# Polychromatic Ramsey Properties of Ideals

Jonah Nan
Mentored by Zach Norwood

August 2021

**Abstract**

Ramsey's theorem states that if pairs of natural numbers are colored using finitely many colors, then there is an infinite monochromatic subset. We consider for which ideals $I$ on the natural numbers and for which $n$ the "infinite monochromatic" clause can be modified to "$I$-positive $n$-chromatic". The case $n = 1$ was studied by Hrusak et al. (2017). We generalize this to consider arbitrary values of $n$. We introduce a family of ideals $ED_{m,n}$ which exhibit nice polychromatic Ramsey properties. We prove exactly which Ramsey properties for colorings of pairs imply which others. We also consider generalizing from colorings of pairs to colorings of $k$-element sets.

## 1 Introduction

We begin with some basic definitions and explanations of notation to be used throughout the paper.

As is standard in set theory, $\omega$ is the natural numbers, $m = \{0, 1, ..., m-1\}$, and $[S]^k$ is the collection of $k$-element subsets of $S$.

**Definition 1.1.** *An* ***ideal*** *$I$ on a set $S$ is a collection of subsets $I \subsetneq \mathcal{P}(S)$ such that:*

- *$I$ contains all finite subsets of $S$*
- *If $B \subseteq A$ and $A \in I$, then $B \in I$.*
- *If $A, B \in I$, then $A \cup B \in I$.*
- *$I \neq \mathcal{P}(S)$.*

*We write $I^+ := \mathcal{P}(S) \setminus I$. Members of $I^+$ are called* ***I-positive*** *sets.*
*We say $I$ is* ***tall*** *if for every infinite $A \subseteq S$ there is an infinite $B \subseteq A$ such that $B \in I$.*

We're interested in ideals on countable sets, usually $\omega$ or $\omega^m$. If necessary, we can view any ideal on a countably infinite $S$ as an ideal on $\omega$ via some bijection between $\omega$ and $S$.

When we talk about the complexity of an ideal ($F_\sigma$, analytic, etc.) we're using the product topology on $\mathcal{P}(\omega) \approx 2^\omega$.

**Definition 1.2.** *A **coloring** of $k$-tuples from $S$ using $m$ colors is a function $\varphi : [S]^k \to m$. (We can think of $\varphi$ as assigning one of $m$ different colors to each size-$k$ set.) When we use the term "$k$-tuple" here, we mean an unordered set with exactly $k$ elements.*

*Given a coloring $\varphi$, a set $A$ is called $n$-**chromatic** if $\varphi$ restricted to $[A]^k$ attains at most $n$ distinct values. (This means if we restrict attention to just the set $A$, at most $n$ colors are used). When $n = 1$, these are called monochromatic sets.*

When $k = 1$, each point receives a color. This is a partition of $S$ into $m$ parts.

For the case $k = 2$, we color unordered pairs. You can think of this as coloring edges from a complete graph.

The cases for larger $k$ are more difficult to visualize.

**Definition 1.3.** *$\omega \to (I^+)^k_{m,n}$ means that given any coloring $\varphi : [\omega]^k \to m$, there is an $I$-positive $n$-chromatic set $A$.*

*We call Ramsey properties of this form "local" Ramsey properties because they only guarantee the existence of a single $n$-chromatic $I$-positive set.*

*$J^+ \to (I^+)^k_{m,n}$ means that given any coloring $\varphi : [\omega]^k \to m$, and given any $J$-positive set $B \subseteq \omega$, there is an $I$-positive $n$-chromatic set $A \subseteq B$.*

*When $J = I$, we call Ramsey properties of the above form "global" Ramsey properties because they guarantee the existence of an $n$-chromatic $I$-positive set within every $I$-positive set.*

*When $J \supsetneq I$, properties of the above form have a strength somewhere between local and global – we'll call them "regional" properties, for lack of a better term.*

*When $n = 1$, we'll call these "monochromatic" properties because they guarantee the existence of a monochromatic set. For $n > 1$, we'll call them "polychromatic" properties.*

Hrusak et al. [1] studied these notions of Ramsey properties for ideals, mostly in the case $k = 2, m = 2, n = 1$. They demonstrated an $F_\sigma$ ideal satisfying $\omega \to (I^+)^2_{2,1}$ but not satisfying $\omega \to (I^+)^2_{3,1}$, proving that these properties are not equivalent. Pelayo-Gomez [2] generalized this to find a family of ideals $ED_m$ satisfying $\omega \to (ED^+_m)^2_{m-1,1}$ but not $\omega \to (ED^+_m)^2_{m,1}$. This proves that all of the local properties with $k = 2, n = 1$ are distinct. We're interested in generalizing to the case $n > 1$.

# 2 Introducing $ED_{m,n}$

We will now introduce a family of ideals with nice Ramsey properties. To do so, we define a special $m$-coloring of pairs from $\omega^m$.

**Definition 2.1.** *The **type** of a pair $\{(x_1, ..., x_m), (y_1, ..., y_m)\} \in [\omega^m]^2$ is the least $i$ such that $x_i \neq y_i$. (Since $x \neq y$, such an $i$ exists.)*

Note that there are $m$ possible types, from type 1 through type $m$. Consider coloring pairs from $\omega^m$ according to their type. This gives an $m$-coloring of pairs from $\omega^m$, which we will call the "type coloring".

**Definition 2.2.** *$ED_{m,n}$ is the ideal generated by sets which are $n$-chromatic with respect to the type coloring. ($ED_{m,n}$ is an ideal on $\omega^m$.) (We assume $m > n$.) (In other words, a set is in $ED_{m,n}$ iff it can be written as a finite union of sets which are $n$-chromatic with respect to the type coloring.)*

Note that $ED_{2,1} = ED$ and $ED_{3,1}$ is the $\tilde{ED}$ ideal introduced by Hrusak et al. The ideals $ED_{m,1}$ are the generalizations introduced by Pelayo-Gomez.

**Proposition 2.3.** *$ED_{m,n}$ is an $F_\sigma$, tall, proper ideal. (For $m > n$.)*

*Proof.* $F_\sigma$: We define a lower semicontinuous submeasure $\psi$ where $\psi(X)$ is the least number of sets needed to write $X$ as a union of $n$-chromatic sets with respect to the type coloring. Then $fin(\psi) = ED_{m,n}$ by definition. So $ED_{m,n}$ is $F_\sigma$. (The fact that this submeasure is lower semicontinuous can be seen via a compactness argument.)

Tall: This follows from Ramsey's theorem. Given any infinite $A \subseteq \omega^m$, there is an infinite monochromatic $B \subseteq A$ with respect to the type coloring. $B$ is monochromatic, so it's $n$-chromatic for any $n$. So $B \in ED_{m,n}$. So $ED_{m,n}$ is tall.

Proper: This follows from a later lemma. Note that $\omega^m$ is an infinite $m$-wedge, so $\omega^m \notin ED_{m,n}$.

□

**Proposition 2.4.** $\omega \not\to (ED^+_{m,n})^2_{m,n}$

*Proof.* The type coloring is a counterexample. By definition, each $n$-chromatic set with respect to the type coloring is in $ED_{m,n}$. So, there is no $ED_{m,n}$-positive $n$-chromatic set for this coloring. □

Next, we will provide a characterization of $ED_{m,n}$-positive sets. To do so, we need to define some new terms.

$\exists^n x\ \psi(x)$ means there are at least $n$ distinct $x$ such that $\psi(x)$ holds.

**Definition 2.5.** *We'll work in $\omega^m$ for this definition.*

*Consider the following property:*

$\forall n\ \exists^n x_1\ \exists^n x_2\ \exists^n x_3 ... \exists^n x_m\ (x_1, ..., x_m) \in A$

*We will call a set $A \subseteq \omega^m$ satisfying this formula an **infinite $m$-wedge**.*

*Consider weakening this property so that some of the $\exists^n$ quantifiers are replaced by $\exists$. If the $i$-th quantifier is $\exists^n$, we call coordinate $i$ an **active coordinate**. Otherwise, the $i$-th quantifier is $\exists$ and coordinate $i$ is an **inactive coordinate**.*

*We will call a set $A \subseteq \omega^m$ satisfying a version of the formula with at least $k$ active coordinates an **infinite $k$-wedge**.*

*For example, if $A \subseteq \omega^5$ satisfies $\forall n\ \exists^n x_1\ \exists x_2\ \exists^n x_3\ \exists^n x_4\ \exists x_5\ (x_1, x_2, x_3, x_4, x_5) \in A$ then we call $A$ an infinite 3-wedge with active coordinates 1, 3, and 4.*

*We can further weaken the property by eliminating the $\forall n$. We will call a set $A$ satisfying this further weakening with at least $k$ active coordinates a **finite $k$-wedge** of length $n$.*

*For example, if $A \subseteq \omega^5$ satisfies $\exists^8 x_1\ \exists x_2\ \exists^8 x_3\ \exists^8 x_4\ \exists x_5\ (x_1, x_2, x_3, x_4, x_5) \in A$ then we call $A$ a finite 3-wedge of length 8 with active coordinates 1, 3, and 4.*

*Another way of looking at these definitions: an infinite $k$-wedge with active coordinates $c_1 < ... < c_k$ is a set containing $k$-wedges of all finite lengths with active coordinates $c_1 < ... < c_k$.*

*Note that if a wedge (infinite or finite with length $> 1$) has coordinate $i$ active, then the wedge has at least one pair of points of type $i$.*

*It will be useful for us to think of $\omega^m$ as consisting of $\omega$-many distinct copies of $\omega^{m-1}$. For each $a$, we will call the set $\{a\} \times \omega^{m-1}$ a "fixed-$x_1$ slice" of $\omega^m$ with $x_1$-coordinate $a$.*

*As a special degenerate case, a 0-wedge is just any nonempty set.*

**Proposition 2.6.** *An equivalent definition in the case where coordinate 1 is active: $A \subseteq \omega^m$ is an infinite $k$-wedge with active coordinates $C = \{1 < c_2 ... < c_k\}$ iff there exists an infinite sequence $a_1 < a_2 < ...$ such that $\forall i$ the $m-1$-dimensional slice of $A$ with fixed $x_1 = a_i$ is a finite $(k-1)$-wedge of length $i$ with active coordinates $c_2 < ... < c_k$.*

*Proof.* **If:** Suppose there exists an infinite sequence $a_1 < a_2 < ...$ such that $\forall i$ the $m-1$-dimensional slice of $A$ with fixed $x_1 = a_i$ is a finite $(k-1)$-wedge of length $i$ with active coordinates $c_2 < ... < c_k$.

