Using Mathematics and Computing to Understand How Bees Use Physics to Keep Their Hives Cool

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Abstract

In this project, we explored a model equation which characterizes airflow in and out of the beehive. First, we solved the model equation analytically, then we implemented two numerical methods: finite-difference method and Greens's function method. Comparisons of plots are also presented. We can see from our numerical result that both of them have $O(h^2)$ convergence. The application of this research is to devise new strategies for more sustainable human architecture and to create new ventilation systems in buildings.

Contents

NOMENCLATURE

x hive entrance coordinate

t time

$v(x,t)$ local flow velocity

$\rho(x,t)$ density of fanning bees

v_b outward air flow generated by each bee

D_v scaled momentum diffusivity

L_x hive entrance length

l_b characteristic length scale derived from fanning pressure gradient and fluid friction

Figure 1: Honeybee activity at a hive's entrance [1].

1 Beehive Problem Description

How does a beehive stay cool on a hot day? The bees move to one side of the entrance and create a fanning flow that pulls hot air out of the hive and draws cool air into the hive as seen in Fig. 1 [1]. Mahadevan's model [2] couples this fanning behavior to a minimal equation (1) for fluid mass conservation across the hive entrance. Equation (1) constitutes 2-point boundary value problem for the air velocity $v(x,t)$ as a function of the hive entrance coordinate x and time t.

$$v(x,t) = l_b v_b \left[\rho(x,t) - \frac{1}{L_x} \int_0^{L_x} \rho(x,t)\, dx \right] + D_v \frac{\partial^2 v(x,t)}{\partial x^2}, 0 \le x \le 1, v(0) = v(1) = 0 \quad (1)$$

We suppose ρ is a simple step function independent of time (t):

$$\rho(x,t) = \begin{cases} 1, & x < \frac{1}{4}, \\ 0.5, & x = \frac{1}{4}, \\ 0, & x > \frac{1}{4}, \end{cases} \quad (2)$$

as depicted in Fig. 2. In this model, the density function ρ is derived from a separate equation.

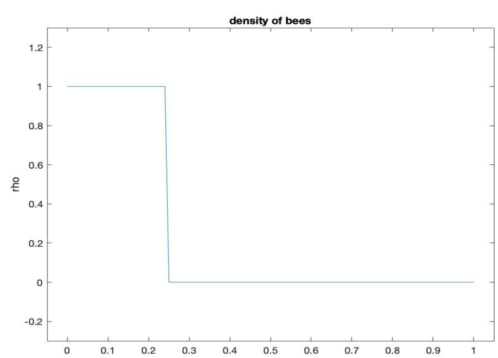


Figure 2: density of fanning bees

2 Exact Solution for 2-point BVP

Let's solve Eq. (1) analytically. We assume all parameters have the value 1 in the model equation, and v is independent of time. Then we have

$$v(x) = \rho - \frac{1}{4} + v''(x), \quad 0 \leq x \leq 1. \tag{3}$$

Since ρ is a step function, we can separate Eq. (3) into two sub-equations:

$$-v_1(x)'' + v_1(x) = \frac{3}{4}, \quad x \in \left[0, \frac{1}{4}\right], \tag{4}$$

$$-v_2(x)'' + v_2(x) = -\frac{1}{4}, \quad x \in \left[\frac{1}{4}, 1\right]. \tag{5}$$

Then we solve for v_1, v_2 separately, and meld them together at the end. In order to make the final $v(x,t)$ continuous, v_1 and v_2 have to be equal at the point $x = \frac{1}{4}$, moreover, the first derivative of v_1 and v_2 also have to be equal at the point $x = \frac{1}{4}$, in other words, $v_1(\frac{1}{4}) = v_2(\frac{1}{4})$, $v_1'(\frac{1}{4}) = v_2'(\frac{1}{4})$.

