AIM Qualifying Review Exam in Differential Equations & Linear Algebra

August 2024

There are five (5) problems in this examination.

There should be sufficient room in this booklet for all your work. But if you use other sheets of paper, be sure to mark them clearly and staple them to the booklet. No credit will be given for answers without supporting work and/or reasoning.

Problem 1

(a) Let $\mathbf{A} = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$. Prove or disprove: $\mathbf{x}^T\mathbf{A}\mathbf{x} > 0$ for all $\mathbf{x} \in \mathbb{R}^2$ such that $\mathbf{x} \neq 0$.

(b) Find the determinant of the matrix $\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 6 & 6 & 1 \\ -1 & 0 & 0 & 3 \\ 0 & 2 & 0 & 7 \end{bmatrix}$.

(c) Show that the nonzero singular values of any matrix and its transpose are the same.

Problem 1

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Problem 2

(a) Suppose the only eigenvectors of $\mathbf{A}$ are multiples of $\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

(i) Could $\mathbf{A}$ be invertible?

(ii) Does $\mathbf{A}$ have a repeated eigenvalue?

(iii) Is $\mathbf{A}$ diagonalizable?

(b) Write the most general matrix that has eigenvectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

(c) Prove or disprove: there exists a matrix $\mathbf{B}$ such that $\mathbf{B}\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ has a solution and $\mathbf{B}^T \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

Problem 2

Problem 2

Problem 2

Problem 3

(a) Show that the origin is a stable fixed point for the system

$$x' = -x^3 + 2y^3 \ ; \ y' = -2xy^2. \tag{1}$$

Hint: consider $V(x, y) = ax^2 + cy^2$.

(b) Find the general solution of $y'''' + 2y'' + y = 0$.

(c) Find the general solution of $\mathbf{x}' = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \mathbf{x}$.

Problem 3

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Problem 4

(a) Find the general solution of $x^2y'' + xy' - \frac{1}{4}y = 0$

(b) Find the first two terms in a power series solution of $x^2y'' + xy' + \left(x - \frac{1}{4}\right)y = 0$ about $x = 0$ that is bounded at $x = 0$ and nonzero for some $x \neq 0$.

(c) Find the exact general solution to $y' + \dfrac{2}{t}y = \dfrac{\cos t}{t^2}$ in terms of elementary functions, not power series.

Problem 4

Problem 4

Problem 4

Problem 5

Solve the PDE

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

for $u(x, y)$ in the rectangle $\{0 < x < 1\ ;\ 0 < y < 2\}$ with the boundary conditions:

$$u(x, 0) = 0\ ,\ u(x, 2) = 0$$
$$u(0, y) = 0\ ,\ u(1, y) = \sin(2\pi y).$$

Problem 5

Problem 5

Problem 5