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Real Analysis Qualifying Review Exam

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Problem 1. Let $f_1, f_2, \ldots, g$ be measurable functions on a measure space $(X, \mathcal{A}, \mu)$. Assume that $f_n \to f$ in measure and that $f_n \le g$ a.e. Prove that $f \le g$ a.e.

Solution. As $f_n \to f$ in measure, we know that $\lim_{n\to\infty} \mu(\{x \in X : f_n \le f - \varepsilon\}) = 0$ for each $\varepsilon > 0$. Therefore, $\mu(\{x \in X : g \le f - \varepsilon\}) = 0$ since the latter set is contained in the set $\{x \in X : f_n \le f - \varepsilon\}$ for all n. One easily concludes that

$$\mu(\{x \in X : g < f\}) \;=\; \sup_{k\in\mathbb{N}} \mu(\{x \in X : g \le f - 2^{-k}\}) \;=\; 0.$$

By definition, this means that $f \le g$ a.e.

Problem 2. Let $B = \{(x,y) \in \mathbb{R}^2 \,|\, x^2 + y^2 < 1\}$ and $f : [0,1) \to [0,+\infty]$ be a measurable function. For $(x,y) \in B$ set $F(x,y) := f(\sqrt{x^2+y^2})$. Prove that

$$F \in L^1(B, \lambda_2) \;\text{ if and only if }\; \sum_{n,m=1}^{+\infty} 2^{n-m} \lambda_1(\{r \in [2^{-m}, 1) \,|\, f(r) \ge 2^n\}) < +\infty\,,$$

where λ_2 (resp., λ_1) stands for the two-(resp., one-)dimensional Lebesgue measure.

Solution. To start with, note that the polar change of the coordinates gives

$$\int_B F d\lambda_2 = 2\pi \int_0^1 f(r)\, r dr\,.$$

For $m, n, p, q \in \mathbb{N}$, denote

$$A_{m,n} := \{r \in [2^{-m}, 1) : f(r) \ge 2^n\},$$
$$B_{p,q} := \{r \in [2^{-p}, 2^{-p+1}) : f(r) \in [2^q, 2^{q+1})\}.$$

By definition, $A_{m,n} = \bigcup_{p=1}^{m} \bigcup_{q=n}^{+\infty} B_{p,q}$. Moreover, it is easy to see that

$$2^{q-p}\lambda_1(B_{p,q}) \;\le\; \int_{B_{p,q}} f(r)\, r dr \;\le\; 4 \cdot 2^{q-p}\lambda_1(B_{p,q}).$$

Note that $\bigcup_{p,q=1}^{+\infty} B_{p,q} = B := \{r \in [0,1) : f(r) \ge 2\}$ and that this union is disjoint. Therefore,

$$\int_{[0,1)} f(r)\, r dr < +\infty \;\Leftrightarrow\; \int_B f(r)\, r dr < +\infty \;\Leftrightarrow\; \sum_{p,q=1}^{+\infty} 2^{q-p}\lambda_1(B_{p,q}) < +\infty\,.$$

At the same time,

$$\sum_{n,m=1}^{+\infty} 2^{n-m}\lambda_1(A_{n,m}) \;=\; \sum_{n,m=1}^{+\infty} 2^{n-m} \sum_{p=1}^{m} \sum_{q=n}^{+\infty} \lambda_1(B_{p,q})$$
$$= \sum_{p,q=1}^{+\infty} \lambda_1(B_{p,q}) \sum_{m=p}^{+\infty} \sum_{n=1}^{q} 2^{n-m} = \sum_{p,q=1}^{+\infty} \lambda_1(B_{p,q}) \cdot 2^{1-p}(2^{q+1} - 1).$$

It remains to note that $2^{q-p} \le 2^{1-p}(2^{q+1} - 1) \le 4 \cdot 2^{q-p}$ for all $p, q \in \mathbb{N}$.

Problem 3. Let $f : \mathbb{R} \to \mathbb{R}$ be a measurable function such that $\int_{\mathbb{R}} |f(x)|dx < +\infty$. Prove that the sequence

$$h_n(x) = \frac{1}{n}\sum_{k=1}^{n} f\left(x + \frac{k}{n}\right)$$

converges in $L^1(\mathbb{R})$ and find its limit.

Solution. Assume first that $f = \mathbb{1}_{[a;b)}$ is the characteristic function of a finite segment $[a;b) \subset \mathbb{R}$. In this case, it is easy to see that for each fixed $x \in \mathbb{R}$ one has $h_n(x) \to h(x)$ as $n \to \infty$, where $h(x) := \int_x^{x+1} f(t)dt$. Moreover,

$$\begin{array}{ll} |h_n(x) - h(x)| = 0 & \text{if } x < a-1 \text{ or } x \geq b, \\ |h_n(x) - h(x)| \leq \frac{1}{n} & \text{if } x \in [a-1; b). \end{array}$$

Therefore, $\|h_n - h\|_1 \leq \frac{1}{n}(b - a + 1)$ and the convergence $h_n \to h$ holds in $L^1(\mathbb{R})$. By additivity, the same result ($h_n \to h$ in $L^1(\mathbb{R})$, where $h(x) = \int_x^{x+1} f(t)dt$) holds for all step functions f.

