exist a polynomial $p(n)$ such that $I^{\langle p(n)\rangle} \subseteq I^n$ for all $n \in \mathbb{Z}_{\geq 0}$. $\square$

The natural next step is finding a value of c such that $I^{\langle cn\rangle} \subseteq I^n$, where c necessarily depends on I.

Proposition 4.22. *Let α be an ordered subset of $\{1, 2, \ldots, z\} = [z]$ with $|\alpha| = m \leq z$ where $z, m \in \mathbb{Z}_{>0}$. Denote the i-th element in the set α as α_i. Consider an ideal I of R where $I = (\prod_{i\in[m]} x_{\alpha_i}^{\beta_i})$ where each β_is are positive integers. $I^{\langle cn\rangle} \subseteq I^n$ where $c = \frac{(n-1)\beta_{\max}+1}{n}$ and $\beta_{\max} = \max\{\beta_i \mid \forall i \in [m]\}$ for all n.*

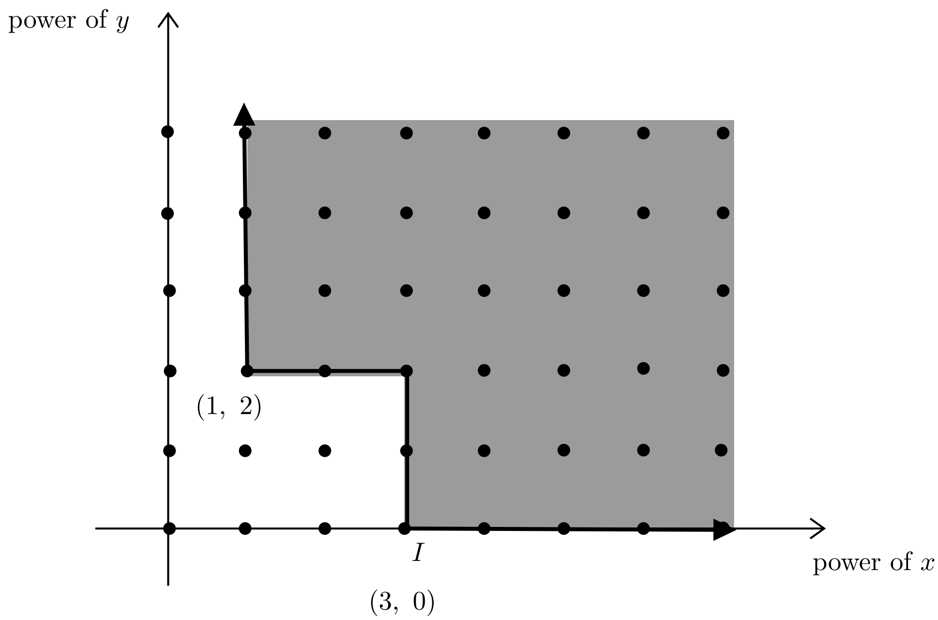
Proof. By Proposition 4.11, $I^{\langle cn\rangle} = (\prod_{i\in[m]} x_{\alpha_i}^{\beta_i+cn-1})$, and since $I^n = (\prod_{i\in[m]} x_{\alpha_i}^{\beta_i n})$, we want to show that $\prod_{i\in[m]} x_{\alpha_i}^{\beta_i+cn-1} \in I^n$. This is equivalent in showing that $\beta_i + cn - 1 \geq \beta_i n$ for all $i \in [m]$. Consider

$$\begin{aligned} \beta_i + cn - 1 &= \beta_i + \frac{(n-1)\beta_{\max}+1}{n} n - 1 \\ &= \beta_i + (n-1)\beta_{\max} \\ &\geq \beta_i n. \end{aligned}$$

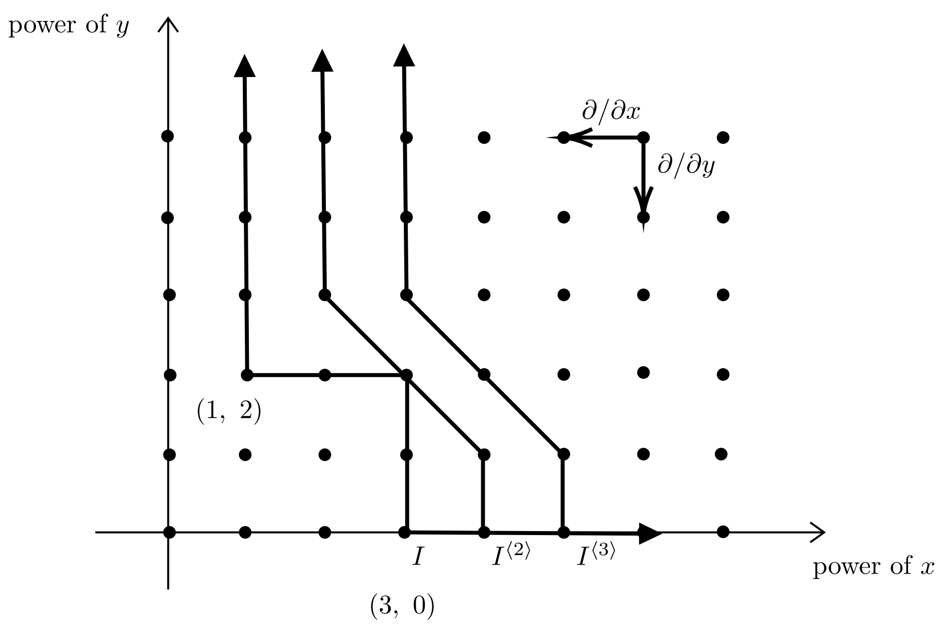
Thus, $I^{\langle cn\rangle} \subseteq I^n$. $\square$

Remark 4.23. Take the same ideal I from Proposition 4.22, $I^{\langle \beta_{\max} n\rangle} \subseteq I^n$ for all $n \in \mathbb{Z}_{>0}$.

4.5. **Differential Powers which are Eventually Principal.** In this section, we let the ring $R = \mathsf{k}[x, y]$. Notice that we can create a graphical representation for every monomial ideal in R using an integer lattice where the x-axis is the power of x and y-axis is the power of y. For example, the point $(2, 3)$ would represent the set of monomials cx^2y^3 where $c \in \mathsf{k}$ and the ideal $I = (xy^2, x^3)$ would be all the points inside the shaded area.



Taking the partial derivative $\partial/\partial y$ would be a translation down in the y direction, and taking the partial derivative $\partial/\partial x$ would be a translation left in the x direction. Now we can also represent the differential powers of monomial ideals graphically. For example, the 2nd and 3rd differential power of the ideal $I = (xy^2, x^3)$ can be represented as



For the rest of this section, let $I = (x^{\beta_{11}}y^{\beta_{12}}, \dots, x^{\beta_{m1}}y^{\beta_{m2}})$.

Remark 4.24. Notice that we can rearrange the generators of I to satisfy the ineqality $\beta_{11} < \cdots < \beta_{m1}$ because $\beta_{i1} = \beta_{(i+1)1}$ implies that one of them is redundant. This inequality implies another ineqality

$\beta_{12} > \cdots > \beta_{m2}$ because $\beta_{j2} \leq \beta_{(j+1)2}$ and $\beta_{j1} < \beta_{(j+1)1}$ implies that $x^{\beta_{(j+1)1}}y^{\beta_{(j+1)2}}$ is redundant. For the rest of this section, we assume that the ideal I has this structure that satisfies both inequalities.