To show that $A$ is an infinite $k$-wedge with active coordinates $1 < c_2 ... < c_k$, we need to show that it satisfies $\forall n \exists^n x_1 \exists^{b_2} x_2 ... \exists^{b_m} x_m \ (x_1, ..., x_m) \in A$ where $b_i = n$ when $i \in C$ and $b_i = 1$ otherwise. Fix some $n$. The $n$-many values of $x_1$ that we use will be $a_{n+1}, a_{n+2}, ..., a_{2n}$. We claim that each of these $a_{n+j}$ satisfies $\exists^{b_2} x_2 ... \exists^{b_m} x_m \ (a_{n+j}, ..., x_m) \in A$ where $b_i = n$ when $i \in C$ and $b_i = 1$ otherwise. Why? Because this expression is exactly what it means for the slice of $A$ with $x_1 = a_{n+j}$ to be a finite $(k-1)$-wedge of length $n$ with active coordinates $c_2 < ... < c_k$. Note that a wedge of length $n + j$ is in particular a wedge of length $n$, so by assumption this formula holds.

**Only if:** Suppose $A \subseteq \omega^m$ is an infinite $k$-wedge with active coordinates $1 < c_2 ... < c_k$. Then $A$ satisfies $\forall n \exists^n x_1 \exists^{b_2} x_2 ... \exists^{b_m} x_m \ (x_1, ..., x_m) \in A$ where $b_i = n$ when $i \in C$ and $b_i = 1$ otherwise. We'll define the sequence $a_i$ recursively.

Note that any nonempty set is a finite wedge of length 1, so for $a_1$ just pick the $x_1$-coordinate of any nonempty slice of $A$.

To define $a_{i+1}$ given $a_i$: set $n = a_i + 1$ in the above formula. $A$ satisfies

$\exists^{a_i+1} x_1 \exists^{b_2} x_2 ... \exists^{b_m} x_m \ (x_1, ..., x_m) \in A$ where $b_i = a_i + 1$ when $i \in C$ and $b_i = 1$ otherwise.

Of the $a_i + 1$-many values of $x_1$ obtained from the above formula, at least one must be greater than $a_i$. So we may let $a_{i+1} > a_i$ such that $\exists^{b_2} x_2 ... \exists^{b_m} x_m \ (a_{i+1}, ..., x_m) \in A$ where $b_i = a_i + 1$ when $i \in C$ and $b_i = 1$ otherwise. This formula means exactly that the slice of $A$ with fixed $x_1 = a_{i+1}$ is a finite $(k-1)$-wedge of length $a_i + 1$ with active coordinates $c_2 < ... < c_k$. Since $a_i + 1 >= i + 1$, this slice is in particular a wedge of length $i + 1$. So $a_{i+1}$ is as desired. □

**Lemma 2.7.** *Let $A \subseteq \omega^m$ be an infinite $k$-wedge with set of active coordinates $C = \{1 < c_2 < ... < c_k\}$. Then there's an infinite $k$-wedge $B \subseteq A$ with the same active coordinates such that $\forall j \notin C$, $B$ contains no pairs of type $j$.*

Note that in this lemma, it is important that type 1 is active. The conclusion is not necessarily true without that assumption!

*Proof.* Fix $j \notin C$. Note $j > 1$. We will show that we can find an infinite $k$-wedge $B \subseteq A$ with the same active coordinates containing no pairs of type $j$. Repeating this for each of the finitely many $j \notin C$ yields the result.

Throughout this proof, $b_i = n$ when $i \in C$ and $b_i = 1$ otherwise. From the previous proposition, we obtain a sequence $a_1 < a_2 < ...$ such that $\forall n$ the $m-1$-dimensional slice of $A$ with fixed $x_1 = a_n$ is a finite $(k-1)$-wedge of length $n$ with active coordinates $c_2 < ... < c_k$. This means for all $n$, $\exists^{b_2} x_2 ... \exists^{b_m} x_m \ (a_n, x_2, ..., x_m) \in A$.

For each $n$, we will define a (partial) function $f_n : \omega^{j-1} \to \omega$ which takes an initial segment of length $j-1$ and returns an $x_j$-coordinate. We define $f_n$ as follows: $f_n(x_1, ..., x_{j-1})$ is the least $x_j$ satisfying $\exists^{b_{j+1}} x_{j+1} ... \exists^{b_m} x_m \ (x_1, x_2, ..., x_m) \in A$. (Such an $x_j$ may not exist for some inputs, so this is a partial function.)

We define $B \subseteq A$ to consist of all points in $A$ of the form

$(a_n, ..., x_{j-1}, f_n(a_n, ..., x_{j-1}), x_{j+1}, ..., x_m)$. Note that this $B$ avoids pairs of type $j$, as whenever two elements of $B$ have the same initial segment $(a_n, ..., x_{j-1})$ their next entry must be $f_n(a_n, ..., x_{j-1})$.

It remains to show that $B$ has the same active coordinates as $A$ does. We need to show that $\forall n \exists^n x_1 \exists^{b_2} x_2 ... \exists^{b_m} x_m \ (x_1, ..., x_m) \in B$. Fix $n$.

The values of $x_1$ we use will be $a_{n+1}, a_{n+2}, ..., a_{2n}$. For each such $a_{n+l}$, we need to show $\exists^{b_2} x_2 ... \exists^{b_m} x_m \ (a_{n+l}, ..., x_m) \in B$.

By definition of $f$, if $f_{n+l}$ of some initial segment $(a_{n+1}, ..., x_{j-1})$ exists then

$\exists^{b_{j+1}} x_{j+1} ... \exists^{b_m} x_m \ (a_{n+l}, ..., x_{j-1}, f_{n+l}(a_{n+l}, ..., x_{j-1}), x_{j+1}, ..., x_m) \in A$. Note that since these points are of the correct form, this is equivalent to

$\exists^{b_{j+1}} x_{j+1} ... \exists^{b_m} x_m \ (a_{n+l}, ..., x_{j-1}, f_{n+l}(a_{n+l}, ..., x_{j-1}), x_{j+1}, ..., x_m) \in B$.

Because $j \notin C$, $b_j = 1$ and thus it suffices to show $\exists^{b_2} x_2 ... \exists^{b_{j-1}} x_{j-1}$ such that $f_{n+l}(a_{n+1}, ..., x_{j-1})$ exists.

We know the slice of $A$ at each fixed $x_1 = a_{n+l}$ is a wedge of length $n$ and thus satisfies $\exists^{b_2} x_2 ... \exists^{b_m} x_m \ (a_{n+l}, ..., x_m) \in A$. This is equivalent to

$\exists^{b_2} x_2 ... \exists^{b_{j-1}} x_{j-1}$ such that $f_{n+l}(a_{n+1}, ..., x_{j-1})$ exists. So, the proof is complete. □

**Lemma 2.8.** *If $A \subseteq \omega^m$ is an infinite $(n+1)$-wedge, then $A$ is $ED_{m,n}$-positive.*

*Proof.* Let $A$ be an infinite $(n+1)$-wedge with set of active coordinates $C = \{c_1 < ... < c_{n+1}\}$.

Consider a collection of finitely many, say $l$-many $n$-chromatic sets with respect to the type coloring: $\{S_1, ..., S_l\}$ where each $S_i$ is $n$-chromatic with respect to the type coloring. We claim that $A \setminus (S_1 \cup ... \cup S_l) \neq \emptyset$.

Since $A$ is an infinite wedge, it contains a finite wedge of length $l+1$ with the same active coordinates $c_1 < ... < c_{n+1}$. That is, $A$ satisfies $\exists^{a_1} x_1 \ \exists^{a_2} x_2 \ ... \ \exists^{a_m} x_m \ (x_1, ..., x_m) \in A$, where $a_i = l+1$ if $i \in C$ and $a_i = 1$ otherwise.

Consider what happens to this formula when we cut away $S_1$ from $A$. $S_1$ attains at most $n$ types, so there must be some $c_* \in C$ such that $S_1$ contains no pairs of type $c_*$. This means for any $(x_1, ..., x_{c_*-1})$, there is at most one value of $x_{c_*}$ such that $\exists (y_1, ..., y_m) \in S_1$ where $y_1 = x_1, ..., y_{c_*} = x_{c_*}$.

So, $A \setminus S_1$ will satisfy a modified version of the above formula with $a_{c_*}$ reduced by 1: $A \setminus S_1$ satisfies $\exists^{a_1} x_1 \ \exists^{a_2} x_2 \ ... \ \exists^{a_{c_*}-1} x_{c_*} ... \ \exists^{a_m} x_m \ (x_1, ..., x_m) \in A \setminus S_1$.

Note $c_* \in C$, so $a_{c_*} = l+1$.

By the same argument, $(A \setminus S_1) \setminus S_2 = A \setminus (S_1 \cup S_2)$ satisfies a modified version of the formula with a total of 2 subtracted from the superscripts that began as $l+1$'s. (We might subtract 2 from a single superscript, or 1 from 2 different superscripts, depending whether the $c_*$ from $S_1$ is the same as that from $S_2$.)

Repeating this $l$-many times, we conclude that $A \setminus (S_1 \cup ... \cup S_l)$ satisfies a version of the above formula with a total of $l$ subtracted from the superscripts that began as $(l+1)$'s. So, none of the superscripts ever reach 0 during this process. This means that $A \setminus (S_1 \cup ... \cup S_l) \neq \emptyset$, as it contains at least 1 point.

