Algebra II QR
August 2024

Problem 1. Let G be a finite group and Z its center.

(a) Prove that if G is nonabelian, then $|G/Z|$ is a composite number.
(b) Assume that G is a p-group for a prime p. Prove that Z is nontrivial.
(c) Give an example of a nonabelian finite p-group.

Problem 2.

(a) Prove that A_5 is the unique index-2 subgroup of S_5. (You may use without proof that A_5 is simple.)
(b) Show that the action of $\mathrm{SL}_2(\mathbb{F}_4)$ on the set of 5 lines through the origin in $\mathbb{F}_4^2$ is faithful. Deduce that $\mathrm{SL}_2(\mathbb{F}_4) \cong A_5$.

Problem 3. Let f be an irreducible separable polynomial of degree 5 in $\mathbb{Q}[X]$ and let F be the splitting field of f. If f has exactly two nonreal roots, show that $\mathrm{Gal}(F/\mathbb{Q}) \cong S_5$.

Problem 4. Let $F = \mathbb{Q}(\sqrt{2}, \sqrt{3})$ and $E = F(\alpha)$ for $\alpha = \sqrt{(\sqrt{2}+2)(\sqrt{3}+3)}$. You may use without proof that $[F : \mathbb{Q}] = 4$.

(a) Prove that $[E : F] = 2$. (*Hint:* Calculate $\sigma(\alpha^2)/\alpha^2$ for some $\alpha \in \mathrm{Aut}(F/\mathbb{Q})$.) Prove that $E/\mathbb{Q}$ is a degree-8 extension.
(b) Prove that E is Galois over $\mathbb{Q}$ and $\mathrm{Gal}(E/\mathbb{Q})$ has two non-commuting elements of order 4.

Problem 5. Let p be a prime such that $p \equiv 3 \pmod 4$ so that we may write $\mathbb{F}_{p^2} = \mathbb{F}_p[\delta]$ where $\delta^2 = -1$. Set $G = \mathrm{GL}_2(\mathbb{F}_p)$.

(a) Prove that
$$\varphi \colon \mathbb{F}_{p^2}^\times \to G, \qquad a + b\delta \mapsto \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$
defines a homomorphism of groups.
(b) Let ℓ be an odd prime such that $p \equiv -1 \pmod{\ell}$. Give an example of an ℓ-Sylow subgroup of G and compute how many ℓ-Sylow subgroups G has. You may use without proof the fact that the multiplicative group of a finite field is cyclic.