# Attempts to Approximate the Partition Function of Hamiltonian Cycles

James Walrad and Lily Wang

August 2021

**Abstract**

In [Bar19], Barvinok successfully applies complex-analytic methods to approximate permanents of a class of diagonally dominant matrices quickly, using a method that relies on bounding the roots of all polynomials per($A$), where $A$ ranges over that class of matrices, away from the origin. Here we investigate whether the same method can be applied to computing a related function, the *partition function of Hamiltonian cycles*, and conclude in the negative by presenting counterexamples. We then apply methods from [BR19] and [Bar19] to approximate a polynomial which allows us to learn about cycle covers without short cycles in a graph.

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# 1 Preliminaries

We first introduce the partition function of Hamiltonian cycles.

**Definition 1.0.1.** Let $G$ be a directed graph on $n$ vertices, for $n$ a natural number. The *partition function of Hamiltonian cycles* $\mathrm{ham}(G;W)$ is the polynomial $\sum_H \prod_{e \in H} w_e$, where the sum runs over all Hamiltonian cycles in $G$, and the multiplicands in the product are weights of the edges of $G$, interpreting the complex-valued $n \times n$ matrix $W = [w_{i,j}]$ as a weighted adjacency matrix for $G$.

Note that the domain of $\mathrm{ham}(G;W)$ changes based on the choice of graph $G$, because in order for the variable weight matrix $W$ to be a weighted adjacency matrix for $G$, we must have $w_e = 0$ when $e$ is not an edge of $G$.

In this paper we consider Hamiltonian directed graphs $G$. Whenever a generic graph $G$ is mentioned, assume it has these properties, as well as the following traits. For a given $G$ we say $G$ has $n$ vertices, where $n$ is an integer. Without loss of generality, the sequence of edges $(1,2),(2,3),\ldots,(n-1,n),(n,1)$ gives a Hamiltonian cycle $H_0$ in $G$. When thinking of the graph $G$, we also restrict ourselves to weight matrices $W = [w_{i,j}]$ such that $w_e = 1$ for each edge $e$ in $H_0$. Thus in our case $\mathrm{ham}(G;W)$ measures how much the Hamiltonian cycles in $G$ differ from $H_0$. We also say that $\Delta_{\mathrm{in}}$ is an integer greater than or equal to the indegree of any vertex in $G$, and we say that $\Delta_{\mathrm{out}}$ is an integer greater than or equal to the outdegree of any vertex in $G$.

Recall that for a complex $n \times n$ matrix $A = [a_{i,j}]$, $\mathrm{per}(A)$ is the polynomial $\sum_{\sigma \in S_n} \prod_{i=1}^n a_{i,\sigma(i)}$. We can think of $A$ as the weighted adjacency matrix of an appropriate graph; then $\mathrm{per}(A)$ is the polynomial $\sum_C \prod_{e \in C} a_e$, where the sum runs over the cycle covers $C$ of the graph. (We say a subgraph of a graph $\Gamma$ is a *cycle cover* if it includes all the vertices in $\Gamma$ and is a disjoint union of cycles.) Thus we can think of $\mathrm{ham}(G;W)$ as an analogue to the permanent where we only consider connected cycle covers; these are precisely the Hamiltonian cycles in $G$. Our requirement that $H_0$ exists and has edges of weight 1 is then analogous to the requirement in [Bar19] that the matrices in question are diagonally dominant.

In particular [Bar19] establishes that, because $\mathrm{per}(I + A) \neq 0$ when the rows of $A$ are sufficiently short vectors, $\log \mathrm{per}(I + A)$ can be approximated within error $\epsilon$ in time $n^{f(n,\epsilon)}$, where $f(n,\epsilon) \in O(\log n - \log \epsilon)$, provided that $A$ again has short enough rows. We can apply the same methods to $\mathrm{ham}(G;W)$ if the following conjecture holds:

**Conjecture 1.0.1.** *There exists a function $\gamma : \mathbf{Z}^2_{>0} \to \mathbf{R}_{>0}$ such that for any directed graph $G$ and any complex weighted adjacency matrix $W$ for $G$ as we specify above, if $|w_e| < \gamma(\Delta_{\mathrm{in}}, \Delta_{\mathrm{out}})$ for each edge $e$ of $G$ not contained in $H_0$, then* $\mathrm{ham}(G;W) \neq 0$.

In fact, the conjecture does not hold, including under nice conditions.

# 2 Hamiltonian Cycles

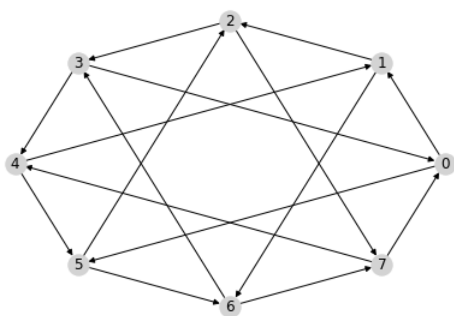
## 2.1 The "Star" Graphs

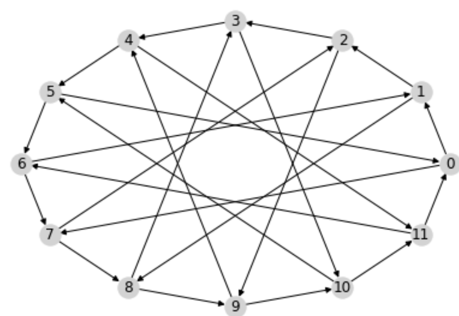
There is a class of graphs reminiscent of stars which provide a coutnerexample to Conjecture 1.0.1.

**Definition 2.1.1.** For a given positive integer $k$, let $G_{4k}$ be the graph with vertex set $\mathbf{Z}/4k\mathbf{Z}$, edges $(i, i+2k+1)$ as $i$ ranges in $\mathbf{Z}/4k\mathbf{Z}$, and edges $(1,2),(2,3),\ldots,(4k-1,4k),(4k,1)$. For each complex number $z$, let the matrix $M_{4k}(z) = [m_{i,j}(z)]$ be the complex-valued $4k \times 4k$ matrix with

