Department of Mathematics, University of Michigan
Complex Analysis Qualifying Exam
August 15, 2023; Morning Session

Problem 1: Let f be an analytic function in the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ such that $f(0) = 0$ and $|f(z)| < 2023$ for all $z \in \mathbb{D}$. Assume also that f satisfies the property $f(iz) = f(z)$ for all $z \in \mathbb{D}$. Prove that $|f(\frac{1}{7})| < 1$.

Problem 2: Let $\mathbb{H} = \{z \in \mathbb{C} : \Im z > 0\}$ be the upper half-plane. Find a conformal mapping from the domain

$$\mathbb{H} \setminus \{\, z \in \mathbb{H} : \ z = e^{i\theta},\ \theta \in (0, \tfrac{\pi}{2}]\,\}$$

(i.e., $\mathbb{H}$ slit along a circular arc) back onto $\mathbb{H}$. You may write your solution as a composition of simpler maps.

Problem 3: Use contour integration to evaluate the integral

$$\int_{-1}^{1} \sqrt{\frac{1+x}{1-x}} \cdot \frac{dx}{1+x^2}\,.$$

[*Simplification:* If you experience difficulties, you can first change the variable of integration to $t = (1+x)/(1-x)$ and use contour integration for the new integral.]

Problem 4: Let $\alpha \in \mathbb{C}$ satisfy $|\alpha| = 1$. Consider the equation $\sin z = \dfrac{\alpha}{z^2}$ for $z \in \mathbb{C}$.

(a) Prove that for each $k \in \mathbb{Z} \setminus \{0\}$ this equation has exactly one solution inside the vertical strip $|\Re z - \pi k| < \frac{\pi}{2}$.

(b) How many solutions (counted with multiplicities) does this equation have inside the vertical strip $|\Re z| < \frac{\pi}{2}$?

Problem 5: Let $a_k \in \mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ for all $k \in \mathbb{N}$. Consider functions

$$B_n(z) := \prod_{k=1}^{n} \frac{z - a_k}{1 - \overline{a}_k z}, \quad z \in \mathbb{D}.$$

(a) Prove that the sequence $\{B_n\}_{n=1}^{\infty}$ contains a subsequence that converges uniformly on compact subsets of the unit disc $\mathbb{D}$.

(b) Assume that $\limsup_{n\to\infty}(1 - |a_n|) > 0$. Prove that each subsequential limit of the functions B_n is identically zero in $\mathbb{D}$.

(c) Prove that the same holds if $\sum_{n=1}^{\infty}(1 - |a_n|) = +\infty$.