By using method of undetermined coefficients,

$$v_1 = a\sinh(x) + b\cosh(x) + \frac{3}{4} \tag{6}$$

$$v_2 = c\sinh(1-x) + d\cosh(1-x) - \frac{1}{4} \tag{7}$$

By boundary condition, $v_1(0) = 0$:

$$v_1(0) = a\sinh(0) + b\cosh(0) + \frac{3}{4} \tag{8a}$$

$$= b + \frac{3}{4} = 0 \tag{8b}$$

$$\Rightarrow b = -\frac{3}{4} \tag{8c}$$

By boundary condition at the other side, $v_2(1) = 0$:

$$v_2(1) = c\sinh(0) + d\cosh(0) - \frac{1}{4} \tag{9a}$$

$$= d - \frac{1}{4} = 0 \tag{9b}$$

$$\Rightarrow d = \frac{1}{4} \tag{9c}$$

Therefore we get,

$$v_1(x) = a\sinh(x) - \frac{3}{4}\cosh(x) + \frac{3}{4} \tag{10a}$$

$$v_2(x) = c\sinh(x) + \frac{1}{4}\cosh(x) - \frac{1}{4} \tag{10b}$$

And also by taking derivative of v_1, v_2:

$$v_1'(x) = a\cosh(x) - \frac{3}{4}\sinh(x) \tag{11a}$$

$$v_2'(x) = -c\cosh(1-x) - \frac{1}{4}\sinh(1-x) \tag{11b}$$

By $v_1(\frac{1}{4}) = v_2(\frac{1}{4})$ and Eq. (10a) & Eq. (11a):

$$a\sinh(\frac{1}{4}) - c\sinh(\frac{3}{4}) = \frac{1}{4}\cosh(\frac{3}{4}) + \frac{3}{4}\cosh(\frac{1}{4}) - 1 \tag{12}$$

By $v_1'(\frac{1}{4}) = v_2'(\frac{1}{4})$ and Eq. (10b) & Eq. (11b):

$$a\cosh(\frac{1}{4}) + c\cosh(\frac{3}{4}) = -\frac{1}{4}\sinh(\frac{3}{4}) + \frac{3}{4}\sinh(\frac{1}{4}) \tag{13}$$

solve Eq. (12) and Eq. (13), we get

$$a = \frac{1}{\sinh(1)}(\frac{1}{4} + \frac{3}{4}\cosh(1) - \cosh(\frac{3}{4})) \tag{14a}$$

$$c = \frac{1}{\sinh(1)}(-\frac{3}{4} - \frac{1}{4}\cosh(1) + \cosh(\frac{1}{4})) \tag{14b}$$

Our final exact solution is

$$v(x) = \begin{cases} v_1 = \frac{1}{\sinh(1)}(\frac{1}{4} + \frac{3}{4}\cosh(1) - \cosh(\frac{3}{4}))\sinh(x) - \frac{3}{4}\cosh(x) + \frac{3}{4} & x \in \left[0, \frac{1}{4}\right] \\ v_2 = \frac{1}{\sinh(1)}(-\frac{3}{4} - \frac{1}{4}\cosh(1) + \cosh(\frac{1}{4}))\sinh(x) + \frac{1}{4}\cosh(x) - \frac{1}{4} & x \in \left[\frac{1}{4}, 1\right] \end{cases} \tag{15}$$

And the plot is shown in Fig. (3).

3 Finite-Difference Method

Let's take a look at how finite-difference method works. Here is our differential equation and boundary conditions.

$$-v(x)'' + v(x) = f(x), \quad 0 \le x \le 1, v(0) = \alpha, v(1) = \beta \tag{16}$$

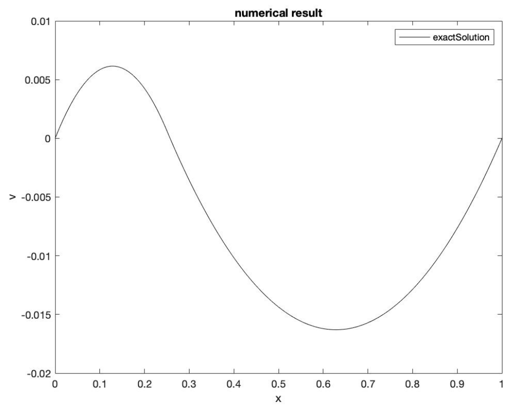


Figure 3: Exact Solution

We first divide the interval [0,1] into $n+1$ subintervals. Then each subinterval has length $h = \frac{1}{n+1}$. There are $n+1$ mesh points: $x_i = ih, i = 0, 1, ..., n+1$, as shown in Fig. 4.