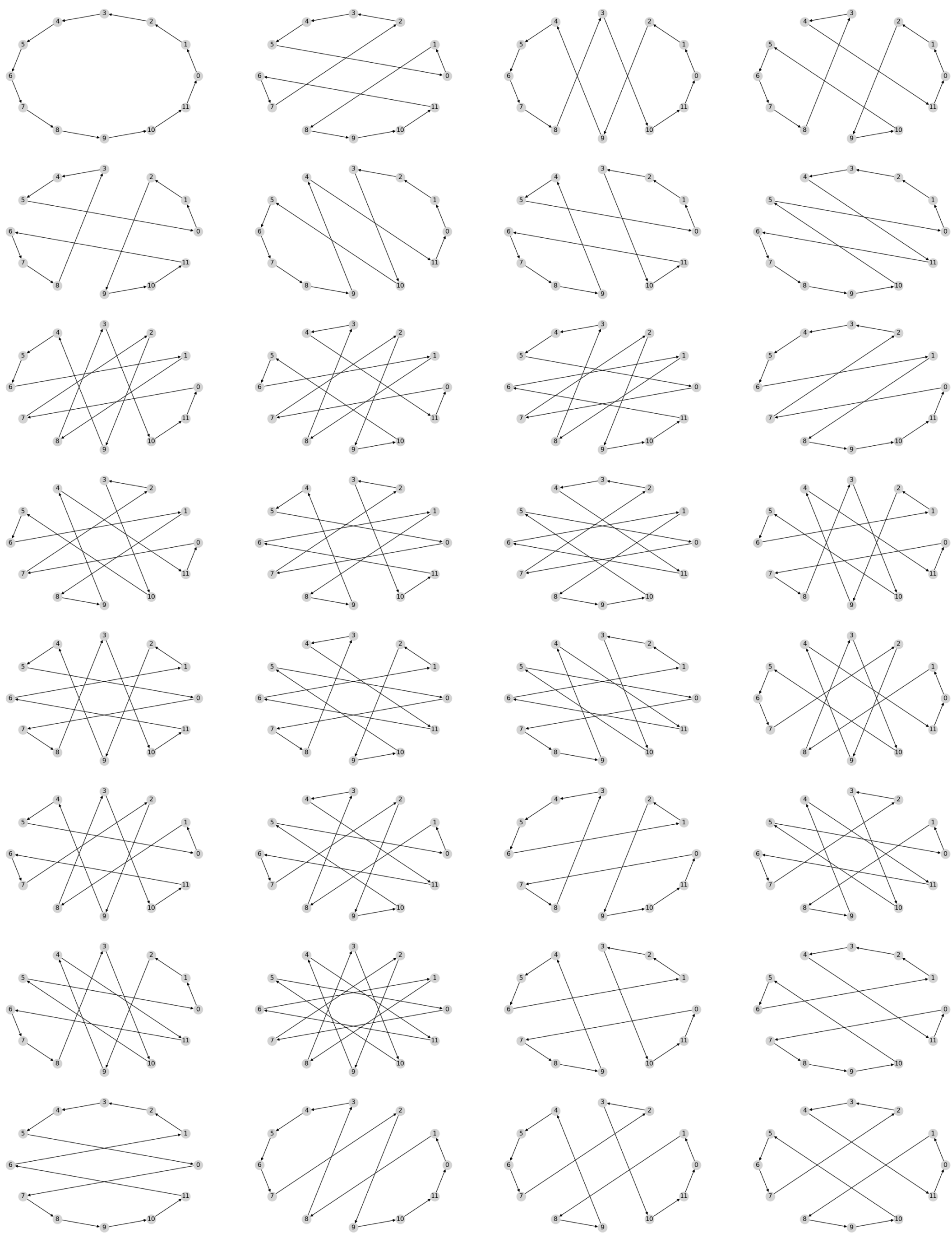
$$m_{ij}(z) = \begin{cases} 1 & \text{if } j - i = 1 \pmod{4k}, \\ 0 & \text{if } i = j, \\ z & \text{otherwise.} \end{cases}$$

For example, the below figures represent $G_8$ and $G_{12}$.



Additionally, this page features figures representing all Hamiltonian cycles in $G_{12}$.



Inspecting these cycles will convince the reader that $\mathrm{ham}(G_{12}; M_{12}(z)) = 1 + 15z^4 + 15z^8 + z^{12}$.

We need to use specialized language to prove what we want to know about the graphs $G_{4k}$.

**Definition 2.1.2.** For a fixed $k$, we call the subgraph of $G_{4k}$ which contains all vertices of $G_{4k}$ and the edges of the form $(i, i+2k+1)$ the *inner tour* of $G_{4n}$.

**Lemma 2.1.1.** *The inner tour of $G_{4k}$ is a Hamiltonian cycle.*

*Proof.* For any positive integer $k$, $\gcd(4k, 2k+1) = \gcd(2k+1, -2) = 1$. Since $4k$ and $2k+1$ are relatively prime, following the inner tour of $G_{4k}$, we visit every vertex once before returning to the vertex we started from. □

**Proposition 2.1.1.** *For $k \in \mathbf{N}$ and $z \in \mathbf{C}$,* $\mathrm{ham}(G_{4k}; M_{4k}(z)) = \sum_{i=0}^{k} \binom{2k}{2i} z^{4i}$.

*Proof.* It is evident from our construction of the graph $G_{4k}$ that for any Hamiltonian cycle $H$ in $G_{4k}$, if $H$ contains the edge $(a, a+2k+1)$, then necessarily $H$ must also contain the edge $(a+2k, a+1)$. Specifically, since every vertex in $G_{4k}$ has indegree and outdegree both equal to 2, and since $H$ is Hamiltonian, if $(a, a+2k+1)$ is an edge in $H$, then $H$ cannot contain $(a, a+1)$. However, $H$ must contain an incoming edge for $a+1$; the only such edge in $G_{4k}$ is $(a+2k, a+1)$.

We call a set of edges in $G_{4k}$ of the form $\{(a, a+2k+1), (a+2k, a+1)\}$ a *star-edge-pair*. By construction, there are $2k$ star-edge-pairs in $G_{4k}$. We also call a set of edge in $G_{4k}$ of the form $\{(a, a+1), (a+2k, a+2k+1)\}$ an *exterior-edge-pair*. There is a bijective correspondence between star-edge-pairs and exterior-edge-pairs, which maps

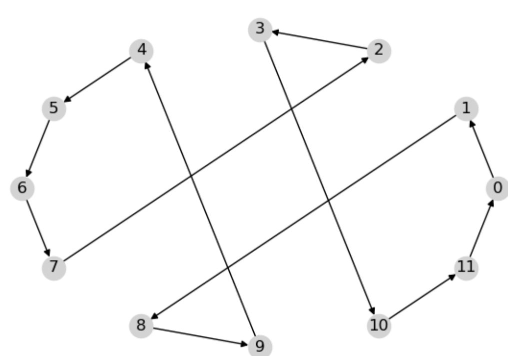
$$\{(a, a+2k+1), (a+2k, a+1)\} \mapsto \{(a, a+1), (a+2k, a+2k+1)\}$$

for each $a \in \mathbf{Z}/4k\mathbf{Z}$. We say $a \in \mathbf{Z}/4k\mathbf{Z}$ is a *representative* of the star-edge-pair $\{(a, a+2k+1), (a+2k, a+1)\}$.

For each choice of any subset $C$ of star-edge-pairs in $G_{4k}$, we can construct a subgraph $K_C$ of $G_{4k}$ as follows. We have a unique set of representatives $S_C = \{a_1, a_2, \ldots a_i\}$ for the $i$ star-edge-pairs in $C$ such that the minimal positive integer representative of each $a_j$ is less than $2k$ for $1 \le j \le i$. We define $K_C$ to contain the same vertices as $G_{4k}$; the edges of $K_C$ are the edges $(a, b)$ such that for each $a \in \mathbf{Z}/4k\mathbf{Z}$,

$$b = \begin{cases} a+2k+1 & \text{if } a \in S_C \text{ or } a+2k \in S_C, \\ a+1 & \text{otherwise.} \end{cases}$$

Below is a figure representing $K_{\{\{(1,8),(7,2)\},\{(3,10),(9,4)\}\}}$ for $G_{12}$.



Note that each vertex of $G_{4k}$ has one outgoing edge and one incoming edge in $K_C$, since for any $a$, if $a-1 \in S_C$ or $a-1+2k \in S_C$, then $(a-1+2k, a)$ is an edge of $K_C$, and otherwise $K_C$ has the edge $(a-1, a)$.

