AIM Qualifying Review Exam in Differential Equations & Linear Algebra

August 2023

There are five (5) problems in this examination.

There should be sufficient room in this booklet for all your work. But if you use other sheets of paper, be sure to mark them clearly and staple them to the booklet. No credit will be given for answers without supporting work and/or reasoning.

Problem 1

Consider the set of real 2-by-2 matrices $\mathbf{A}$ such that $\mathbf{A} = \mathbf{A}^T$.

(a) Let $\mathbf{x}$ and $\mathbf{y}$ be independent eigenvectors of $\mathbf{A}$, i.e. $\mathbf{x} \neq \alpha\mathbf{y}$ for any scalar α. For which $\mathbf{A}$ in the set mentioned above is $\mathbf{x}^T\mathbf{y}$ always zero, and for which $\mathbf{A}$ in the set could $\mathbf{x}^T\mathbf{y}$ be nonzero? Justify your answer and give an example of a matrix for each of the two cases.

(b) Show that any $\mathbf{A}$ in the set can be written $\mathbf{x}_1\mathbf{x}_1^T + \mathbf{x}_2\mathbf{x}_2^T$ for some $\mathbf{x}_1$ and $\mathbf{x}_2 \in \mathbb{C}^2$.

Solution

(a) Let $\lambda_{\mathbf{x}}$ and $\lambda_{\mathbf{y}}$ be the eigenvalues for $\mathbf{x}$ and $\mathbf{y}$ respectively. Then $\lambda_{\mathbf{y}}\mathbf{x}^T\mathbf{y} = \mathbf{x}^T\mathbf{A}\mathbf{y} = \mathbf{x}^T\mathbf{A}^T\mathbf{y} = (\mathbf{A}\mathbf{x})^T\mathbf{y}$ $= \lambda_{\mathbf{x}}\mathbf{x}^T\mathbf{y}$, so $(\lambda_{\mathbf{x}} - \lambda_{\mathbf{y}})\mathbf{x}^T\mathbf{y} = 0$. Thus $\mathbf{x}^T\mathbf{y}$ is always zero for those $\mathbf{A}$ that have distinct eigenvalues (i.e. without repetition), e.g. $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$. Whereas $\mathbf{x}^T\mathbf{y}$ could be nonzero for $\mathbf{A}$ with repeated eigenvalues, such as the identity matrix.

(b) Such an $\mathbf{A}$ has an orthogonal eigendecomposition, $\mathbf{A} = \mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^T$ with a diagonal matrix $\mathbf{\Lambda}$. Therefore

$$\mathbf{A} = \begin{bmatrix} \mathbf{q}_1 & \mathbf{q}_2 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} \mathbf{q}_1^T \\ \mathbf{q}_2^T \end{bmatrix} = \begin{bmatrix} \lambda_1\mathbf{q}_1 & \lambda_2\mathbf{q}_2 \end{bmatrix} \begin{bmatrix} \mathbf{q}_1^T \\ \mathbf{q}_2^T \end{bmatrix}$$

This last product can be done by the rule that each column of the product is the first matrix times the corresponding column of the second matrix. Thus,

$$\begin{aligned} \mathbf{A} &= \begin{bmatrix} q_{11}\lambda_1\mathbf{q}_1 + q_{21}\lambda_2\mathbf{q}_2 & q_{12}\lambda_1\mathbf{q}_1 + q_{22}\lambda_2\mathbf{q}_2 \end{bmatrix} = \begin{bmatrix} q_{11}\lambda_1\mathbf{q}_1 & q_{12}\lambda_1\mathbf{q}_1 \end{bmatrix} + \begin{bmatrix} q_{21}\lambda_2\mathbf{q}_2 & q_{22}\lambda_2\mathbf{q}_2 \end{bmatrix} \\ &= \lambda_1\mathbf{q}_1\mathbf{q}_1^T + \lambda_2\mathbf{q}_2\mathbf{q}_2^T. \end{aligned}$$

Now let $\mathbf{x}_1 = \sqrt{\lambda_1}\mathbf{q}_1$ and $\mathbf{x}_2 = \sqrt{\lambda_2}\mathbf{q}_2$. $\mathbf{x}_1$ and $\mathbf{x}_2$ may be complex. One can also find them by direct computation, but the steps are somewhat complicated in general.

Problem 2

Find a basis for each of these subspaces of $\mathbb{R}^4$ (or $\mathbb{R}^2$ and $\mathbb{R}^5$ in part d). Justify your answers.

(a) All vectors whose components are equal.

(b) All vectors whose components add to zero.

(c) All vectors that are perpendicular to $(1,1,0,0)^T$ and $(1,0,1,1)^T$.

(d) The column space (in $\mathbb{R}^2$) and the null space (in $\mathbb{R}^5$) of $\mathbf{U} = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$.

Solution

(a) All such vectors are proportional to $(1,1,1,1)^T$, so this is a basis.

(b) The subspace is the space orthogonal to $(1,1,1,1)^T$, and it has dimension 3. Any linearly independent set of 3 vectors in the subspace will do. One choice is: $(1,-1,0,0)^T$, $(1,0,-1,0)^T$, and $(1,0,0,-1)^T$. Any nontrivial linear combination of these will have at least one of the second, third, or fourth components nonzero.

