THE UNIVERSITY OF MICHIGAN
DEPARTMENT OF MATHEMATICS

Qualifying Review examination in Differential Topology
Solutions

August 2024

1. Consider in $\mathbb{R}^5$ the locus M of solutions of the equations

$$x^2 + y^2 + z^2 + t^2 + u^2 = 1,$$

$$x^2 + 2y^2 + 3z^2 + 4t^2 + 5u^2 = 6.$$

Is M a smooth submanifold? Explain.

Solution: The maximum (i.e. 2×2) determinants are non-zero multiples of a product of two of variables. So points where at least two of the variables are non-zero are regular. When at most one of the variables is non-zero, its value has to be one of the values $-1, 1$ by the first equation, but then we see that it does not satisfy the second equation, so no such point is in the locus. Therefore, all points are regular and M is a closed smooth submanifold.

2. For each $n \in \mathbb{N}_0$, prove or disprove the following statement:

"If M is a smooth (=infinitely differentialble) real n-manifold, ω is a (smooth) nowhere vanishing differential 1-form on M and $x, y \in M$, then there exists open neighborhoods U of x and V of y and a diffeomorphism $f : U \to V$ such that $f^*(\omega_V) = \omega|_U$."

Solution: For $n = 0$, it is true since there are no non-zero 1-forms. For $n = 1$, a non-where zero 1-form determines length, so the statement is true due to parametrization by arc length. For $n > 1$, it is false since we can choose $d\omega$ to be non-zero at x, but 0 at y.

3. Let M be a smooth real n-manifold. Call M *orientable* if there exists an open cover (U_i) of M together with diffeomorphisms h_i from U_i to open subsets of $\mathbb{R}^n$ such that $Det(D(h_j h_i^{-1})) > 0$ where defined (where D is the total differential). Using this definition, prove or disprove the following statement:

"M is orientable if and only if there exists a nowhere vanishing (smooth) diffeential n-form on M."

Solution: The statement is true. Using the definition given, take the form $\omega_0 = dx_1 \wedge \ldots \wedge dx_n$, pull it back to U_i via h_i, and glue using smooth partition of unity. Conversely, on the atlas of the manifold, use either h_i or h_i followed by a hyperplane reflection based on whether $h_i^* \omega_0$ is a positive or negative multiple of ω.

4. Let GL_n denote the Lie group of all invertible $n \times n$ real matrices (made into a smooth manifold as an open subset of $\mathbb{R}^{n^2}$). Fix an $n \times n$ matrix M considered as an element of the tangent space to GL_n at the identity matrix I. Let v_M be the left-invariant vector field on GL_n containing M. Compute the derivative of the determinant function $det : GL_n \to \mathbb{R}$ by the vector field v_M.

   Solution: Since v_M is a left-invariant vector field, at a point $A \in GL_n$, it is given by differentiation along the vector AM. Now the total derivative of the determinant function at A is $(A^{-1})^T det(A)$ by the Cramer rule, so the answer is $tr(AMA^{-1} det(A)) = det(A) tr(M)$.

5. Denote by Sp_n the subset of $\mathbb{R}^{4n^2}$ consisting of quadruples of $n \times n$ real matrices A, B, C, D satisfying the conditions

$$C^T A = A^T C,$$

$$D^T B = B^T D,$$

$$A^T D - C^T B = I$$

   (M^T is the transpose of the matrix M, I is the identity matrix).

   (a) Prove that Sp_n is a smooth submanifold of $\mathbb{R}^{4n^2}$.

   (b) Describe the tangent space of Sp_n at the point where $A = D = I$, $B = C = 0$.

   Solution: (a) The conditions describe the split real form of the symplectic group of rank n (specifically, all matrices preserving the bilinear form $\left(\begin{array}{cc} 0 & I \\ -I & 0 \end{array} \right)$). Consequently, we have a closed subgroup of the Lie group GL_{2n}, which is therefore a closed Lie sugroup, in particular a closed submanifold of GL_{2n}, consequently a submanifold of $\mathbb{R}^{4n^2}$.

   (b) Linearizing the conditions, we get $A = -D^T$, $B = B^T$, $C = C^T$.