We now proceed to show that if $i = |C|$ is even, $K_C$ is a Hamiltonian cycle in $G_{4k}$. If $i = 0$ then this holds automatically, so assume $i > 1$. If $i = 2k$ then by lemma 2.1.1, $K_C$ is a Hamiltonian cycle. Fix a set $C$ of star-edge-pairs as above such that $|C|$ is even. We already know $K_C$ is a cycle cover of $G_{4k}$. To show that there's only one cycle in $K_C$, it suffices to show that $K_C$ can be transformed into a Hamiltonian cycle via contraction of edges. Take the set of representatives $S_C = \{a_1, a_2, \ldots, a_i\}$ which we used in constructing $K_C$, and let $S'_C = S_C \cup \{a + 2k : a \in S_C\}$. $S'_C$ is the set of sources for edges contained in the chosen star-edge-pairs. By contracting all the edges of $K_C$ which are not contained in any star-edge-pair, we identify each vertex $v \in S'_C$ with all $u \in \mathbf{Z}/4k\mathbf{Z}$ such that there exists a path in $K_C$ of sequential vertices from $u$ to $v$. No elements of the equivalence class of $v$ obtained via this transformation other than $v$ itself are elements of $S'_C$, since for $w \in S'_C$, $(w, w+1)$ is not an edge in $K_C$. Thus this contraction results in a graph $K'$ with $2i$ vertices given by the equivalence classes $[a_1], [a_1+2k], [a_2], [a_2+2k], \ldots, [a_i], [a_i+2k]$, and edges given by $([b], [b+2k+1])$ for $b \in \mathbf{Z}/4k\mathbf{Z}$. Without loss of generality, when following the edges of $G_{4k}$ which connect consecutive vertices, one first encounters the vertices $a_1, a_2, \ldots, a_i, 2n$ in the listed order. Since a minimal integer representative of $a_i$ is less than $2n$, when following the edges of $G_{4k}$ which connect consecutive vertices, one encounters the following vertices as listed:

$$a_1, a_1+1, a_2, a_2+1, \ldots, a_i, a_i+1, a_1+2k, a_1+2k+1, a_2+2k, a_2+2k+1, \ldots, a_i+2k, a_i+2k+1, a_1.$$

Thus, $[a_j+2k+1] = [a_{j+1}+2k]$ and $[a_j+1] = [a_{j+1}]$ for $1 \le j < i$, while $[a_i+1] = [a_1+2k]$ and $[a_i+2k+1] = [a_1]$.

Then, letting $b_j = [a_j]$ for $1 \le j \le k$, and letting $b_j = [a_{j-i}+2n]$ for $i < j \le 2i$, we have that the edges of $K'$ are precisely $(b_j, b_{j+i+1})$ for $1 \le j \le 2i$, where the indices are taken modulo $2i$. This graph is connected by lemma 2.1.1, since it is isomorphic to the inner tour of $G_{4(i/2)}$. Since $K'$ was obtained by contracting edges in $K_C$, $K_C$ must be connected.

Furthermore, suppose there exists a Hamiltonian cycle $H$ in $G_{4k}$ such that $H$ contains an odd number of star-edge-pairs. Let $C$ be the set of these star-edge-pairs. By removing the edges of one star-edge-pair, and replacing those edges with the edges $(a, a+1), (a+2k, a+2k+1)$ of the corresponding exterior-edge-pair, we obtain the unique Hamiltonian cycle $H'$ in $G_{4k}$ which contains all star-edge-pairs from $C \setminus \{\{(a, a+2k+1), (a+2k, a+1)\}\}$, and no other star-edge-pairs. $H$ must necessarily contain a path $P$ from $a+1$ to $a$. If $P$ hits $a+2k$ then $H$ contains a non-Hamiltonian cycle; this is impossible, so $P$ does not hit $a+2k$, so $(a+2k, a+1)$ is not in $P$. Furthermore $(a, a+2k+1)$ is not in $P$, as $a$ is the endpoint of $P$. Thus $P$ is also a path in $H'$, and is not a Hamiltonian path in $H'$. Following $P$ and then $(a, a+1)$ in $H'$ yields a non-Hamiltonian cycle in $H'$: this cycle does not hit $a+2k$. Thus $H'$ is not a Hamiltonian cycle in $G_{4k}$. This is impossible. Thus there exists no Hamiltonian cycle $H$ in $G_{4k}$ such that $H$ contains an odd number of star-edge-pairs.

Is is clear that $[z^m]\,\mathrm{ham}(G_{4k}; M_{4k}(z))$ is the number of Hamiltonian cycles in $G_{4k}$ with $m$ edges that do not connect consecutive vertices. We have shown that each Hamiltonian cycle in $G_{4k}$ is uniquely determined by its set of edges which do not connect consecutive vertices. We have also shown that for any Hamiltonian cycle $H$ in $G_{4k}$, the set of edges in $H$ which do not connect consecutive vertices can be partitioned into an even number of disjoint star-edge-pairs. Thus $\mathrm{ham}(G_{4k}; M_{4k}(z)) = \sum_{i=0}^{k} c_i z^{4i}$, where $c_i$ is the number of Hamiltonian cycles in $G$ whose corresponding sets of star-edge-pairs have cardinality $2i$. We have established that any choice of $2ki$ star-edge-pairs out of the set of $2k$ star-edge-pairs in $G_{4k}$ yields a corresponding Hamiltonian cycle; thus $c_i = \binom{2k}{2i}$ for $1 \leq i \leq k$. The proposition follows. □

We are interested in the roots of $\mathrm{ham}(G_{4k}, M_{4k})$, because information about these roots could disprove Conjecture 1.0.1. In particular, if the magnitude of the smallest root of $\mathrm{ham}(G_{4k}, M_{4k})$ vanishes as $k \to \infty$, then Conjecture 1.0.1 is false.

**Proposition 2.1.2.** *The roots of* $\mathrm{ham}(G_{4k}, M_{4k})$ *are the* $4k$ *complex numbers* $z$ *such that* $z^2 = \frac{1-w}{1+w}$ *for some* $2k^{th}$ *complex root* $w$ *of* $-1$.

*Proof.* Let $y = z^2$. Then

$$\mathrm{ham}(G_{4k}, M_{4k}(z)) = \sum_{i=0}^{k} \binom{2k}{2i} y^{2i} = \frac{(1+y)^{2k} + (1-y)^{2k}}{2}.$$

We then have $\mathrm{ham}(G_{4k}, M_{4k}(z)) = 0$ if and only if $\left(\frac{1-y}{1+y}\right)^{2k} = -1$. Then $\frac{1-y}{1+y}$ is a $2k^{\mathrm{th}}$ root of $-1$. Let $w = \frac{1-y}{1+y}$. Then $y = \frac{1-w}{1+w}$, and the proposition follows. □

Two corollaries illustrate why this result contradicts Conjecture 1.0.1.

**Corollary 2.1.1.** *There exists a sequence* $(z_k)$ *of complex numbers such that* $\mathrm{ham}(G_{4k}, M_{4k}(z_k)) = 0$ *for all positive integers* $k$ *and* $\lim z_k = 0$.