We claim that we can find a positive integer N for all such ideal I such that $I^{\langle N\rangle}$ is principal. We need to first show a lemma.

Lemma 4.25. *Let $\beta_{(i+1)1}, \beta_{i2}$ be the pair of βs such that $\beta_{(i+1)1} + \beta_{i2} = \max\{\beta_{(j+1)1} + \beta_{j2} \mid 1 \leq j < m\}$. If $a + b \geq \beta_{(i+1)1} + \beta_{i2} - 1$ and $a \geq \beta_{11}$, $b \geq \beta_{m2}$, then $x^a y^b \in I$.*

Proof. Suppose there exists such a, b satisfying the condition, and $x^a y^b \notin I$. Since $a \geq \beta_{11}$ and $b \geq \beta_{m2}$, then there exists an index k such that $\beta_{k1} \leq a < \beta_{(k+1)1}$ and $\beta_{k2} > b$. Since $\beta_{k1}, \beta_{(k+1)1}, \beta_{k2}, a, b$ are positive integers, then we have $\beta_{(k+1)1} \geq a + 1$ and $\beta_{k2} \geq b + 1$. This implies that $\beta_{(k+1)} + \beta_{k2} \geq a + b + 2 \geq \beta_{(i+1)1} + \beta_{i2} + 1$ which contradicts the assumption that $\beta_{(i+1)1} + \beta_{i2} = \max\{\beta_{(j+1)1} + \beta_{j2} \mid 1 \leq j < m\}$. Thus, $x^a y^b \in I$. $\square$

Theorem 4.26. *For all such ideals I, there exists a positive integer N such that $I^{\langle N\rangle}$ is principal.*

Proof. If $m = 1$, then by Proposition 4.11, $I^{\langle n\rangle}$ is always a principal monomial ideal for all positive integer n. Now assume $m > 1$. Let $\beta_{(i+1)1}, \beta_{i2}$ be a pair of βs such that $\beta_{(i+1)1} + \beta_{i2} = \max\{\beta_{(j+1)1} + \beta_{j2} \mid 1 \leq j < m\}$. We want to show that $I^{\langle N\rangle}$ is principal where $N = \beta_{(i+1)1} + \beta_{i2} - \beta_{m2} - \beta_{11}$. Let $a = \beta_{(i+1)1} + \beta_{i2} - \beta_{m2} - 1$, and $b = \beta_{(i+1)1} + \beta_{i2} - \beta_{11} - 1$. We claim that $I^{\langle N\rangle} = (x^a y^b)$. We want to show that $x^a y^b$ is an element of $I^{\langle N\rangle}$ and $x^{a-1}y^e$ and $x^f y^{b-1}$ are not elements of $I^{\langle N\rangle}$ for all positive integer e, f. To show that $x^a y^b \in I^{\langle N\rangle}$, it is sufficient to check the differential operators $D = \frac{\partial^{N-1}}{\partial x^d \partial y^{N-1-d}}$ for all nonnegative integers d less than or equal to $N-1$ because $\frac{\partial^{N-1}}{\partial x^{N-1}}(x^a y^b), \frac{\partial^{N-1}}{\partial y^{N-1}}(x^a y^b) \notin \mathsf{k}$. Since $\beta_{12} > \beta_{m2}$ and $\beta_{11} < \beta_{m1}$, then $N > 0$. Notice

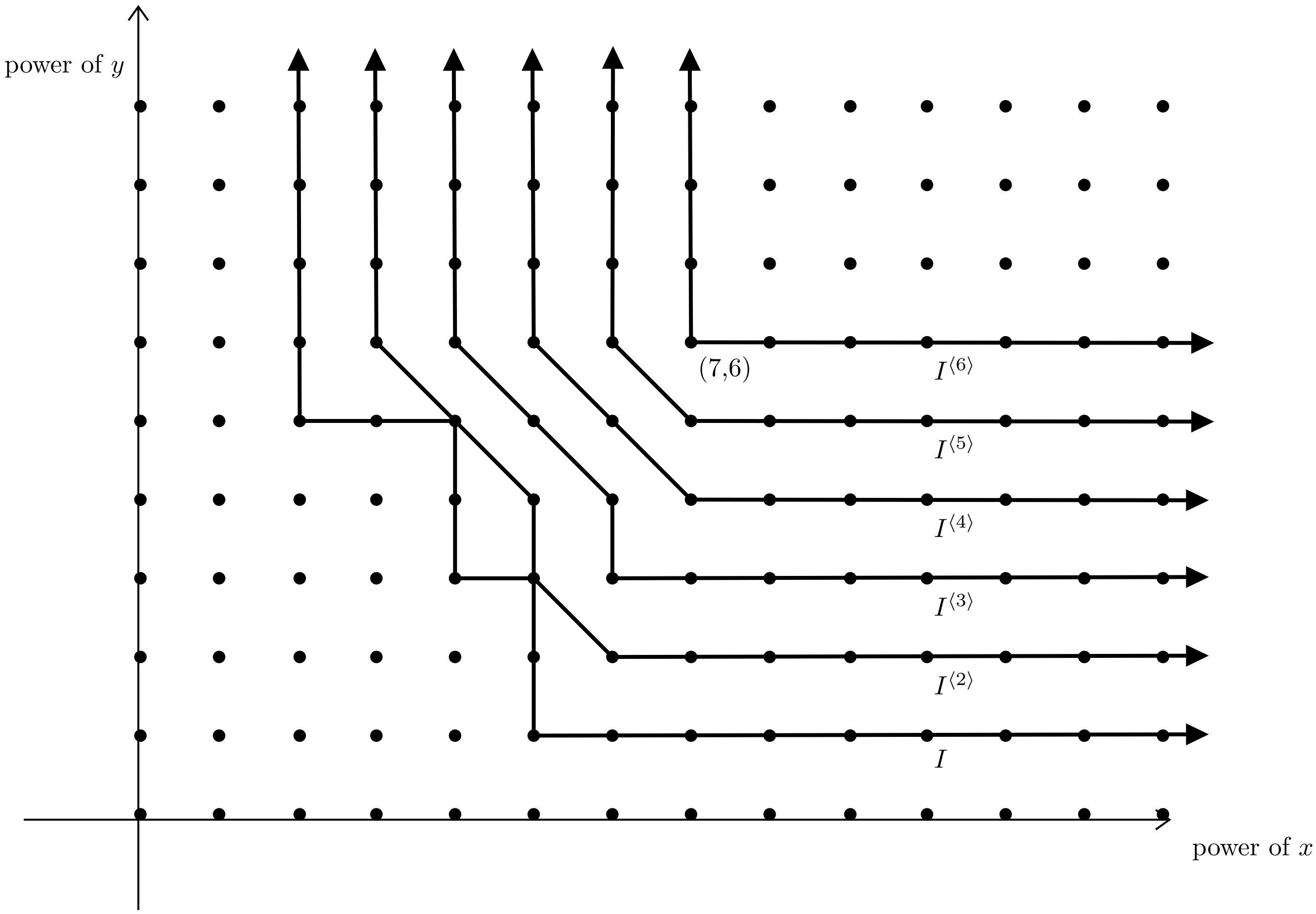
$$\begin{aligned} D(x^a y^b) &= \frac{\partial^{N-1}}{\partial x^d \partial y^{N-1-d}}(x^a y^b) \\ &= c_1 x^{a-d} c_2 y^{b-N+1+d} \\ &= c_1 c_2 x^{a-d} y^{\beta_{m2}+d} \end{aligned}$$

where c_1, c_2 are some elements in k. Now we need to check when $0 \leq d \leq N-1$. By Lemma 4.25, since $a - d + \beta_{m2} + d = a + \beta_{m2} = \beta_{(i+1)1} + \beta_{i2} - 1$ and $a - d \geq a - N + 1 = \beta_{11}$, and $\beta_{m2} + d \geq \beta_{m2}$, then we know that $D(x^a y^b) \in I$.

