AIM Qualifying Review Exam in Differential Equations & Linear Algebra

August 2025

There are five (5) problems in this examination.

There should be sufficient room in this booklet for all your work. But if you use other sheets of paper, be sure to mark them clearly and staple them to the booklet. No credit will be given for answers without supporting work and/or reasoning.

Problem 1

(a) Show that a strictly-positive-definite real matrix cannot have a zero or a negative number on its main diagonal. (If you get stuck, think about some simple examples).

(b) Find a 2-by-3 matrix A and a vector b such that the general solution to $Ax = b$ is $x = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} + c \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$ where c is any real number.

(c) Assume that $B \in \mathbb{R}^{n\times n}$ has an orthogonal set of n eigenvectors. Prove that $BB^T = B^T B$.

Solution outline

(a) If $0 \geq A_{jj} = e_j^T A e_j$ then A is not positive definite.

(b) The null space of A is spanned by $\begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$, so the two rows of A are orthogonal to this space and independent. Let us choose $A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$. Now $Ax = b = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$. Other solutions are possible.

(c) B has an eigendecomposition $B = V\Lambda V^{-1} = V\Lambda V^T$ where the columns of V are orthogonal and Λ is diagonal. Thus $BB^T = V\Lambda V^T V\Lambda V^T = V\Lambda^2 V^T = B^T B$.

Problem 2

(a) A permutation matrix has a single 1 in each row and in each column, and zeros elsewhere. Show that for a 3-by-3 permutation matrix, 1 is always an eigenvalue and may have multiplicity one, two, or three.

(b) Show that for an n-by-n permutation matrix, 1 is always an eigenvalue and may have multiplicity equal to any integer from 1 to n.

(c) Suppose P is the projection matrix onto the subspace S and Q is the projection onto the orthogonal complement $S^\perp$. Show that $\|P - Q\|_2 = 1$.

Solution outline

(a) For any 3-by-3 permutation matrix, $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is an eigenvector with eigenvalue 1. For $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, and $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, 1 has multiplicity 1, 2, and 3, respectively.

(b) For any n-by-n permutation matrix, the n-vector with all ones is an eigenvector with eigenvalue 1. Let an n-by-n permutation matrix be block diagonal with two blocks. The upper left block is the j-by-j identity. The lower right block is the $(n - j)$-by-$(n - j)$ permutation matrix that shifts each entry of a vector forward by one row, and the last entry to the first row, under left multiplication. Then 1 has multiplicity j for the upper block plus 1 for the lower block, and j may range from 0 to $n - 1$. The reason 1 has multiplicity 1 for the lower block is that the eigenvalues of the last block are the $(n - j)$th roots of unity, as may be seen by writing the characteristic equation for this block.

(c) We can decompose any vector v as $v = Pv + Qv$. So $\|v\|^2 = \|Pv+Qv\|^2 = \|Pv\|^2 + \|Qv\|^2 = \|Pv - Qv\|^2$. So $\|(P - Q)v\|_2 / \|v\|_2 = 1$ for all $v \neq 0$.

Problem 3

(a) Given that the general solution to $\dfrac{du}{dt} = Au$ is $u(t) = c_1 e^{3t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, find A.

(b) Show that the origin is an asymptotically stable critical point for the system of ODEs

$$x' = -x^3 + xy^2 \quad ; \quad y' = -2x^2 y - y^3. \tag{1}$$

Solution outline

(a) $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & -8 \\ 4 & -5 \end{bmatrix}$

(b) Consider the Liapunov function $V = 2x^2 + y^2$. $dV/dt = 4x dx/dt + 2y dy/dt = -4x^4 - 2y^4 < 0$ in any neighborhood of the origin and $V > 0$ except for the origin, where they are both zero. Thus the origin is an asymptotically stable critical point.

Problem 4

(a) Give an example of $A \in \mathbb{R}^{2\times 2}$ such that the origin is a center for the linear system $x' = Ax$. For your example, show that arbitrarily small perturbations to the entries of A can change the origin to a stable or unstable critical point.

(b) Give the form of a particular solution of the ODE $y'''' + 2y'' + y = \sin t + \cos t$ as a linear combination of functions of t but do not evaluate the constants.

Solution outline

(a) One example is $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. For this case we may take $A_\epsilon = \begin{bmatrix} \epsilon & 1 \\ -1 & \epsilon \end{bmatrix}$, but similar examples work in other cases. The eigenvalues of A_ϵ are $\epsilon \pm i$, so we have a spiral source or sink for arbitrarily small ϵ.

(b) The homogeneous equation has solution $y_c = A\sin t + B\cos t + Ct\sin t + Dt\cos t$. Thus we multiply our initial guess $a\sin t + b\cos t$ by t^2, so $y_p(t) = at^2\sin t + bt^2\cos t$.

Problem 5

(a) Find the general form of the solution to the PDE

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0 \tag{2}$$

for $u(x,t)$ with boundary conditions

$$u(0,t) = 0\ ,\ u(1,t) = 0$$

and general initial conditions.

(b) For any of the solutions in part (a), let $E(t) = \frac{1}{2}\int_0^1 (\partial_t u)^2 + (\partial_x u)^2 dx$. Show that for any such solution with a nonzero initial condition, $dE/dt < 0$ and give a physical interpretation for the two terms in the integrand of E in terms of an elastic string.

Solution outline

(a) First, we write $u = X(x)T(t)$ and obtain

$$\frac{X''}{X} = \frac{T''}{T} + \frac{T'}{T} = -k^2.$$

The separation constant k^2 has been chosen to be positive so that there are nontrivial solutions that satisfy the boundary conditions $X(0) = X(1) = 0$. Such is the case for $k = n\pi$ for integers n, in which case $X = A\sin(n\pi x)$ and $T = Be^{-t/2}e^{it\sqrt{4(n\pi)^2-1}} + Ce^{-t/2}e^{-it\sqrt{4(n\pi)^2-1}}$. The general solution can be written

$$u = \sum_{n=1}^{\infty} a_n \sin(n\pi x)e^{-t/2}\sin\left(t\sqrt{4(n\pi)^2-1}\right) + b_n\sin(n\pi x)e^{-t/2}\cos\left(t\sqrt{4(n\pi)^2-1}\right).$$

(b)

$$dE/dt = \int_0^1 \partial_t u \partial_{tt} u + \partial_{xt} u \partial_x u dx \tag{3}$$

$$= \partial_t u \partial_x u|_0^1 dx + \int_0^1 \partial_t u \partial_{tt} u - \partial_t u \partial_{xx} u dx \tag{4}$$

$$= -\int_0^1 (\partial_t u)^2\, dx. \tag{5}$$

For any nonzero initial condition, at least one of the coefficients in the solution to part (a) is nonzero, and by orthogonality, the expression in (5) must be strictly negative. One can also derive the result by working directly with the solution for u in part (a). The first term in the integrand of E is the kinetic energy density of the string, and the second is the elastic potential energy density.