AIM Qualifying Review Exam in Differential Equations & Linear Algebra

January 2025

There are five (5) problems in this examination.

There should be sufficient room in this booklet for all your work. But if you use other sheets of paper, be sure to mark them clearly and staple them to the booklet. No credit will be given for answers without supporting work and/or reasoning.

Problem 1

(a) The vectors q_1, q_2, and q_3 are mutually orthogonal. What linear combination of q_1 and q_2 is closest to q_3? Prove your answer.

(b) Let $q_1, \ldots, q_n \in \mathbb{R}^n$ be an orthonormal set (mutually perpendicular, each with norm 1). Prove that $A = q_1 q_1^T + \ldots + q_n q_n^T$ is the n-by-n identity matrix.

(c) Find an orthonormal basis for the subspace spanned by $v_1 = (1, -1, 0)$, $v_2 = (0, 1, -1)$, and $v_3 = (1, 0, -1)$.

Problem 1

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Problem 2

(a) Let the eigenvalues of A be 2, 2, 5. For each of (i), (ii), and (iii), answer true or false and prove your answer:

(i) A must be invertible.

(ii) A must be diagonalizable.

(iii) A cannot be diagonalizable.

(b) If the vectors x_1 and x_2 are the first and second columns of S, what are the eigenvalues and eigenvectors of $A = S \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} S^{-1}$ and $B = S \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix} S^{-1}$?

(c) Suppose the first row of A is $[7, 6]$ and its eigenvalues are i and $-i$. Find A.

Problem 2

Problem 2

Problem 2

Problem 3

(a) Solve the initial value problem

$$y' + 3y = e^{-2t}\ ;\ y(0) = 0 \tag{1}$$

without using Laplace Transforms.

(b) Solve the problem in part (a) using the Laplace Transform method. Recall: the Laplace Transform is defined as

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)dt. \tag{2}$$

(c) Solve the differential equation

$$\frac{x}{(x^2+y^2)^{3/2}} + \frac{y}{(x^2+y^2)^{3/2}}\frac{dy}{dx} = 0 \tag{3}$$

Problem 3

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Problem 3

Problem 4

Consider the system of equations

$$x' = 1 - xy \; ; \; y' = x - y^3. \tag{4}$$

(a) Find all the critical points of the system.

(b) For each critical point, classify its type and stability.

(c) Sketch the phase portrait in the neighborhood of each critical point.

Problem 4

Problem 4

Problem 4

Problem 5

(a) Consider the PDE

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0 \tag{5}$$

for $u(x,t)$ with boundary conditions

$$u(0,t) = 0 \ , \ u(1,t) = 0$$

and general initial conditions. Use separation of variables to show that the general solution can be written

$$u(x,t) = F(x+ct) + G(x-ct). \tag{6}$$

Useful identities:

$$\sin A \sin B = \frac{1}{2}\left(\cos(A-B) - \cos(A+B)\right) \ ; \ \sin A \cos B = \frac{1}{2}\left(\sin(A-B) + \sin(A+B)\right)$$

(b) It turns out that solutions to (5) can also have the form in (6) when the domain is unbounded: $\{-\infty < x < \infty \ ; \ t > 0\}$. In this case, solve equation (5) with the initial conditions:

$$u(x,0) = 0 \ , \ \frac{\partial u}{\partial t}(x,0) = 2xe^{-x^2}, \quad -\infty < x < \infty.$$

Problem 5

Problem 5

Problem 5