Department of Mathematics, University of Michigan
Complex Analysis Qualifying Review Exam
August 16, 2024

Problem 1. Prove that there is no polynomial $P(z)$ such that

$$\left| P(z) - \frac{1}{2z^2+1} \right| < \frac{1}{9} \quad \text{for all } z \text{ with } 1 < |z| < 2.$$

Problem 2. Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, $\mathbb{S} = \{z \in \mathbb{C} : |\text{Im} z| < \frac{\pi}{2}\}$, and $f : \mathbb{S} \to \mathbb{D}$ be an analytic function such that $f(0) = 0$. Prove that $|f'(0)| \leq \frac{1}{2}$.

Problem 3. Evaluate the integral

$$\int_\gamma \left(\frac{1}{z^2+4} - \frac{1}{(z+2)^2} \right) dz \, ,$$

where the curve γ is given by the parametrization $z(t) = te^{it}$, $t \in [0, 2\pi]$.

Problem 4. Let $\{z_n\}_{n=1}^\infty \subset \mathbb{C} \setminus \{0\}$ be such that $z_n \to 0$ as $n \to \infty$, and $f : \mathbb{C} \setminus \{z_n\}_{n=1}^\infty \setminus \{0\}$ be an analytic function. Assume that the function f has a pole at each of the points z_n. Prove that for each $a \in \mathbb{C}$ there exists a sequence $\{w_n\}_{n=1}^\infty$ such that $w_n \to 0$ and $f(w_n) \to a$ as $n \to \infty$.

Problem 5. Let $\mathbb{H} = \{z \in \mathbb{C} : \text{Im} z > 0\}$ and $f : \mathbb{H} \to \mathbb{C}$ be an analytic function. Assume that

$$\sup_{y>0} \int_{-\infty}^{+\infty} |f(x+iy)| dx \, < \, +\infty \, .$$

(a) Prove that the integral

$$\int_{-\infty}^{+\infty} f(x+iy) dx$$

does not depend on $y > 0$.

(b) Prove that

$$\int_{-\infty}^{+\infty} \frac{f(x+iy)}{x+iy} dx = 0 \quad \text{for all } y > 0.$$