So, $A$ cannot be covered by a finite union of $n$-chromatic sets with respect to the type coloring. So infinite $(n+1)$-wedges are $ED_{m,n}$-positive. □

We are most interested in $m$-wedges in $\omega^m$. For these maximal wedges, there is no need to specify which coordinates are active because all $m$ coordinates must be active.

**Lemma 2.9.** *If $A \subseteq \omega^m$ is $ED_{m,m-1}$-positive, then $A$ is an infinite $m$-wedge.*

*Proof.* We'll prove the contrapositive: if $A$ is not an infinite $m$-wedge, than it is in the ideal $ED_{m,m-1}$.

If $A$ isn't an infinte $m$-wedge, then $A$ does NOT satisfy

$\forall k\ \exists^k x_1 \exists^k x_2 \exists^k x_3 ... \exists^k x_m\ (x_1, ..., x_m) \in A$.

So, there is some $k$ for which $\exists^k x_1 \exists^k x_2 \exists^k x_3 ... \exists^k x_m\ (x_1, ..., x_m) \in A$ fails.

There are fewer than $k$ values of $x_1$ so that $\exists^k x_2 \exists^k x_3 ... \exists^k x_m\ (x_1, ..., x_m) \in A$.

We want to cover $A$ by finitely many sets $S_i$ so that each $S_i$ attains at most $m-1$ types. In other words, each $S_i$ should avoid using at least 1 of the $m$ possible types.

A set $S$ avoids edges of type 1 iff all points in $S$ share the same $x_1$-coordinate.

A set $S$ avoids edges of type 2 iff whenever two points $(a_1, ..., a_m), (b_1, ..., b_m) \in S$ share the same initial segment up to the $x_2$-coordinate (exclusive), their next coordinates also agree: $a_1 = b_1$ implies $a_2 = b_2$. In other words, there is a function $f$ so that members of $S$ have the form $(a_1, f(a_1), ..., a_m)$.

In general, a set $S$ avoids edges of type $i$ iff whenever two points $(a_1, ..., a_m), (b_1, ..., b_m) \in S$ share the same initial segment up to the $x_i$-coordinate (exclusive), their next coordinates also agree: $(a_1, ..., a_{i-1}) = (b_1, ..., b_{i-1})$ implies $a_i = b_i$. In other words, there is a function $f$ so that members of $S$ have the form $(a_1, a_2, ..., a_{i-1}, f(a_1, a_2, ..., a_{i-1}), ..., a_m)$.

It follows that a set $S$ can be written as the union of $l$-many sets $S_1, ..., S_l$, each of which avoids edges of type $i$, iff for each ordered $(i-1)$-tuple $(a_1, ..., a_{i-1})$ there is a set $C$ of size at most $l$ so that whenever $(b_1, ..., b_m) \in S$ satisfies $(a_1, ..., a_{i-1}) = (b_1, ..., b_{i-1})$, we have $b_i \in C$. In other words, there is a relation $R$ from ordered $(i-1)$-tuples to singletons so that each input has at most $l$-many distinct outputs and for each $(b_1, ..., b_m) \in S$, we have $(b_1, ..., b_{i-1})\ R\ b_i$.

Recall that since $A$ is not an infinite $m$-wedge, there are fewer than $k$ values of $x_1$ so that $\exists^k x_2 \exists^k x_3 ... \exists^k x_m\ (x_1, ..., x_m) \in A$. Say the set of $x_1$-values satisfying the above is $C = \{c_1, ..., c_{k-1}\}$.

For each $c_i$, the set $\{(x_1, ..., x_m) | x_1 = c_i\}$ avoids edges of type 1. The union $S_1 := \bigcup_{i=1}^{k-1} \{(x_1, ..., x_m) | x_1 = c_i\}$ contains all points whose $x_1$-coordinates satisfy

$\exists^k x_2 \exists^k x_3 ... \exists^k x_m\ (x_1, ..., x_m) \in A$. This $S_1$ is a union of finitely many sets, each of which avoids edges of type 1.

Now consider $A \setminus S_1$. The $x_1$-coordinates of points in

$A \setminus S_1$ must NOT satisfy $\exists^k x_2 \exists^k x_3 ... \exists^k x_m \ (x_1, ..., x_m) \in A$. This means for each value of $x_1$ attained by points in $A \setminus S_1$, there are fewer than $k$-many values of $x_2$ so that $\exists^k x_3 ... \exists^k x_m \ (x_1, ..., x_m) \in A$.

Consider the relation $a_1 \ R_2 \ a_2$ iff $a_1$ is the $x_1$-coordinate of some point in $A \setminus S_1$ such that $\exists^k x_3 ... \exists^k x_m \ (a_1, a_2, x_3, ..., x_m) \in A$. By the above, this relation attains at most $(k-1)$ outputs for each input. Define the set $S_2$ to contain all points of the form $(b_1, ..., b_m)$ such that $b_1 R_2 b_2$. By the above discussion, $S_2$ can be written as a union of $(k-1)$-many sets, each of which avoids edges of type 2.

Now note that the $(x_1, x_2)$ initial segments of points in $A \setminus (S_1 \cup S_2)$ do NOT satisfy $\exists^k x_3 ... \exists^k x_m \ (x_1, ..., x_m) \in A$.

We repeat this same argument inductively. At the $i$-th step, we consider $A \setminus (S_1 \cup ... \cup S_{i-1})$. The $(x_1, x_2, ... x_{i-1})$ initial segments of points in $A \setminus (S_1 \cup ... \cup S_{i-1})$ do NOT satisfy $\exists^k x_i ... \exists^k x_m \ (x_1, ..., x_m) \in A$. This means for each initial segment $(x_1, x_2, ... x_{i-1})$ attained by points in $A \setminus (S_1 \cup ... \cup S_{i-1})$, there are fewer than $k$-many values of $x_i$ so that $\exists^k x_{i+1} ... \exists^k x_m \ (x_1, ..., x_m) \in A$.

Consider the relation $(a_1, ..., a_{i-1}) \ R_i \ a_i$ iff $(a_1, ..., a_{i-1})$ is the initial segment of some point in $A \setminus (S_1 \cup ... \cup S_{i-1})$ such that $\exists^k x_{i+1} ... \exists^k x_m \ (a_1, ..., a_i, x_{i+1}, ..., x_m) \in A$. By the above, this relation attains at most $(k-1)$ outputs for each input. Define the set $S_i$ to contain all points of the form $(b_1, ..., b_m)$ such that $(b_1, ..., b_{i-1}) \ R_i \ b_i$. By the above discussion, $S_i$ can be written as a union of $(k-1)$-many sets, each of which avoids edges of type $i$.

Now note that the $(x_1, ..., x_i)$ initial segments of points in $A \setminus (S_1 \cup ... \cup S_i)$ do NOT satisfy $\exists^k x_{i+1} ... \exists^k x_m \ (x_1, ..., x_m) \in A$.

After the $m$-th and final step of this process, we find that if a point $(x_1, ..., x_m)$ is in $A \setminus (S_1 \cup ... \cup S_m)$, the point does NOT satisfy $(x_1, ..., x_m) \in A$. In other words, $A \setminus (S_1 \cup ... \cup S_m) = \emptyset$. Thus, we have covered $A$ by the union of finitely many sets, each of which attains at most $m-1$ distinct types. So $A \in ED_{m,m-1}$.

By contraposition, if $A \subseteq \omega^m$ is $ED_{m,m-1}$-positive, then $A$ is an infinite $m$-wedge. □

This gives a nice description of the positive sets for the ideal $ED_{m,m-1}$. Since the only way to have $m$ active coordinates is to have every coordinate be active, a set $A$ is $ED_{m,m-1}$ positive iff $A$ satisfies $\forall n \ \exists^n x_1 \ \exists^n x_2 \ ... \ \exists^n x_m \ (x_1, ..., x_m) \in A$.

We conjecture that this characterization extends to all $ED_{m,n}$ ideals, not just $ED_{m,m-1}$:

**Conjecture 2.10.** *If $A \subseteq \omega^m$ is $ED_{m,n}$-positive, then $A$ is an infinite $(n+1)$-wedge.*

# 3 The Wedge Lemma and Global Ramsey Properties

In this section, we prove an important lemma which will be central to many proofs in the remainder of the paper. We use it to prove some global Ramsey properties of $ED_{m,m-1}$ ideals.

**Lemma 3.1.** *Finite wedge lemma for colorings of singletons*

*Let $m, n, p \in \omega$. There exists $l$ such that for all $p$-colorings of singletons from any finite $m$-wedge $A \subseteq \omega^m$ of length $l$, there is a monochromatic $B \subseteq A$ such that $B$ is a finite $m$-wedge of length $n$.*

*Proof.* By induction on $m$. The base case $m = 1$ is just the pigeonhole principle, as finite 1-wedges in $\omega^1$ are just sets of specified size.

Suppose the $m$ case holds, consider $m+1$. We can view a finite $m+1$-wedge in $\omega^{m+1}$ of length $l$ as a sequence of $l$-many different finite $m$-wedges of length $l$ in different copies of $\omega^m$. (Each has a different $x_1$ coordinate.) Applying the $m$-dimensional case of this lemma, by taking $l$ sufficiently large we can guarantee that each of these $m$-dimensional slices contains a monochromatic finite $m$-wedge of length $n$. The color of these monochromatic wedges may depend on which slice we're looking at, though. This gives a coloring of $x_1$ coordinates. By taking $l$ sufficiently large, by pigeonhole we can find at least $n$ distinct $x_1$ coordinates sharing a color. Taking these slices, we get a monochromatic finite $m+1$-wedge of length $n$. □

**Lemma 3.2.** *Infinite wedge lemma for colorings of singletons*

*For all $m, p$, for all $p$-colorings of singletons from any infinite $m$-wedge $A \subseteq \omega^m$, there exists an infinite monochromatic $m$-wedge $A' \subseteq A$.*

*Proof.* When $m = 1$, this is just the pigeonhole principle. Suppose $m > 1$.