(c) This is a subspace of dimension 2. One vector in the subspace is $(0,0,1,-1)^T$. To get another independent vector, one could perform the Gram-Schmidt process starting with simple random vectors until the result is nonzero. Or a faster approach: let the third and fourth components be 1; this is automatically orthogonal to $(0,0,1,-1)^T$. Now let the first component be -2 so the vector is orthogonal to $(1,0,1,1)^T$. Now let the second component be 2 so it's also orthogonal to $(1,1,0,0)^T$. The second basis vector is thus $(-2,2,1,1)^T$.

(d) The column space is all of $\mathbb{R}^2$, so take the standard basis $(1,0)^T$ and $(0,1)^T$ for example. The null space is orthogonal to $(1,0,1,0,1)^T$ and $(0,1,0,1,0)^T$, and has dimension 3. It consists of vectors whose 1st, 3rd and 5th components sum to zero and second and fourth components also sum to zero, and is thus a direct sum of a two-dimensional and a one-dimensional space. For a basis take $(0,1,0,-1,0)^T$ (for the one-dimensional space) and $(1,0,-1,0,0)^T$ and $(0,0,1,0,-1)^T$ (for the two-dimensional space).

Problem 3

Consider the system of ODEs

$$dx/dt = (1+x)\sin y \ \ ; \ \ dy/dt = 1 - x - \cos y. \tag{1}$$

(a) Determine all critical points.

(b) Find the corresponding linear system near each critical point.

(c) Find the eigenvalues of each linear system. What conclusions can you then draw about the nonlinear system?

Solution

(a) The critical points are $(x,y) = (0, 2n\pi)$ or $(2, (2n+1)\pi)$, where n ranges over the integers.

(b) The Jacobian matrix $J(x,y) = \begin{bmatrix} \sin y & (1+x)\cos y \\ -1 & \sin y \end{bmatrix}$, so we linear systems with matrices given by $J(0, 2n\pi) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $J(2, (2n+1)\pi) = \begin{bmatrix} 0 & -3 \\ -1 & 0 \end{bmatrix}$.

(c) The first case of J is a center with eigenvalues $\pm i$ and the second has eigenvalues that are roots of $\lambda^2 - 3 = 0$, i.e. $\pm\sqrt{3}$, so it is a saddle. The stability of the nonlinear system is undetermined by the linear system in the first case (the center), and is unstable in the second (the saddle).

Problem 4

Consider the initial value problem

$$y^{(4)} + 2y'' + y = g(t),\ y(0) = y'(0) = 0,\ y''(0) = y'''(0) = 0.$$

(a) What is the most general class of functions $g(t)$ that guarantees the solution exists for all real t?

(b) Solve the initial value problem in the special case $g(t) = 3t + 4$.

Solution

(a) If we transform the ODE to a first-order system, we can apply the existence and uniqueness theorem for linear systems. The theorem says that a unique solution exists for all t where the ODE coefficients and g are continuous. The coefficients are constants, so we require that g be continuous for all t.

(b) First we find the homogeneous solution y_h, using the roots of the characteristic equation $r^4+2r^2+1 = 0$: $r = \pm i$ with multiplicity 2. So $y_h = A\sin t + B\cos t + Ct\sin t + Dt\cos t$. For the particular solution y_p, we use a polynomial of the same degree as the right hand side. We don't need to add powers of t since there is not overlap with the terms of y_h. We get $y_p = 3t + 4$. So the general solution is $y = A\sin t + B\cos t + Ct\sin t + Dt\cos t + 3t + 4$. To satisfy the initial conditions, one can compute derivatives rapidly using two-term Taylor expansions of sine and cosine at 0. We get $A = -4$, $B = -4$, $C = -3/2$, and $D = 1$.

Problem 5

Solve the PDE

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0$$

for $u(x,t)$ in the domain $\{t > 0\ ;\ 0 < x < 1\}$ with the boundary conditions and initial conditions:

$$u(0,t) = 0\ ,\ u(1,t) = 1$$
$$u(x,0) = x + \sin(\pi x)\ ,\ \frac{\partial u}{\partial t}(x,0) = \sin(\pi x).$$

Solution

First, we write $u = x + u_h(x,t)$, so u_h satisfies the same equation with homogeneous boundary conditions and one initial condition is modified to $u_h(x,0) = \sin(\pi x)$. Next, we plug in a separation of variables solution $u_h = X(x)T(t)$ and obtain

$$\frac{X''}{X} = \frac{T''}{T} = -\lambda^2.$$

The separation constant $-\lambda^2$ has been chosen to be negative so that there are nontrivial solutions that satisfy the boundary conditions $X(0) = X(1) = 0$. Such is the case for $\lambda = n\pi$ for integers n, in which case $X = A\sin(n\pi x)$ and $T = B\cos(n\pi t) + C\sin(n\pi t)$. The general solution is

$$u_h = \sum_{n=1}^{\infty} (a_n\cos(n\pi t) + b_n\sin(n\pi t))\sin(n\pi x)$$

We can determine the constants by matching the initial conditions. We find $a_1 = 1$, $b_1 = 1/\pi$, and all other constants are zero. The solution is

$$u = x + \left(\cos(\pi t) + \frac{1}{\pi} \sin(\pi t) \right) \sin(\pi x).$$