Since $\frac{\partial^{N-1}}{\partial x^{N-1}}(x^{a-1}y^e) \notin I$ and $\frac{\partial^{N-1}}{\partial y^{N-1}}(x^f y^{b-1}) \notin I$ for all positive integers e, f which implies that $x^{a-1}y^e, x^f y^{b-1} \notin I^{\langle N\rangle}$ for all positive integers e, f, then $I^{\langle N\rangle} = (x^a y^b)$. This shows that there exists a positive integer N such that $I^{\langle N\rangle}$ is principal. $\square$

Remark 4.27. Not only we know that there exists an integer N, we know that $N = \beta_{(i+1)1} + \beta_{i2} - \beta_{m2} - \beta_{11}$ where $\beta_{(i+1)1}$ and β_{i2} is a pair of β such that $\beta_{(i+1)1} + \beta_{i2} = \max\{\beta_{(j+1)1} + \beta_{j2} \mid 1 \leq j < m\}$.

Example 4.28. Let $I = (x^2y^5, x^4y^3, x^5y)$, then we know that $I^{\langle N\rangle}$ is principal where $N = 5+4-1-2 = 6$. This means that $I^{\langle 6\rangle} = (x^7y^6)$. We can check the result with the integer lattice



Notice that from our example, N is the smallest n such that $I^{\langle n\rangle}$ is principal. Here we can prove that N is always the smallest n such that $I^{\langle n\rangle}$ is principal for all I.

Proposition 4.29. *Let $I = (x^{\beta_{11}}y^{\beta_{12}}, \dots, x^{\beta_{m1}}y^{\beta_{m2}})$ which follows the ordering of β's specified at the beginning of the section. Let $\beta_{(i+1)1}$ and β_{i2} be a pair of β such that $\beta_{(i+1)1} + \beta_{i2} = \max\{\beta_{(j+1)1} + \beta_{j2} \mid 1 \leq j < m\}$, then $N = \beta_{(i+1)1} + \beta_{i2} - \beta_{m2} - \beta_{11}$ is the smallest n such that $I^{\langle n\rangle}$ is principal.*

Proof. This is equivalent in showing that $I^{\langle N-1\rangle}$ is not principal because by Proposition 4.11 $I^{\langle m\rangle}$ is principal implies that $I^{\langle m+1\rangle}$ is also principal. Let $a = \beta_{(i+1)1} + \beta_{i2} - \beta_{m2} - 1$, and $b = \beta_{(i+1)1} + \beta_{i2} - \beta_{11} - 1$, we can show that $x^{a-1}y^b, x^a y^{b-1} \in I^{\langle N-1\rangle}$ and $x^{a-1}y^{b-1} \notin I^{\langle N-1\rangle}$ which implies that $I^{\langle N-1\rangle}$ is not principal. By Lemma 3.2, we only need to check the differential operators of the form $\frac{\partial^d}{\partial x^{d_1}\partial y^{d_2}}$ where $d \leq N-2$ and $d_1 + d_2 = d$. Now consider

$$\begin{aligned} a - 1 - d_1 + b - d_2 &\geq a + b - 1 - N + 2 \\ &= \beta_{(i+1)1} + \beta_{i2} - 1. \end{aligned}$$

By Lemma 4.25, $\frac{\partial^d}{\partial x^{d_1}\partial y^{d_2}}(x^{a-1}y^b) \in I$ which implies that $x^{a-1}y^b \in I^{\langle N-1\rangle}$. Now consider

$$\begin{aligned} a - d_1 + b - 1 - d_2 &\geq a + b - 1 - N + 2 \\ &= \beta_{(i+1)1} + \beta_{i2} - 1. \end{aligned}$$

By Lemma 4.25, $\frac{\partial^d}{\partial x^{d_1}\partial y^{d_2}}(x^a y^{b-1}) \in I$ which implies that $x^a y^{b-1} \in I^{\langle N-1\rangle}$. Since

$$\frac{\partial^{N-2}}{\partial x^{\beta_{i2}-\beta_{m2}-1}\partial y^{\beta_{(i+1)1}-\beta_{11}-1}}(x^{a-1}y^{b-1}) = cx^{\beta_{(i+1)1}-1}y^{\beta_{i2}-1} \notin I,$$

then $x^{a-1}y^{b-1} \notin I^{\langle N-1\rangle}$. Since $I^{\langle N-1\rangle}$ is not principal and $I^{\langle N\rangle}$ is principal, then $N = \beta_{(i+1)1}+\beta_{i2}-\beta_{m2}-\beta_{11}$ is the smallest n such that $I^{\langle n\rangle}$ is principal. □

From Remark 4.23, we know the rule of containment for the ideals generated by single monomials. We can apply Theorem 4.26 to find the containment for the special type of monomial ideal I.

Proposition 4.30. *Let $\beta_{(i+1)1}, \beta_{i2}$ be a pair of βs such that $\beta_{(i+1)1}+\beta_{i2} = \max\{\beta_{(j+1)1}+\beta_{j2} \mid 1 \geq j < m\}$. Define $a = \beta_{(i+1)1} + \beta_{i2} - \beta_{m2} - 1$, $b = \beta_{(i+1)1} + \beta_{i2} - \beta_{11} - 1$, $N = \beta_{(i+1)1} + \beta_{i2} - \beta_{m2} - \beta_{11}$, and $\beta_{\max} = \max\{a, b\}$. Then $I^{\langle \beta_{\max}n+N-1\rangle} \subseteq I^n$*

Proof. By Theorem 4.26, $I^{\langle N\rangle}$ is a principal monomial ideal generated by $x^a y^b$. By Remark 4.23, we have ${I^{\langle N\rangle}}^{\langle \beta_{\max}n\rangle} \subseteq (I^{\langle N\rangle})^n$. Since $I^{\langle N\rangle} \subseteq I$, then we have $(I^{\langle N\rangle})^n \subseteq I^n$ which implies that ${I^{\langle N\rangle}}^{\langle \beta_{\max}n\rangle} \subseteq I^n$. By Proposition 3.10, we have ${I^{\langle N\rangle}}^{\langle \beta_{\max}n\rangle} = I^{\langle \beta_{\max}n+N-1\rangle}$. Thus, by transitivity of containment $I^{\langle \beta_{\max}n+N-1\rangle} \subseteq I^n$. □

4.6. **Differential Powers which are Eventually Principal in 3 variables.** In this section, we let the ring $R = \mathsf{k}[x, y, z]$, $I = (x^{\beta_{11}}y^{\beta_{12}}z^{\beta_{13}}, \ldots, x^{\beta_{m1}}y^{\beta_{m2}}z^{\beta_{m3}})$ with some positive integer m and each β_{ij} is positive.