*Proof.* Define a sequence $(w_k)$ of complex numbers by the rule $w_k = e^{\frac{\mathrm{i}\pi}{2k}}$. Fix a positive real number $\epsilon$. Then there exists a positive integer $N$ such that $N > \frac{\pi}{4\epsilon}$. Then for any integer $i$ with $i \geq N$, we have $\arg w_i = \frac{\pi}{2i} \leq \frac{\pi}{2N} < 2\epsilon$. Since $w_i$ is on the unit circle, the arc length along the unit circle between $w_i$ and 1 is equal to $\arg w_i$. Since $|1 - w_i|$ is at most equal to this arc length, and since $w_i$ lies on the unit circle, we have $|1 - w_i| < 2\epsilon = (|1| + |w_i|)\epsilon \leq |1 + w_i|\epsilon$. Thus, $\left|\frac{1-w_i}{1+w_i}\right| < \epsilon$. Thus for each positive real number $\epsilon$, there exists a positive integer $N$ such that for each integer $i$ with $i \geq N$, $\left|\frac{1-w_i}{1+w_i}\right| < \epsilon$. Thus $\lim \frac{1-w_k}{1+w_k} = 0$. Furthermore, $w_k^{2k} = e^{\mathrm{i}\pi} = -1$.

For each positive integer $k$, let $z_k$ be a complex number such that $z_k^2 = \frac{1-w_k}{1+w_k}$. Then $\lim z_k = 0$ and $\mathrm{ham}(G_{4k}, M_{4k}(z_k)) = 0$ for all positive integers $k$. □

**Corollary 2.1.2.** *Conjecture 1.0.1 is false.*

*Proof.* Suppose Conjecture 1.0.1 were true. Throughout this proof we will use terms introduced in Conjecture 1.0.1. By construction, for each positive integer $k$, $G_{4k}$ and $M_{4k}(z)$ satisfy the conditions of Conjecture 1.0.1, with a constant $\Delta_{\mathrm{in}}$ and $\Delta_{\mathrm{out}}$. By construction, due to Corollary 2.1.1, we then have that there exists a positive real number $\eta$ such that $\mathrm{ham}(G_{4k}, M_{4k}(z)) \neq 0$ for each positive integer $k$ and $z \in \mathbf{C}$ with $|z| < \eta$. However, by Corollary 2.1.1 there exists a complex number $v$ such that $|v| < \gamma$ and $\mathrm{ham}(G_{4k}, M_{4k}(v)) = 0$ for some natural number $k$. This is impossible, so Conjecture 1.0.1 must be false. □

## 2.2 The "Tortoise Shell" Graphs

Since Conjecture 1.0.1 is false, we now ask ourselves for what types of graphs the conjecture holds. The properties we posit for the graphs $G$ should be properties that are preserved under deletion of edges, since this would allow us to apply methods of reducing and simplifying the problem at hand from [Bar19]. One natural way to construct graph properties which are preserved under deletion of edges is to pick a set of forbidden graph minors. The most famous property that arises by forbidding graph minors in undirected graphs is planarity. We will use an equivalent of planarity for directed graphs as follows.

**Definition 2.2.1.** Let $G$ be a directed graph, where $G = (V, E)$. For a given vertex set $V$ let $\pi_V : V^2 \to \binom{V}{2}$ be the function mapping $(u, v) \mapsto \{u, v\}$ and $(v, u) \mapsto \{u, v\}$ for all choices of $u, v \in V$. We say $G$ is *planar* if the graph $(V, \pi_V(E))$ is planar.

This definition is useful because we can check if a directed graph $G$ is planar by applying the forgetful function $\pi_V$ and then checking planarity for the resultant undirected graph. Inspired by this train of thought, we investigate the following conjecture:

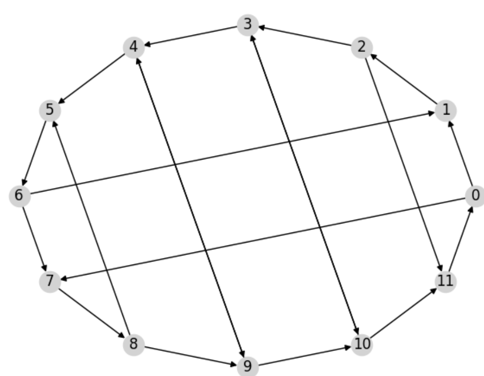
**Conjecture 2.2.1.** *There exists a function* $\gamma : \mathbf{Z}^2_{>0} \to \mathbf{R}_{>0}$ *such that for any planar directed graph $G$ and any complex weighted adjacency matrix $W$ for $G$ as we specify in conjecture 1.0.1, if* $|w_e| < \gamma(\Delta_{\text{in}}, \Delta_{\text{out}})$ *for each edge $e$ of $G$ not contained in $H_0$, then* $\text{ham}(G; W) \neq 0$.

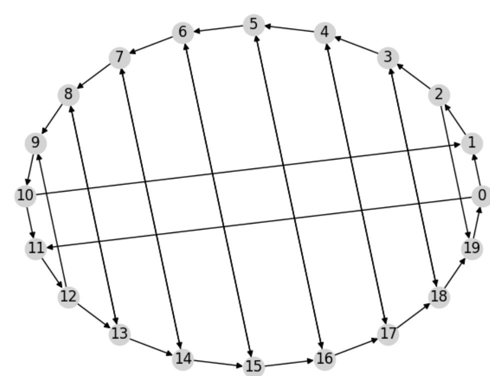
Sadly, Conjecture 2.2.1 is also false. We prove this claim using a class of graphs reminiscent of tortoise shells.

**Definition 2.2.2.** For a given positive integer $k$, Let $P_{2k+4}$ be the graph with $2k + 4$ vertices given by the elements of the set $[2k + 4]$, and edges of the form $(i, i + 1)$ for $0 \leq i < 2k + 3$ and $(2k + 3, 0)$ making up a Hamiltonian cycle $H_{2k+4}$, as well as edges $(k + 2, 1)$, $(0, k + 3)$, as well as $(k + 4 + j, k + 1 - j)$ and $(j + 2, 2k + 3 - j)$ for $0 \leq j < k - 1$, which edges we call the elements of *chord set* $B_k$.

We call $\{(k + 2, 1), (0, k + 3)\}$ the *chord set* $A_k$, and we call $\{(k + 4 + j, k + 1 - j) : 0 \leq j < k - 1\} \cup \{(j + 2, 2k + 3 - j) : 0 \leq j < k - 1\}$ the *chord set* $B_k$.

