AIM Qualifying Review Exam in Differential Equations & Linear Algebra

August 2024

There are five (5) problems in this examination.

There should be sufficient room in this booklet for all your work. But if you use other sheets of paper, be sure to mark them clearly and staple them to the booklet. No credit will be given for answers without supporting work and/or reasoning.

Problem 1

(a) Let $\mathbf{A} = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$. Prove or disprove: $\mathbf{x}^T\mathbf{A}\mathbf{x} > 0$ for all $\mathbf{x} \in \mathbb{R}^2$ such that $\mathbf{x} \neq 0$.

(b) Find the determinant of the matrix $\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 6 & 6 & 1 \\ -1 & 0 & 0 & 3 \\ 0 & 2 & 0 & 7 \end{bmatrix}$.

(c) Show that the nonzero singular values of any matrix and its transpose are the same.

Solution

(a) True. Sketch of proof: compute the eigenvalues, find they are both positive, and note that $\mathbf{x}^T\mathbf{A}\mathbf{x}$ can be written as a sum of squares times the eigenvalues.

(b) The matrix can be reduced to upper triangular by adding multiples of rows to other rows, which does not change the determinant, and by switching rows, which multiplies the determinant by -1. One obtains 36 for the answer.

(c) If $A = U\Sigma V^T$ is an SVD for A then $A^T = V\Sigma^T U^T$ is an SVD for A^T. Σ and Σ^T have the same diagonal entries, so the singular values are the same (except possibly for additional zeros if one considers the full SVD vs the reduced SVD).

Problem 2

(a) Suppose the only eigenvectors of $\mathbf{A}$ are multiples of $\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

  (i) Could $\mathbf{A}$ be invertible?
  (ii) Does $\mathbf{A}$ have a repeated eigenvalue?
  (iii) Is $\mathbf{A}$ diagonalizable?

(b) Write the most general matrix that has eigenvectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

(c) Prove or disprove: there exists a matrix $\mathbf{B}$ such that $\mathbf{Bx} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ has a solution and $\mathbf{B}^T \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

Solution

(a) Such an A is similar to a 3-by-3 Jordan block, of form $\begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}$. If $\lambda \neq 0$ then det A $\neq 0$ and thus A is invertible. Yes, A has a repeated eigenvalue, λ with multiplicity 3. No, A is not diagonalizable. If it were, A would have a basis of three eigenvectors.

(b) Write a 2-by-2 matrix with general entries that satisfies the eigenvalue equation with each of the given eigenvectors and arbitrary eigenvalues. By performing algebra on these equations, one finds that it is of the form $\begin{bmatrix} a & b \\ b & a \end{bmatrix}$.

(c) False. Proof: $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ lies in the column space of B, but $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is perpendicular to the column space of B. Since these vectors are not orthogonal, we have a contradiction.

Problem 3

(a) Show that the origin is a stable fixed point for the system

$$x' = -x^3 + 2y^3 \; ; \; y' = -2xy^2. \tag{1}$$

Hint: consider $V(x, y) = ax^2 + cy^2$.

(b) Find the general solution of $y'''' + 2y'' + y = 0$.

(c) Find the general solution of $\mathbf{x}' = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \mathbf{x}$.

Solution

(a) Compute dV/dt. By choosing $a = c > 0$, we have $V' \leq 0$. Since V is positive definite and dV/dt is negative semidefinite on any neighborhood of the origin, the origin is a stable fixed point.

(b) Substitute $y = e^{rt}$ and obtain $r = \pm 1$ with multiplicity 2. The general solution is thus a linear combination of $\{e^{\pm it}, te^{\pm it}\}$ or $\{\sin t, \cos t, t \sin t, t \cos t\}$.

(c) The eigenvectors are (1,1) and (1,-1) with eigenvalues 2 and 0 respectively, so the general solution is $c_1(1,1)e^{2t} + c_2(1,-1)$.

Problem 4

(a) Find the general solution of $x^2y'' + xy' - \frac{1}{4}y = 0$

(b) Find the first two terms in a power series solution of $x^2y'' + xy' + \left(x - \frac{1}{4}\right)y = 0$ about $x = 0$ that is bounded at $x = 0$ and nonzero for some $x \neq 0$.

(c) Find the exact general solution to $y' + \dfrac{2}{t}y = \dfrac{\cos t}{t^2}$ in terms of elementary functions, not power series.

Solution

(a) Plug in x^r and find $r = \pm 1/2$. The general solution is a linear combination of these two solutions.

(b) The bounded solution is proportional to $y = x^{1/2} + a_1x^{3/2} + \ldots$. Plugging in and equating like terms, we obtain $a_1 = -1/2$.

(c) Multiply both sides by the integrating factor $e^{\int 2dt/t} = t^2$ and integrate. Obtain $y = \dfrac{\sin t}{t^2} + \dfrac{c}{t^2}$.

Problem 5

Solve the PDE

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

for $u(x,y)$ in the rectangle $\{0 < x < 1\ ;\ 0 < y < 2\}$ with the boundary conditions:

$$u(x,0) = 0\ ,\ u(x,2) = 0$$
$$u(0,y) = 0\ ,\ u(1,y) = \sin(2\pi y).$$

Solution

First, we write $u = X(x)Y(y)$ and obtain

$$\frac{X''}{X} = -\frac{Y''}{Y} = \lambda^2.$$

The separation constant λ^2 has been chosen to be positive so that there are nontrivial solutions that satisfy the boundary conditions $Y(0) = Y(2) = 0$. Such is the case for $\lambda = n\pi/2$ for integers n, in which case $Y = A\sin(n\pi y/2)$ and $X = B\cosh(n\pi x/2) + C\sinh(n\pi x/2)$. In order to have $u(0,y) = 0$ we need $X(0) = 0$, so $B = 0$. The general solution is

$$u = \sum_{n=1}^{\infty} a_n \sinh(n\pi x/2)\sin(n\pi y/2)$$

We can determine the a_n by matching the nonhomogeneous boundary condition. We find $a_4 = 1/\sinh(2\pi)$ and all other a_n are zero. The solution is

$$u = \frac{1}{\sinh(2\pi)}\sinh(2\pi x)\sin(2\pi y).$$