Theorem 4.31. *For all such ideal I, there exists a positive integer N such that $I^{\langle N\rangle}$ is principal.*

Proof. Let

$$\beta_{xmax} = \max\{\beta_{i1} \mid i \in [m]\} \qquad \beta_{ymax} = \max\{\beta_{i2} \mid i \in [m]\} \qquad \beta_{zmax} = \max\{\beta_{i3} \mid i \in [m]\}$$
$$\beta_{xmin} = \min\{\beta_{i1} \mid i \in [m]\} \qquad \beta_{ymin} = \min\{\beta_{i2} \mid i \in [m]\} \qquad \beta_{zmin} = \min\{\beta_{i3} \mid i \in [m]\}$$

Define

$$\begin{aligned} N_{xy} &= \beta_{xmax} + \beta_{ymax} - \beta_{xmin} - \beta_{ymin} \\ N_{yz} &= \beta_{ymax} + \beta_{zmax} - \beta_{ymin} - \beta_{zmin} \\ N_{xz} &= \beta_{xmax} + \beta_{zmax} - \beta_{xmin} - \beta_{zmin} \end{aligned}$$

Let $N = \max\{N_{xy}, N_{yz}, N_{xz}\}$ and

$$a = N - 1 + \beta_{xmin} \qquad b = N - 1 + \beta_{ymin} \qquad c = N - 1 + \beta_{zmin}$$

We need to show that $x^a y^b z^c$ is an element of $I^{\langle N\rangle}$ and $x^{a-1}y^d z^e$, $x^d y^{b-1} z^e$, $x^d y^e z^{c-1} \notin I^{\langle N\rangle}$ for all positive integer d, e. Since $\frac{\partial^{N-1}}{\partial x^{N-1}} x^a y^b z^c = C_1 x^{\beta_{xmin}} y^b z^c \notin \mathsf{k}$, $\frac{\partial^{N-1}}{\partial y^{N-1}} x^a y^b z^c = C_2 x^a y^{\beta_{ymin}} z^c \notin \mathsf{k}$, and $\frac{\partial^{N-1}}{\partial z^{N-1}} x^a y^b z^c = C_3 x^a y^b z^{\beta_{zmin}} \notin \mathsf{k}$, then it is sufficient to check all of the differential operators in the form $\frac{\partial^{N-1}}{\partial x^{k_1}\partial y^{k_2}\partial z^{k_3}}$ where $k_1 + k_2 + k_3 = N - 1$. Now suppose that $x^{a-k_1}y^{b-k_2}z^{c-k_3} \notin I$ which is equivalent in $x^a y^b z^c \notin I^{\langle N\rangle}$. Since any two of $a - k_1 \geq \beta_{xmax}, b - k_2 \geq \beta_{ymax}, c - k_3 \geq \beta_{zmax}$ being true implies $x^{a-k_1}y^{b-k_2}z^{c-k_3} \in I$, then by the contrapositive $x^{a-k_1}y^{b-k_2}z^{c-k_3} \notin I$ implies that two of $a - k_1 < \beta_{xmax}, b - k_2 < \beta_{ymax}, c - k_3 < \beta_{zmax}$ are true. Without the loss of generality, let $a - k_1 < \beta_{xmax}$ and $b - k_2 < \beta_{ymax}$. This implies that $a - k_1 \leq \beta_{xmax} - 1$ and $b - k_2 \leq \beta_{ymax} - 1$. This means that $a - k_1 + b - k_2 \leq \beta_{xmax} + \beta_{ymax} - 2$. However, we know that

$$\begin{aligned} a - k_1 + b - k_2 &\geq a + b - k_1 - k_2 - k_3 \\ &= a + b - N + 1 \\ &= 2N - 2 + \beta_{xmin} + \beta_{ymin} - N + 1 \\ &= N - 1 + \beta_{xmin} + \beta_{ymin} \\ &\geq \beta_{xmax} + \beta_{ymax} - 1. \end{aligned}$$

This is a contradiction, so $x^{a-k_1}y^{b-k_2}z^{c-k_3} \in I$ which implies that $x^a y^b z^c \in I^{\langle N\rangle}$.

Since $\frac{\partial^{N-1}}{\partial x^{N-1}} x^{a-1}y^d z^e = C_1 x^{\beta_{xmin}-1} y^d z^e \notin I$, $\frac{\partial^{N-1}}{\partial y^{N-1}} x^d y^{b-1} z^e = C_2 x^d y^{\beta_{ymin}-1} z^e \notin I$, and $\frac{\partial^{N-1}}{\partial z^{N-1}} x^d y^e z^{c-1} = C_3 x^d y^e z^{\beta_{zmin}-1} \notin I$, then we know that $I^{\langle N \rangle} = (x^a y^b z^c)$. □

We conclude this subsection with a list of examples with omitted computations.

Example 4.32.

(1) $I = (x^5y^4z, x^2yz^{10}, xy^7z^3)$. $N = 15$, smallest actual $N = 12$. $I^{\langle 12 \rangle} = (x^{12}y^{12}z^{12})$.
(2) $I = (x^4yz, xy^4z, xyz^4)$. $N = 6$, smallest actual $N = 5$. $I^{\langle 5 \rangle} = (x^5y^5z^5)$.
(3) $I = (x^2y^7z^3, x^7y^2z^5, xy^5z^7)$. $N = 11$, smallest actual $N = 9$. $I^{\langle 9 \rangle} = (x^9y^{10}z^{11})$.
(4) $I = (x^7y^9z^3, x^5y^5z^7, x^{10}y^4z^8)$. $N = 10$, smallest actual $N = 9$. $I^{\langle 9 \rangle} = (x^{13}y^{12}z^{11})$.

Remark 4.33. Notice that this N that we get from Theorem 4.31 is not the smallest n such that $I^{\langle n \rangle}$ is principal.

Remark 4.34. Define $I_{xy} = (x^{\beta_{11}}y^{\beta_{12}}, \dots, x^{\beta_{m1}}y^{\beta_{m2}})$, $I_{xz} = (x^{\beta_{11}}z^{\beta_{13}}, \dots, x^{\beta_{m1}}z^{\beta_{m3}})$, and $I_{yz} = (y^{\beta_{12}}z^{\beta_{13}}, \dots, y^{\beta_{m2}}z^{\beta_{m3}})$. To study the ideal I, it is not sufficient to study only the ideals I_{xy}, I_{xz}, I_{yz}. For example, take $I = (x^3y^2z^7, x^5y^2z^2, x^2y^6z^3, x^4y^5z^4)$ and the point $x^4y^4z^5$. Notice that $x^4y^4 \in I_{xy}$, $x^4z^5 \in I_{xz}$, and $y^4z^5 \in I_{yz}$, but $x^4y^4z^5 \notin I$.