Since $A$ is an infinite $m$-wedge in $\omega^m$, for any $b$ we can find an infinite sequence $a_1 < a_2 < ...$ such that the fixed-$x_1$ slices of $A$ at $x_1 = a_1, a_2, ...$ are each finite $m-1$-wedges of length $b$. This means that for any fixed $t$, there will always be some value of $x_1$ with $x_1 > t$ whose slice of $A$ is a finite $m-1$-wedge of length $b$.

Fix any $q, t$. By lemma 3.1 (finite wedge lemma for colorings of singletons), there's some $b$ such that any finite $m-1$-wedge of length $b$ contains a monochromatic finite $m-1$-wedge of length $q$. By the above, there must be some value of $x_1$ with $x_1 > t$ whose slice of $A$ is a finite $m-1$-wedge of length $b$. Thus, there's some value of $x_1$ with $x_1 > t$ whose slice of $A$ contains a monochromatic finite $m-1$-wedge of length $q$.

Apply the above with $q = 1, t = 0$ to find a monochromatic finite $m-1$-wedge of length 1 in some fixed $x_1$ slice, say $x_1 = t_1$. Then use $q = 2, t = t_1$ to find a monochromatic finite $m-1$-wedge of length 2 in some further fixed $x_1$ slice, say $x_1 = t_2 > t_1$. Then use $q = 3, t = t_2$ to find a monochromatic finite $m-1$-wedge of length 3 in some further fixed $x_1$ slice, say $x_1 = t_3 > t_2 > t_1$.... Repeat this construction recursively.

This gives an infinite sequence of $x_1$ coordinates $t_1 < t_2 < ...$ such that the fixed-$x_1$ slice of $A$ for each value of $t_i$ contains a monochromatic finite $m-1$-wedge of length $i$, call it $W_i$.

This is close to being an infinite monochromatic $m$-wedge, but the color of the each monochromatic $W_i$ may depend on $i$. There are only finitely many colors, though, so some color must appear for infinitely many $W_i$. Taking this infinite subsequence gives an infinite monochromatic $m$-wedge $A' \subseteq A$ as desired. □

**Lemma 3.3.** *Wedge-thinning lemma*

*Let $m, n, p \in \omega$. There exists $l$ with the following property: fix any $p$-coloring of $[\omega^m]^2$, fix any infinite $m$-wedge $D \subseteq \omega^m$, and fix any finite $(m-1)$-wedge $C \subseteq \omega^{m-1}$ of length $l$ contained in some fixed-$x_1$ slice whose fixed $x_1$ coordinate is strictly less than the $x_1$-coordinate of any point in $D$. Then there is an infinite $m$-wedge $D' \subseteq D$ and there is a finite $m-1$-wedge $C' \subseteq C$ of length $n$ such that each edge from $C'$ to $D'$ shares a color.*

*Proof.* We will specify requirements on $l$ later. For now, say $|C| = z$ and $C = \{c_1, c_2..., c_z\}$. We will construct a decreasing sequence of wedges $D \supseteq D_1 \supseteq D_2... \supseteq D_z = D'$ such that each edge from $c_i$ to $D_i$ shares a color (which may depend on $i$).

To construct $D_1$: We want all the edges from $c_1$ to $D_1$ to share a color. Consider coloring singletons from $D$ based on the color of their edge to $c_1$. We want $D_1$ to be monochromatic with respect to this coloring of singletons. The existence of such a monochromatic $D_1 \subseteq D$ is guaranteed by lemma 3.2 (infinite wedge lemma for colorings of singletons).

To build $D_{i+1}$, apply the same argument with $c_{i+1}$ and $D_i$ instead of $c_1$ and $D$. Do this $z$-many times to build the decreasing sequence $D \supseteq D_1 \supseteq D_2... \supseteq D_z = D'$. By construction, the color of any edge from $c_i$ to $D'$ depends only on $i$.

This gives a coloring of singletons from $C$: the color of $c_i$ is the shared color of all edges from $c_i$ to $D'$. By lemma 3.1 (finite wedge lemma for colorings of singletons), if we take $l$ to be large enough then we can guarantee the existence of a monochromatic $m-1$-wedge $C' \subseteq C$ of length $n$. By construction, all edges from $C'$ to $D'$ share a color, which is exactly what we wanted. □

**Lemma 3.4.** *Infinite wedge lemma*

*For all $m, p$, for all $p$-colorings of pairs from any infinite $m$-wedge $A \subseteq \omega^m$, there exists an infinite $m$-wedge $A' \subseteq A$ on which any two pairs of the same type have the same color.*

**Lemma 3.5.** *Finite wedge lemma*

*Let $m, n, p \in \omega$. There exists $l$ such that for all $p$-colorings of pairs from any finite $m$-wedge $A \subseteq \omega^m$ of length $l$, there is $A' \subseteq A$ which is a finite $m$-wedge of length $n$ on which any two pairs of the same type have the same color.*

We will prove both of these lemmas concurrently, using the following two propositions:

**Proposition 3.6.** *Infinite wedge lemma in dimension $m$ $\implies$ finite wedge lemma in dimension $m$*

*Proof.* Suppose the infinite wedge lemma holds in dimension $m$, but the finite wedge lemma fails in dimension $m$. This means there are some $n, p$ such that for all $l$, there is a finite $m$-wedge $A_l \subseteq \omega^m$ of length $l$ and a $p$-coloring $\varphi_l$ of pairs from $\omega^m$ such that there is no $A'_l \subseteq A_l$ which is a finite $m$-wedge of length $n$ on which any two pairs of the same type have the same color (with respect to the coloring $\varphi_l$).

By definition, each $A_l$ satisfies $\exists^l x_1 ... \exists^l x_m$ $(x_1, ..., x_m \in A_l)$. WLOG suppose each $A_l$ satisfies $\exists !^l x_1 ... \exists !^l x_m$ $(x_1, ..., x_m \in A_l)$. Here $\exists !^l$ means "there exist *exactly* $l$-many". We can do this WLOG since each $A_l$ has a subset satisfying this, and each such subset still satisfies the defining property of $A_l$: there is no $A'_l \subseteq A_l$ which is a finite $m$-wedge of length $n$ on which any two pairs of the same type have the same color (with respect to the coloring $\varphi_l$).

Note that any set satisfying $\exists !^l x_1 ... \exists !^l x_m$ $(x_1, ..., x_m \in A_l)$ is naturally isomorphic to $\{1, 2, ..., l\}^m$. And, this isomorphism preserves exactly which subsets are $m$-wedges of length $n$.

So, WLOG we may assume $A_l = \{1, 2, ..., l\}^m$ since under this isomorphism, $\{1, 2, ..., l\}^m$ satisfies the defining property of $A_l$. (The coloring $\varphi_l$ is transformed under the isomorphism as well.) If there were an $A'_l \subseteq \{1, 2, ..., l\}^m$ with the forbidden property, then under the isomorphism it would become a forbidden subset of the actual $A_l$.

So, we may WLOG take the $A_l$ to be nested with $A_l = \{1, 2, ..., l\}^m$.

Fix some enumeration of $\omega^m$. For convenience, in the following paragraph we'll write $\omega^m = \{1, 2, 3, ...\}$.

There are only finitely many $p$-colorings of pairs from $\{1, 2\}$, so infinitely many $\varphi_l$ must agree on $\{1, 2\}$. Call this infinitely-agreed-upon coloring $\psi_2$. Consider only the infinitely many $\varphi_l$ which agree with $\psi_2$ and discard the rest. There are only finitely many $p$-colorings of pairs from $\{1, 2, 3\}$, so infinitely many of the remaining $\varphi_l$ which agree with $\psi_2$ must further agree on $\{1, 2, 3\}$. Call this infinitely-agreed-upon coloring $\psi_3$. Consider only the infinitely many $\varphi_l$ which agree with $\psi_3$ and discard the rest. There are only finitely many $p$-colorings of pairs from $\{1, 2, 3, 4\}$... continue this process recursively. This gives a sequence of $p$-colorings $\psi_l$ of pairs from $\{1, 2, ..., l\}$ such that for $i < j$, $\psi_j$ agrees with $\psi_i$ on $\{1, 2, ..., i\}$. We define a coloring $\psi$ of pairs on all of $\omega^m \approx \omega$ by $\psi(i, j) = \psi_j(i, j)$ for $i < j$.

Since $\omega^m$ itself is an infinite $m$-wedge, we may apply the infinite wedge lemma in dimension $m$ to obtain an infinite $m$-wedge $A$ on which any two pairs of the same type have the same color with respect to $\psi$. In particular, we obtain a finite $m$-wedge $A' \subseteq A$ of length $n$ on which any two pairs of the same type have the same color with respect to $\psi$. WLOG suppose $A'$ is actually a finite set (our nomenclature technically allows infinite sets to be "finite wedges"). Since $A'$ is finite, it is contained within some large enough $A_q = \{1, 2, ..., q\}^m$. Under our isomorphism $\omega^m \approx \omega$, say $\max(A_q) = t$. Then the coloring $\psi$ restricted to $A_q \supseteq A'$ is simply $\psi_t$, and infinitely many $\varphi_l$ agree with $\psi_t$ on its domain. This means some $\varphi_{l^*}$ for $l^* > q$ agrees with $\psi_t$ on its domain. Since $l^* > q$, $A_q \subseteq A_{l^*}$ so $A' \subseteq A_{l^*}$.

We now have a finite $m$-wedge $A' \subseteq A_{l^*}$ of length $n$ on which any two pairs of the same type have the same color with respect to $\varphi_{l^*}$. This contradicts the defining property of $\varphi_{l^*}$, and the proof is complete.

□

**Proposition 3.7.** *Finite wedge lemma in dimension $m \implies$ Infinite wedge lemma in dimension $m + 1$*

*Proof.* This proof proceeds in two steps. In the first step, we deal with edges of type 1. In the second step, we deal with all other types.