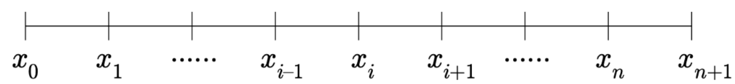


Figure 4: mesh points

Let $v_i = v(x_i)$, $f_i = f(x_i)$. By boundary conditions, we know $v_0 = v(x_0) = \alpha$, $v_{n+1} = v(x_{n+1}) = \beta$. The second derivative of v_i can be approximated discretely:

$$D_+D_-v_i = \frac{v_{i+1} - 2v_i + v_{i-1}}{h^2} = v_i'' + \frac{h^2}{12}v_i^{(4)} + O(h^4) \tag{17}$$

Denote u_i as the numerical solution, s.t. $u_i \approx v_i$, $u_0 = \alpha$, $u_{n+1} = \beta$. We have $-(\frac{u_{i+1}-2u_i+u_{i-1}}{h^2}) + u_i = f_i, i = 1, ..., n$. After we transform it to a linear system, we get:

$$\frac{1}{h^2}\begin{pmatrix} 2+h^2 & -1 & & & & \\ -1 & 2+h^2 & -1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & \ddots & \ddots \\ & & & -1 & 2+h^2 & -1 \\ & & & & -1 & 2+h^2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ \vdots \\ \vdots \\ u_{n-1} \\ u_n \end{pmatrix} = \begin{pmatrix} f_1 + \alpha/h^2 \\ f_2 \\ \vdots \\ \vdots \\ \vdots \\ f_{n-1} \\ f_n + \beta/h^2 \end{pmatrix}$$

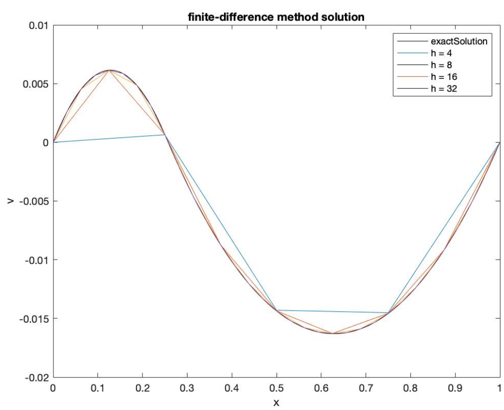
Figure 5: matrix form of finite-difference scheme: $A_h u_h = f_h$

$A_h u_h = f_h$, A_h : tridiagonal, symmetric. We solve this linear system by tridiagonal LU factorization: $A_h = L_h U_h$, where L is a lower triangular matrix, U is a upper triangular matrix.(see figure 6)

$$\begin{pmatrix} a_1 & c_1 & & & \\ b_2 & a_2 & c_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & c_{n-1} \\ & & & b_n & a_n \end{pmatrix} = \begin{pmatrix} 1 & & & & \\ \beta_2 & 1 & & & \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ & & & \beta_n & 1 \end{pmatrix} \begin{pmatrix} \alpha_1 & c_1 & & & \\ & \alpha_2 & c_2 & & \\ & & \ddots & \ddots & \\ & & & \ddots & c_{n-1} \\ & & & & \alpha_n \end{pmatrix}$$

Figure 6: LU factorization

Procedure to solve $A_h u_h = f_h$:
1) find L_h, U_h
2) solve $L_h z = f_h$
3) solve $U_h u = z$.
u is our final numerical solution. After applying finite difference method to equation (1), our result is shown in figure 7.