5. General Results

5.1. **Results on More General Ideals.** To begin studying results on more general ideals, we can consider ideals which are isomorphic to monomial ideals through some k-algebra homomorphism. First, we look at a special type of isomorphism.

Lemma 5.1. *Let k be a field of characteristic zero, and φ be a ring isomorphism from $R = \mathsf{k}[x_1, \dots, x_z]$ to $S = \mathsf{k}[a_1, \dots, a_z]$ where φ is defined by $\varphi(x_i) = c_{i1}a_i + c_{i2}$ for all i where c_{i1}, c_{i2} are elements of k. Then,*

$$\varphi\left(\left(\sum_{i=1}^{d} \mathbf{D}^{\alpha_i}\right) r\right) = \sum_{i=1}^{d} \left(\frac{\mathbf{D}'}{\mathbf{C}}\right)^{\alpha_i} (\varphi(r))$$

where $\mathbf{D} = \prod_{j=1}^{z} \frac{\partial}{\partial x_j}$, $\mathbf{D}' = \prod_{j=1}^{z} \frac{\partial}{\partial a_j}$, $\alpha_i \in \mathbb{Z}^z_{\geq 0}$ and $\sum_{j=1}^{z} \alpha_{ij} < n$ for all $i \in [d]$, $\mathbf{C} = \prod_{j=1}^{z} c_{j1}$. Let $\mathbf{D}^{\alpha_i}$ denote $\prod_{j=1}^{z} \frac{\partial^{\alpha_{ij}}}{\partial x_j^{\alpha_{ij}}}$.

Proof. Since φ is a ring homomorphism which preserves addition, this means that we only need to check

$$\varphi\left(\mathbf{D}^{\alpha} \cdot r\right) = \left(\frac{\mathbf{D}'}{\mathbf{C}}\right)^{\alpha} (\varphi(r))$$

where $\mathbf{D} = \prod_{j=1}^{z} \frac{\partial}{\partial x_j}$, $\mathbf{D}' = \prod_{j=1}^{z} \frac{\partial}{\partial a_j}$, $\alpha \in \mathbb{Z}^z_{\geq 0}$ and $\sum_{j=1}^{z} \alpha_j < n$, $\mathbf{C} = \prod_{j=1}^{z} c_{j1}$.

Since $r \in R$, then we can write $r = \sum_{i=1}^{m} u_i \mathbf{x}^{\beta_i}$ for some positive integer m, where $\mathbf{x} = x_1 \cdots x_z$, $u_i \in \mathsf{k}$, and $\beta_i \in \mathbb{Z}^z_{\geq 0}$ for all i. Since φ is a ring homomorphism which preserves addition, then we only need to check

$$\varphi\left(\mathbf{D}^{\alpha} \cdot u\mathbf{x}^{\beta}\right) = \left(\frac{\mathbf{D}'}{\mathbf{C}}\right)^{\alpha} (\varphi(u\mathbf{x}^{\beta}))$$

where $\mathbf{x} = x_1 \cdots x_z$, $u \in \mathsf{k}$, and $\beta \in \mathbb{Z}^z_{\geq 0}$. If there exists an index $j \in [k]$ such that $\alpha_j > \beta_j$, then $\mathbf{D}^{\alpha} \cdot u\mathbf{x}^{\beta} = 0$ which means the equality holds, so we can assume that $\alpha_j \leq \beta_j$ for all $j \in [z]$. Now consider the left hand side

$$\begin{aligned}\varphi\left(\mathbf{D}^{\alpha} \cdot u\mathbf{x}^{\beta}\right) &= \varphi\left(u \prod_{i=1}^{z} \frac{\beta_i!}{(\beta_i - \alpha_i)!} x_i^{\beta_i - \alpha_i}\right) \\ &= u \prod_{i=1}^{z} \frac{\beta_i!}{(\beta_i - \alpha_i)!} (c_{i1}a_i + c_{i2})^{\beta_i - \alpha_i},\end{aligned}$$

and the right hand side is

$$\begin{aligned}\left(\frac{\mathbf{D}'}{\mathbf{C}}\right)^{\alpha}(\varphi(u\mathbf{x}^{\beta})) &= \frac{\mathbf{1}}{\mathbf{C}^{\alpha}}\mathbf{D}'^{\alpha}u\prod_{i=1}^{z}(c_{i1}a_i+c_{i2})^{\beta_i}\\ &= \frac{\mathbf{1}}{\mathbf{C}^{\alpha}}u\prod_{i=1}^{z}\frac{\beta_i!}{(\beta_i-\alpha_i)!}c_{i1}^{\alpha_i}(c_{i1}a_i+c_{i2})^{\beta_i-\alpha_i}\\ &= \frac{\mathbf{1}}{\mathbf{C}^{\alpha}}u\mathbf{C}^{\alpha}\prod_{i=1}^{z}\frac{\beta_i!}{(\beta_i-\alpha_i)!}(c_{i1}a_i+c_{i2})^{\beta_i-\alpha_i}\\ &= u\prod_{i=1}^{z}\frac{\beta_i!}{(\beta_i-\alpha_i)!}(c_{i1}a_i+c_{i2})^{\beta_i-\alpha_i}\\ &= \varphi\left(\mathbf{D}^{\alpha}\cdot u\mathbf{x}^{\beta}\right).\end{aligned}$$

Thus, $\varphi\left(\mathbf{D}^{\alpha}\cdot r\right) = (\frac{\mathbf{D}'}{\mathbf{C}})^{\alpha}(\varphi(r))$. □

Using this lemma, we can make a statement about differential powers of ideals of the form $\varphi(I)$ for φ of the form above.

Proposition 5.2. *Let φ be a ring isomorphism and k-algebra isomorphism from $R = k[x_1, x_2, \dots, x_z]$ to $S = k[a_1, a_2, \dots, a_z]$ defined by $\varphi(x_i) = c_{i1}a_i + c_{i2}$, where k is a field of characteristic zero. $\varphi(I)^{\langle n\rangle} = \varphi(I^{\langle n\rangle})$.*

Proof. We want to show that $\varphi(I)^{\langle n\rangle} \supseteq \varphi(I^{\langle n\rangle})$. Take an element r from $I^{\langle n\rangle}$, by definition, we have $D\cdot r \in I$ for all differential operators D with order less than n. Let D be a composition of partial derivatives, $D = \prod_{j=1}^{z}\frac{\partial^{\alpha_j}}{\partial x_j^{\alpha_j}}$ for $\alpha = (\alpha_1, \dots, \alpha_z) \in \mathbb{N}^z$ such that $\sum_{j=1}^{z}\alpha_j < n$. Since φ is an isomorphism, then we have $\varphi(D\cdot r) \in \varphi(I)$. By Lemma 5.1, we know that

$$\varphi(D\cdot r) = \frac{1}{C}D'\cdot\varphi(r) \in \varphi(I)$$

where $D' = \prod_{j=1}^{z}\frac{\partial^{\alpha_j}}{\partial^{\alpha_j}a_j}$, and $C = \prod_{j=1}^{z}c_{j1}$.