**Step 1:** Consider an arbitrary infinite $m + 1$-wedge $A \subseteq \omega^{m+1}$. We will begin by constructing an infinite $m + 1$-wedge $B \subseteq A$ on which all edges of type 1 (i.e. edges between different $x_1$ coordinates) share a color.

We will build $B$ recursively, one fixed-$x_1$ slice at a time. We will also build a decreasing sequence of infinite $m + 1$-wedges $A = A_1 \supseteq A_2 \supseteq A_3$.... The $i$-th slice of $B$ will be taken from $A_i$, and all edges from this slice to $A_{i+1}$ will share a color (which may depend on $i$).

For the first step of the construction, we will apply lemma 3.3 (the wedge-thinning lemma) with $n = 1$.

For $C$, use any large enough fixed-$x_1$ slice of $A$. (The slices of $A$ become arbitrarily large finite $m$-wedges for large enough $x_1$.)

For $D$, use any infinite $m + 1$-wedge $D \subseteq A$ such that the $x_1$-coordinate of any point in $D$ is greater than the fixed $x_1$-coordinate of $C$.

Let $A_2$ be the $D'$ obtained from lemma 3.3, and let the first fixed-$x_1$ slice of $B$ be the $C'$ obtained from lemma 3.3. Lemma 3.3 guarantees that all edges from this $C'$ to $A_2 = D'$ share a color, as desired.

For further steps, apply lemma 3.3 (wedge-thinning lemma) again but increase $n$ and work in the latest $A_i$ instead of the original $A$.

At the $i-th$ step of the construction, we add a finite $m$-wedge of length $i$ to $B$ to obtain its $i$-th fixed-$x_1$ slice. By this construction, $B$ becomes an infinite $m+1$-wedge.

This gives $B$ and $A = A_1 \supseteq A_2 \supseteq A_3 ...$ such that the $i$-th slice of $B$ lies in $A_i$ and all edges from this slice to $A_{i+1}$ share a color. This means that for $i < j$, the color of an edge from the $i$-th slice of $B$ to the $j$-th depends only on $i$.

The color may depend on $i$, but some color must appear for infinitely many $i$ by pigeonhole. Ignore other slices; we're left with an infinite $m+1$-wedge such that all of the type-1 edges share one color.

**Step 2:** Consider the infinite $m+1$-wedge from step 1 on which all of the type-1 edges share one color. Call it $B$. Because the fixed-$x_1$ slices of $B$ contain arbitrarily large finite $m$-wedges for large enough values of $x_1$, by the finite wedge lemma in dimension $m$, for each $n \in \omega$ we can find some fixed-$x_1$ slice of $B$ which contains a finite $m$-wedge of length $n$ on which any two pairs of the same type have the same color. We do this recursively over all $n$, requiring that the $x_1$ coordinate increases as $n$ does. (This procedure is exactly analogous to the proof of lemma 3.2, infinite wedge lemma for colorings of singletons.) Call the set obtained from this construction $B'$. Note that $B'$ is an infinite $m+1$-wedge.

By construction, within a given fixed-$x_1$ slice of $B'$ each type of pair uses only one color. The assignment of colors to types may vary with $x_1$, but since there are finitely many colors and finitely many types there are finitely many possible assignments of colors to types and thus one appears infinitely often. Considering only slices using this assignment, we can trim $B'$ to obtain an infinite $m+1$-wedge where color depends only on type, except perhaps for edges of type 1 (edges between different slices). But those edges were dealt with in step 1, so in fact the proof is complete. □

With these propositions, the proof of the two lemmas is simple:

*Proof.* We induct on $m$. Our base case is the infinite wedge lemma in dimension 1, which is exactly Ramsey's theorem. Combining the prior two propositions, we have:

Infinite wedge lemma in dimension $m \implies$ finite wedge lemma in dimension $m \implies$ Infinite wedge lemma in dimension $m+1$.

By induction, both lemmas hold in every dimension. □

Armed with the infinite wedge lemma, we can now state our first theorem about Ramsey properties of the $ED_{m,n}$ ideals:

**Theorem 3.8.** $\forall m, p, \quad ED^+_{m,m-1} \to (ED^+_{m,m-1})^2_{p,m}$

*Proof.* This follows from the infinite wedge lemma and the lemma characterising $ED_{m,m-1}$-positive sets. Fix a $p$-coloring of pairs from an $ED_{m,m-1}$-positive set $A$. $A$ must be an infinite $m$-wedge, so by the infinite wedge lemma we can find an infinite $m$-wedge $B \subseteq A$ on which color depends only on type. There are only $m$ possible types, and all infinite $m$-wedges are $ED_{m,m-1}$-positive, so $B \subseteq A$ is an $m$-chromatic $ED_{m,m-1}$-positive set. □

# 4 Regional Ramsey properties

In this section, we use the wedge lemma to prove regional Ramsey properties of $ED_{m,n}$ ideals. This first theorem generalizes the previous theorem:

**Theorem 4.1.** *For any $m, n, k$, $ED^+_{m,m-1} \to (ED^+_{m,n})^2_{k,n+1}$*

*Proof.* Fix any $ED_{m,m-1}$-positive set $A$. Fix any $k$-coloring of pairs from $A$. From a prior lemma, we know that $A$ is an infinite $m$-wedge. Applying the infinite wedge lemma, we get another infinite $m$-wedge $B \subseteq A$ on which color depends only on type. In particular, $B$ is an infinite $n+1$-wedge with active coordinates $1, 2, ..., n+1$. By another prior lemma, we can find a subset $B' \subseteq B$ which is still an infinite $n+1$-wedge that contains only pairs of types $1, 2, ...,$ and $n+1$. Since color depends only on type on $B'$ and pairs from $B'$ attain only $n+1$-many types, pairs from $B'$ attain at most $n+1$-many colors. So $B'$ is $(n+1)$-chromatic. Since $B'$ is an infinite $n+1$-wedge, $B'$ is $ED_{m,n}$-positive. So we've found an $(n+1)$-chromatic, $ED_{m,n}$-positive set. □

Perhaps these could be upgraded to proofs of global properties?

**Conjecture 4.2.** *For any $m, n, k$, $ED^+_{m,n} \to (ED^+_{m,n})^2_{k,n+1}$*

**Theorem 4.3.** *$ED^+_{m,m-1} \to (ED^+_{m,n})^2_{a,b}$ whenever $m, n, a, b$ satisfy the following combinatorial property:*

*Consider dividing $m$-many objects, one of which is distinguished from the rest, into $a$-many piles. We require that for any such assignment of objects to piles, there is a collection of $b$-many piles which contains the distinguished object and contains at least $n+1$ objects in total.*

*Proof.* Think of the $m$-many types of pairs as the objects, and think of the $a$-many colors used as the piles. By the wedge lemma and the fact that $ED_{m,m-1}$-positive sets are infinite $m$-wedges, given any $ED_{m,m-1}$-positive set $A$ we can find a subset $A'$ that's an infinite $m$-wedge on which color depends only on type. This gives us an assignment of the $m$-many types to the $a$-many colors. Each type uses only one color, though the same color may be used for multiple types. The distinguished object is type 1. By assumption, there is a collection of $b$-many colors which contains the type 1 and contains at least $n+1$-many types in total. Call these colors $c_1, ..., c_b$ and types $1, t_2, ..., t_{n+1}$. Given a pair of any of these $(n+1)$ types, the pair must have one of these $b$-many colors. Since $A'$ is an infinite $m$-wedge, in particular it is an infinite $(n+1)$-wedge with active coordinates $1, t_2, ..., t_{n+1}$. By a previous lemma, we can find a subset $B \subseteq A'$ which is an infinite $(n+1)$-wedge with active coordinates $1, t_2, ..., t_{n+1}$ such that pairs from $B$ only attain types from among $1, t_2, ..., t_{n+1}$. Since $B$ is an infinite $(n+1)$-wedge, $B$ is $ED_{m,n}$-positive. Since pairs from $B$ only attain types from among $1, t_2, ..., t_{n+1}$ and pairs of these types only attain colors among $c_1, ..., c_b$, pairs from $B$ only attain colors from among $c_1, ..., c_b$. So $B$ is a $b$-chromatic positive set. □

Under what circumstances does this combinatorial requirement hold? In the worst case, the distinguished object is assigned to a pile with no other members. We then need to divide the remaining $m-1$-many objects into $a-1$-many piles. We require that for any such assignment of objects to piles, there is a collection of $b-1$-many piles containing at least $n$ objects in total.

In the worst case, the $m-1$-many indistinguishable objects are distributed as evenly as possible among the remaining $a-1$-many piles. Each of the $b-1$-many piles we choose has at least $\left\lfloor \frac{m-1}{a-1} \right\rfloor$-many objects in it. $m-1$ may not divide $a-1$ evenly, in which case some piles will have 1 more object than the rest do. We will always choose these larger piles when possible. We can choose at most $b-1$ piles. The number of larger piles available is $(m-1)$ $mod$ $(a-1)$. So, the number of larger piles we get to choose is $min(b-1,(m-1)$ $mod$ $(a-1))$. We want to have at least $n$-many objects total in the piles we choose.

Putting this all together, the requirement holds iff

$\left\lfloor \frac{m-1}{a-1} \right\rfloor (b-1) + min(b-1,(m-1)\ mod\ (a-1)) \geq n.$

**Theorem 4.4.** *$ED^+_{m,m-1} \rightarrow (ED^+_{m,n})^2_{a,b}$ whenever $m,n,a,b$ satisfy:*

$\left\lfloor \frac{m-1}{a-1} \right\rfloor (b-1) + min(b-1,(m-1)\ mod\ (a-1)) \geq n$

# 5 Local Ramsey properties

In this section, we prove *exactly* which local Ramsey properties each $ED_{m,n}$ has. As a consequence, we derive exactly which properties imply which others.