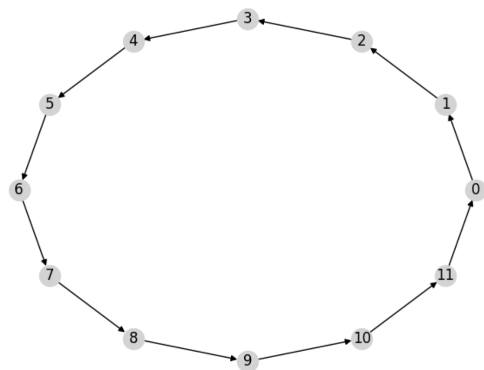
For example, the below figures represent $P_{12}$ and $P_{20}$:

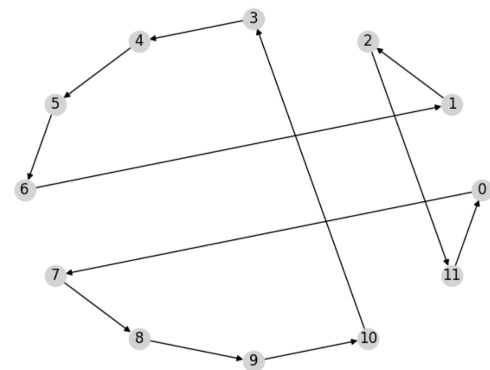


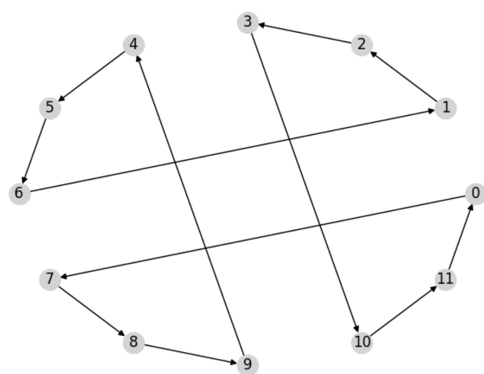



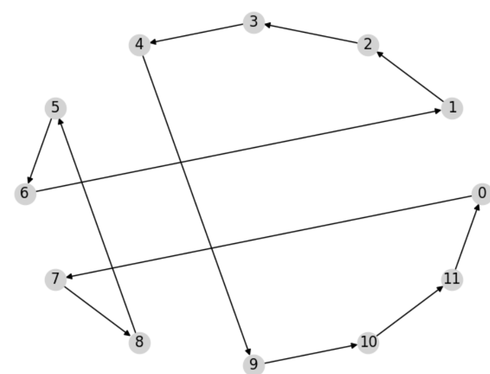
By drawing the edges in $A_k$ on the exterior of $H_{2k+4}$ in the drawing of $P_{2k+4}$ where all edges not in $H_{2k+4}$ are chords across the center of $P_{2k+4}$, we attain a planar drawing of $P_{2k+4}$. Thus $P_{2k+4}$ is planar. It is evident that no vertex of $P_{2k+4}$ has indegree or outdegree exceeding 2.

Here are the four distinct Hamiltonian cycles in $P_{12}$:




Again we require a specialized weight matrix.

**Definition 2.2.3.** For a given positive integer $k$ and complex number $z$, let $W_{2k+4}(z)$ be the weighted adjacency matrix for $P_{2k+4}$ which assigns weight 1 to the edges in $H_{2k+4}$ and weight $z$ to the remaining edges of $P_{2k+4}$.

We can see that $\mathrm{ham}(P_{12}, W_{12}(z)) = 1 + 3z^4$. This pattern persists as $k$ varies.

**Proposition 2.2.1.** *For each integer $k$ with $k > 1$,* $\mathrm{ham}(P_{2k+4}, W_{2k+4}(z)) = 1 + (k-1)z^4$.

*Proof.* Throughout this proof, weight the edges of $P_{2k+4}$ according to the entries of $W_{2k+4}(z)$. Clearly $H_{2k+4}$ is a Hamiltonian cycle in $P_{2k+4}$ of weight 1, since all of the edges of $H_{2k+4}$ have weight 1. Since only the edges of $H_{2k+4}$ in $P_{2k+4}$ have weight other than $z$, any other Hamiltonian cycle in $P_{2k+4}$ will have weight equal to a positive power of $z$.

For $0 \leq j < k-1$, let the pair of edges $\{(2+j, 2k+3-j), (2k+2-j, 3+j)\}$ be the $j^{\text{th}}$ $B_k$*-edge-pair*. Clearly there are $k-1$ such pairs, and the pairs partition $B_k$. We claim that each

$B_k$*-edge-pair* is contained within a unique Hamiltonian cycle in $P_{2k+4}$, and that furthermore, these are the only Hamiltonian cycles in $P_{2k+4}$ other than $H_{2k+4}$.

For some $j$ with $0 \leq j < k-1$, let $G_j$ be the cycle cover of $P_{2k+4}$ with edges $(0, k+3), (k+2, 1), (2+j, 2k+3-j), (2k+2-j, 3+j), (2k+3, 0)$, and $(n, n+1)$ for $n \notin \{0, k+2, 2+j, 2k+2-j, 2k+3\}$. It is easy to verify that $G_j$ is in fact a cycle cover, since every vertex in $G_j$ has indegree and outdegree 1. If $G_j$ is not a Hamiltonian cycle, then $G_j$ is disconnected. If $G_j$ is disconnected, then $G_j$ remains disconnected under contraction of edges. We contract all edges of $G_j$ of the form $(n, n+1)$ for $n$ an integer. Since $1 < 2+j < 3+j < k+2 < k+3 < 2k+2-j < 2k+3-j \leq 2k+3$, the resultant graph $G'_j$ is a graph whose vertices are the equivalence classes under contraction $[1], [2k+3-j], [0], [k+3], [3+j]$, and whose edges are $([1], [2k+3-j]), ([2k+3-j], [0]), ([0], [k+3]), ([k+3], [3+j]), ([3+j], [1])$. Clearly $G'_j$ is a 5-cycle, so $G'_j$ is connected, so $G_j$ must also be connected. Thus $G_j$ is a Hamiltonian cycle. To show that $G_j$ is the unique Hamiltonian cycle in $P_{2k+4}$ which contains the $j^{\text{th}}$ $B_k$-edge-pair, it suffices to show both that no Hamiltonian cycle in $P_{2k+4}$ contains more than two $B_k$-edge-pairs, and that every Hamiltonian cycle in $P_{2k+4}$ other than $H_{2k+4}$ contains the edges from $A_k$.

Suppose $H$ is a Hamiltonian cycle in $P_{2k+4}$ which does not contain both edges from $A_k$. If $H$ contains only $(k+2, 1)$ from $A_k$, then $H$ contains the edge $(0, 1)$, so 1 has indegree at least 2 in $H$, which is impossible, as $H$ is a cycle. If $H$ contains only $(0, k+3)$ from $A_k$, then $H$ contains the edge $(k+2, k+3)$, so $k+3$ has indegree at least 2 in $H$, which is impossible, as $H$ is a cycle. Thus $H$ contains no edges of $A_k$, so $H$ contains $(0, 1)$ and $(k+2, k+3)$. Furthermore, we know $H$ contains the edges $(2k+3, 0), (1, 2), (k+1, k+2)$, and $(k+3, k+4)$. If $H \neq H_{2k+4}$ then there exists an integer $j$ with $0 \leq j < k-1$ such that either $(2+j, 2k+3-j)$ or $(2k+2-j, 3+j)$ is in $H$. Suppose the former holds. Then, $(2k+2-j, 2k+3-j)$ is not in $H$, since then $2k+3-j$ would have indegree at least 2 in $H$, which is impossible as $H$ is a cycle. Thus $(2k+2-j, 3+j)$ must be in $H$. If the latter holds, then $(2+j, 3+j)$ is not in $H$, since in that case $3+j$ would have indegree at least 2 in $H$, which is impossible as $H$ is a cycle. Thus, in full generality, any Hamiltonian cycle in $P_{2k+4}$ which contains any edge $e \in B_k$ must contain both edges in the unique $B_k$-edge-pair containing $e$. In particular, we have a least $j$ such that the $j^{\text{th}}$ $B_k$-edge-pair is contained in $H$. Then there is a non-Hamiltonian cycle in $H$ given by the edge $(2+j, k+3-j)$ and the path in $H$ from $k+3-j$ to $2+j$. Thus $H$ is not a Hamiltonian cycle, which contradicts our hypothesis. Thus $H = H_{2k+4}$ necessarily. Therefore, any Hamiltonian cycle in $P_{2k+4}$ other than $H_{2k+4}$ contains the edges from $A_k$.