This implies that $C\cdot\frac{1}{C}D'\cdot\varphi(r) = D'\cdot\varphi(r) \in \varphi(I)$ because $\varphi(I)$ is an ideal. By Lemma 3.2, we know that D' is any differential operator in the ring S of order less than n. By definition, $\varphi(r) \in \varphi(I)^{\langle n\rangle}$. Thus, $\varphi(I)^{\langle n\rangle} \supseteq \varphi(I^{\langle n\rangle})$.
Now we want to show that $\varphi(I)^{\langle n\rangle} \subseteq \varphi(I^{\langle n\rangle})$. Take an element $s \in \varphi(I)^{\langle n\rangle}$, by definition, we have $D\cdot s \in \varphi(I)$ for all differential operators D in S with order less than n. Since φ is an isomorphism, then there exists a ring isomorphism ψ such that ψ is the inverse of φ which is defined by

$$\psi(a_i) = \frac{1}{c_{i1}}x_i + \left(-\frac{c_{i2}}{c_{i1}}\right)$$

for all $i \in [z]$. By Lemma 5.1, we know that

$$\psi(D\cdot s) = CD'\cdot\psi(s) \in \psi(\varphi(I)) = I.$$

This implies that $D'\cdot\psi(s) \in I$ because I is an ideal. Then by Lemma 3.2, $\psi(s) \in I^{\langle n\rangle}$.
Since we showed both ways of containment, we have $\varphi(I)^{\langle n\rangle} = \varphi(I^{\langle n\rangle})$. □

This is especially important since it allows us to explicitly write out the differential power for non-monomial ideals which are isomorphic to a monomial ideal under a mapping satisfying the properties in the proposition.

Example 5.3. Let $R = \mathbb{Q}[a, b, c]$ and $I = ((a+1)^3, (b-3)^5)$. Consider the map φ from $\mathbb{Q}[x, y, z]$ to R which takes x to $a+1$, y to $b-3$, and z to c. This map is a ring isomorphism and k-algebra homomorphism. By Proposition 5.2, we can say $I^{\langle n\rangle} = \varphi((x, y))^{\langle n\rangle} = \varphi((x, y)^{\langle n\rangle})$. Performing this calculation gives us $I^{\langle n\rangle} = ((a+1)^{n+2}, (b-3)^{n+4}, \{(a+1)^i(b-3)^j : i+j = n+6\})$.

We can also show that, in general, ring isomorphisms which are also k-algebra homomorphisms respect differential operators and their order.

Proposition 5.4. *For any differential operator D of order η and any ring isomorphism and k-algebra homomorphism φ, $\Psi = \varphi \circ D \circ \varphi^{-1}$ is a differential operator of order η.*

Proof. Proof by Induction.

Base case, $\eta = 1$. We want to check that $[a, \Psi]$ has order zero for all $a \in R$.

$$\begin{aligned}
[a, \Psi](f) &= (a \circ \Psi)(f) - (\Psi \circ a)(f) \\
&= (a \circ \varphi \circ D \circ \varphi^{-1})(f) - (\varphi \circ D \circ \varphi^{-1} \circ a)(f) \\
&= (a \circ \varphi \circ D \circ \varphi^{-1})(f) - (\varphi \circ D \circ \varphi^{-1})(af) \\
&= (a \circ \varphi \circ D \circ \varphi^{-1})(f) - (\varphi \circ D)(\varphi^{-1}(a) \cdot \varphi^{-1}(f)) \\
&= (a \circ \varphi \circ D \circ \varphi^{-1})(f) - \varphi(\varphi^{-1}(a) \cdot (D \circ \varphi^{-1})(f) + (D \circ \varphi^{-1})(a) \cdot \varphi^{-1}(f)) \\
&= (a \circ \varphi \circ D \circ \varphi^{-1})(f) - (a \cdot (\varphi \circ D \circ \varphi^{-1})(f) + (\varphi \circ D \circ \varphi^{-1})(a) \cdot f) \\
&= -(\varphi \circ D \circ \varphi^{-1})(a) \cdot f
\end{aligned}$$

This last line is a ring element $-(\varphi \circ D \circ \varphi^{-1})(a)$ multiplied with f. All ring elements are order 0, so we are done.

Inductive hypothesis: Assume that for all differential operators D' of order $\eta - 1$, $\Psi' = \varphi \circ D' \circ \varphi^{-1}$ has order $\eta - 1$.

We want to show that this implies all differential operators D of order η, $\Psi = \varphi \circ D \circ \varphi^{-1}$ has order η. So we can show that $[a, \Psi]$ has order less than η for all $a \in R$. We can write $D = (D_1 \circ d) + D_2$ for a differential operator D_1 of order $\eta - 1$, a differential operator D_2 of order less than or equal to $\eta - 1$, and a differential operator d of order 1. Let $\Psi_1 = \varphi \circ D_1 \circ \varphi^{-1}$, $\Psi_2 = \varphi \circ D_2 \circ \varphi^{-1}$, and $\psi = \varphi \circ d \circ \varphi^{-1}$. Then, $\Psi = \Psi_1 \circ \psi + \Psi_2$. Calculating $[a, \Psi]$,

$$\begin{aligned}
[a, \Psi] &= (a \circ (\Psi_1 \circ \psi + \Psi_2)) - ((\Psi_1 \circ \psi + \Psi_2) \circ a) \\
&= (a \circ \Psi_1 \circ \psi) - (\Psi_1 \circ \psi \circ a) + (a \circ \Psi_2) - (\Psi_2 \circ a) \\
&= ((\Psi_1 \circ a) + [a, \Psi_1]) \circ \psi - (\Psi_1 \circ \psi \circ a) + [a, \Psi_2] \\
&= (\Psi_1 \circ a \circ \psi) + ([a, \Psi_1] \circ \psi) - (\Psi_1 \circ \psi \circ a) + [a, \Psi_2] \\
&= \Psi_1 \circ ((\psi \circ a) + [a, \psi]) + [a, \Psi_1] \circ \psi - (\Psi_1 \circ \psi \circ a) + [a, \Psi_2] \\
&= (\Psi_1 \circ \psi \circ a) + \Psi_1 \circ [a, \psi] + [a, \Psi_1] \circ \psi - (\Psi_1 \circ \psi \circ a) + [a, \Psi_2] \\
&= \Psi_1 \circ [a, \psi] + [a, \Psi_1] \circ \psi + [a, \Psi_2]
\end{aligned}$$

From our inductive hypothesis and base case, we have Ψ_1 has order $\eta - 1$, Ψ_2 has order less than or equal to $\eta - 1$, and ψ has order 1. By the definition of commutators, this implies $[a, \Psi_1]$ has order less than $\eta - 1$, $[a, \Psi_2]$ has order less than $\eta - 1$, and $[a, \psi]$ has order 0. Thus, $[a, \Psi]$ has order less than or equal to $\eta - 1$ for all a. This proves Ψ is a differential operator of order η. ☐

This helps us understand the relationship between differential powers and ring isomorphisms/k-algebra homomorphisms.

5.2. **Simple $\mathcal{D}$-Modules.** In this section, we generalize our results on polynomial rings to finitely generated k-algebras that are either regular characteristic zero, or any characteristic p.

We begin with the necessary definitions.