**Theorem 5.1.** *$\omega \rightarrow (ED^+_{m,n})^2_{a,b}$ iff $m,n,a,b$ satisfy:*

$\left\lfloor \frac{m}{a} \right\rfloor (b) + min(b,\ m\ mod\ a) \geq n+1$

*Note that this expression is equivalent to the following combinatorial property:*

*Consider dividing $m$-many objects into $a$-many piles. We require that for any such assignment of objects to piles, there is a collection of $b$-many piles containing at least $n+1$ objects in total.*

*Proof.* See the previous section for the proof that the combinatorial property matches the algebraic formula. In this proof, we'll reason based on the combinatorial property.

As in the proof of the prior theorem, the $m$-many types will be our objects and the $a$-many colors will be our piles.

**Only if:** Suppose this combinatorial property fails. There is some mapping $f$ taking the $m$ types to the $a$ colors so that the inverse image of any collection of $b$-many colors contains at most $n$ types in total.

Consider the following $a$-coloring of pairs from $\omega^m$: if a pair is type $i$, it is assigned color $f(i)$. What do the $b$-chromatic sets with respect to this coloring look like? Since the inverse image of any collection of $b$-many colors contains at most $n$ types in total, any $b$-chromatic set will attain at most $n$ types. Thus, any $b$-chromatic set lies in the ideal $ED_{m,n}$. So when the combinatorial property fails, $\omega \not\rightarrow (ED^+_{m,n})^2_{a,b}$.

**If:** Suppose the combinatorial property holds.

Fix an $a$-coloring of pairs from $\omega^m$. For each fixed initial segment $(x_1,...,x_{m-n-1})$, consider the set $(x_1,...,x_{m-n-1}) \times \omega^{n+1}$. Each of these sets is an isomorphic copy of $\omega^{n+1}$. Consider one of these copies. We have an $a$-coloring of pairs from $\omega^{n+1}$, so there is an infinite $n+1$-wedge $W_1$ within which color depends only on type. We can think of this as dividing the $n+1$ types into $a$-many piles representing colors (piles may be empty). Remember that because we've fixed the initial segment $(x_1, ..., x_{m-n-1})$, the $n+1$-many types under consideration here are type $m-n$, type $m-n+1, ...,$ type $m$.

If any collection of $b$ color piles contains all $n+1$ types, this gives a $b$-chromatic $n+1$-wedge $W_1$. Remembering that we're working with some fixed initial segment $(x_1, ..., x_{m-n-1})$, the set $(x_1, ..., x_{m-n-1}) \times W_1$ gives a $b$-chromatic $n+1$-wedge in $\omega^m$ (the active coordinates are $m-n, m-n+1, ..., m$). This gives a $b$-chromatic $ED_{m,n}$-positive set, and we're done.

Otherwise, no collection of $b$-many color piles contains all $n+1$ types, and this holds for all initial segments $(x_1, ..., x_{m-n-1})$.

Now for each fixed $(x_1, ..., x_{m-n-2})$, consider letting $x_{m-n-1}$ vary. This gives infinitely many isomorphic copies of $\omega^{n+1}$, each of which has some assignment of colors to types. There are only finitely many possible assignments, so one must appear for infinitely many values of $x_{m-n-1}$. We restrict our attention to these values of $x_{m-n-1}$.

We now have infinitely many copies of $\omega^{n+1}$, each of which contains an infinite $n+1$-wedge in which color depends only on type, and the assignment of colors to types is the same across different copies. These infinitely many wedges linked together across different $x_{m-n-1}$-coordinates give an infinite $n+2$-wedge. Applying the wedge lemma, we can find a subset $W_2$ which is an infinite $n+2$-wedge within which color depends only on type. The only new information here is which color type $m-n-1$ gets – the assignment of colors for types $m-n, m-n+1..., m$ must be the same as it was in each of the infinitely many $n+1$-wedges we just linked together. We had an assignment of $n+1$ types to $a$-many piles, then we added one new type (type $m-n-1$) to some pile.

Now suppose some collection of $b$ color piles contains $n+1$-many types. Since we assumed this wasn't true of our distribution of $n+1$ types before we added the new one (type $n-m-1$), there must be a collection of $b$ color piles containing $n+1$-many types and containing our new type (type $n-m-1$). Say these types are $n-m-1, t_2, ..., t_{n+1}$. Note that since we're working with some fixed initial segment $(x_1, ..., x_{m-n-2})$, our new type $m-n-1$ is isomorphic to type 1 under the isomorphism $(x_1, ..., x_{m-n-2}) \times \omega^{n+2} \approx \omega^{n+2}$. So by an old lemma, we can find a subset $W_2' \subseteq W_2$ which is an $n+1$-wedge with active coordinates $n-m-1, t_2, ..., t_{n+1}$ that only attains pairs of types from among $n-m-1, t_2, ..., t_{n+1}$. Since these types are contained in a union of $b$-many color piles, this means $W_2'$ is $b$-chromatic. This gives a $b$-chromatic $ED_{m,n}$-positive set, and we're done.

We repeat that exact same argument for every initial segment $(x_1, ..., x_{m-n-2})$. If none of them give a $b$-chromatic $ED_{m,n}$-positive set, then each of their assignments of $n+2$ types to $a$-many color piles fails to have the property that some collection of $b$ color piles contains $n+1$-many types.

As long as we have not found the desired $b$-chromatic positive set, we continue this same procedure inductively. The next step is to look at any fixed initial segment $(x_1, ..., x_{m-n-3})$ and allow $x_{m-n-2}$ to vary...

In the $i$-th step of the procedure, we look at any fixed initial segment $(x_1, ..., x_{m-n-i})$ and allow $x_{m-n-i+1}$ to vary. This gives infinitely many isomorphic copies of $\omega^{n+i-1}$, each of which has some assignment of colors to types. There are only finitely many possible assignments, so one must appear for infinitely many values of $x_{m-n-i+1}$. We restrict our attention to these values of $x_{m-n-i+1}$.

We now have infinitely many copies of $\omega^{n+i-1}$, each of which contains an infinite $n+i-1$-

wedge in which color depends only on type, and the assignment of colors to types is the same across different copies. These infinitely many wedges linked together across different $x_{m-n-i+1}$-coordinates give an infinite $n+i$-wedge. Applying the wedge lemma, we can find a subset $W_i$ which is an infinite $n+i$-wedge within which color depends only on type. The only new information here is which color type $m-n-i+1$ gets – the assignment of colors for types $m-n-i+2, ..., m$ must be the same as it was in each of the infinitely many $n+i-1$-wedges we just linked together. We had an assignment of $n+i-1$ types to $a$-many piles, then we added one new type (type $m-n-i+1$) to some pile.

Now suppose some collection of $b$ color piles contains $n+1$-many types. Since we assumed this wasn't true of our distribution of $n+i-1$ types before we added the new one (type $n-m-i+1$), there must be a collection of $b$ color piles containing $n+1$-many types and containing our new type (type $n-m-i+1$). Say these types are $n-m-i+1, t_2, ..., t_{n+1}$. Note that since we're working with some fixed initial segment $(x_1, ..., x_{m-n-i})$, our new type $m-n-i+1$ is isomorphic to type 1 under the isomorphism $(x_1, ..., x_{m-n-i}) \times \omega^{n+i} \approx \omega^{n+i}$. So by an old lemma, we can find a subset $W_i' \subseteq W_i$ which is an $n+1$-wedge with active coordinates $n-m-i+1, t_2, ..., t_{n+1}$ that only attains pairs of types from among $n-m-i+1, t_2, ..., t_{n+1}$. Since these types are contained in a union of $b$-many color piles, this means $W_i'$ is $b$-chromatic. This gives a $b$-chromatic $ED_{m,n}$-positive set, and we're done.

We repeat that exact same argument for every initial segment $(x_1, ..., x_{m-n-i})$. If none of them give a $b$-chromatic $ED_{m,n}$-positive set, then each of their assignments of $n+i$ types to $a$-many color piles fails to have the property that some collection of $b$ color piles contains $n+1$-many types.

The inductive process continues as long as we have not found the desired $b$-chromatic positive set.

(If we get all the way to step $i = m-n$ before finding the desired set, at this final step there is no initial segment to fix and no need to pass through an isomorphism – we're finally analyzing the coloring on all of $\omega^m$.)

Our combinatorial assumption tells us exactly that this search must terminate by the time we finish the final step $i = m-n$: this step involves assigning all $m$-many types to $a$-many color piles, and by assumption there *must* be some union of $b$-many piles which contains $n+1$-many types. Since the search didn't terminate at the prior step, there must be some union of $b$-many piles which contains $n+1$-many types and also contains type 1. We can then use these types to build a $b$-chromatic $ED_{m,n}$-positive set.

So this procedure for finding a $b$-chromatic $ED_{m,n}$-positive set will always terminate, and the proof is complete. □

As a corollary, we learn exactly which local Ramsey properties imply which others:

**Corollary 5.2.** *The property $\omega \to (I^+)^2_{a,b}$ implies the property $\omega \to (I^+)^2_{m,n}$ for all ideals $I$ iff $m, n, a, b$ satisfy:*
$\left\lfloor \frac{m}{a} \right\rfloor (b) + min(b,\ m \ mod\ a) \leq n$

*Note this is the opposite of the property from the previous theorem. Its combinatorial interpretation is:*

*Consider dividing $m$-many objects into $a$-many piles. We require that there is some such assignment of objects to piles such that every collection of $b$-many piles contains at most $n$ objects in total.*

*Proof.* See the previous section for the proof that the combinatorial property matches the algebraic formula. In this proof, we'll reason based on the combinatorial property.

**If:** Suppose the combinatorial property holds and suppose we have $\omega \to (I^+)^2_{a,b}$. Consider some $m$-coloring of pairs from $\omega$. We'll call these $m$ colors the "old colors", and use them to develop an $a$-coloring which we'll call the "new coloring". The old colors will be our objects, and the new colors will be our piles.

By the combinatorial property, there is some way to divide the $m$ old colors into $a$ new-color piles such that every collection of $b$ new colors contains at most $n$ old colors. Use this division to define the new $a$-coloring: given a pair, look at its old color and then check which new-color pile that old color is assigned to. This gives the pair's new color.