Let $H$ be a Hamiltonian cycle in $P_{2k+4}$ which contains at least 2 distinct $B_k$-edge-pairs. Let $a$ be the least integer such that $H$ contains the $a^{\text{th}}$ $B_k$-edge-pair, and let $b$ be the greatest integer such that $H$ contains the $b^{\text{th}}$ $B_k$-edge-pair. Then, there exist paths of consecutive vertices from $3+b$ to $k+2$, from 1 to $2+a$, from $k+3$ to $2k+2-b$, and from $2k+3-a$ to 0. Thus there is a cycle in $H$ which we attain by following these paths and the edges $(k+2, 1)$, $(2+a, 2k+3-a)$, $(0, k+3)$, and $(2k+2-b, 3+b)$. This cycle cannot hit the vertices $v$, of which there is at least one, such that $2+a < v < 3+b$. Thus $H$ contains a non-Hamiltonian cycle, so $H$ is not a Hamiltonian cycle in $P_{2k+4}$. This is a contradiction; thus any Hamiltonian cycle in $P_{2k+4}$ hay contain at most one $B_k$-edge-pair.

By now we have shown that each Hamiltonian cycle in $P_{2k+4}$ is one of the following:

1. $H_{2k+4}$, of weight 1,
2. The Hamiltonian cycle $G_j$ for an integer $j$ with $0 \leq j < k-1$, of weight $z^4$,
3. A Hamiltonian cycle $G$ which contains both the edges from $A_k$, but no edges from $B_k$, of weight $z^2$.

We will prove that the third case is not possible. Pick such a Hamiltonian cycle $G$. Then, for $k+3 \leq j < 2k+3$, $G$ has an edge $(j, j+1)$. Thus there is a path from $k+3$ to $2k+3$ in $G$ which does not hit 1. By following this path and the edges $(2k+3, 0)$ and $(0, k+3)$, we attain a cycle in $G$ which does not hit 1. Thus $G$ cannot be a Hamiltonian cycle, so the third case is not possible.

The proposition follows from the fact that each $G_j$ is a distinct Hamiltonian cycle in $P_{2k+4}$. □

Thus, Conjecture 2.2.1 does not hold, as the following corollary establishes.

**Corollary 2.2.1.** *Conjecture 2.2.1 is false.*

*Proof.* Suppose Conjecture 2.2.1 were true. Throughout this proof we will use terms introduced in Conjecture 2.2.1. By construction, for each integer $k$ with $k > 1$, $P_{2k+4}$ satisfies the conditions of Conjecture 2.2.1, with a fixed $\Delta_{\text{in}}$ and $\Delta_{\text{out}}$. We then have a positive real number $\eta$ such that for $z \in \mathbf{C}$ with $|z| < \eta$, $\text{ham}(P_{2k+4}, W_{2k+4}(z)) \neq 0$ for each integer $k$ with $k > 1$. By Proposition 2.2.1,

$$\text{ham}\left(P_{2k+4}, W_{2k+4}\left(\frac{1+\mathrm{i}}{\sqrt[4]{4k-4}}\right)\right) = 1 + (k-1)\left(\frac{1+\mathrm{i}}{\sqrt[4]{4k-4}}\right)^4 = 0$$

for each integer $k$ with $k > 1$. Clearly $\frac{1+\mathrm{i}}{\sqrt[4]{4k-4}} \to 0$ as $k \to \infty$, so there exists an integer $K$ with $K > 1$ such that $\left|\frac{1+\mathrm{i}}{\sqrt[4]{4K-4}}\right| < \eta$, and yet

$$\text{ham}\left(P_{2K+4}, W_{2K+4}\left(\frac{1+\mathrm{i}}{\sqrt[4]{4K-4}}\right)\right) = 0.$$

This contradicts Conjecture 2.2.1, so Conjecture 2.2.1 must be false. □

# 3 Cycle Covers Without Short Cycles

Since we now know both of our two conjectures about the roots of $\text{ham}(G; W)$ do not hold, we decide to think about subgraphs of a graph $G$ that come close to being Hamiltonian cycles. These graphs should be cycle covers, but should not contain short cycles. The following definition gives a precise treatment of this condition.

**Definition 3.0.1.** For a cycle cover $C$ of a graph $G$, $C$ is a *$\kappa$-cycle cover* of $G$ if the length of any cycle in $C$ is at least $\kappa$.

Due to a reduction of Papadimitriou published in [CP80], we know that the problem of finding a $\kappa$-cycle cover in a given graph $G$ is $\mathsf{NP}$-Complete whenever $\kappa \geq 5$. We now introduce yet another partition function.

**Definition 3.0.2.** Let $G$ be a directed graph on $n$ vertices, for $n$ a natural number. Let $\kappa$ be a positive integer. The *partition function of $\kappa$-cycle covers* $p_{G,\kappa}(W)$ is the polynomial $\sum_C \prod_{e \in C} w_e$, where the sum runs over all $\kappa$-cycle covers of $G$, and the multiplicands in the product are weights of the edges of $G$, interpreting the complex-valued $n \times n$ matrix $W = [w_{i,j}]$ as a weighted adjacency matrix for $G$.

Similarly to $\mathrm{ham}(G;W)$, the domain of $p_{G,\kappa}(W)$ changes based on the choice of graph $G$, because in order for the variable weight matrix $W$ to be a weighted adjacency matrix for $G$, we must have $w_e = 0$ when $e$ is not an edge of $G$.

Our desired result is that we can use methods from [Bar19] to approximate $p_{G,\kappa}(W)$ for a fixed $\kappa$, $\Delta_{\text{in}}$ and $\Delta_{\text{out}}$, where the respective indegrees of the vertices in any graph $G$ under our consideration do not exceed $\Delta_{\text{in}}$ and the respective outdegrees of the vertices in any graph $G$ under our consideration do not exceed $\Delta_{\text{out}}$. While this goal remains out of reach, we can use the same methods, under the same conditions, to approximate a polynomial $q_{G,\kappa}(W)$ with which we can learn information about $p_{G,\kappa}(W)$. Accordingly, in this section the generic graph $G$ has a natural number $n$ vertices, none of which vertices have indegree exceeding $\Delta_{\text{in}}$ or outdegree exceeding $\Delta_{\text{out}}$. We also posit that each graph $G$ with a given number of vertices must have the same cycle cover $C_0$, and that the edges of $C_0$ must all have weight 1.

Our aim in this section is to apply the methods of [BR19] to the problem of weighted counting of $\kappa$-cycle covers for a fixed $\kappa$ and $\Delta$. Our approach will result in a family of polynomials with which we can learn about $p_{G,\kappa}(W)$.

## 3.1 A System of Linear Equations

We begin with some lemmata which are relevant to the coming work.