P	H	Error	ErrorOverH2
2	0.25	7.03586713679469e-05	0.00112573874188715
3	0.125	1.96628569184738e-05	0.00125842284278233
4	0.0625	4.92304743207259e-06	0.00126030014261058
5	0.03125	1.23122123082617e-06	0.001260770540366
6	0.015625	3.07834034849602e-07	0.00126088820674397
7	0.0078125	7.69889796185808e-08	0.00126138744207083

Figure 7: Numerical results of v(x,t)

4 Green's Function Method

Then we tried another numerical method, Green's function method, to solve the model equation. Let's look at a general equation:

$$-\phi''(x,y) + \sigma^2 \phi(x,y) = f, \quad \phi(0) = 0, \quad \phi(1) = 0 \tag{18}$$

Let g be the Green's function of Eq. (18). Then g has to satisfy 4 properties:
1. $-g_{xx}(x,y) + \sigma^2 g(x,y) = 0$ for $x \neq y$

2. $g(y^+, y) = g(y^-, y); g_x(y^+, y) - g_x(y^-, y) = -1$
3. $g(0, y) = 0, g(1, y) = 0$
4. $\phi(x) = \int_0^1 g(x, y) f(y)\, dy$

By the first three properties, we can solve the Green's function$g(x, y)$ for Eq. (18):

$$g(x,y) = \begin{cases} \dfrac{\sinh(\sigma(1-y))}{\sigma \sinh(\sigma)} \sinh(\sigma x) & x < y \\ \dfrac{\sinh(\sigma y)}{\sigma \sinh(\sigma)} \sinh(\sigma(1-x)) & x > y \end{cases} \tag{19}$$

Then we can apply property 4 to get $\phi(x)$.

4.1 Derivation

For our beehive problem, we have two sub-equations Eq. (4) & Eq (5). Therefore, we have to separate our integral into two parts as well.

$$\phi(x) = \int_0^{\frac{1}{4}} g(x,y) * \frac{3}{4}\, dy + \int_{\frac{1}{4}}^1 g(x,y) * -\frac{1}{4}\, dy \tag{20}$$

For the first part, when $0 \le x \le \frac{1}{4}$:

$$\phi(x) = \frac{3}{4}\left[\int_0^x \frac{\sinh(\sigma y)}{\sigma \sinh(\sigma)} \sinh(\sigma(1-x))\, dy + \int_x^{\frac{1}{4}} \frac{\sinh(\sigma(1-y))}{\sigma \sinh(\sigma)} \sinh(\sigma x)\, dy\right] \tag{21a}$$

$$- \frac{1}{4}\int_{\frac{1}{4}}^1 \frac{\sinh(\sigma(1-y))}{\sigma \sinh(\sigma)} \sinh(\sigma x)\, dy \tag{21b}$$

$$= \frac{3}{4}\left[\frac{\sinh(\sigma(1-x))}{\sigma \sinh(\sigma)} \frac{1}{\sigma}(\cosh(\sigma x) - \cosh(0)) + \frac{\sinh(\sigma x)}{\sigma \sinh(\sigma)}(-\frac{1}{\sigma})(\cosh(\frac{3}{4}\sigma) - \cosh(\sigma(1-x)))\right] \tag{21c}$$

$$- \frac{1}{4}\frac{\sinh(\sigma x)}{\sigma \sinh(\sigma)}(-\frac{1}{\sigma})(\cosh(0) - \cosh(\frac{3}{4}\sigma))) \tag{21d}$$

For the second part, when $\frac{1}{4} \le x \le 1$:

$$\phi(x) = \frac{3}{4}\int_0^{\frac{1}{4}} \frac{\sinh(\sigma y)}{\sigma\sinh(\sigma)}\sinh(\sigma(1-x))\,dy \tag{22a}$$

$$-\frac{1}{4}\left[\int_{\frac{1}{4}}^{x} \frac{\sinh(\sigma y)}{\sigma\sinh(\sigma)}\sinh(\sigma(1-x))\,dy + \int_x^1 \frac{\sinh(\sigma(1-y))}{\sigma\sinh(\sigma)}\sinh(\sigma x)\,dy\right] \tag{22b}$$