Since $\omega \to (I^+)^2_{a,b}$, there is some positive $b$-chromatic set $S$ with respect to our new $a$-coloring. Since $S$ is $b$-chromatic with respect to the new coloring and every collection of $b$ new colors contains at most $n$ old colors, $S$ is $n$-chromatic with respect to the old coloring. So $S$ is a positive $n$-chromatic set with respect to the old coloring. So $\omega \to (I^+)^2_{m,n}$.

**Only if:** Remember that this corollary's assumption is the opposite of the previous theorem's. So if this corollary's assumption fails, we can apply the previous theorem to conclude $\omega \to (ED^+_{m,n})^2_{a,b}$. Recall that $\omega \not\to (ED^+_{m,n})^2_{m,n}$. So the ideal $ED_{m,n}$ serves as a counterexample: $\omega \to (ED^+_{m,n})^2_{a,b}$, but $\omega \not\to (ED^+_{m,n})^2_{m,n}$. □

# 6 Colorings of $k$-tuples

In this section, we consider Ramsey properties for colorings of $k$-tuples on $ED_{m,m-1}$.

Note that by "$k$-tuple", we mean an *unordered* set with exactly $k$ elements. We only consider colorings of *unordered* sets. We use the term "$k$-tuple" because it's more succinct than "$k$-element set".

In the case $k = 2$, the Ramsey properties of $ED_{m,m-1}$ can be understood by assigning each pair to one of $m$ types, then proving that for every coloring of pairs, every infinite $m$-wedge $A$ has $B \subseteq A$ such that $B$ is still an infinite $m$-wedge and within $B$ color depends only on type. (This is the wedge lemma.) Also, we note that any infinite $m$-wedge contains pairs of all $m$ types.

When generalizing to arbitrary $k$-tuples, we begin by generalizing this notion of type.

**Definition 6.1.** *Consider members of $[\omega^m]^k$, that is, $k$-element subsets of $\omega^m$. We will assign each such $\{(a^1_1, a^1_2, ..., a^1_m), ..., (a^m_1, a^m_2, ..., a^m_m)\}$ a **type**. This type will be an element of $\{1, ..., m\}^{k-1}$.*

*Since we're given an unordered set of $k$ points, we can WLOG suppose the points $\{(a^1_1, a^1_2, ..., a^1_m), ..., (a^m_1, a^m_2, ..., a^m_m)\}$ satisfy $(a^1_1, a^1_2, ..., a^1_m) < ... < (a^m_1, a^m_2, ..., a^m_m)$ when ordered lexicographically. (First ordered by $x_1$-coordinate, with ties broken by $x_2$ coordinate, etc.)*

*We define the type of this set of $k$-points to be $(t_1, ..., t_{k-1})$ where $t_i$ is the first coordinate in which $(a^i_1, a^i_2, ..., a^i_m)$ and $(a^{i+1}_1, a^{i+1}_2, ..., a^{i+1}_m)$ differ.*

*Additionally, for $k = 1$ we declare all singletons to have the same type, the "empty type" or "null type".*

In other words, we find the type of a $k$-element set by ordering the $k$ points lexicographically then recording the type of each of the $k-1$-many pairs of points adjacent in this ordering. This gives an ordered $(k-1)$-tuple of integers from $\{1, ..., m\}$, which is the type of our $k$-element set.

Note that this definition agrees with our original definition of type for $k = 2$.

Note that there are $m^{k-1}$-many possible types.

Now, we claim that this definition of type generalizes the following properties from the $k = 2$ case: for every coloring of $k$-tuples, every infinite $m$-wedge $A$ has $B \subseteq A$ such that $B$ is still an infinite $m$-wedge and within $B$ color depends only on type. (This is a generalization of the wedge lemma.) Also, we claim that any infinite $m$-wedge contains $k$-tuples of all $m^{k-1}$ types. These are the main results of this section.

**Proposition 6.2.** *Any finite $m$-wedge $A \subseteq \omega^m$ of length $k$ contains $k$-element sets of each of the $m^{k-1}$-many possible types.*

*Proof.* We induct on $m$.

The base case is $m = 1$. There's only 1 possible type for any value of $k$ (because $1^{k-1} = 1$). So the only requirement is that our 1-wedge have at least $k$ elements, which is exactly what it means for a 1-wedge in $\omega^1$ to have length $k$.

Now, suppose the $m$ case holds and consider the $m+1$ case. We can view a finite $m+1$-wedge $A \subseteq \omega^{m+1}$ of length $k$ as a sequence $\{a_1\} \times W_1 \cup ... \cup \{a_k\} \times W_k$ of $k$-many finite $m$-wedges each of length $k$ in different copies of $\omega^m$ with different $x_1$ coordinates. Say the order of ascending $x_1$ coordinates is $W_1 < W_2 < ... < W_k$.

Suppose we want to find a $k$-element set of type $(t_1, ..., t_{k-1})$ in $A$. If 1 never appears in the desired type (no $t_i = 1$), then we can simply work in some fixed-$x_1$ $m$-dimensional slice $W_1$. Applying the $m$-dimensional case of the proposition, we get a set of the desired type within the slice. If 1 appears exactly once in the desired type, re-write the desired type $(t_1, ..., t_{k-1})$ as $(\vec{t_a}, 1, \vec{t_b})$. Here $\vec{t_a}, \vec{t_b}$ are themselves ordered tuples, neither of which contains a 1. We now work in 2 different fixed-$x_1$ $m$-dimensional slices, $W_1$ and $W_2$. Find a set $S_a$ of type $\vec{t_a}$ in $W_1$, then a set $S_b$ of type $\vec{t_b}$ in $W_2$. This is possible by inductive hypothesis. (If either of $\vec{t_a}, \vec{t_b}$ is empty, just choose any singleton – we can view a singleton a set of "null type".) Everything in $W_1$ precedes everything in $W_2$ in the lexicographic order, and any pair with one member from each of $W_1, W_2$ has type 1, so $S_a \cup S_b$ has the desired type $(\vec{t_a}, 1, \vec{t_b})$.

This procedure easily generalizes to any number of 1's in the desired type. If there are $i$-many 1's in the desired type, we work across $i+1$-many $m$-dimensional slices. There are at most $k-1$-many 1's, so we need at most $k$-many slices, which are provided by the assumption that our $m+1$-dimensional wedge $A$ has length $k$. □

We obtain the following corollary immediately:

**Corollary 6.3.** *Any infinite $m$-wedge $A \subseteq \omega^m$ contains $k$-element sets of each of the $m^{k-1}$-many possible types.*

Now, we generalize the wedge lemma to colorings of $k$-tuples. The statements of the lemmas are almost identical, but with $k$-tuples instead of pairs. The structure of the proof is somewhat similar, but now we induct on both $m$ and $k$.

Note that we proved both the $k = 1$ and the $k = 2$ cases in section 3. The $k = 1$ case is somewhat special, but the proof of the $k = 2$ case strongly resembles the general proof.

**Lemma 6.4.** *Generalized infinite wedge lemma*

*For all $m, k, p$, for all $p$-colorings of $k$-tuples from any infinite $m$-wedge $A \subseteq \omega^m$, there exists an infinite $m$-wedge $A' \subseteq A$ on which any two $k$-tuples of the same type have the same color.*

**Lemma 6.5.** *Generalized finite wedge lemma*

*Let $m, k, n, p \in \omega$. There exists $l$ such that for all $p$-colorings of $k$-tuples from any finite $m$-wedge $A \subseteq \omega^m$ of length $l$, there is $A' \subseteq A$ which is a finite $m$-wedge of length $n$ on which any two $k$-tuples of the same type have the same color.*

**Proposition 6.6.** *Generalized infinite wedge lemma for fixed $m, k \implies$ Generalized finite wedge lemma for the same $m, k$*

*Proof.* This is identical to the proof of proposition 3.6, which covered the case $k = 2$. Nothing in that proof relied upon the specific value of $k$. □

With that proposition out of the way, we now prove the generalized finite and infinite wedge lemmas at the same time.

*Proof.* The proof is by induction on both $m$ and $k$. The base cases $k = 1$ and $k = 2$ were already dealt with in section 3. The base case $m = 1$ is Ramsey's theorem for colorings of $k$-tuples.

Our outer induction will be on $m$. The base case $m = 1$ is already known, so in light of the prior proposition we just need to prove the infinite $m + 1$-dimensional case for all $k$ assuming the infinite and finite $m$-dimensional cases for all $k$.

Assume the infinite and finite $m$-dimensional cases hold for all $k$. We will use strong induction on $k$ to show that the infinite $m + 1$-dimensional case holds for all $k$. The base cases $k = 1$ and $k = 2$ have already been shown. Suppose $k > 2$ and that the theorem holds for every previous value of $k$.

Let $A \subseteq \omega^{m+1}$ be an infinite $m + 1$-wedge, and fix some coloring of $k$-tuples from $\omega^{m+1}$.

We will show that for each fixed type, we can find a subset $A^* \subseteq A$ that's still an infinite $m + 1$-wedge within which all $k$-tuples of the specified type share a color. Applying this result sequentially for each of the finitely many possible types gives the desired $A' \subseteq A$.

There are two cases.

For the rest of this proof, "slice" means "fixed-$x_1$ $m$-dimensional slice".

**Case 1:** The case where our specified type $(t_1, ..., t_{k-1})$ includes at least one 1. Split this ordered tuple at the first 1 to write the type as $(\vec{t_a}, 1, \vec{t_b})$ where $\vec{t_a}, \vec{t_b}$ are themselves ordered tuples of lengths $a - 1, b - 1$ respectively, $a, b < k$, $a + b = k$, $\vec{t_a}$ does NOT contain any 1's. $\vec{t_a}, \vec{t_b}$ represent specific types of $a$ and $b$ tuples. Note these are tuples of size strictly less than $k$. In case $a = 1$ or $b = 1$, they represent the "empty type" of a singleton.