**Lemma 3.1.1.** *Pick $x, y \in \{0,1\}$. The equation $x + y = 2z + s$ has a solution for $z, s \in \{0,1\}$ if and only if $z = xy$. Furthermore, if such a solution exists, it is given by $s = \mathrm{xor}(x,y)$.*

*Proof.* We have that $\mathrm{xor}(x,y) = (x-y)(x-y) = x^2 - 2xy + y^2 = x - 2xy + y$ for $x, y \in \{0,1\}$. Then $x + y = 2xy + \mathrm{xor}(x,y)$. If $z = xy$, then $x + y = 2z + s$ has a solution, which is unique. Otherwise either $z = 1$, in which case $1 \geq x + y = 2z + s \geq 2$, which is impossible, or $z = 0$, in which case $2 \leq x + y = s \leq 1$, which is also impossible. □

**Lemma 3.1.2.** *Pick $x_1, \dots, x_m \in \{0,1\}$. The equation $x_m = \sum_{i=1}^{m-1} 2^{m-1-i}(x_i - s_i)$ has a solution with $s_i \in \{0,1\}$ for $1 \leq i \leq m-1$ if and only if $\prod_{k=1}^{m} x_k = 0$. Furthermore, if such a solution exists, we have $s_i = \mathrm{xor}(x_i, \prod_{j=i+1}^{m} x_j)$.*

*Proof.* This lemma results from applying lemma 3.1.1 to $\prod_{k=1}^{m} x_k = 0$. □

Now, fix a directed graph $G$. Assign variable weights $w_e$ to the edges $e$ of $G$ not contained in $C_0$, and fix a positive integer $\kappa$.

For each edge $e$ of $G$, let $x_e$ be a 0-1 variable. For each cycle $K$ in $G$ of length $m$ such that $m < \kappa$, pick an ordering $e_{K,1}, \dots, e_{K,m}$ for the edges in $K$, and let $s_{K,i}$ be a 0-1 variable for $1 \leq i \leq m-1$. Let $\omega$ be a function mapping each cycle $K$ to the ordering of its own edges which we chose. Then, we have three families of linear equations in 0-1 variables with integer coefficients:

$$\sum_{e \text{ has target } v} x_e = 1, \tag{1}$$

$$\sum_{e \text{ has source } v} x_e = 1, \tag{2}$$

$$x_{e_{K,m}} - \sum_{i=1}^{m-1} 2^{m-1-i}(x_{e_{K,i}} - s_{K,i}) = 0, \tag{3}$$

where the families of equations (1) and (2) respectively range over all vertices $v$ in $G$, and the family of equations (3) ranges over all cycles $K$ in $G$ of any length $m$ such that $m < \kappa$. By taking together all of the families of linear equations above, we have a system of linear equations.

By lemma 3.1.2, we have a bijection between the set of solutions to the above system of linear equations and the set of $\kappa$-cycle covers of $G$, given by mapping a cycle cover $C$ to the unique solution such that $x_e = 1$ for $e \in C$ and $x_e = 0$ for $e \notin C$. For each edge $e$, let $w_{x_e} = w_e$; for each variable $s_{K,i}$, let $w_{s_{K,i}}$ be a complex weight. Using the convention that $0^0 = 1$, we define the weight $\mathrm{wt}(S)$ of a solution $S$ to our system of linear equations to be $\prod w_x^x$, where the product runs over each 0-1 variable $x$. We let $q_{G,\kappa,\omega}(W, L) = \sum_S \mathrm{wt}(S)$, where $W$ is an admissible weight matrix for $G$, $L$ is a list of weights for the variables $s_{K,i}$, and the sum runs over all solutions of the above system of equations.

## 3.2 Sparsity

In the above system of equations, any equation in family (1) can feature at most $\Delta_{\text{in}}$ variables, any equation in family 2 can feature at most $\Delta_{\text{out}}$ variables, and any equation in family (3) can feature at most $2\kappa-3$ variables. Thus any equation in the above system features at most $\max\{\Delta_{\text{in}}, \Delta_{\text{out}}, 2\kappa-3\}$ variables. Knowing this sparsity condition, the following proposition establishes that our system of equations is sufficiently sparse to approximate $q_{G,\kappa,\omega}(W, L)$ using methods from [BR19].

**Proposition 3.2.1.** *Let $x$ be a variable in the above system of equations and let $\Delta = \min\{\Delta_{\text{in}}, \Delta_{\text{out}}\}$. At most $2 + \sum_{m=1}^{\kappa-1} \Delta^{m-1}$ equations in the system feature $x$.*

*Proof.* Take $x$ and $\Delta$ as in the proposition. Then $x$ is in at most one equation in family (1) and at most one equation in family (2). If $x = s_{K,i}$ for any integer $i$ and cycle $K$ in $G$ of length less than $\kappa$, then $x$ is in at most one equation in family (3) and the proposition follows. Otherwise $x = x_e$ for some edge $e$ of $G$. Otherwise $x = x_e$ for some edge in $G$. Thus to prove the proposition it suffices to show that for any edge $e$ of $G$, and any integer $m$ with $m > 0$, $e$ features in at most $\Delta^m$ cycles of length $m$. We proceed to prove the stronger condition that for any vertex $v$ of $G$ and any integer $m$ with $m > 0$, $v$ features in at most $\Delta^{m-1}$ cycles of length $m$.

Let $X$ be the $n \times n$ variable matrix whose $i, j^{\text{th}}$ entry is $x_{i,j}$ if $(i, j)$ is an edge in $G$ and whose $i, j^{\text{th}}$ entry is zero if $(i, j)$ is not an edge in $G$. Fix a vertex $v$ of $G$. Without loss of generality $v$ is the $i^{\text{th}}$ vertex of $G$. Then the cycles in $G$ of length $m$ containing $v$ appear as monomials in the polynomial $X^m(i, i)$. Let $\mu(m, i, j)$ be the number of monomials in $X^m(i, j)$ for $i, j \in [n]$ and $m > 0$. We now show that $\mu(m, i, j) < \Delta^{m-1}$. Clearly $X^1(i, j)$ contains at most one monomial, so $\mu(1, i, j) \leq \Delta^0$. Fix $m > 1$ and suppose $\mu(m-1, i, j) \leq \Delta^{m-2}$. Then $X^{m-1}(i, j)$ contains at most $\Delta^{m-2}$ monomials for any $i, j \in [n]$. By construction,

$$\mu(m, i, j) \leq \left(\max_{k\in[n]} \mu(m-1, k, j)\right) \left(\max_{\ell\in[n]} \mu(1, i, \ell)\right) \Delta_{\text{out}} = \Delta_{\text{out}} \max_{k\in[n]} \mu(m-1, k, j) \leq \Delta_{\text{out}}\Delta^{m-2}$$

and

$$\mu(m, i, j) \leq \left(\max_{k\in[n]} \mu(1, k, j)\right) \left(\max_{\ell\in[n]} \mu(m-1, i, \ell)\right) \Delta_{\text{in}} = \Delta_{\text{in}} \max_{k\in[n]} \mu(m-1, i, k) \leq \Delta_{\text{in}}\Delta^{m-2},$$

so $\mu(m, i, j) \leq \min\{\Delta_{\text{out}}\Delta^{m-2}, \Delta_{\text{in}}\Delta^{m-2}\} = \Delta^{m-1}$. Thus by induction, $\mu(m, i, j) < \Delta^{m-1}$ for $m > 1$ and $i, j \in [n]$. Thus $X^m(i, i)$ has at most $\Delta^{m-1}$ monomials for $m > 1$, so any vertex in $G$ is contained in at most $\Delta^{m-1}$ cycles of length $m$. The proposition follows. □

The result from [BR19] that we will use is as follows:

**Proposition.** *([BR19]) Let $A$ be an $j \times k$ integer matrix, such that $A$ has at most $r$ nonzero entries in each row and at most $c$ nonzero entries in each column. Pick $b \in \mathbf{Z}^j$. Let*

$$X = \left\{x \in \{0,1\}^k : Ax = b\right\}$$

*and suppose $y \in X$. For each $x \in X$, let $x = (x_1, \ldots, x_k)$. Let $y = (y_1, \ldots, y_k)$. There exists some positive number $\beta$, where it suffices to let $\beta = 0.45$, such that we can efficiently (in $j$ and $k$) approximate the polynomial $\sum_{x \in X, x=(x_1,\ldots,x_k)} \prod_{i: x_i \neq y_i} w_i$ whenever $|w_i| \leq \frac{\beta}{r\sqrt{c}}$ for $1 \leq i \leq k$.*

With this proposition, we can now establish the conclusion we wanted.

