AIM Qualifying Review Exam in Differential Equations & Linear Algebra

January 2025

There are five (5) problems in this examination.

There should be sufficient room in this booklet for all your work. But if you use other sheets of paper, be sure to mark them clearly and staple them to the booklet. No credit will be given for answers without supporting work and/or reasoning.

Problem 1

(a) The vectors q_1, q_2, and q_3 are mutually orthogonal. What linear combination of q_1 and q_2 is closest to q_3? Prove your answer.

(b) Let $q_1, \ldots, q_n \in \mathbb{R}^n$ be an orthonormal set (mutually perpendicular, each with norm 1). Prove that $A = q_1 q_1^T + \ldots + q_n q_n^T$ is the n-by-n identity matrix.

(c) Find an orthonormal basis for the subspace spanned by $v_1 = (1, -1, 0)$, $v_2 = (0, 1, -1)$, and $v_3 = (1, 0, -1)$.

Solution outline

(a) The zero vector. $\|a_1 q_1 + a_2 q_2 - q_3\|^2 = \|a_1 q_1 + a_2 q_2\|^2 + \|q_3\|^2$ is minimum with $a_1 = a_2 = 0$.

(b) $(q_1 q_1^T + \ldots + q_n q_n^T) q_j = q_j$ for each $j = 1$ to n, since all products except the jth are zero. Thus $A q_j = q_j = I q_j$ for all $j = 1$ to n, and also for all linear combinations (all of $\mathbb{R}^n$).

(c) Apply the Gram-Schmidt process and obtain $b_1 = (1/\sqrt{2}, -1/\sqrt{2}, 0)$, $b_2 = (1/\sqrt{6}, 1/\sqrt{6}, -2/\sqrt{6})$ as the first two basis vectors. The third vector from the process is zero, so $\{b_1, b_2\}$ is the basis.

Problem 2

(a) True or false the following: If the eigenvalues of A are 2, 2, 5, then the matrix is certainly

(i) invertible.

(ii) diagonalizable.

(iii) not diagonalizable.

Prove your answer to each of (i), (ii) and (iii).

(b) If the vectors x_1 and x_2 are the first and second columns of S, what are the eigenvalues and eigenvectors of $A = S\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}S^{-1}$ and $B = S\begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}S^{-1}$?

(c) Suppose the first row of A is $[7, 6]$ and its eigenvalues are i and $-i$. Find A.

Solution outline

(a) (i) True. A is invertible iff the determinant (the product of the eigenvalues) is nonzero.

(ii) False. Eigenvalue 2 could have geometric multiplicity 1, strictly less than the algebraic multiplicity, giving a Jordan form with a 1 above the main diagonal.

(iii) False. Eigenvalue 2 could have geometric multiplicity 2, so A would be similar to the diagonal matrix with 2, 2, and 5 on the main diagonal.

(b) If v is an eigenvector of C with eigenvalue λ, then Sv is an eigenvector of SCS^{-1} with the same eigenvalue. Hence the eigenvectors of A are $Se_1 = x_1$ and $Se_2 = x_2$ with eigenvalues 2 and 1 respectively. The eigenvectors of $\begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$ are $[1, 0]^T$ and $[3, -1]^T$ with eigenvalues 2 and 1 respectively. Hence the eigenvectors of B are $Se_1 = x_1$ and $S[3, -1]^T = 3x_1 - x_2$ with eigenvalues 2 and 1 respectively.

(c) For a 2-by-2 matrix, the trace is the sum of the eigenvalues and the determinant is their product. The trace is 0, so the lower right entry of A is -7. Since the determinant is 1, the lower left entry is -25/3.

Problem 3

(a) Solve the initial value problem

$$y' + 3y = e^{-2t} \; ; \; y(0) = 0 \tag{1}$$

without using Laplace Transforms.

(b) Solve the problem in part (a) using the Laplace Transform method. Recall: the Laplace Transform is defined as

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t) dt. \tag{2}$$

(c) Solve the differential equation

$$\frac{x}{(x^2+y^2)^{3/2}} + \frac{y}{(x^2+y^2)^{3/2}} \frac{dy}{dx} = 0 \tag{3}$$

Solution outline

(a) Multiply both sides by the integrating factor e^{3t} and get $(ye^{3t})' = e^t$. Integrate and apply the initial condition to get $y = e^{-2t} - e^{-3t}$.