We will want to use a generalized version of the wedge-thinning lemma from section 3:

**Claim:**

Let $n \in \omega$. There exists $l$ with the following property: for any infinite $m+1$-wedge $D \subseteq \omega^{m+1}$ and any finite $m$-wedge $C \subseteq \omega^m$ of length $l$ contained in some fixed-$x_1$ slice whose fixed $x_1$ coordinate is strictly less than the $x_1$-coordinate of any point in $D$, there is an infinite $m+1$-wedge $D' \subseteq D$ and there is a finite $m$-wedge $C' \subseteq C$ of length $n$ such that if we choose any $a$-many points from $C'$ of type $\vec{t_a}$ and any $b$-many points from $D'$ of type $\vec{t_b}$, then their union (which is a $k$-tuple of the type we care about) will have some fixed color regardless of which points we choose.

**Proof of claim:**

This is similar to the proof of the wedge-thinning lemma from section 3.

Note that any pair with one member in $C$ and another in $D$ will have type 1, since $C$ is contained in some fixed-$x_1$ slice whose fixed $x_1$ coordinate is strictly less than the $x_1$-coordinate of any point in $D$.

We'll use $\mathcal{A}$ to represent any $a$-tuple of type $\vec{t_a}$ from $C$ and $\mathcal{B}$ to represent any $b$-tuple of type $\vec{t_b}$ from $D$.

WLOG assume $C$ is actually finite. Enumerate all $a$-tuples of type $\vec{t_a}$ from $C$ (there are finitely many of them). Consider the first such $a$-tuple, let's call it $\mathcal{A}_1$.

We get a coloring of $b$-tuples from $D$ based on the color of the $k$-tuple obtained when each $b$-tuple $\mathcal{B}$ is unioned with $\mathcal{A}_1$. Since $b < k$, we can use our strong induction hypothesis to find an infinite sub-wedge $D_1 \subseteq D$ on which every $b$-tuple of type $\vec{t_b}$ shares a color. This means that on $D_1$, if we choose any $b$-many points of type $\vec{t_b}$ then when we union them with $\mathcal{A}_1$ the color of the resulting $k$-tuple is independent of our selection of $b$ points.

We repeat this same procedure, considering each subsequent $a$-tuple from $C$, to further thin our wedge $D_1$ into a sequence $D \supseteq D_1 \supseteq D_2 \supseteq ... \supseteq D_z = D'$ in a manner similar to the proof from section 3. Now if we choose any $\mathcal{A}$ from $C$ and any $\mathcal{B}$ from $D'$ of type $\vec{t_b}$, the color of the resulting $k$-tuple depends only on our choice of $\mathcal{A}$.

This gives a coloring of $a$-tuples from $C$ – the color of an $a$-tuple $\mathcal{A}$ is the color of the $k$-tuple obtained when unioning on any $\mathcal{B}$ from $D'$ of type $\vec{t_b}$. Since $a < k$, we apply the finite version of our strong induction hypothesis to conclude that if we make $l$ large enough, there must be some $C' \subseteq C$ of length $n$ within which all $a$-tuples of type $\vec{t_a}$ share a color. Now this $C', D'$ are as desired.

With this generalized wedge-thinning lemma, the proof proceeds similarly to Step 1 in the proof from section 3.

We will find a subset $A^* \subseteq A$ that's still an infinite $m+1$-wedge within which all $k$-tuples of the specified type share a color. We'll build $A^*$ recursively, one fixed-$x_1$ slice at a time. We will also build a decreasing sequence of infinite $m+1$-wedges $A = A_1 \supseteq A_2 \supseteq A_3 ....$ The $i$-th slice of $A^*$ will be taken from $A_i$. When unioning any $a$-tuple of type $\vec{t_a}$ from the $i$-th slice with any $b$-tuple of type $\vec{t_b}$ from $A_{i+1}$, the result will be a $k$-tuple of the type we care about. We will construct the $i$-th slice and $A_{i+1}$ so that all such $k$-tuples share one color (which may depend on $i$).

For the first step of the construction, we will apply the generalized wedge-thinning claim with $n = 1$.

For $C$, use any large enough fixed-$x_1$ slice of $A$. (The slices of $A$ become arbitrarily large finite $m$-wedges for large enough $x_1$.)

For $D$, use any infinite $m+1$-wedge $D \subseteq A$ such that the $x_1$-coordinate of any point in $D$ is greater than the fixed $x_1$-coordinate of $C$.

Let $A_2$ be the $D'$ obtained from the thinning process, and let the first fixed-$x_1$ slice of $A^*$ be the $C'$ obtained from the thinning process. The claim guarantees that all $k$-tuples consisting of an $a$-tuple of type $\vec{t_a}$ from this first slice and a $b$-tuple of type $\vec{t_b}$ from $A_2$ share a color, as desired.

For further steps, apply the thinning claim again but increase $n$ and work in the latest $A_i$ instead of the original $A$.

At the $i-th$ step of the construction, we add a finite $m$-wedge of length $i$ to $A^*$ to obtain its $i$-th fixed-$x_1$ slice. By this construction, $A^*$ becomes an infinite $m+1$-wedge.

This gives $A^*$ and $A = A_1 \supseteq A_2 \supseteq A_3 ...$ such that the $i$-th slice of $A^*$ lies in $A_i$ and all $k$-tuples consisting of an $a$-tuple of type $\vec{t_a}$ from this $i$-th slice and a $b$-tuple of type $\vec{t_b}$ from $A_{i+1}$ share a color.

Now if we fix any $k$-tuple of the type we care about from $A^*$, it must consist of an $a$-tuple of type $\vec{t_a}$ from some $i$-th slice and a $b$-tuple of type $\vec{t_b}$ from $A_{i+1}$. (The $b$-tuple must lie in $A_{i+1}$ because each of its members must have greater $x_1$-coordinate than that of the $a$-tuple, so each must lie in some $i+l$-th slice. Each such slice is contained in $A_{i+l} \subsetneq A_{i+1}$.) So the color of such a $k$-tuple depends only on which $i$-th slice the $a$-tuple of type $\vec{t_a}$ lies in.

The color may depend on $i$, but some color must appear for infinitely many $i$ by pigeonhole. Ignore other slices; we're left with an infinite $m+1$-wedge on which all $k$-tuples of the specified type share one color.

**Case 2:** The case where our specified type $(t_1, ..., t_{k-1})$ includes no 1's. This is almost identical to Step 2 in the proof of the $k = 2$ case from section 3.

In this case, each $k$-tuple of the type we care about is contained entirely within a single slice. We assumed the finite $m$-dimensional case in our inductive hypothesis, so we apply it recursively to build an infinite $m+1$-wedge such that within each slice, color depends only on type. The assignment of colors to types may vary from slice to slice, but there are finitely many possibilities so one must appear infinitely often. Taking these infinitely many slices gives the desired infinite $m+1$-wedge $A^* \subseteq A$ within which any two $k$-tuples of the specified type share a color.

□

From this generalized infinite wedge lemma, we obtain the following theorem:

**Theorem 6.7.** *For any $m, k, p$, $ED^+_{m,m-1} \to (ED^+_{m,m-1})^k_{p,m^{k-1}}$.*

*Proof.* Recall that positive sets for $ED_{m,m-1}$ are exactly infinite $m$-wedges. Fix some $p$-coloring of any positive set $A$. $A$ is an infinite $m$-wedge. Apply the generalized wedge lemma to find $A' \subseteq A$, another infinite $m$-wedge within which color depends only on type. $A'$ is positive since it's an infinite $m$-wedge. There are $m^{k-1}$-many types, so $A'$ is $m^{k-1}$-chromatic. So $A' \subseteq A$ is a positive $m^{k-1}$-chromatic set. □

## 7 Future Work

We conclude by discussing potential directions for future work.

Having answered the question of exactly when $\omega \to (I^+)^2_{a,b}$ implies $\omega \to (I^+)^2_{m,n}$, it would be natural to ask what happens if we change the superscript: when does $\omega \to (I^+)^c_{a,b}$ implies $\omega \to (I^+)^k_{m,n}$?

One potential method for answering this question could be to generalize the $ED_{m,n}$ construction to define some $ED^k_{m,n}$. We would want $ED^k_{m,n}$ to not satisfy $\omega \to ((ED^k_{m,n})^+)^k_{m,n}$, but we would want these ideals to have all of the "next-best" Ramsey properties. Perhaps the generalized definition of "type" in section 6 could be used for this definition.

We might be interested in studying the "Ramsey number" of an ideal $I$: the least $n$ such that $\forall p$, we have $\omega \to (I^+)^2_{p,n}$. We saw in section 3 that $ED_{m,m-1}$ has Ramsey number $m$. We could generalize this to depend on $k$ – what is the least $n$ such that $\forall p$, we have $\omega \to (I^+)^k_{p,n}$? In this case, from section 6 we know that $ED_{m,m-1}$ exhibits exponential growth in Ramsey number as a function of $k$. This raises the question: are there ideals for which this sequence grows differently? Faster than exponential? Slower?

Based on the above notion of "Ramsey number", do all $F_\sigma$ ideals have finite Ramsey numbers?

Hrusak's question of whether any tall analytic ideal satisfies $I^+ \to (I^+)^2_{2,1}$ remains unanswered. Hrusak et al. proved that no tall $F_\sigma$ ideal satisfies the above. However, we have demonstrated the "next-best thing" – $ED$ is tall, $F_\sigma$, and satisfies $ED^+ \to (ED^+)^2_{3,2}$. This suggests that polychromatic Ramsey properties behave somewhat differently to monochromatic ones. Future work might aim to apply the methods of this paper to attempt to answer Hrusak's question.

## 8 Acknowledgments

This research was supported by the University of Michigan's REU program. I'd like to thank my mentor, Zach Norwood, for his support and guidance throughout the project. I'd also like to thank the organizers of the REU program for offering me this opportunity.

## 9 References

[1] M. Hrušák, D. Meza-Alcántara, E. Thümmel, C. Uzcátegui, *Ramsey Type Properties of Ideals*, Annals of Pure and Applied Logic
[2] José de Jesús Pelayo Gómez, *Combinatorics of Borel Ideals*, arXiv:1808.09088