**Corollary 3.2.1.** *Define functions $f$ and $g$ by the rules*

$$f(\Delta_1, \Delta_2, k) = 2 + \sum_{m=1}^{k-1} (\min\{\Delta_1, \Delta_2\})^{m-1},$$
$$g(\Delta_1, \Delta_2, k) = \max\{\Delta_1, \Delta_2, 2k-3\}.$$

*For a fixed $\Delta_{\text{in}}$, $\Delta_{\text{out}}$ and $\kappa$, as the graph $G$ and $\omega$ varies, $q_{G,\kappa,\omega}(W, L)$ can be efficiently approximated as long as every free variable $x$ in $W$ or $L$ satisfies*

$$|x| \leq \frac{0.45}{g(\Delta_{\text{in}}, \Delta_{\text{out}}, \kappa)\sqrt{f(\Delta_{\text{in}}, \Delta_{\text{out}}, \kappa)}}.$$

*Proof.* Take $f$ and $g$ as in the proposition. We know due to Proposition 3.2.1 that in the system of equations we use to define $q_{G,\kappa,\omega}(W, L)$, each variable appears in at most $f(\Delta_{\text{in}}, \Delta_{\text{out}}, \kappa)$ equations and each equation features at most $g(\Delta_{\text{in}}, \Delta_{\text{out}}, \kappa)$ variables. The result then follows from an application of the above result from [BR19]. □

Specifically, $q_{G,\kappa,\omega}(W, L)$ is a polynomial that one obtains by multiplying the monomials in $p_{G,\kappa}(W)$ by disjoint respective products of variables in $L$. Since our construction of $q_{G,\kappa,\omega}(W, L)$ does not address which variables in $L$ will be multiplied with which monomials of $p_{G,\kappa}(W)$, we do not know the difference between $p_{G,\kappa}(W)$ and $q_{G,\kappa,\omega}(W, L)$. However, if we can limit the difference (for instance, by evaluating $q_{G,\kappa,\omega}(W, L)$ at a fixed $L$), we can learn some qualitative information about $p_{G,\kappa}(W)$ from $q_{G,\kappa,\omega}(W, L)$. For instance, we might learn that there is an exponential in $n$ number of cycle covers in the vicinity of the fixed cover $C_0$. However, this fails if the difference in weights of cycle covers in $q_{G,\kappa,\omega}(W, L)$ depends on $\omega$ or the number $n$ of vertices in $G$, since then it would not be possible to ascertain the exponential trend as $\omega$ and $n$ vary. Luckily, this is not the case.

**Proposition 3.2.2.** *Let $C$ be a cycle cover of $G$ which differs from the fixed cycle cover $C_0$ by $j$ edges. Let $\Delta = \min\{\Delta_{\text{in}}, \Delta_{\text{out}}\}$. Let $\sigma$ be the solution of the linear equation which defines $q_{G,\kappa,\omega}(W, L)$ corresponding to $C$, and let $\sigma_0$ be the solution of the same system corresponding to $C_0$. The Hamming distance between $\sigma$ and $\sigma_0$ is at least $2j$ and at most*

$$2j \cdot \left(1 + \sum_{m=1}^{\kappa-1}(m-1)\Delta^{m-1}\right).$$

*Proof.* Take $C$, $\sigma$, $\sigma_0$, and $j$ as in the proposition. Suppose $\sigma$ and $\sigma_0$ differ at a variable $x$. Suppose $x = x_e$ for $e$ an edge in $G$. Then either $e$ is one of the $j$ edges in $C$ which are not in $C_0$, or $e$ is one of the $j$ edges in $C_0$ which are not in $C$. Thus $\sigma$ and $\sigma_0$ differ at a maximum of $2j$ variables of the form $x_e$, where $e$ is an edge in $G$. Otherwise $x = s_{K,i}$ for some cycle $K$ in $G$ of length less than $\kappa$ and some integer $i$. Each linear equation in family (3) has a respective cycle $M$, such that each choice of edges in $M$ which does not cover all of $M$ yields a unique solution to the equation corresponding to $M$. Thus $K$ must have an edge $e$ which is in $C$ and not in $C_0$, or which is in $C_0$ and not in $C$. There are at most $2j$ such edges; we know from the proof of Proposition 3.2.1 that for each edge $e$ of $G$, $x_e$ features in at most $(\min\{\Delta_{\text{in}}, \Delta_{\text{out}}\})^{m-1}$ cycles $K$ of length $m$. For a fixed cycle $K$ of length $m$, the possible variables $s_{K,i}$ are $s_{K,1}, \ldots, s_{K,m-1}$. Thus $x$ could be one of at most $2j \cdot \sum_{m=1}^{\kappa-1}(m-1)\Delta^{m-1}$ variables. The proposition follows. □

Thus if two cycle covers in $G$ differ at a fixed number of edges, the difference between the weights of the cycle covers does not depend on the number of vertices in $G$. Therefore we can indeed use $q_{G,\kappa,\omega}(W, L)$ to qualitatively learn about $p_{G,\kappa}(W)$ as we planned earlier.

# 4 Conclusion

Our work attempting to develop applications for the techniques from [Bar19] and [BR19] yields two results. First, it is not possible to use these methods to approximate the partition function of Hamiltonian cycles. Second, we can use these methods to approximate a polynomial with which we can learn qualitatively about the partition function of $\kappa$-cycle covers. Knowing that our chosen methods can approximate permanents quickly, this indicates that there may be a trend where they are more useful for more local problems; one verifies a cycle cover by checking that each of its vertices has indegree and outdegree 2, whereas one must check the length of cycles to verify a $\kappa$-cycle cover and one must check connectedness to verify a Hamiltonian cycle. This would be an interesting trend to investigate with other graph problems.

We still have hope that we may be able to approximate $\text{ham}(G; W)$ using other methods. We have this hope by analogy with Kirchoff's Matrix-tree theorem, which for a fixed graph $G$ gives an efficient way to compute the polynomial $\sum_T \prod_{e\in T} w_e$, where the sum runs over all spanning trees of $G$ and the multiplicands in the product are weights of the edges of $G$.

## 4.1 Acknowledgements

We thank our research mentor, Prof. Alexander Barvinok, for his guidance and insight into our chosen problems. We also thank the men and women of the University of Michigan Mathematics REU program for making this work possible with their funding, time, attention and training.

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