AIM Qualifying Review Exam in Differential Equations & Linear Algebra

August 2025

There are five (5) problems in this examination.

There should be sufficient room in this booklet for all your work. But if you use other sheets of paper, be sure to mark them clearly and staple them to the booklet. No credit will be given for answers without supporting work and/or reasoning.

Problem 1

(a) Show that a strictly-positive-definite real matrix cannot have a zero or a negative number on its main diagonal. (If you get stuck, think about some simple examples).

(b) Find a 2-by-3 matrix A and a vector b such that the general solution to $Ax = b$ is $x = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} + c \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$

where c is any real number.

(c) Assume that $B \in \mathbb{R}^{n \times n}$ has an orthogonal set of n eigenvectors. Prove that $BB^T = B^T B$.

Problem 1

Problem 1

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Problem 2

(a) A permutation matrix has a single 1 in each row and in each column, and zeros elsewhere. Show that for a 3-by-3 permutation matrix, 1 is always an eigenvalue and may have multiplicity one, two, or three.

(b) Show that for an n-by-n permutation matrix, 1 is always an eigenvalue and may have multiplicity equal to any integer from 1 to n.

(c) Suppose P is the projection matrix onto the subspace S and Q is the projection onto the orthogonal complement $S^{\perp}$. Show that $\|P - Q\|_2 = 1$.

Problem 2

Problem 2

Problem 2

Problem 3

(a) Given that the general solution to $\frac{du}{dt} = Au$ is $u(t) = c_1 e^{3t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, find A.

(b) Show that the origin is an asymptotically stable critical point for the system of ODEs

$$x' = -x^3 + xy^2 \quad ; \quad y' = -2x^2 y - y^3. \tag{1}$$

Problem 3

Problem 3

Problem 3

Problem 4

(a) Give an example of $A \in \mathbb{R}^{2\times 2}$ such that the origin is a center for the linear system $x' = Ax$. For your example, show that arbitrarily small perturbations to the entries of A can change the origin to a stable or unstable critical point.

(b) Give the form of a particular solution of the ODE $y'''' + 2y'' + y = \sin t + \cos t$ as a linear combination of functions of t but do not evaluate the constants.

Problem 4

Problem 4

Problem 4

Problem 5

(a) Find the general form of the solution to the PDE

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0 \tag{2}$$

for $u(x,t)$ with boundary conditions

$$u(0,t) = 0 \ , \ u(1,t) = 0$$

and general initial conditions.

(b) For any of the solutions in part (a), let $E(t) = \frac{1}{2}\int_0^1 (\partial_t u)^2 + (\partial_x u)^2 dx$. Show that for any such solution with a nonzero initial condition, $dE/dt < 0$ and give a physical interpretation for the two terms in the integrand of E in terms of an elastic string.

Problem 5

Problem 5

Problem 5