Department of Mathematics, University of Michigan
Real Analysis Qualifying Review Exam
August 16, 2024

Problem 1. Let $f_1, f_2, \ldots, g$ be measurable functions on a measure space $(X, \mathcal{A}, \mu)$. Assume that $f_n \to g$ in measure and that $f_n \le g$ a.e. Prove that $f \le g$ a.e.

Problem 2. Let $B = \{(x,y) \in \mathbb{R}^2 \,|\, x^2 + y^2 < 1\}$ and $f : [0,1) \to [0,+\infty]$ be a measurable function. For $(x,y) \in B$ set $F(x,y) := f(\sqrt{x^2+y^2})$. Prove that

$$F \in L^1(B, \lambda_2) \quad \text{if and only if} \quad \sum_{n,m=1}^{+\infty} 2^{n-m} \lambda_1(\{r \in [2^{-m}, 1) \,|\, f(r) \ge 2^n\}) < +\infty\,,$$

where λ_2 (resp., λ_1) stands for the two-(resp., one-)dimensional Lebesgue measure.

Problem 3. Let $f : \mathbb{R} \to \mathbb{R}$ be a measurable function such that $\int_{\mathbb{R}} |f(x)| dx < +\infty$. Prove that the sequence

$$h_n(x) = \frac{1}{n} \sum_{k=1}^{n} f\left(x + \frac{k}{n}\right)$$

converges in $L^1(\mathbb{R})$ and find its limit.

Problem 4. Let $K = \{f : (0,+\infty) \to \mathbb{R} \,|\, \int_0^{+\infty} (f(x))^4 dx < 1\}$. Find

$$\sup_{f \in K} \int_0^{+\infty} \frac{(f(x))^3}{1+x} dx\,.$$

Problem 5. Let $f_n : \mathbb{R} \to [0,1]$ be measurable functions such that $\sup_{x \in \mathbb{R}} f_n \le \frac{1}{n}$ and $\int_{\mathbb{R}} f_n(x) dx = 1$. Set $F(x) = \sup_{n \in \mathbb{N}} f_n(x)$. Prove that $\int_{\mathbb{R}} F(x) dx = +\infty$.