Department of Mathematics, University of Michigan
Complex Analysis Qualifying Review Exam
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Problem 1. Prove that there is no polynomial $P(z)$ such that

$$\left|P(z) - \frac{1}{2z^2+1}\right| < \frac{1}{9} \quad \text{for all } z \text{ with } 1 < |z| < 2.$$

Solution. It is easy to see from the triangle inequality that $3 \le |2z^2+1| \le 5$ if $|z| = \sqrt{2}$. Therefore, one can apply the Rouché theorem for meromorphic functions, which claims that the number of zeroes minus the number of poles of P (both counted with multiplicity) inside the circle $|z| = \sqrt{2}$ equals the same quantity for $1/(2z^2+1)$, which is -2. This is a contradiction as P does not have poles and hence this quantity has to be nonnegative.

If you do not know this generalized form of the Rouché theorem, it can be avoided by saying that

$$|(2z^2+1)P(z) - 1| < 1 \quad \text{for} \quad |z| = \sqrt{2},$$

which means – due to the usual Rouché theorem – that the number of zeroes of the function $(2z^2+1)P(z)$ inside the circle $|z| = \sqrt{2}$, counted with multiplicities, equals the number of zeroes of the polynomial 1. In other words, the polynomial $(2z^2+1)P(z)$ has to have no zeroes inside this circle, which is again impossible because of the presence of two zeroes at the points $z = \pm i/\sqrt{2}$.

Problem 2. Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, $\mathbb{S} = \{z \in \mathbb{C} : |\mathrm{Im} z| < \frac{\pi}{2}\}$, and $f : \mathbb{S} \to \mathbb{D}$ be an analytic function such that $f(0) = 0$. Prove that $|f'(0)| \le \frac{1}{2}$.

Solution. Note that the mapping $z \mapsto e^z$ bijectively sends $\mathbb{S}$ onto the right half-plane $\{z \in \mathbb{C} : \Re z > 0\}$. Therefore, the Riemann uniformization map $\phi : \mathbb{S} \to \mathbb{D}$ of $\mathbb{S}$ onto the unit disc $\mathbb{D}$ that sends 0 to 0 is given by $\phi(z) = \dfrac{e^z - 1}{e^z + 1}$. We can now apply the Schwarz lemma to the mapping $f \circ \phi^{-1} : \mathbb{D} \to \mathbb{D}$:

$$\left|(f \circ \phi^{-1})'(0)\right| = |f'(0)| \cdot |\phi'(0)|^{-1} \le 1,$$

which is equivalent to saying that $|f'(0)| \le \frac{1}{2}$ since $|\phi'(0)| = \frac{1}{2}$.

Problem 3. Evaluate the integral

$$\int_\gamma \left(\frac{1}{z^2+4} - \frac{1}{(z+2)^2} \right) dz\,,$$

where the curve γ is given by the parametrization $z(t) = te^{it}$, $t \in [0, 2\pi]$.

Solution. Note that the integrand has singularities at $z = \pm 2i$ and $z = -2$ and that only $z = -2$ and $z = -2i$ lie inside the closed contour obtained from γ and the segment $[0; 2\pi]$ traveled in the opposite direction. Further,

$$\operatorname*{res}_{z=-2i} \frac{1}{z^2+4} = \left.\frac{1}{z-2i}\right|_{z=-2i} = \frac{i}{4} \quad \text{and} \quad \operatorname*{res}_{z=-2} \frac{1}{(z+2)^2} = 0\,.$$

Therefore, the Cauchy formula gives

$$\int_\gamma \left(\frac{1}{z^2+4} - \frac{1}{(z+2)^2} \right) dz - \int_0^{2\pi} \left(\frac{1}{x^2+4} - \frac{1}{(x+2)^2} \right) dx = 2\pi i \cdot \frac{i}{4} = -\frac{\pi}{2}\,.$$

Since

$$\int_0^{2\pi}\left(\frac{1}{x^2+4}-\frac{1}{(x+2)^2}\right)dx=\left(\frac{1}{2}\arctan\frac{x}{2}+\frac{1}{x+2}\right)\Big|_0^{2\pi}=\frac{1}{2}\left(\arctan\pi-\frac{\pi}{\pi+1}\right),$$

the answer is $\frac{1}{2}\left(\arctan\pi-\frac{\pi(\pi+2)}{\pi+1}\right)$.

Problem 4. Let $\{z_n\}_{n=1}^{\infty}\subset\mathbb{C}\smallsetminus\{0\}$ be such that $z_n\to 0$ as $n\to\infty$, and $f:\mathbb{C}\smallsetminus\{z_n\}_{n=1}^{\infty}\smallsetminus\{0\}$ be an analytic function. Assume that the function f has a pole at each of the points z_n. Prove that for each $a\in\mathbb{C}$ there exists a sequence $\{w_n\}_{n=1}^{\infty}$ such that $w_n\to 0$ and $f(w_n)\to a$ as $n\to\infty$.