Assume now that $f \in L^1(\mathbb{R})$ is an arbitrary function. Let $h(x) := \int_x^{x+1} f(x)dx$ and note that

$$\int_{\mathbb{R}} |h(x)|dx \leq \int_{\mathbb{R}}\int_x^{x+1} |f(t)|dt\,dx = \int_{\mathbb{R}} |f(t)|dt = \|f\|_1$$

due to the Tonelli theorem; in particular $h \in L^1(\mathbb{R})$. For each $\varepsilon > 0$ one can find a step function $f^{(\varepsilon)} \in L^1(\mathbb{R})$ such that $\|f - f^{(\varepsilon)}\|_1 < \frac{1}{3}\varepsilon$ and then $N(\varepsilon) \in \mathbb{N}$ such that $\|h_n^{(\varepsilon)} - h^{(\varepsilon)}\|_1 < \frac{1}{3}\varepsilon$ for all $n \geq N(\varepsilon)$, where $h^{(\varepsilon)}(x) := \int_x^{x+1} f^{(\varepsilon)}(t)dt$ and $h_n^{(\varepsilon)}(x) := \frac{1}{n}\sum_{k=1}^{n} f^{(\varepsilon)}(x + \frac{k}{n})$. We have

$$\big\|h_n^{(\varepsilon)} - h_n\big\|_1 \leq \frac{1}{n}\sum_{k=1}^{n} \big\|f^{(\varepsilon)}(\cdot + \tfrac{k}{n}) - f(\cdot + \tfrac{k}{n})\big\|_1 = \big\|f^{(\varepsilon)} - f\big\|_1 < \tfrac{1}{3}\varepsilon$$

and

$$\begin{aligned} \big\|h^{(\varepsilon)} - h\big\|_1 &= \int_{\mathbb{R}} \left|\int_x^{x+1} f^{(\varepsilon)}(t)dt - \int_x^{x+1} f(t)dt\right| dx \leq \int_{\mathbb{R}}\int_x^{x+1} \big|f^{(\varepsilon)}(t) - f(t)\big|dt\,dx \\ &= \int_{\mathbb{R}} \big|f^{(\varepsilon)}(t) - f(t)\big|dt = \big\|f^{(\varepsilon)} - f\big\|_1 < \tfrac{1}{3}\varepsilon. \end{aligned}$$

Therefore,

$$\|h_n - h\|_1 \leq \big\|h_n - h_n^{(\varepsilon)}\big\|_1 + \big\|h_n^{(\varepsilon)} - h^{(\varepsilon)}\big\|_1 + \big\|h^{(\varepsilon)} - h\big\|_1 < \varepsilon \quad \text{for all } n \geq N(\varepsilon),$$

which means that $h_n \to h$ in $L^1(\mathbb{R})$, where $h(x) := \int_x^{x+1} f(x)dx$.

Problem 4. Let $K = \{f : (0, +\infty) \to \mathbb{R} \mid \int_0^{+\infty} (f(x))^4 dx < 1\}$. Find

$$\sup_{f \in K} \int_0^{+\infty} \frac{(f(x))^3}{1+x} dx\,.$$

Solution. It follows from the Hölder inequality that

$$\int_0^{+\infty} \frac{(f(x))^3}{1+x} dx \leq \left(\int_0^{+\infty} |(f(x))^3|^{\frac{4}{3}} dx\right)^{\frac{3}{4}} \left(\int_0^{+\infty} \frac{dx}{(1+x)^4}\right)^{\frac{1}{4}} < 3^{-\frac{1}{4}}$$

for all $f \in K$. Hence, the supremum over all $f \in K$ cannot be greater than $3^{-\frac{1}{4}}$. In order to prove that it actually equals this number one can, e.g., consider functions $f_\varepsilon(x) := (1-\varepsilon)\cdot 3^{\frac{1}{4}}(1+x)^{-1}$ with $\varepsilon \downarrow 0$ for which the Hölder inequality becomes the equality, $\int_0^{+\infty}(f_\varepsilon(x))^4 dx = (1-\varepsilon)^4$, and $\int_0^{+\infty}(f_\varepsilon(x))^3(1+x)^{-1}dx = (1-\varepsilon)^3 \cdot 3^{-\frac{1}{4}}$.

Problem 5. Let $f_n : \mathbb{R} \to [0,1]$ be measurable functions such that $\sup_{x\in\mathbb{R}} f_n \le \frac{1}{n}$ and $\int_\mathbb{R} f_n(x)dx = 1$. Set $F(x) = \sup_{n\in\mathbb{N}} f_n(x)$. Prove that $\int_\mathbb{R} F(x)dx = +\infty$.

Solution. Clearly, $f_n \to 0$ a.e. as $n \to \infty$ and $f_n(x) \le F(x)$ for all $n \in \mathbb{N}$ and $x \in \mathbb{R}$. If we had $\int_\mathbb{R} F(x)dx < +\infty$, then the dominated convergence theorem would imply that $\lim_{n\to\infty} \int_\mathbb{R} f_n(x)dx = \int_\mathbb{R} \lim_{n\to\infty} f_n(x)dx = 0$, which is a contradiction.

There is also a more 'constructive' solution. Suppose that $\int_\mathbb{R} F(x)dx < +\infty$. Then one can find a (big) segment $[-a;a] \subset \mathbb{R}$ such that $\int_{\mathbb{R}\setminus[-a;a]} F(x)dx < \frac{1}{2}$. Let us now take $n > 4a$. As $f_n \le \frac{1}{n} < \frac{1}{4a}$ everywhere and, in particular, on the segment $[-a;a]$, we have $\int_{[-a;a]} f_n(x)dx < \frac{1}{2}$. Therefore,

$$\int_{\mathbb{R}\setminus[-a;a]} F(x)dx > \int_{\mathbb{R}\setminus[-a;a]} f_n(x)dx = 1 - \int_{[-a;a]} f_n(x)dx > \tfrac{1}{2},$$

a contradiction.