(b) Get $(s+3)Y(s) = 1/(s+2)$. Then get $Y(s) = 1/(s+2) - 1/(s+3)$. An inverse Laplace transform gives the answer in part (a).

(c) This is a first order equation that is neither linear nor separable, so we check if it's exact. We seek ψ such that $\partial_x \psi = \frac{x}{(x^2+y^2)^{3/2}}$ and $\partial_y \psi = \frac{y}{(x^2+y^2)^{3/2}}$. We can integrate either of these equations to obtain $\psi = -(x^2+y^2)^{-1/2}$, and the differential equation tells us $\psi = -(x^2+y^2)^{-1/2} = \text{const.}$ is a solution.

Problem 4

Consider the system of equations

$$x' = 1 - xy \; ; \; y' = x - y^3. \tag{4}$$

(a) Find all the critical points of the system.

(b) For each critical point, classify its type and stability.

(c) Sketch the phase portrait in the neighborhood of each critical point.

Solution outline

(a) (-1, -1) and (1, 1).

(b) For (-1, -1) the linearized system is $u' = u + v, v' = u - 3v$; the eigenvalues are $-1 \pm \sqrt{5}$, so it's an unstable saddle point. For (1, 1) the linearized system is $u' = -u - v, v' = u - 3v$; the eigenvalues are -2 with multiplicity 2, so it's a stable sink, possibly degenerate.

(c) For (-1, -1), the trajectories move outward along $[-1, 2-\sqrt{5}]$, the eigenvector for the eigenvalue $-1+\sqrt{5}$, and inward along $[2-\sqrt{5}, 1]$, the eigenvector for the eigenvalue $-1-\sqrt{5}$. For (1, 1), there is a single eigenvector $[1, 1]$, so it's a degenerate sink. All trajectories approach the critical point along the eigenvector direction. If they start off of this line, they adopt an S-shape centered on the critical point, e.g. fig. 7.8.2a in Boyce and DiPrima.

Problem 5

(a) Consider the PDE

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0 \tag{5}$$

for $u(x,t)$ with boundary conditions

$$u(0,t) = 0 \; , \; u(1,t) = 0$$

and general initial conditions. Use separation of variables to show that the general solution can be written

$$u(x,t) = F(x+ct) + G(x-ct). \tag{6}$$

Useful identities:

$$\sin A \sin B = \frac{1}{2}\left(\cos(A-B) - \cos(A+B)\right) \; ; \; \sin A \cos B = \frac{1}{2}\left(\sin(A-B) + \sin(A+B)\right)$$

(b) It turns out that the solution (6) can also hold for $u(x,t)$ in the domain $\{-\infty < x < \infty \; ; \; t > 0\}$. Solve equation (5) with the initial conditions:

$$u(x,0) = 0 \; , \; \frac{\partial u}{\partial t}(x,0) = 2xe^{-x^2}, \quad -\infty < x < \infty.$$

Solution outline

(a) First, we write $u = X(x)T(t)$ and obtain

$$\frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T} = -\lambda^2.$$

The separation constant λ^2 has been chosen to be positive so that there are nontrivial solutions that satisfy the boundary conditions $X(0) = X(1) = 0$. Such is the case for $\lambda = n\pi$ for integers n, in which case $X = A\sin(n\pi x)$ and $T = B\sin(n\pi ct) + C\cos(n\pi ct)$. The general solution is

$$u = \sum_{n=1}^{\infty} a_n \sin(n\pi x)\sin(n\pi ct) + b_n \sin(n\pi x)\cos(n\pi ct).$$

The identities $\sin A \sin B = \frac{1}{2}\left(\cos(A-B) - \cos(A+B)\right)$ and $\sin A \cos B = \frac{1}{2}\left(\sin(A-B) + \sin(A+B)\right)$ show that all the terms in the sum are functions of $x - ct$ and $x + ct$.

(b) The first initial condition implies $F(x) + G(x) = 0$. The second is $cF'(x) - cG'(x) = 2cF'(x) = 2xe^{-x^2}$. Thus

$$2cF(x) = -e^{-x^2} + D.$$

and the solution is

$$u(x,t) = -\frac{1}{2c}\left(e^{-(x+ct)^2} - e^{-(x-ct)^2}\right).$$