$$= \frac{3}{4}\frac{\sinh(\sigma(1-x))}{\sigma\sinh(\sigma)}\frac{1}{\sigma}(\cosh(\frac{1}{4}\sigma) - \cosh(0))) \tag{22c}$$

$$-\frac{1}{4}\left[\frac{\sinh(\sigma(1-x))}{\sigma\sinh(\sigma)}\frac{1}{\sigma}(\cosh(\sigma x) - \cosh(\frac{1}{4}\sigma)) + \frac{\sinh(\sigma x)}{\sigma\sinh(\sigma)}(-\frac{1}{\sigma})(\cosh(0) - \cosh(\sigma(1-x)))\right] \tag{22d}$$

4.2 Numerical solution

For our beehive problem, $\sigma = 1$ in Eq. (18). We can use midpoint method to solve Eq. (21a)(21b), (22a)(22b). First we discretize this equation, $\Delta y = \frac{1}{n} = h, y_i = ih, i = 0, ..., n$, we can then transform the integral to Riemann sum,

$$\phi(x) = \int_0^1 g(x,y)f(y)\,dy = \sum_{i=1}^{n} g(x, \frac{y_{i-1}+y_i}{2})f(\frac{y_{i-1}+y_i}{2}) * h \tag{23}$$

Although we can solve these equations directly using midpoint method, it is generally impossible to find the Green's function for a 2-point boundary problem. Instead, we first find Green's function (call it g_0) of

$$-\widehat{\phi}''(x,y) = f, \widehat{\phi}(0) = 0, \widehat{\phi}(1) = 0 \tag{24}$$

and then do some smart tricks to solve $-\phi''(x,y) + \sigma^2\phi(x,y) = f, \phi(0) = 0, \phi(1) = 0$. By doing the same procedures as solving Eq. (19),

1. $-g_{xx}(x,y) = 0$ for $x \neq y$
2. $g(y^+,y) = g(y^-,y); g_x(y^+,y) - g_x(y^-,y) = -1$
3. $g(0,y) = 0, g(1,y) = 0$
4. $\widehat{\phi}(x) = \int_0^1 g(x,y)f(y)\,dy$

we get $g_0(x,y)$.

$$g_0(x,y) = \begin{cases} (1-y)x & x < y \\ x(1-y) & x > y \end{cases} \tag{25}$$

Here is the trick. We can rewrite Eq. (18) as

$$-\phi''(x, y) = f - \sigma^2 \phi(x, y), \phi(0), \phi(1) = 0 \tag{26}$$

Then we see it has the same form as Eq. (24). By using g_0,

$$\phi(x) = \int_0^1 g_0(x, y)(f(y) - \sigma^2 \phi(x))\, dy \tag{27a}$$

$$\rightarrow \phi(x) + \sigma^2 \int_0^1 g_0(x, y)\phi(y)\, dy = \int_0^1 g_0(x, y) f(y)\, dy = \widehat{\phi}(x) \tag{27b}$$

Let $x_i = i * h$, $y_i = i * h$, $x_i^* = \frac{x_{i-1}+x_i}{2} = \frac{y_{i-1}+y_i}{2}$, $\phi_i = \phi(x_i)$, $\phi_i^* = \phi(x_i^*)$, $\widehat{\phi}_i^* = \widehat{\phi}(x_i^*)$. x_i is end point and x_i^* is midpoint, see Fig. 8 below.

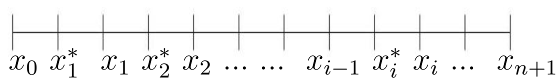


Figure 8: mesh points2

By boundary conditions, we have $\phi_0 = \phi(x_0) = 0 \quad \phi_n = \phi(x_n) = 0$.