Solution. Suppose that such a sequence w_n does not exist. Then, one can find $\varepsilon>0$ such that $|f(z)-a|>\varepsilon$ for all $z\in\Omega:=\mathbb{C}\smallsetminus\{z_n\}_{n=1}^{\infty}\smallsetminus\{0\}$ such that $|z|<\varepsilon$. This means that the function $g(z):=1/(f(z)-a)$ is analytic and bounded in a small neighborhood of 0 in Ω. Consider now the behavior of g near z_n. The function f has a pole at z_n, which means that $|f(z)|\to\infty$ and hence $g(z)\to 0$ as $z\to z_n$. Hence, each z_n is a removable singularity of g. Therefore, g is a bounded analytic function the punctured disc $\{z\in\mathbb{C}:0<|z|<\varepsilon\}$, which means that g is actually an analytic function in a vicinity of the origin. However, this contradicts to the uniqueness principle for analytic functions since $g(z_n)=0$ and $z_n\to 0$ as $n\to\infty$.

Remark. Note that one *cannot* apply the Casorati–Weierstrass theorem to f since 0 is *not* an isolated singularity of f.

Problem 5. Let $\mathbb{H}=\{z\in\mathbb{C}:\mathrm{Im}z>0\}$ and $f:\mathbb{H}\to\mathbb{C}$ be an analytic function. Assume that

$$\sup_{y>0}\int_{-\infty}^{+\infty}|f(x+iy)|dx\;<\;+\infty\,.$$

(a) Prove that the integral

$$\int_{-\infty}^{+\infty}f(x+iy)dx$$

does not depend on $y>0$.

(b) Prove that

$$\int_{-\infty}^{+\infty}\frac{f(x+iy)}{x+iy}dx=0\quad\text{for all } y>0.$$

Solution. (a) Let $0<y_1<y_2<+\infty$ and $R>0$. As the function $f:\mathbb{H}\to\mathbb{C}$ is analytic, we know that

$$\int_{-R}^{R}f(x+iy_1)dx-\int_{-R}^{R}f(x+iy_2)dx\;=\;\int_{-R+iy_1}^{R+iy_1}f(z)dz+\int_{R+iy_2}^{-R+iy_2}f(z)dz$$

$$=-\int_{R+iy_1}^{R+iy_2}f(z)dz-\int_{-R+iy_2}^{-R+iy_1}f(z)dz=i\int_{y_1}^{y_2}f(-R+iy)dy-i\int_{y_1}^{y_2}f(R+iy)dy\,,$$

where the second equality follows from the Cauchy theorem applied to the boundary of the rectangle $\{z\in\mathbb{C}:|\mathrm{Re}(z)|\le R,\ \mathrm{Im}(z)\in[y_1;y_2]\}$.

We now want to pass to the limit as $R\to+\infty$. Since $\int_{-\infty}^{+\infty}|f(x+iy)|dx<+\infty$ for both $y=y_1,y_2$, the l.h.s. converges to $\int_{-\infty}^{+\infty}f(x+iy_1)dx-\int_{-\infty}^{+\infty}f(x+iy_2)dx$.

However, it is a priori not clear why the r.h.s. converges to 0. The easiest way to overcome this difficulty is to note that

$$\int_{-\infty}^{+\infty}\left|\int_{y_1}^{y_2} f(x+iy)dy\right|dx \;\le\; \int_{-\infty}^{+\infty}\int_{y_1}^{y_2}|f(x+iy)|dydx$$
$$=\;\int_{y_1}^{y_2}\int_{-\infty}^{+\infty}|f(x+iy)|dxdy \;\le\; (y_2-y_1)\cdot\sup_{y>0}\int_{-\infty}^{+\infty}|f(x+iy)|dx \;<\; +\infty\,.$$

This guarantees the existence of a sequence $R_n \to +\infty$ such that both integrals $\int_{y_1}^{y_2} f(\pm R+iy)dy$ tend to zero as $n\to\infty$. One can then pass to the limit in the identity coming from the Cauchy theorem along this sequence $R=R_n\to+\infty$.

(b) Fix $y_0>0$. Let

$$g(z)=\frac{f(z+\frac{i}{2}y_0)}{z+\frac{i}{2}y_0}.$$

This is again an analytic function in the upper half-plane $\mathbb{H}$ and it is easy to see that

$$\sup_{y>0}\int_{-\infty}^{+\infty}|g(x+iy)|dx \;\le\; \frac{2}{y_0}\cdot\sup_{y>0}\int_{-\infty}^{+\infty}|f(x+iy)|dx<+\infty.$$

Therefore, the integral $\int_{-\infty}^{+\infty} g(x+iy)dx$ does not depend on y. It remains to note that for any $y>0$,

$$\left|\int_{-\infty}^{+\infty}\frac{f(x+iy_0)}{x+iy_0}dx\right| \;=\left|\int_{-\infty}^{+\infty} g(x+\tfrac{i}{2}y_0)dx\right| \;=\left|\int_{-\infty}^{+\infty} g(x+iy)dx\right|$$
$$\le\; \frac{1}{y+\frac{1}{2}y_0}\int_{-\infty}^{+\infty}|f(x+i(y+\tfrac{1}{2}y_0))|dx \;\to\; 0 \;\text{ as }\; y\to+\infty$$

since the latter integrals are uniformly bounded by our assumption.