Definition 5.5. A $\mathcal{D}_R$-module is a module over the ring of differential operators, $\mathcal{D}_R$ over the ring R.

Definition 5.6. A simple $\mathcal{D}_R$-module is a $\mathcal{D}_R$-module that has no proper $\mathcal{D}_R$-submodule.

Note, we consider only regular rings of characteristic 0 because of the following lemma.

Lemma 5.7 ([QG]). *If R is a regular k-algebra, where k is a field of characteristic zero, then $D_{\mathsf{k}}(R)$ is generated by R and its derivations.*

By this lemma, we generalize Proposition 3.5.1, so we know that $\overline{I}^{\text{diff}}$ is an ideal when R is regular characteristic 0.

Our following proposition further generalizes our prior result, Proposition 3.5.

Proposition 5.8. *Let R be a ring and $r \in R$. For all $m > l$, $r^{m-l}|D_{\leq l}(r^m)$ where $0 \leq l \leq m$ and $D_{\leq l}$ denotes an arbitrary differential operator of order less than or equal to l.*

Proof. Proof by induction on l.

We begin by considering the base case $l = 0$. The proof for this case is the same as Proposition 3.5.

Next we form our inductive hypothesis. Assume that for all $m > l$, r^{m-l} divides $D_{\leq l}(r^m)$ (IH1). We want to show for all $m' > l+1$, $r^{m'-l-1}$ divides $D_{\leq l+1}(r^{m'})$. The inequality $m > l$ implies $m+1 > l+1$. So, we can rewrite m' as $m+1$ to use the same m as the inductive hypothesis. After this substitution, we want to show for all $m > l$, r^{m-l} divides $D_{\leq l+1}(r^{m+1})$. To prove this statement, we use another proof by induction.

* * * * *

Proof by induction on m when l is fixed.

We begin by considering the base case $m = l+1$. In this case, we want to show $r|D_{\leq l+1}(r^{m+1})$ which has been proved in Proposition 3.5.

Next we form our inductive hypothesis. Assume r^{m-l} divides $D_{\leq l+1}(r^{m+1})$ (IH2). We want to show r^{m+1-l} divides $D_{\leq l+1}(r^{m+2})$.

Let D be an operator of order less than or equal to $l+1$. Then, $[D, r]$ is a differential operator of order less than or equal to l. Thus, we can use IH1 to say $r^{(m+1)-l}$ divides $[D, r](r^{m+1})$. (We are replacing the m in IH1 with $m+1$ because the hypothesis holds for all $m > l$). We also use IH2 to say r^{m-l} divides $D(r^{m+1})$. Let $A, B \in R$ such that $[D, r](r^{m+1}) = r^{m+1-l} \cdot A$ and $D(r^{m+1}) = r^{m-l} \cdot B$. Then,

$$\begin{aligned} r^{m+1-l} \cdot A &= [D, r](r^{m+1}) \\ &= D(r^{m+2}) - rD(r^{m+1}) \\ &= D(r^{m+2}) - r(r^{m-l} \cdot B) \\ r^{m+1-l}(A+B) &= D(r^{m+2}) \end{aligned}$$

for some $A, B \in R$.

So, r^{m+1-l} divides $D(r^{m+2})$ for all differential operators D of order less than or equal to $l+1$.

* * * * *

Thus, we have shown for all $m > l$, r^{m-l} divides $D_{\leq l+1}(r^{m+1})$ and we are done. □

We are now able to prove one direction of containment for the radical and the differential closure.

Proposition 5.9. *Let R be a ring, and let $I \subseteq R$ be a ideal. Then $\sqrt{I} \subseteq \overline{I}^{diff}$.*

Proof. Let $r \in \sqrt{I}$. Then there exists a $k > 0$ such that $r^k \in I$. We want to show $r \in \overline{I}^{\text{diff}}$, or equivalently, there exists a nonzero c such that for all n sufficiently large and for all D differential operators of order less than n $D(cr^n) \in I$.

If we are able to show $r^k | D(cr^n)$ then we are done. To do this, we use Proposition 5.8 to show $r^{m-l} | D_{\leq l}(r^m)$ and then set m to satisfy $m \geq k + l$. Let $c = r^k$. So, we want to show $r^k | D(r^{k+n})$. Since D order strictly less than n, apply inductive proof to say $r^{(k+n)-n} | D(cr^n)$.

□

We will now show the other containment, first by the fact that for maximal ideals equal their own differential closures. Using that fact, we will then show that for any radical ideal the differential closure is also equivalent, leading us to the other containment.

Proposition 5.10. *Let $\mathfrak{m} \subseteq R$ a maximal ideal of the ring R, and let R is a regular k-algebra and a simple $\mathcal{D}-$module then $\mathfrak{m} = \overline{\mathfrak{m}}^{diff}$.*

Proof. Following same idea as for polynomial rings. We know $\mathfrak{m} \subseteq \overline{\mathfrak{m}}^{\text{diff}}$, since it holds for all ideals. Since $\mathfrak{m}$ is maximal, $\overline{\mathfrak{m}}^{\text{diff}}$ must either be the whole ring or equal to $\mathfrak{m}$.

If $\overline{\mathfrak{m}}^{\text{diff}}$ is the whole ring, then it contains some unit u. Thus, there exists a nonzero c such that for all n sufficiently large, $cu^n \in \mathfrak{m}^{\langle n \rangle}$. This implies there exists a nonzero c suc that for all n sufficiently large and for all D differential operators of order less than n, $D(cu^n) \in \mathfrak{m}$ note u^n is still a unit. Then, $D(cu^n) = u^n D(c) \in \mathfrak{m}$. R is a simple $\mathcal{D}-$module if and only if for all $r \in R \setminus \{0\}$ there exists some $D \in D_{R/\mathsf{k}}$ such that $D(r) = 1$. Then $u^n D(c)$ is a unit in $\mathfrak{m}$. This is not possible since $\mathfrak{m}$ is maximal, so $\overline{\mathfrak{m}}^{\text{diff}}$ must be equal to $\mathfrak{m}$. □

Proposition 5.11. *If R is a $\mathsf{k}-$algebra and a regular Jacobson ring, and $I \subseteq R$ is a radical ideal, then*

$$\overline{I}^{diff} = I$$

Proof. Using Proposition 3.9, the fact that R is Jacobson, and Proposition 5.10 write

$$\overline{I}^{\text{diff}} = \overline{\bigcap_{I \subseteq \mathfrak{m}_\alpha} \mathfrak{m}_\alpha}^{\text{diff}} \subseteq \bigcap_{I \subseteq \mathfrak{m}_\alpha} \overline{\mathfrak{m}_\alpha}^{\text{diff}} = \bigcap_{I \subseteq \mathfrak{m}_\alpha} \mathfrak{m}_\alpha = I$$

Hence we see, $\overline{I}^{\text{diff}} \subseteq I$. We already know $I \subseteq \overline{I}^{\text{diff}}$ from Proposition 3.7. So we have equality of $I = \overline{I}^{\text{diff}}$ for some radical ideal I. □

Idempotence of the differential operator follows directly from this proposition.