For a mesh point x_i^*, it satisfies Eq. (27):

$$\phi(x_i^*) + \sigma^2 \int_0^1 g_0(x_i^*, y)\phi(y)\, dy = \widehat{\phi}_i^* \tag{28}$$

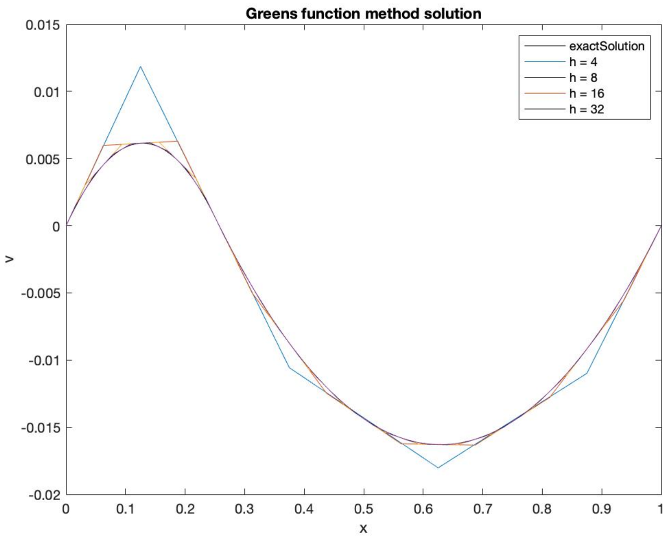
Using midpoint method, we transform it to Riemann sum:

$$\phi(x_i^*) + \sigma^2 \sum_{i=1}^{n} g_0(x_i^*, \frac{y_{i-1}+y_i}{2})\phi(\frac{y_{i-1}+y_i}{2}) * h = \widehat{\phi}_i^* \tag{29}$$

Transform it to a linear system:

$$\begin{pmatrix} 1+\sigma^2 h g_0(x_1^*, x_1^*) & \sigma^2 h g_0(x_1^*, x_2^*) & ..... & \sigma^2 h g_0(x_1^*, x_n^*) \\ \sigma^2 h g_0(x_2^*, x_1^*) & 1+\sigma^2 h g_0(x_2^*, x_2^*) & ..... & \sigma^2 h g_0(x_2^*, x_n^*) \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ \sigma^2 h g_0(x_n^*, x_1^*) & \sigma^2 h g_0(x_n^*, x_2^*) & ..... & 1+\sigma^2 h g_0(x_n^*, x_n^*) \end{pmatrix} \begin{pmatrix} \phi_1^* \\ \phi_2^* \\ . \\ . \\ . \\ \phi_n^* \end{pmatrix} = \begin{pmatrix} \widehat{\phi}_1^* \\ \widehat{\phi}_2^* \\ . \\ . \\ . \\ \widehat{\phi}_n^* \end{pmatrix}$$

We use LU factorization to solve these ϕ_i^*. After applying this method to equation(1), our result is shown in figure(9).



P	H	Error	ErrorOverH2
2	0.25	0.0057114	0.091383
3	0.125	0.0014457	0.092527
4	0.0625	0.00036379	0.093129
5	0.03125	9.1248e-05	0.093438
6	0.015625	2.285e-05	0.093593
7	0.0078125	5.7173e-06	0.093672

Figure 9: Green's function results of $v(x,t)$

5 Summary and Future Work

We described the beehive problem and solved both exact and numerical solution. From these plots, we found both of these two numerical method converges and have second order accuracy. Our future work is to explore how to solve

$$-\phi''(x) + 2a\phi'(x) + \sigma^2\phi(x) = f \quad \phi(0) = 0, \phi(1) = 0 \tag{30}$$

by using Green's function method. And eventually we want to consider two dimensional boundary value problems using finite-difference method and Green's function method.

References

[1] Lakshmi Chandrasekaran. (2019). How Bees Use Physics to Keep Hives Cool.

[2] Peters, J., Peleg, O., & Mahadevan, L. (2019). Collective ventilation in honeybee nests. J. Roy. Soc. Inter., 16(150), 20180561.

[3] Ocko, S.A., & Mahadevan, L. (2014). Collective thermoregulation in bee clusters. J. Roy. Soc. Inter., 11(91), 20131033.