Corollary 5.11.1. *If R is a k-algebra and a regular Jacobson ring, and $I \subseteq R$ is a radical ideal, then*

$$\overline{\overline{I}^{diff}}^{diff} = \overline{I}^{diff}$$

Proposition 5.12. *For any regular characteristic 0 ring, that is is a simple $\mathcal{D}_R-$module, and any ideal $I \subseteq R$, we have $\overline{I}^{diff} = \sqrt{I}$.*

Proof. By definition of the radical, $I \subseteq \sqrt{I}$. By properties of closure operations, $\overline{I}^{\text{diff}} \subseteq \overline{\sqrt{I}}^{\text{diff}}$. By Proposition 5.11, we have $\overline{\sqrt{I}}^{\text{diff}} = \sqrt{I}$. So, $\overline{I}^{\text{diff}} \subseteq \sqrt{I}$. From Proposition 5.9 $\sqrt{I} \subseteq \overline{I}^{\text{diff}}$. Putting this all together, we have $\sqrt{I} = \overline{I}^{\text{diff}}$. □

Finally, we will show that when $\overline{I}^{\text{diff}}$ is an ideal, then we have a direct correspondence between $\sqrt{I} = \overline{I}^{\text{diff}}$ and our ring being a simple $\mathcal{D}_R$-module.

Theorem 5.13. *Let R be a a characteristic 0 regular ring, and I is an ideal of R, then $\sqrt{I} = \overline{I}^{diff}$ if and only if R is a simple $\mathcal{D}_R-$module.*

Proof. One direction was proved in Proposition 5.12 For the other direction, we use the contrapositive. Assume R is not a simple $\mathcal{D}_R$-module, then there exists some proper $\mathcal{D}_R-$submodule, M, which implies that for all $D \in \mathcal{D}_R$, $D(M) \subseteq M$. Note, M is an ideal of R. Now consider that $r \in \overline{M}^{\text{diff}}$ if $D(cr^n) \in M$ for D some n-th differential operator where n is sufficiently large and some $c \in R$. Now, since M has the property $D(M) \subseteq M$ for all D, then all we need is that $cr^n \in M$ in order to know $r \in \overline{M}^{\text{diff}}$. However, for all r, we may allow $c \in M$, then it follows that for all $r \in R$, then $r \in \overline{M}^{\text{diff}}$, or $\overline{M}^{\text{diff}} = R$. We know though, that $\sqrt{M} \neq R$ since M is a proper submodule. In this case, we can conclude that $\sqrt{I} \neq \overline{I}^{\text{diff}}$ when R is not a simple $\mathcal{D}_R-$module. By the contrapositive, our proposition follows. □

We will close with one example, and one non-example.

Example 5.14. Consider that $R = \mathsf{k}[x, y, z]/(z - xy)$ is a regular ring, where k is of positive characteristic. By [Smi95], will be a simple $\mathcal{D}_R$-module since it is a domain and strongly $F-$regular.

Example 5.15. Non-example, consider the ring $R = \mathsf{k}[x, y]/(xy)$, we have that $(0), (x), (y), (xy)$ are $\mathcal{D}_R$-ideals. In general, Stanley-Reisner Rings are not simple $\mathcal{D}_R$-modules [Jef20]. In this case, for example, $\sqrt{(x)} \neq \overline{(x)}^{\text{diff}}$.

6. Open Questions

Our final section consists of topics that could be explored in the future. We begin with the differential Rees Algebra.

Definition 6.1. The differential Rees Algebra is

$$\langle\mathcal{R}\rangle(It) = \bigoplus_{n\geq 0} I^{\langle n\rangle}t^n$$

We know the algebra is not necessarily finitely generated since there exists a prime ideal such that the symbolic Rees Algebra is not finitely generated. By Zariski-Nagata, we then know that the differential Rees algebra is not necessarily finitely generated. Hence, we have the following question.

Question 6.1. *Under what conditions is the differential Rees algebra finitely generated?*

We also believe that Theorem 5.13 can be generalized. In other words,

Question 6.2. *For all rings R with an ideal I, is it true that, $\sqrt{I} = \overline{I}^{diff}$ if and only if R is a simple $\mathcal{D}_R$-module?*

In Theorem 4.26 we prove that there exists an N such that $I^{\langle N\rangle}$ is principal for ideals of a particular form. We also want to know when the differential power is principal in more than two variables, which prompts the following question.

Question 6.3. *Given an ideal I in a ring $\mathsf{k}[x_1, \ldots, x_z]$ where each generator of the ideal contains each of the variables, what is the smallest n such that $I^{\langle n\rangle}$ is principal?*

We also want to generalize our results of the generators of the differential power, to ideals besides principal and monomial ideals.

Question 6.4. *When can we write an explicit formula for the differential power of a non-monomial ideal?*

Moreover, in [BJNB19], they give an explicit formulation of the differential power in polynomial rings of characteristic 0, so generalizing to broader rings is a future goal.

Question 6.5. *Is there a formulation of the differential power in polynomial rings of characteristic p? Non-polynomial rings?*

7. Acknowledgments

The authors would like to thank our mentors, Jennifer Kenkel, Janet Page, and Daniel Smolkin for their invaluable advice and guidance. They would also like to acknowledge the University of Michigan Department of Mathematics, and in particular David Speyer, for organizing the REU. The research was supported by the executive committee funds, NSF grants DMS-1855135, and DMS-1801697.

References

[AM69] Michael Francis Atiyah and I. G. MacDonald. *Introduction to commutative algebra.* Addison-Wesley-Longman, 1969.

[BJNB19] Holger Brenner, Jack Jeffries, and Luis Núñez-Betancourt. Quantifying singularities with differential operators. *Advances in Mathematics*, 358:106843, 2019.

[Cou95] S. C. Coutinho. *A primer of algebraic D-modules*, volume 33 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 1995.

[DDSG+18] Hailong Dao, Alessandro De Stefani, Eloísa Grifo, Craig Huneke, and Luis Núñez Betancourt. Symbolic powers of ideals. In *Singularities and foliations. geometry, topology and applications*, volume 222 of *Springer Proc. Math. Stat.*, pages 387–432. Springer, Cham, 2018.

[ELS01] Lawrence Ein, Robert Lazarsfeld, and Karen E. Smith. Uniform bounds and symbolic powers on smooth varieties. *Invent. Math.*, 144(2):241–252, 2001.

[HH89] Melvin Hochster and Craig Huneke. Tight closure. In *Commutative algebra (Berkeley, CA, 1987)*, volume 15 of *Math. Sci. Res. Inst. Publ.*, pages 305–324. Springer, New York, 1989.

[HH11] Jürgen Herzog and Takayuki Hibi. *Monomial ideals*, volume 260 of *Graduate Texts in Mathematics*. Springer-Verlag London, Ltd., London, 2011.

[Hoc07] Melvin Hochster. Foundations of tight closure theory, 2007.

[HS06] Craig Huneke and Irena Swanson. *Integral closure of ideals, rings, and modules*, volume 336 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2006.

[Jef20] Jack Jefferies. Differential operators spring 2020, May 2020.

[QG] Eamon Quinlan-Gallego. Rings of differential operators.

[Smi95] Karen E. Smith. The D-module structure of F-split rings. *Math. Res. Lett.*, 2(4):377–386, 1995.